# ElEMENTARY AND INTERMEDIATE AlGEBRA 

Charles P. McKeague EIGHTH EDITION

A COMBINED COURSE


## How to Be Successful in a Math Course

Being successful in algebra depends most on having good study skills. We list study skills and other good habits to develop to be successful in this course at the beginning of the first few chapters. Here's a recap.

## Make a Good Start (Chapter 1)

## Put Yourself on a Schedule

Make a schedule for yourself in which you set aside 2 hours each day to work on algebra.

## Find Your Mistakes and Correct Them

Always check your answers against the answers in the back of the book.

## Gather Information about Available Resources and Help

You should anticipate that you will need extra help sometime during the course, so gather information about the resources (tutors, study groups, online practice) available to you.

## Organize Your Work-And You Won’t Get Nervous! (Chapter 2)

## Imitate Success

Your work should look like the work you see in this book and the work your instructor shows.

## List Difficult Problems

Begin to make lists of problems that give you the most difficulty.

## Begin to Develop Confidence with Word Problems

Instead of telling yourself in advance that you can't do word problems, decide to do whatever it takes to master them.

## Be Attentive (Chapter 3)

## Review with the Exam in Mind

Each day you should review material that will be covered on the next exam.

## Pay Attention to Instructions

Train yourself to pay attention to the instructions that accompany a problem as you work through your homework problems.

## Focus on Being a Success in the Course (Chapter 4)

## Be Focused, Not Distracted

Don't dwell on questions and worries about the class that can become excuses for not doing well.

## Be Resilient

Don't let setbacks keep you from your goals.

## Intend to Succeed

Set out to master the material, no matter what it takes.

## Develop Good Habits (Chapter 5)

## Continue to Set and Keep a Schedule

Even if you are doing well in the course, you should not slacken your efforts.

## Increase Effectiveness

Spend time on study activities that produce the best results for you, so that you can use success to build success.

## Continue to List Difficult Problems

Don't let down your guard. List difficult problems so that you can prepare for the next exam effectively.

The Blueprint for Problem Solving is a general outline that you can use to help you solve application problems.

## BLUEPRINT FOR <br> PROBLEM SOLVING

STEP 1: Read the problem, and then mentally list the items that are known and the items that are unknown.

STEP 2: Assign a variable to one of the unknown items. (In most cases this will amount to letting $x=$ the item that is asked for in the problem.) Then translate the other information in the problem to expressions involving the variable.

STEP 3: Reread the problem, and then write an equation, using the items and variables listed in steps 1 and 2, that describes the situation.

STEP 4: Solve the equation found in step 3.
STEP 5: Write your answer using a complete sentence.
STEP 6: Reread the problem, and check your solution with the original words in the problem.

Here is a Blueprint for Problem Solving that you can use specifically for problems about systems of equations (beginning in Chapter 3).

## BLUEPRINT FOR PROBLEM SOLVING USING A SYSTEM OF EQUATIONS

STEP 1: Read the problem, and then mentally list the items that are known and the items that are unknown.
STEP 2: Assign variables to each of the unknown items; that is, let $x=$ one of the unknown items and $y=$ the other unknown item. Then translate the other information in the problem to expressions involving the two variables.

STEP 3: Reread the problem, and then write a system of equations, using the items and variables listed in steps 1 and 2, that describes the situation.
STEP 4: Solve the system found in step 3.
STEP 5: Write your answers using complete sentences.
STEP 6: Reread the problem, and check your solution with the original words in the problem.

## Resources

By gathering resources early in the term, before you need help, the information about these resources will be available to you when they are needed.

## INSTRUCTOR

Knowing the contact information for your instructor is very important. You may already have this information from the course syllabus. It is a good idea to write it down again.


## TUTORING CENTER

Many schools offer tutoring, free of charge to their students. If this is the case at your school, find out when and where tutoring is offered.

| Tutoring Location |  | Phone Number |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| Available Hours: M | T | W | TH | ext ___ |

## COMPUTER LAB

Many schools offer a computer lab where students can use the online resources and software available with their textbook. Other students using the same software and websites as you can be very helpful. Find out where the computer lab at your school is located.


## VIDEO LESSONS

A complete set of video lessons is available to your school. These videos feature the author of your textbook presenting full-length, 15 - to 20 -minute lessons from every section of your textbook. If you miss class, or find yourself behind, these lessons will prove very useful.


## CLASSMATES

Form a study group and meet on a regular basis. When you meet try to speak to each other using proper mathematical language. That is, use the words that you see in the definition and property boxes in your textbook.

| Name | Phone | E-mail |
| :--- | :--- | :--- |
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| Name | Phone | E-mail |

## Elementary and Intermediate Algebra A Combined Course

THIRD EDITION

## Charles P. McKeague <br> CUESTA COLLEGE



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## Preface to the Instructor

I have a passion for teaching mathematics. That passion carries through to my textbooks. My goal is a textbook that is user-friendly for both students and instructors. For the instructor, I build features into the text that reinforce the habits and study skills we know will bring success to our students. For students, this combined book forms a bridge between the elementary and intermediate algebra courses. Students benefit, too, from clear, concise writing; continuous review; and foreshadowing of topics to come.

The third edition of Elementary and Intermediate Algebra: A Combined Course builds upon these strengths. In addition, the combined book offers the required course topics in a more efficient physical format. Many instructors like the convenience of having two courses in one book.

In this edition, renewal of the problem sets, along with the continued emphasis on foreshadowing of later topics, a program of continuous cumulative review, and the focus on applications, make this the best edition of Elementary and Intermediate Algebra yet.

## Renewal and Reorganization of Exercise Sets

This edition of Elementary and Intermediate Algebra contains over 2,000 new problems in the reorganized problem sets, roughly 30 percent of the problems. Most of these new problems have been added to enhance the midrange of exercises. Doing so gives our problem sets a better bridge from easy problems to more difficult problems. Many of these midrange problems cover more than one concept or technique in a new and slightly more challenging way. In short, they start students thinking mathematically and working with algebra productively.

This enhanced midrange of exercises also underscores our series' appeal to the middle level of rigor for the course. We think instructors will use many of these midrange problems as classroom examples, so we have labeled some of them as chalkboard problems.

As part of our revamping of the problem sets in this edition, we have also reordered the categories of problems to make a more logical bridge between sections.

## A Better Progression of Categories of Problems

The categories in our problem sets now appear in the following order.
General (Undesignated) Exercises These "starter" exercises normally do not have labels. They involve a certain amount of the drill necessary to master basic techniques. These problems then progress in difficulty so that students can begin to put together more than one concept or idea. It is here that you will find
the foreshadowing problems. Instead of drill for the sake of drill, we have students work the problems that they will need later in the course-hence the description foreshadowing. This category is also where you will find the midrange of problems discussed earlier. As in previous editions, we have kept the oddeven similarity of the problems in this part of the problem set.

Applying the Concepts Students are always curious about how the algebra they are learning can be applied, so we have included applied problems in most of the problem sets in the book and have labeled them to show students the array of uses of mathematics. These applied problems are written in an inviting way, to help students overcome some of the apprehension associated with application problems. We have a number of new applications under the heading Improving Your Quantitative Literacy that are particularly accessible and inviting.

Maintaining Your Skills One of the major themes of our book is continuous review. We strive to continuously hone techniques learned earlier by keeping the important concepts in the forefront of the course. The Maintaining Your Skills problems review material from the previous chapter, or they review problems that form the foundation of the course-the problems that you expect students to be able to solve when they get to the next course.

Getting Ready for the Next Section Many students think of mathematics as a collection of discrete, unrelated topics. Their instructors know that this is not the case. The new Getting Ready for the Next Section problems reinforce the cumulative, connected nature of this course by showing how the concepts and techniques flow one from another throughout the course. These problems review all of the material that students will need in order to be successful, forming a bridge to the next section and gently preparing students to move forward.

## The Combined Course: A Natural Bridge

Elementary algebra is a bridge course. One of its main goals is to get students ready to make a successful start in intermediate algebra. Intermediate algebra, too, is a bridge. The course and its syllabus bring the student to the level of ability to do quantitative work in his or her major. Many students will go on after the intermediate algebra course to precalculus, calculus, and majors requiring quantitative literacy. At many schools, a large number of students enroll in the elementary-intermediate sequence. Instructors at those schools want some flexibility regarding which topics they cover and in what order.

Chapters 1 through 6 cover concepts and techniques traditionally associated with the elementary course. Chapters 8 through 13 would normally be in an intermediate algebra course. Chapter 7, the transitional chapter, allows greater flexibility in the design of the courses and in teaching the sequence.

The combined book is ideal for adapting the developmental math sequence to a school's specific needs. The comprehensive coverage that a combined book offers gives instructors a chance to end the elementary course at a point suitable for their students or to begin the intermediate course at a conceptual level more appropriate for the kinds of students who take intermediate algebra at their schools.

## Our Proven Commitment to Student Success

After two successful editions, we have developed several interlocking, proven features that will improve students' chances of success in the course. We placed practical, easily understood study skills in the first six chapters (look for them on the page after the chapter opener). Here are some of the other, important success features of the book.

Getting Ready for Class Just before each problem set is a list of four questions under the heading Getting Ready for Class. These problems require written responses from students and are to be done before students come to class. The answers can be found by reading the preceding section. These questions reinforce the importance of reading the section before coming to class.

Linking Objectives and Examples At the end of each section we place a small box that shows which examples support which learning objectives. This feature helps students to understand how the section and its examples are built around objectives. We think that this feature helps to make the structure of exposition of concepts clearer.

Blueprint for Problem Solving Found in the main text, the Blueprint for Problem Solving is a detailed outline of steps needed to successfully attempt application problems. Intended as a guide to problem solving in general, the blueprint takes the student through the solution process to various kinds of applications.

Maintaining Your Skills We believe that students who consistently work review problems will be much better prepared for class than students who do not engage in continuous review. The Maintaining Your Skills problems cumulatively review the most important concepts from the previous chapter as well as concepts that form the foundation of the course.

Getting Ready for the Next Section At the ends of section problem sets, you will find Getting Ready for the Next Section, a category of problems that students can work to prepare themselves to navigate the next section successfully. These problems polish techniques and reinforce the idea that all topics in the course are built on previous topics.

## End-of-Chapter Summary, Review, and Assessment

We have learned that students are more comfortable with a chapter that sums up what they have learned thoroughly and accessibly through a well-organized presentation that reinforces concepts and techniques well. To help students grasp concepts and get more practice, each chapter ends with the following features that together give a comprehensive reexamination of the chapter.

Chapter Summary The chapter summary recaps all main points from the chapter in a visually appealing grid. In a column next to each topic is an example that illustrates the type of problem associated with the topic being reviewed. Our way of summarizing shows students that concepts in mathematics do
relate-and that mastering one concept is a bridge to the next. When students prepare for a test, they can use the chapter summary as a guide to the main concepts of the chapter.

Chapter Review Test Following the chapter summary in each chapter is the chapter review test. It contains an extensive set of problems that review all the main topics in the chapter. This feature can be used flexibly-as assigned review, as a recommended self-test for students as they prepare for examinations, or as an in-class quiz or test.

Chapter Projects Each chapter closes with a pair of projects. One is a group project, suitable for students to work on in class. Group projects list details about number of participants, equipment, and time, so that instructors can determine how well the project fits into their classroom. The second project is a research project for students to do outside of class and tends to be open ended.

## Additional Features of the Book

Facts from Geometry Many of the important facts from geometry are listed under this heading. In most cases, an example or two accompanies each of the facts to give students a chance to see how topics from geometry are related to the algebra they are learning.

Unit Analysis Chapter 6 contains problems requiring students to convert from one unit of measure to another. The method used to accomplish the conversions is the method they will use if they take a chemistry class. Since this method is similar to the method we use to multiply rational expressions, unit analysis is covered in Section 6.2 as an application of multiplying rational expressions.

Chapter Openings Each chapter opens with an introduction in which a realworld application is used to spark interest in the chapter. We expand on each of these opening applications later in the chapter.

## Supplements

Enhanced WebAssign (0534495826) Instant feedback and ease of use are just two reasons why WebAssign is the most widely used homework system in higher education. WebAssign's Homework Delivery System lets you deliver, collect, grade, and record assignments using the web. This proven system has been enhanced to include end-of-chapter problems from McKeague's Elementary and Intermediate Algebra, Third Edition-incorporating figures, videos, examples, PDF pages of the text, and quizzes to promote active learning and provide the immediate, relevant feedback students want.

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Text-Specific Videos (0495382981) These text-specific DVD sets completed by Pat McKeague are available at no charge to qualified adopters of the text. They feature 10- to 20-minute problem-solving lessons that cover each section of every chapter.

Student Solutions Manual (0495383007) The Student Solutions Manual provides worked-out solutions to the odd-numbered problems in the text.

Annotated Instructor's Edition (0495383031) The Instructor's Edition provides the complete student text with answers next to each respective exercise.

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## Website www.thomsonedu.com/math/mckeague

The book's website offers instant access to the Student Resource Center, a rich array of teaching and learning resources that offers videos, chapter-by-chapter online tutorial quizzes, a final exam, chapter outlines, chapter reviews, chapter-by-chapter web links, flash cards, and more interactive options.

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## The Basics

uch of what we do in mathematics is concerned with recognizing patterns. If you recognize the patterns in the following two sequences, then you can easily extend each sequence.

Sequence of odd numbers $=1,3,5,7,9, \ldots$
Sequence of squares $=1,4,9,16,25, \ldots$
Once we have classified groups of numbers as to the characteristics they share, we sometimes discover that a relationship exists between the groups. Although it may not be obvious at first, there is a relationship that exists between the two sequences shown. The introduction to The Book of Squares, written in 1225 by the mathematician known as Fibonacci, begins this way:

I thought about the origin of all square numbers and discovered that they arise out of the increasing sequence of odd numbers.

The relationship that Fibonacci refers to is shown visually in Figure 1.


FIGURE 1
Many times we describe a relationship or pattern in a number of different ways. Figure 1 is a visual description of a relationship. In this chapter we will work on describing relationships numerically and verbally (in writing).

## CHAPTER OUTLINE

1.1 Notation and Symbols
1.2 Real Numbers
1.3 Addition of Real Numbers
1.4 Subtraction of Real Numbers
1.5 Properties of Real Numbers
1.6 Multiplication of Real Numbers
1.7 Division of Real Numbers
1.8 Subsets of the Real Numbers
1.9 Addition and Subtraction with Fractions

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## Notation and Symbols

## OBJECTIVES

A Translate between phrases written in English and expressions written in symbols.

B Simplify expressions containing exponents.

C Simplify expressions using the rule for order of operations.

D Read tables and charts.
E Recognize the pattern in a sequence of numbers.

Suppose you have a checking account that costs you \$15 a month, plus \$0.05 for each check you write. If you write 10 checks in a month, then the monthly charge for your checking account will be

$$
15+10(0.05)
$$

Do you add 15 and 10 first and then multiply by 0.05 ? Or do you multiply 10 and 0.05 first and then add 15 ? If you don't know the answer to this question, you will after you have read through this section.
Because much of what we do in algebra involves comparison of quantities, we will begin by listing some symbols used to compare mathematical quantities. The comparison symbols fall into two major groups: equality symbols and inequality symbols.
We will let the letters $a$ and $b$ stand for (represent) any two mathematical quantities. When we use letters to represent numbers, as we are doing here, we call the letters variables.

## Variables: An Intuitive Look

When you filled out the application for the school you are attending, there was a space to fill in your first name. "First name" is a variable quantity because the value it takes depends on who is filling out the application. For example, if your first name is Manuel, then the value of "First Name" is Manuel. However, if your first name is Christa, then the value of "First Name" is Christa.
If we denote "First Name" as FN, "Last Name" as $L N$, and "Whole Name" as $W N$, then we take the concept of a variable further and write the relationship between the names this way:

$$
F N+L N=W N
$$

(We use the + symbol loosely here to represent writing the names together with a space between them.) This relationship we have written holds for all people who have only a first name and a last name. For those people who have a middle name, the relationship between the names is

$$
F N+M N+L N=W N
$$

A similar situation exists in algebra when we let a letter stand for a number or a group of numbers. For instance, if we say "let $a$ and $b$ represent numbers," then $a$ and $b$ are called variables because the values they take on vary. We use the variables $a$ and $b$ in the following lists so that the relationships shown there are true for all numbers that we will encounter in this book. By using variables, the following statements are general statements about all numbers, rather than specific statements about only a few numbers.

## Note

In the past you may have used the notation $3 \times 5$ to denote multiplication. In algebra it is best to avoid this notation if possible, because the multiplication symbol $\times$ can be confused with the variable $x$ when written by hand.

## Comparison Symbols

$$
\begin{array}{lll}
\text { Equality: } & a=b & a \text { is equal to } b \text { ( } a \text { and } b \text { represent the same number) } \\
& a \neq b & a \text { is not equal to } b
\end{array} \text { Inequality: } \begin{array}{ll}
a<b & a \text { is less than } b \\
& a \nless b \\
a>b & a \text { is not less than } b \\
& a \text { is greater than } b \\
& \ngtr b
\end{array} a \text { is not greater than } b \begin{array}{ll}
a \text { is greater than or equal to } b \\
& a \geq b \\
& a \text { is less than or equal to } b
\end{array}
$$

The symbols for inequality, $<$ and $>$, always point to the smaller of the two quantities being compared. For example, $3<x$ means 3 is smaller than $x$. In this case we can say " 3 is less than $x$ " or " $x$ is greater than 3 "; both statements are correct. Similarly, the expression $5>y$ can be read as " 5 is greater than $y$ " or as " $y$ is less than 5 " because the inequality symbol is pointing to $y$, meaning $y$ is the smaller of the two quantities.
Next, we consider the symbols used to represent the four basic operations: addition, subtraction, multiplication, and division.

## Operation Symbols

Addition: $\quad a+b \quad$ The sum of $a$ and $b$
Subtraction: $\quad a-b$
Multiplication: $\quad a \cdot b,(a)(b), a(b),(a) b, a b$

Division:

$$
a \div b, a / b, \frac{a}{b}, b \sqrt{a}
$$ The difference of $a$ and $b$ The product of $a$ and $b$ The quotient of $a$ and $b$

When we encounter the word sum, the implied operation is addition. To find the sum of two numbers, we simply add them. Difference implies subtraction, product implies multiplication, and quotient implies division. Notice also that there is more than one way to write the product or quotient of two numbers.

Grouping Symbols Parentheses ( ) and brackets [ ] are the symbols used for grouping numbers together. (Occasionally, braces \{ \} are also used for grouping, although they are usually reserved for set notation, as we shall see.)

The following examples illustrate the relationship between the symbols for comparing, operating, and grouping and the English language.

## EXAMPLES

Mathematical Expression

1. $4+1=5$
2. $8-1<10$
3. $2(3+4)=14$

English Equivalent
The sum of 4 and 1 is 5 .
The difference of 8 and 1 is less than 10 .
Twice the sum of 3 and 4 is 14 .
4. $3 x \geq 15$
5. $\frac{y}{2}=y-2$

The product of 3 and $x$ is greater than or equal to 15 . The quotient of $y$ and 2 is equal to the difference of $y$ and 2 .

The last type of notation we need to discuss is the notation that allows us to write repeated multiplications in a more compact form-exponents. In the expression $2^{3}$, the 2 is called the base and the 3 is called the exponent. The exponent 3 tells us the number of times the base appears in the product; that is,

$$
2^{3}=2 \cdot 2 \cdot 2=8
$$

The expression $2^{3}$ is said to be in exponential form, whereas $2 \cdot 2 \cdot 2$ is said to be in expanded form. Here are some additional examples of expressions involving exponents.

## EXAMPLES

Expand and multiply.
6. $5^{2}=5 \cdot 5=25$

Base 5, exponent 2
7. $2^{5}=2 \cdot 2 \cdot 2 \cdot 2 \cdot 2=32$

Base 2, exponent 5
8. $10^{3}=10 \cdot 10 \cdot 10=1,000$ Base 10 , exponent 3

Notation and Vocabulary Here is how we read expressions containing exponents.

| Mathematical <br> Expression | Written Equivalent |
| :---: | :---: |
| $5^{2}$ | five to the second power |
| $5^{3}$ | five to the third power |
| $5^{4}$ | five to the fourth power |
| $5^{5}$ | five to the fifth power |
| $5^{6}$ | five to the sixth power |

We have a shorthand vocabulary for second and third powers because the area of a square with a side of 5 is $5^{2}$, and the volume of a cube with a side of 5 is $5^{3}$.
$5{ }^{2}$ can be read "five squared."
$5^{3}$ can be read "five cubed."


The symbols for comparing, operating, and grouping are to mathematics what punctuation symbols are to English. These symbols are the punctuation symbols for mathematics.

Consider the following sentence:
Paul said John is tall.
It can have two different meanings, depending on how it is punctuated.

1. "Paul," said John, "is tall."
2. Paul said, "John is tall."

Let's take a look at a similar situation in mathematics. Consider the following mathematical statement:

$$
5+2 \cdot 7
$$

If we add the 5 and 2 first and then multiply by 7 , we get an answer of 49 . However, if we multiply the 2 and the 7 first and then add 5 , we are left with 19 . We have a problem that seems to have two different answers, depending on whether we add first or multiply first. We would like to avoid this type of situation. Every problem like $5+2 \cdot 7$ should have only one answer. Therefore, we have developed the following rule for the order of operations.

Rule (Order of Operations) When evaluating a mathematical expression, we will perform the operations in the following order, beginning with the expression in the innermost parentheses or brackets first and working our way out.

1. Simplify all numbers with exponents, working from left to right if more than one of these expressions is present.
2. Then do all multiplications and divisions left to right.
3. Perform all additions and subtractions left to right.

EXAMPLES Simplify each expression using the rule for order of operations.
9. $5+8 \cdot 2=5+16$

$$
=21
$$

10. $12 \div 4 \cdot 2=3 \cdot 2$

$$
=6
$$

11. $2[5+2(6+3 \cdot 4)]=2[5+2(6+12)]$

$$
=2[5+2(18)]
$$

$$
=2[5+36]
$$

$$
=2[41]
$$

$$
=82
$$

12. $10+12 \div 4+2 \cdot 3=10+3+6$

$$
=19
$$

13. $2^{4}+3^{3} \div 9-4^{2}=16+27 \div 9-16$

$$
=16+3-16
$$

$$
\left.\begin{array}{l}
=19-16 \\
=3
\end{array}\right\}
$$

## Multiply $8 \cdot 2$ first

Work left to right

Simplify within the innermost
grouping symbols first
Next, simplify inside
the brackets
Multiply
Multiply and divide left to right
Add left to right
Simplify numbers with exponents
Then, divide
Finally, add and subtract left to right

## Reading Tables and Bar Charts

The following table shows the average amount of caffeine in a number of beverages. The diagram in Figure 1 is a bar chart. It is a visual presentation of the information in Table 1. The table gives information in numerical form, whereas the chart gives the same information in a geometric way. In mathematics, it is important to be able to move back and forth between the two forms.

TABLE 1
Caffeine Content of Hot Drinks

| Drink <br> (6-ounce cup) | Caffeine <br> (milligrams) |
| :--- | :---: |
| Brewed coffee | 100 |
| Instant coffee | 70 |
| Tea | 50 |
| Cocoa | 5 |
| Decaffeinated coffee | 4 |



FIGURE 1

EXAMPLE 14 Referring to Table 1 and Figure 1, suppose you have 3 cups of brewed coffee, 1 cup of tea, and 2 cups of decaf in one day. Write an expression that will give the total amount of caffeine in these six drinks, then simplify the expression.
SOLUTION From the table or the bar chart, we find the number of milligrams of caffeine in each drink; then we write an expression for the total amount of caffeine:

$$
3(100)+50+2(4)
$$

Using the rule for order of operations, we get 358 total milligrams of caffeine.

## Number Sequences and Inductive Reasoning

Suppose someone asks you to give the next number in the sequence of numbers below. (The dots mean that the sequence continues in the same pattern forever.)

$$
2,5,8,11, \ldots
$$

If you notice that each number is 3 more than the number before it, you would say the next number in the sequence is 14 because $11+3=14$. When we reason in this way, we are using what is called inductive reasoning. In mathematics we use inductive reasoning when we notice a pattern to a sequence of numbers and then use the pattern to extend the sequence.

EXAMPLE 15 Find the next number in each sequence.
a. $3,8,13,18, \ldots$
b. $2,10,50,250, \ldots$
c. $2,4,7,11, \ldots$

SOLUTION To find the next number in each sequence, we need to look for a pattern or relationship.
a. For the first sequence, each number is 5 more than the number before it; therefore, the next number will be $18+5=23$.
b. For the sequence in part (b), each number is 5 times the number before it; therefore, the next number in the sequence will be $5 \cdot 250=1,250$.
c. For the sequence in part (c), there is no number to add each time or multiply by each time. However, the pattern becomes apparent when we look at the differences between the numbers:


Proceeding in the same manner, we would add 5 to get the next term, giving us $11+5=16$.

In the introduction to this chapter we mentioned the mathematician known as Fibonacci. There is a special sequence in mathematics named for Fibonacci. Here it is.

Fibonacci sequence $=1,1,2,3,5,8, \ldots$
Can you see the relationship among the numbers in this sequence? Start with two 1 's, then add two consecutive members of the sequence to get the next number. Here is a diagram.


Sometimes we refer to the numbers in a sequence as terms of the sequence.

EXAMPLE 16 Write the first 10 terms of the Fibonacci sequence.
SOLUTION The first six terms are given above. We extend the sequence by adding 5 and 8 to obtain the seventh term, 13 . Then we add 8 and 13 to obtain 21. Continuing in this manner, the first 10 terms in the Fibonacci sequence are

$$
1,1,2,3,5,8,13,21,34,55
$$

## GETTING READY FOR CLASS

Each section of the book will end with some problems and questions like the ones below. They are for you to answer after you have read through the section but before you go to class. All of them require that you give written responses in complete sentences. Writing about mathematics is a valuable exercise. If you write with the intention of explaining and communicating what you know to someone else, you will find that you understand the topic you are writing about even better than you did before you started writing. As with all problems in this course, you want to approach these writing exercises with a positive point of view. You will get better at giving written responses to questions as you progress through the course. Even if you never feel comfortable writing about mathematics, just the process of attempting to do so will increase your understanding and ability in mathematics.
After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a variable?
2. Write the first step in the rule for order of operations.
3. What is inductive reasoning?
4. Explain the relationship between an exponent and its base.

For each sentence below, write an equivalent expression in symbols.

1. The sum of $x$ and 5 is 14 .
2. The difference of $x$ and 4 is 8 .
3. The product of 5 and $y$ is less than 30 .
4. The product of 8 and $y$ is greater than 16 .
5. The product of 3 and $y$ is less than or equal to the sum of $y$ and 6 .
6. The product of 5 and $y$ is greater than or equal to the difference of $y$ and 16 .
7. The quotient of $x$ and 3 is equal to the sum of $x$ and 2.
8. The quotient of $x$ and 2 is equal to the difference of $x$ and 4 .

Expand and multiply.
9. $3^{2}$
10. $4^{2}$
11. $7^{2}$
12. $9^{2}$
13. $2^{3}$
14. $3^{3}$
15. $4^{3}$
16. $5^{3}$
17. $2^{4}$
18. $3^{4}$
19. $10^{2}$
20. $10^{4}$
21. $11^{2}$
22. $111^{2}$

Use the rule for order of operations to simplify each expression as much as possible.
23. a. $2 \cdot 3+5$
b. $2(3+5)$
25. a. $5+2 \cdot 6$
b. $(5+2) \cdot 6$
27. a. $5 \cdot 4+5 \cdot 2$
b. $5(4+2)$
29. a. $8+2(5+3)$
b. $(8+2)(5+3)$
31. $20+2(8-5)+1$
33. $5+2(3 \cdot 4-1)+8$
35. $4+8 \div 4-2$
37. $3 \cdot 8+10 \div 2+4 \cdot 2$
39. a. $(5+3)(5-3)$
b. $5^{2}-3^{2}$
24. a. $8 \cdot 7+1$
b. $8(7+1)$
26. a. $8+9 \cdot 4$
b. $(8+9) \cdot 4$
28. a. $6 \cdot 8+6 \cdot 3$
b. $6(8+3)$
30. a. $7+3(8-2)$
b. $(7+3)(8-2)$
32. $10+3(7+1)+2$
34. $11-2(5 \cdot 3-10)+2$
36. $6+9 \div 3+2$
38. $5 \cdot 9+10 \div 2+3 \cdot 3$
40. a. $(7+2)(7-2)$
b. $7^{2}-2^{2}$
41. a. $(4+5)^{2}$
b. $4^{2}+5^{2}$
42. a. $(6+3)^{2}$
b. $6^{2}+3^{2}$
43. $2 \cdot 10^{3}+3 \cdot 10^{2}+4 \cdot 10+5$
44. $5 \cdot 10^{3}+6 \cdot 10^{2}+7 \cdot 10+8$
45. $10-2(4 \cdot 5-16)$
46. $15-5(3 \cdot 2-4)$
47. $4[7+3(2 \cdot 9-8)]$
48. $5[10+2(3 \cdot 6-10)]$
49. $3(4 \cdot 5-12)+6(7 \cdot 6-40)$
50. $6(8 \cdot 3-4)+5(7 \cdot 3-1)$
51. $3^{4}+4^{2} \div 2^{3}-5^{2}$
52. $2^{5}+6^{2} \div 2^{2}-3^{2}$
53. $5^{2}+3^{4} \div 9^{2}+6^{2}$
54. $6^{2}+2^{5} \div 4^{2}+7^{2}$

Simplify each expression.
55. $20 \div 2 \cdot 10$
56. $40 \div 4 \cdot 5$
57. $24 \div 8 \cdot 3$
58. $24 \div 4 \cdot 6$
59. $36 \div 6 \cdot 3$
60. $36 \div 9 \cdot 2$
61. $16-8+4$
62. $16-4+8$
63. $24-14+8$
64. $24-16+6$
65. $36-6+12$
66. $36-9+20$

We are assuming that you know how to do arithmetic with decimals. Here are some problems to practice.
Simplify each expression.
-67. $0.08+0.09$ 68. $0.06+0.04$
69. $0.10+0.12$
70. $0.08+0.06$
71. $4.8-2.5$
72. $6.3-4.8$
73. $2.07+3.48$
74. $4.89+2.31$
75. $0.12(2,000)$
76. $0.09(3,000)$
77. $0.25(40)$
78. $0.75(40)$
79. $510 \div 0.17$
80. $400 \div 0.1$
81. $240 \div 0.12$
82. $360 \div 0.12$

Use a calculator to find the following quotients. Round your answers to the nearest hundredth, if necessary.
83. $37.80 \div 1.07$
84. $85.46 \div 4.88$
85. $555 \div 740$
86. $740 \div 108$
87. $70 \div 210$
88. $15 \div 80$
89. $6,000 \div 22$
90. $51,000 \div 17$

## Applying the Concepts

Food Labels In 1993 the government standardized the way in which nutrition information was presented on the labels of most packaged food products. Figure 2 shows a standardized food label from a package of cookies that I ate at lunch the day I was writing the problems for this problem set. Use the information in Figure 2 to answer the following questions.


FIGURE 2
91. How many cookies are in the package?
92. If I paid $\$ 0.50$ for the package of cookies, how much did each cookie cost?
93. If the "calories" category stands for calories per serving, how many calories did I consume by eating the whole package of cookies?
94. Suppose that, while swimming, I burn 11 calories each minute. If I swim for 20 minutes, will I burn enough calories to cancel out the calories I added by eating 5 cookies?
95. Reading Tables and Charts The following table and bar chart give the amount of caffeine in five different soft drinks. How much caffeine is in each of the following?
a. A 6-pack of Jolt
b. 2 Coca-Colas plus 3 Tabs

| Caffeine Content in <br> Soft <br> Drink | Daffeine (milligrams) |
| :--- | :---: |
| Jolt | 100 |
| Tab | 47 |
| Coca-Cola | 45 |
| Diet Pepsi | 36 |
| 7UP | 0 |

Caffeine (mg)

96. Reading Tables and Charts The following table and bar chart give the amount of caffeine in five different nonprescription drugs. How much caffeine is in each of the following?
a. A box of 12 Excedrin
b. 1 Dexatrim plus 4 Excedrin

| Caffeine Content in Nonprescription Drugs <br> Caffeine (milligrams) |  |
| :--- | :---: |
| Nonprescription Drug | 200 |
| Dexatrim | 100 |
| NoDoz | 65 |
| Excedrin | 30 |
| Triaminicin tablets | 16 |
| Dristan tablets |  |

Caffeine (mg)

97. Reading Tables and Charts The following bar chart gives the number of calories burned by a 150-pound person during 1 hour of various exercises. The accompanying table should display the same information. Use the bar chart to complete the table.

| Calories Burned by <br> 150-Pound Person <br> Calories Burned in 1 Hour |  |
| :--- | :---: |
| Activity | 374 |
| Bowling |  |
| Handball |  |
| Jogging |  |
| Skiing |  |

Calories Burned in 1 Hour by a 150-Pound Person

98. Reading Tables and Charts The following bar chart gives the number of calories consumed by eating some popular fast foods. The accompanying table should display the same information. Use the bar chart to complete the table.

| Calories in Fast Food |  |
| :--- | :---: |
| Food | Calories |
| McDonald's Hamburger | 270 |
| Burger King Hamburger |  |
| Jack in the Box Hamburger |  |
| McDonald's Big Mac |  |
| Burger King Whopper |  |

Calories in Fast Food


Find the next number in each sequence.
99. $1,2,3,4, \ldots$ (The sequence of counting numbers)
100. $0,1,2,3, \ldots$ (The sequence of whole numbers)
101. $2,4,6,8, \ldots$ (The sequence of even numbers)
102. $1,3,5,7, \ldots$ (The sequence of odd numbers)
103. $1,4,9,16, \ldots$ (The sequence of squares)
104. $1,8,27,64, \ldots$ (The sequence of cubes)
105. $2,2,4,6, \ldots$ (A Fibonacci-like sequence)
106. $5,5,10,15, \ldots$ (A Fibonacci-like sequence)


## Real Numbers

## OBJECTIVES

Locate and label points on the number line.

B Change a fraction to an equivalent fraction with a new denominator.

C Simplify expressions containing absolute value.

D Identify the opposite of a number.

E Multiply fractions.
F Identify the reciprocal of a number.

G Find the value of an expression.

H Find the perimeter and area of squares, rectangles, and triangles.

Table 1 and Figure 1 give the record low temperature, in degrees Fahrenheit, for each month of the year in the city of Jackson, Wyoming. Notice that some of these temperatures are represented by negative numbers.


FIGURE 1

In this section we start our work with negative numbers. To represent negative numbers in algebra, we use what is called the real number line. Here is how we construct a real number line: We first draw a straight line and label a conve-

## Note

If there is no sign (+ or - ) in front of a number, the number is assumed to be positive (+).

## Note

There are other numbers on the number line that you may not be as familiar with. They are irrational numbers such as $\pi, \sqrt{2}$, $\sqrt{3}$. We will introduce these numbers later in the chapter.

TABLE 1
Record Low Temperatures for Jackson, Wyoming

| Month | Temperature <br> (Degrees Fahrenheit) | Month | Temperature <br> (Degrees Fahrenheit) |
| :--- | :---: | :--- | :---: |
| January | -50 | July | 24 |
| February | -44 | August | 18 |
| March | -32 | September | 14 |
| April | -5 | October | 2 |
| May | 12 | November | -27 |
| June | 19 | December | -49 |

nient point on the line with 0 . Then we mark off equally spaced distances in both directions from 0 . Label the points to the right of 0 with the numbers 1,2 , $3, \ldots$ (the dots mean "and so on"). The points to the left of 0 we label in order, $-1,-2,-3, \ldots$. Here is what it looks like.


The numbers increase in value going from left to right. If we "move" to the right, we are moving in the positive direction. If we move to the left, we are moving in the negative direction. When we compare two numbers on the number line, the number on the left is always smaller than the number on the right. For instance, -3 is smaller than -1 because it is to the left of -1 on the number line.

EXAMPLE 1 Locate and label the points on the real number line associated with the numbers $-3.5,-1 \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 2.5$.
SOLUTION We draw a real number line from -4 to 4 and label the points in question.


DEFINITION The number associated with a point on the real number line is called the coordinate of that point.

In the preceding example, the numbers $\frac{1}{2}, \frac{3}{4}, 2.5,-3.5$, and $-1 \frac{1}{4}$ are the coordinates of the points they represent.

DEFINITION The numbers that can be represented with points on the real number line are called real numbers.

Real numbers include whole numbers, fractions, decimals, and other numbers that are not as familiar to us as these.

## Fractions on the Number Line

As we proceed through Chapter 1, from time to time we will review some of the major concepts associated with fractions. To begin, here is the formal definition of a fraction.

DEFINITION If $a$ and $b$ are real numbers, then the expression

$$
\frac{a}{b} \quad b \neq 0
$$

is called a fraction. The top number $a$ is called the numerator, and the bottom number $b$ is called the denominator. The restriction $b \neq 0$ keeps us from writing an expression that is undefined. (As you will see, division by zero is not allowed.)

The number line can be used to visualize fractions. Recall that for the fraction $\frac{a}{b}$, $a$ is called the numerator and $b$ is called the denominator. The denominator indicates the number of equal parts in the interval from 0 to 1 on the number line. The numerator indicates how many of those parts we have. If we take that part of the number line from 0 to 1 and divide it into three equal parts, we say that we have divided it into thirds (Figure 2). Each of the three segments is $\frac{1}{3}$ (one third) of the whole segment from 0 to 1 .


FIGURE 2
Two of these smaller segments together are $\frac{2}{3}$ (two thirds) of the whole segment. And three of them would be $\frac{3}{3}$ (three thirds), or the whole segment.

Let's do the same thing again with six equal divisions of the segment from 0 to 1 (Figure 3). In this case we say each of the smaller segments has a length of $\frac{1}{6}$ (one sixth).


The same point we labeled with $\frac{1}{3}$ in Figure 2 is now labeled with $\frac{2}{6}$. Likewise, the point we labeled earlier with $\frac{2}{3}$ is now labeled $\frac{4}{6}$. It must be true then that

$$
\frac{2}{6}=\frac{1}{3} \quad \text { and } \quad \frac{4}{6}=\frac{2}{3}
$$

Actually, there are many fractions that name the same point as $\frac{1}{3}$. If we were to divide the segment between 0 and 1 into 12 equal parts, 4 of these 12 equal parts $\left(\frac{4}{12}\right)$ would be the same as $\frac{2}{6}$ or $\frac{1}{3}$; that is,

$$
\frac{4}{12}=\frac{2}{6}=\frac{1}{3}
$$

Even though these three fractions look different, each names the same point on the number line, as shown in Figure 4. All three fractions have the same value because they all represent the same number.


DEFINITION Fractions that represent the same number are said to be equivalent. Equivalent fractions may look different, but they must have the same value.

It is apparent that every fraction has many different representations, each of which is equivalent to the original fraction. The next two properties give us a way of changing the terms of a fraction without changing its value.

Property 1 Multiplying the numerator and denominator of a fraction by the same nonzero number never changes the value of the fraction.

Property 2 Dividing the numerator and denominator of a fraction by the same nonzero number never changes the value of the fraction.

EXAMPLE 2 Write $\frac{3}{4}$ as an equivalent fraction with denominator 20 .
SOLUTION The denominator of the original fraction is 4 . The fraction we are trying to find must have a denominator of 20 . We know that if we multiply 4 by 5 , we get 20 . Property 1 indicates that we are free to multiply the denominator by 5 as long as we do the same to the numerator.

$$
\frac{3}{4}=\frac{3 \cdot 5}{4 \cdot 5}=\frac{15}{20}
$$

The fraction $\frac{15}{20}$ is equivalent to the fraction $\frac{3}{4}$.

## Absolute Values and Opposites

Representing numbers on the number line lets us give each number two important properties: a direction from zero and a distance from zero. The direction from zero is represented by the sign in front of the number. (A number without a sign is understood to be positive.) The distance from zero is called the absolute value of the number, as the following definition indicates.

DEFINITION The absolute value of a real number is its distance from zero on the number line. If $x$ represents a real number, then the absolute value of $x$ is written $|x|$.

## EXAMPLES

3. $|5|=5$

Write each expression without absolute value symbols.
4. $|-5|=5 \quad$ The number -5 is 5 units from zero

The number 5 is 5 units from zero
5. $\left|-\frac{1}{2}\right|=\frac{1}{2} \quad$ The number $-\frac{1}{2}$ is $\frac{1}{2}$ units from zero

The absolute value of a number is never negative. It is the distance the number is from zero without regard to which direction it is from zero. When working with the absolute value of sums and differences, we must simplify the expression inside the absolute value symbols first and then find the absolute value of the simplified expression.

## EXAMPLES Simplify each expression.

6. $|8-3|=|5|=5$
7. $\left|3 \cdot 2^{3}+2 \cdot 3^{2}\right|=|3 \cdot 8+2 \cdot 9|=|24+18|=|42|=42$
8. $|9-2|-|8-6|=|7|-|2|=7-2=5$

Another important concept associated with numbers on the number line is that of opposites. Here is the definition.

DEFINITION Numbers the same distance from zero but in opposite directions from zero are called opposites.

## EXAMPLES

Number
9. 5
10. -3
11. $\frac{1}{4}$
12. -2.3

Give the opposite of each number.
Opposite
-5 5 and -5 are opposites
$3 \quad-3$ and 3 are opposites
$-\frac{1}{4} \quad \frac{1}{4}$ and $-\frac{1}{4}$ are opposites
$2.3-2.3$ and 2.3 are opposites

Each negative number is the opposite of some positive number, and each positive number is the opposite of some negative number. The opposite of a negative number is a positive number. In symbols, if $a$ represents a positive number, then

$$
-(-a)=a
$$

## Note

In past math classes you may have written fractions like $\frac{7}{3}$ (improper fractions) as mixed numbers, such as $2 \frac{1}{3}$. In algebra it is usually better to write them as improper fractions rather than mixed numbers.

Opposites always have the same absolute value. And, when you add any two opposites, the result is always zero:

$$
a+(-a)=0
$$

## Reciprocals and Multiplication with Fractions

The last concept we want to cover in this section is the concept of reciprocals. Understanding reciprocals requires some knowledge of multiplication with fractions. To multiply two fractions, we simply multiply numerators and multiply denominators.

## EXAMPLE 13 Multiply $\frac{3}{4} \cdot \frac{5}{7}$.

SOLUTION The product of the numerators is 15 , and the product of the denominators is 28 :

$$
\frac{3}{4} \cdot \frac{5}{7}=\frac{3 \cdot 5}{4 \cdot 7}=\frac{15}{28}
$$

EXAMPLE 14 Multiply $7\left(\frac{1}{3}\right)$.
SOLUTION The number 7 can be thought of as the fraction $\frac{7}{1}$ :

$$
7\left(\frac{1}{3}\right)=\frac{7}{1}\left(\frac{1}{3}\right)=\frac{7 \cdot 1}{1 \cdot 3}=\frac{7}{3}
$$

## EXAMPLE 15 Expand and multiply $\left(\frac{2}{3}\right)^{3}$.

SOLUTION Using the definition of exponents from the previous section, we have

$$
\left(\frac{2}{3}\right)^{3}=\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}=\frac{8}{27}
$$

We are now ready for the definition of reciprocals.

DEFINITION Two numbers whose product is 1 are called reciprocals.

## EXAMPLES

Number
16. 5
17. 2
$\frac{1}{2}$
3
Because $\frac{1}{3}(3)=\frac{1}{3}\left(\frac{3}{1}\right)=\frac{3}{3}=1$
18. $\frac{1}{3}$

Because $\frac{3}{4}\left(\frac{4}{3}\right)=\frac{12}{12}=1$

Although we will not develop multiplication with negative numbers until later in the chapter, you should know that the reciprocal of a negative number is also a negative number. For example, the reciprocal of -4 is $-\frac{1}{4}$.

Previously we mentioned that a variable is a letter used to represent a number or a group of numbers. An expression that contains any combination of numbers, variables, operation symbols, and grouping symbols is called an algebraic expression (sometimes referred to as just an expression). This definition includes the use of exponents and fractions. Each of the following is an algebraic expression.

$$
3 x+5 \quad 4 t^{2}-9 \quad x^{2}-6 x y+y^{2} \quad-15 x^{2} y^{4} z^{5} \quad \frac{a^{2}-9}{a-3} \quad \frac{(x-3)(x+2)}{4 x}
$$

In the last two expressions, the fraction bar separates the numerator from the denominator and is treated the same as a pair of grouping symbols; it groups the numerator and denominator separately.

## The Value of an Algebraic Expression

An expression such as $3 x+5$ will take on different values depending on what $x$ is. If we were to let $x$ equal 2 , the expression $3 x+5$ would become 11 . On the other hand, if $x$ is 10 , the same expression has a value of 35 :

| When | $x=2$ | When | $x=10$ |
| :---: | :---: | :---: | :---: |
| the expression | $3 x+5$ | the expression | $3 x+5$ |
| becomes | $3(2)+5$ | becomes | $3(10)+5$ |
|  | $=6+5$ |  | $=30+5$ |
|  | $=11$ |  | $=35$ |

Table 2 lists some other algebraic expressions, along with specific values for the variables and the corresponding value of the expression after the variable has been replaced with the given number.

TABLE 2

| Original <br> Expression | Value of <br> the Variable | Value of <br> the Expression |
| :---: | :---: | ---: |
| $5 x+2$ | $x=4$ | $5(4)+2$ $=20+2$ <br>  $=22$ |
| $3 x-9$ | $t=5$ | $3(2)-9$ $=6-9$ <br>  $=-3$ |
| $4 t^{2}-9$ | $a=8$ | $4\left(5^{2}\right)-9$ $=4(25)-9$ <br>  $=100-9$ <br>  $=91$ <br> $a^{2}-9$  <br> $a-3$  |
|  |  | $\frac{8^{2}-9}{8-3}$ $=\frac{64-9}{8-3}$ <br>   |

## Note

The vertical line labeled $h$ in the triangle is its height, or altitude. It extends from the top of the triangle down to the base, meeting the base at an angle of $90^{\circ}$. The altitude of a triangle is always perpendicular to the base. The small square shown where the altitude meets the base is used to indicate that the angle formed is $90^{\circ}$.

## FACTS FROM GEOMETRY

## Formulas for Area and Perimeter

A square, rectangle, and triangle are shown in the following figures. Note that we have labeled the dimensions of each with variables. The formulas for the perimeter and area of each object are given in terms of its dimensions.


The formula for perimeter gives us the distance around the outside of the object along its sides, whereas the formula for area gives us a measure of the amount of surface the object has.

EXAMPLE 20 Find the perimeter and area of each figure.

5 feet
b.


25 meters

SOLUTION We use the preceding formulas to find the perimeter and the area.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1
B 2
C 3-8
D 9-12
E $\quad 13-15$
F $\quad 16-19$
G Table 2
H 20

In each case, the units for perimeter are linear units, whereas the units for area are square units.
a. $\quad$ Perimeter $=4 s=4 \cdot 5$ feet $=20$ feet

Area $=s^{2}=(5 \text { feet })^{2}=25$ square feet
b. Perimeter $=2 l+2 w=2(8$ inches $)+2(6$ inches $)=28$ inches Area $=l w=(8$ inches $)(6$ inches $)=48$ square inches
c. Perimeter $=a+b+c=(20$ meters $)+(25$ meters $)+(15$ meters $)$

$$
=60 \text { meters }
$$

Area $=\frac{1}{2} b h=\frac{1}{2}(25$ meters $)(12$ meters $)=150$ square meters

## GETTING READY FOR CLASS

## After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a real number?
2. Explain multiplication with fractions.
3. How do you find the opposite of a number?
4. Explain how you find the perimeter and the area of a rectangle.

Draw a number line that extends from -5 to +5 . Label the points with the following coordinates.

1. 5
2. -2
3. -4
4. -3
5. 1.5
6. -1.5
7. $\frac{9}{4}$
8. $\frac{8}{3}$

Write each of the following fractions as an equivalent fraction with denominator 24 .
9. $\frac{3}{4}$
10. $\frac{5}{6}$
11. $\frac{1}{2}$
12. $\frac{1}{8}$
13. $\frac{5}{8}$
14. $\frac{7}{12}$

Write each fraction as an equivalent fraction with denominator 60.
15. $\frac{3}{5}$
16. $\frac{5}{12}$
17. $\frac{11}{30}$
18. $\frac{9}{10}$

Fill in the missing numerator so the fractions are equal.
19. $\frac{1}{2}=\frac{-}{4}$
20. $\frac{1}{5}=\frac{}{10}$
21. $\frac{5}{9}=\frac{}{45}$
22. $\frac{2}{5}=\frac{}{45}$
23. $\frac{3}{4}=\frac{-}{8}$
24. $\frac{1}{2}=\frac{-}{8}$

For each of the following numbers, give the opposite, the reciprocal, and the absolute value. (Assume all variables are nonzero.)
25. 10
26. $\frac{3}{4}$
27. -3
28. $-\frac{2}{5}$
29. $x$
30. $a$

Place one of the symbols $<$ or $>$ between each of the following to make the resulting statement true.
31. $-5 \quad-3$
32. $-8-1$
33. $-3 \quad-7$
34. $-6 \quad 5$
35. $|-4| \quad-|-4|$
36. $3-|-3|$
37. $7 \quad-|-7|$
38. $-7|-7|$
39. $-\frac{3}{4} \quad-\frac{1}{4}$
40. $-\frac{2}{3} \quad-\frac{1}{3}$
41. $-\frac{3}{2} \quad-\frac{3}{4}$
42. $-\frac{8}{3} \quad-\frac{17}{3}$

Simplify each expression.
43. $|8-2|$
44. $|6-1|$
45. $\left|5 \cdot 2^{3}-2 \cdot 3^{2}\right|$
46. $\left|2 \cdot 10^{2}+3 \cdot 10\right|$
47. $|7-2|-|4-2|$
48. $|10-3|-|4-1|$
49. $10-|7-2(5-3)|$
50. $12-|9-3(7-5)|$
51. $15-|8-2(3 \cdot 4-9)|-10$
52. $25-|9-3(4 \cdot 5-18)|-20$

Multiply the following.
53. $\frac{2}{3} \cdot \frac{4}{5}$
54. $\frac{1}{4} \cdot \frac{3}{5}$
55. $\frac{1}{2}(3)$
57. $\frac{4}{3} \cdot \frac{3}{4}$
59. $3 \cdot \frac{1}{3}$
61. Multiply.
a. $\frac{1}{2}(4)$
b. $\frac{1}{2}(8)$
c. $\frac{1}{2}(16)$
d. $\frac{1}{2}(0.06)$
63. Multiply.
a. $\frac{3}{2}(4)$
b. $\frac{3}{2}(8)$
c. $\frac{3}{2}(16)$
d. $\frac{3}{2}(0.06)$
56. $\frac{1}{5}(4)$
58. $\frac{5}{7} \cdot \frac{7}{5}$
60. $4 \cdot \frac{1}{4}$
62. Multiply.
a. $\frac{1}{4}(8)$
b. $\frac{1}{4}(24)$
c. $\frac{1}{4}(16)$
d. $\frac{1}{4}(0.20)$
64. Multiply.
a. $\frac{3}{4}(8)$
b. $\frac{3}{4}(24)$
c. $\frac{3}{4}(16)$
d. $\frac{3}{4}(0.20)$

Expand and multiply.
65. $\left(\frac{3}{4}\right)^{2}$
66. $\left(\frac{5}{6}\right)^{2}$
67. $\left(\frac{2}{3}\right)^{3}$
68. $\left(\frac{1}{2}\right)^{3}$
69. Find the value of $2 x-6$ when
a. $X=5$
b. $x=10$
c. $x=15$
d. $x=20$
70. Find the value of $2(x-3)$ when
a. $x=5$
b. $x=10$
c. $x=15$
d. $x=20$
71. Find the value of each expression when $x$ is 10 .
a. $x+2$
b. $2 x$
c. $X^{2}$
d. $2^{x}$
72. Find the value of each expression when $x$ is 3 .
a. $x+3$
b. $3 x$
c. $x^{2}$
d. $3^{x}$
73. Find the value of each expression when $x$ is 4 .
a. $x^{2}+1$
b. $(x+1)^{2}$
c. $x^{2}+2 x+1$
74. Find the value of $b^{2}-4 a c$ when
a. $a=2, b=6, c=3$
b. $a=1, b=5, c=6$
c. $a=1, b=2, c=1$

Find the perimeter and area of each figure.
75.

76.


15 millimeters

15 millimeters
77.

78.

79.

80.


## Applying the Concepts

81. Football Yardage A football team gains 6 yards on one play and then loses 8 yards on the next play. To what number on the number line does a loss of 8 yards correspond? The total yards gained or lost on the two plays corresponds to what negative number?
82. Checking Account Balance A woman has a balance of $\$ 20$ in her checking account. If she writes a check for $\$ 30$, what negative number can be used to represent the new balance in her checking account?
Temperature In the United States, temperature is measured on the Fahrenheit temperature scale. On this scale, water boils at 212 degrees and freezes at 32 degrees. To denote a temperature of 32 degrees on the Fahrenheit scale, we write

## $32^{\circ} \mathrm{F}$, which is read " 32 degrees Fahrenheit"

Use this information for Problems 83 and 84.
83. Temperature and Altitude Marilyn is flying from Seattle to San Francisco on a Boeing 737 jet. When the plane reaches an altitude of 35,000 feet, the temperature outside the plane is 64 degrees below zero Fahrenheit. Represent the temperature with a negative number. If the temperature outside the plane gets warmer by 10 degrees, what will the new temperature be?
84. Temperature Change At 10:00 in the morning in White Bear Lake, Minnesota, John notices the temperature outside is 10 degrees below zero Fahrenheit. Write the temperature as a negative number. An hour later it has warmed up by 6 degrees. What is the temperature at 11:00 that morning?
85. Scuba Diving Steve is scuba diving near his home in Maui. At one point he is 100 feet below the surface. Represent this number with a negative number. If he descends another 5 feet, what negative number will represent his new position?

86. Reading a Chart The chart shows yields for certificates of deposit during one week in 2005. Write a mathematical statement using one of the symbols $<$ or $>$ to compare the following:
a. 6-month yield a year ago to 1-year yield last week
b. $21 / 2$-year yield this week to 5 -year yield a year ago
c. 5-year yield last week to 6-month yield this week


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87. Geometry Find the area and perimeter of an $8 \frac{1}{2}$-by-11-inch piece of notebook paper.
88. Geometry Find the area and perimeter of an $8 \frac{1}{2}$-by$5 \frac{1}{2}$-inch piece of paper.

Calories and Exercise Table 3 gives the amount of energy expended per hour for various activities for a person weighing 120, 150, or 180 pounds. Use Table 3 to answer questions 89-92.

TABLE 3
Energy Expended from Exercising

|  | Calories per Hour |  |  |
| :--- | :---: | :---: | :---: |
| Activity | $\mathbf{1 2 0} \mathbf{~ l b}$ | $\mathbf{1 5 0} \mathbf{~ l b}$ | $\mathbf{1 8 0} \mathbf{~ b}$ |
| Bicycling | 299 | 374 | 449 |
| Bowling | 212 | 265 | 318 |
| Handball | 544 | 680 | 816 |
| Horseback trotting | 278 | 347 | 416 |
| Jazzercise | 272 | 340 | 408 |
| Jogging | 544 | 680 | 816 |
| Skiing (downhill) | 435 | 544 | 653 |

89. Suppose you weigh 120 pounds. How many calories will you burn if you play handball for 2 hours and then ride your bicycle for an hour?
90. How many calories are burned by a person weighing 150 pounds who jogs for $\frac{1}{2}$ hour and then goes bicycling for 2 hours?
91. Two people go skiing. One weighs 180 pounds and the other weighs 120 pounds. If they ski for 3 hours, how many more calories are burned by the person weighing 180 pounds?
92. Two people spend 3 hours bowling. If one weighs 120 pounds and the other weighs 150 pounds, how many more calories are burned during the evening by the person weighing 150 pounds?
93. Improving Your Quantitative Literacy Quantitative literacy is a subject discussed by many people involved in teaching mathematics. The person they are concerned with when they discuss it is you. We are going to work at improving your quantitative literacy; but before we do that, we should answer the question: What is quantitative literacy? Lynn Arthur Steen, a noted mathematics educator, has stated that quantitative literacy is "the capacity to deal effectively with the quantitative aspects of life."
a. Give a definition for the word quantitative.
b. Give a definition for the word literacy.
c. Are there situations that occur in your life that you find distasteful, or that you try to avoid, because they involve numbers and mathematics? If so, list some of them here. (For example, some people find the process of buying a car particu-
larly difficult because they feel that the numbers and details of the financing are beyond them.)
94. Improving Your Quantitative Literacy Use the chart shown here to answer the following questions.
a. How many millions of camera phones were sold in 2004?
b. True or false? The chart shows projected sales in 2005 to be more than 155 million camera phones.
c. True or false? The chart shows projected sales in 2007 to be less than 310 million camera phones.

## Camera Phone Growth: Projected sales in millions of units



## 1.3

## Addition of Real Numbers

Suppose that you are playing a friendly game of poker with some friends, and you lose $\$ 3$ on the first hand and $\$ 4$ on the second hand. If you represent winning with positive numbers and losing with negative numbers, how can you translate this situation into symbols? Because you lost $\$ 3$ and $\$ 4$ for a total of $\$ 7$, one way to represent this situation is with addition of negative numbers:

$$
(-\$ 3)+(-\$ 4)=-\$ 7
$$

From this equation, we see that the sum of two negative numbers is a negative number. To generalize addition with positive and negative numbers, we use the number line.

Because real numbers have both a distance from zero (absolute value) and a direction from zero (sign), we can think of addition of two numbers in terms of distance and direction from zero.

Let's look at a problem for which we know the answer. Suppose we want to add the numbers 3 and 4 . The problem is written $3+4$. To put it on the number line, we read the problem as follows:

1. The 3 tells us to "start at the origin and move 3 units in the positive direction."
2. The + sign is read "and then move."
3. The 4 means " 4 units in the positive direction."

To summarize, $3+4$ means to start at the origin, move 3 units in the positive direction, and then move 4 units in the positive direction.


We end up at 7 , which is the answer to our problem: $3+4=7$.
Let's try other combinations of positive and negative 3 and 4 on the number line.

## EXAMPLE 1 Add $3+(-4)$.

SOLUTION Starting at the origin, move 3 units in the positive direction and then 4 units in the negative direction.


We end up at -1 ; therefore, $3+(-4)=-1$.

## EXAMPLE 2 Add $-3+4$.

SOLUTION Starting at the origin, move 3 units in the negative direction and then 4 units in the positive direction.


We end up at +1 ; therefore, $-3+4=1$.

## EXAMPLE 3 Add $-3+(-4)$.

SOLUTION Starting at the origin, move 3 units in the negative direction and then 4 units in the negative direction.


We end up at -7 ; therefore, $-3+(-4)=-7$.

Here is a summary of what we have just completed:

$$
\begin{aligned}
3+4 & =7 \\
3+(-4) & =-1 \\
-3+4 & =1 \\
-3+(-4) & =-7
\end{aligned}
$$

Let's do four more problems on the number line and then summarize our results into a rule we can use to add any two real numbers.

## EXAMPLE 4 <br> Show that $5+7=12$.

## SOLUTION



## EXAMPLE 5 Show that $5+(-7)=-2$.

## SOLUTION



EXAMPLE 6 Show that $-5+7=2$.
SOLUTION


EXAMPLE 7 Show that $-5+(-7)=-12$.
SOLUTION


If we look closely at the results of the preceding addition problems, we can see that they support (or justify) the following rule.

## Note

This rule is what we have been working toward. The rule is very important. Be sure that you understand it and can use it. The problems we have done up to this point have been done simply to justify this rule. Now that we have the rule, we no longer need to do our addition problems on the number line.

## Rule To add two real numbers with

1. The same sign: Simply add their absolute values and use the common sign. (Both numbers are positive, the answer is positive. Both numbers are negative, the answer is negative.)
2. Different signs: Subtract the smaller absolute value from the larger. The answer will have the sign of the number with the larger absolute value.

This rule covers all possible combinations of addition with real numbers. You must memorize it. After you have worked a number of problems, it will seem almost automatic.

## EXAMPLE $8 \quad$ Add all combinations of positive and negative 10 and 13 .

SOLUTION Rather than work these problems on the number line, we use the rule for adding positive and negative numbers to obtain our answers:

$$
\begin{aligned}
10+13 & =23 \\
10+(-13) & =-3 \\
-10+13 & =3 \\
-10+(-13) & =-23
\end{aligned}
$$

## EXAMPLE 9

Add all possible combinations of positive and negative
12 and 17.
SOLUTION Applying the rule for adding positive and negative numbers, we have

$$
\begin{aligned}
12+17 & =29 \\
12+(-17) & =-5 \\
-12+17 & =5 \\
-12+(-17) & =-29
\end{aligned}
$$

## EXAMPLE 10 Add $-3+2+(-4)$.

SOLUTION Applying the rule for order of operations, we add left to right:

$$
\begin{aligned}
-3+2+(-4) & =-1+(-4) \\
& =-5
\end{aligned}
$$

## EXAMPLE 11 Add $-8+[2+(-5)]+(-1)$.

SOLUTION Adding inside the brackets first and then left to right, we have

$$
\begin{aligned}
-8+[2+(-5)]+(-1) & =-8+(-3)+(-1) \\
& =-11+(-1) \\
& =-12
\end{aligned}
$$

EXAMPLE 12 Simplify $-10+2(-8+11)+(-4)$.
SOLUTION First, we simplify inside the parentheses. Then, we multiply. Finally, we add left to right:

$$
\begin{aligned}
-10+2(-8+11)+(-4) & =-10+2(3)+(-4) \\
& =-10+6+(-4) \\
& =-4+(-4) \\
& =-8
\end{aligned}
$$

## Arithmetic Sequences

The pattern in a sequence of numbers is easy to identify when each number in the sequence comes from the preceding number by adding the same amount each time. This leads us to our next level of classification, in which we classify together groups of sequences with a common characteristic.

DEFINITION An arithmetic sequence is a sequence of numbers in which each number (after the first number) comes from adding the same amount to the number before it.

Here is an example of an arithmetic sequence:

$$
2,5,8,11, \ldots
$$

Each number is obtained by adding 3 to the number before it.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A $1-9$

B $\quad 10-12$
C $\quad 13$
EXAMPLE 13 Each sequence below is an arithmetic sequence. Find the next two numbers in each sequence.
a. $7,10,13, \ldots$
b. $9.5,10,10.5, \ldots$
c. $5,0,-5, \ldots$

SOLUTION Because we know that each sequence is arithmetic, we know to look for the number that is added to each term to produce the next consecutive term.
a. $7,10,13, \ldots$ : Each term is found by adding 3 to the term before it. Therefore, the next two terms will be 16 and 19 .
b. $9.5,10,10.5, \ldots$ : Each term comes from adding 0.5 to the term before it. Therefore, the next two terms will be 11 and 11.5 .
c. $5,0,-5, \ldots$ : Each term comes from adding -5 to the term before it. Therefore, the next two terms will be $-5+(-5)=-10$ and $-10+(-5)=-15$.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Explain how you would add 3 and -5 on the number line.
2. How do you add two negative numbers?
3. What is an arithmetic sequence?
4. Why is the sum of a number and its opposite always $O$ ?
5. Add all combinations of positive and negative 3 and 5. (Look back to Examples 8 and 9.)
6. Add all combinations of positive and negative 6 and 4 .
7. Add all combinations of positive and negative 15 and 20.
8. Add all combinations of positive and negative 18 and 12.

Work the following problems. You may want to begin by doing a few on the number line.
5. $6+(-3)$
6. $7+(-8)$

- 7. $13+(-20)$

8. $15+(-25)$
9. $18+(-32)$
10. $6+(-9)$
11. $-6+3$
12. $-8+7$
13. $-30+5$
14. $-18+6$
15. $-6+(-6)$
16. $-5+(-5)$
17. $-9+(-10)$
18. $-8+(-6)$
19. $-10+(-15)$
20. $-18+(-30)$

Work the following problems using the rule for addition of real numbers. You may want to refer back to the rule for order of operations.
21. $5+(-6)+(-7)$
23. $-7+8+(-5)$
25. $5+[6+(-2)]+(-3)$
26. $10+[8+(-5)]+(-20)$
27. $[6+(-2)]+[3+(-1)]$
28. $[18+(-5)]+[9+(-10)]$
29. $20+(-6)+[3+(-9)]$
30. $18+(-2)+[9+(-13)]$
31. $-3+(-2)+[5+(-4)]$
32. $-6+(-5)+[-4+(-1)]$
33. $(-9+2)+[5+(-8)]+(-4)$
34. $(-7+3)+[9+(-6)]+(-5)$
35. $[-6+(-4)]+[7+(-5)]+(-9)$
36. $[-8+(-1)]+[8+(-6)]+(-6)$
37. $(-6+9)+(-5)+(-4+3)+7$
38. $(-10+4)+(-3)+(-3+8)+6$

The problems that follow involve some multiplication. Be sure that you work inside the parentheses first, then multiply, and, finally, add left to right.
39. $-5+2(-3+7)$
40. $-3+4(-2+7)$
41. $9+3(-8+10)$
42. $4+5(-2+6)$
43. $-10+2(-6+8)+(-2)$
44. $-20+3(-7+10)+(-4)$
45. $2(-4+7)+3(-6+8)$
46. $5(-2+5)+7(-1+6)$

Add.
47. $3.9+7.1$
48. $4.7+4.3$
49. $8.1+2.7$
50. $2.4+7.3$
51. Find the value of $0.06 x+0.07 y$ when $x=7,000$ and $y=8,000$.
52. Find the value of $0.06 x+0.07 y$ when $x=7,000$ and $y=3,000$.
53. Find the value of $0.05 x+0.10 y$ when $x=10$ and $y=12$.
54. Find the value of $0.25 x+0.10 y$ when $x=3$ and $y=11$.

Each sequence below is an arithmetic sequence. In each case, find the next two numbers in the sequence.
55. $3,8,13,18, \ldots$
56. $1,5,9,13, \ldots$
57. $10,15,20,25, \ldots$
58. $10,16,22,28, \ldots$
59. $20,15,10,5, \ldots$
60. $24,20,16,12, \ldots$
61. $6,0,-6, \ldots$
62. $1,0,-1, \ldots$
63. $8,4,0, \ldots$
64. $5,2,-1, \ldots$
65. Is the sequence of odd numbers an arithmetic sequence?
66. Is the sequence of squares an arithmetic sequence?

Recall that the word sum indicates addition. Write the numerical expression that is equivalent to each of the following phrases and then simplify.
67. The sum of 5 and 9

[^0]68. The sum of 6 and -3
69. Four added to the sum of -7 and -5
70. Six added to the sum of -9 and 1
71. The sum of -2 and -3 increased by 10
72. The sum of -4 and -12 increased by 2

Answer the following questions.
73. What number do you add to -8 to get -5 ?
74. What number do you add to 10 to get 4 ?
75. The sum of what number and -6 is -9 ?
76. The sum of what number and -12 is 8 ?

## Applying the Concepts

77. Temperature Change The temperature at noon is 12 degrees below 0 Fahrenheit. By 1:00 it has risen 4 degrees. Write an expression using the numbers -12 and 4 to describe this situation.

78. Stock Value On Monday a certain stock gains 2 points. On Tuesday it loses 3 points. Write an expression using positive and negative numbers with addition to describe this situation and then simplify.
79. Gambling On three consecutive hands of draw poker a gambler wins $\$ 10$, loses $\$ 6$, and then loses another $\$ 8$. Write an expression using positive and negative numbers and addition to describe this situation and then simplify.
80. Number Problem You know from your past experience with numbers that subtracting 5 from 8 results in $3(8-5=3)$. What addition problem that starts with the number 8 gives the same result?
81. Checkbook Balance Suppose that you balance your checkbook and find that you are overdrawn by $\$ 30$; that is, your balance is $-\$ 30$. Then you go to the bank and deposit $\$ 40$. Translate this situation into an addition problem, the answer to which gives the new balance in your checkbook.
82. Improving Your Quantitative Literacy One of our goals is for you to be able to read graphs and charts like the one you see here, and to recognize when the information in the chart is displayed correctly. Although the chart appears exactly as it did in USA Today, this does not mean it is correct. A problem surfaces when we ask ourselves "How many births does one baby carriage represent?" Explain what is wrong with the chart.

USA TODAY Snapshots

## 20-somethings have most babies

About 4 million babies were born in 2002. A breakdown by mothers' ages and number of babies:
Under $15 \underset{\text { §g }}{ } \mathbf{7}, 315$



40-44 हैन्ठ 95,788
45-49 ${ }_{〔}^{f} 5,224$
50-54 263
Source: National Center for Health Statistics
By Shannon Reilly and Alelandro Conzalez, USA TODAY

## 1.4

## Subtraction of Real Numbers

## OBJECTIVES

A Subtract any combination of positive and negative numbers.

B Simplify expressions using the rule for order of operations.

C Translate sentences from English into symbols and then simplify.

D Find the complement and the supplement of an angle.

Suppose that the temperature at noon is $20^{\circ}$ Fahrenheit and 12 hours later, at midnight, it has dropped to $-15^{\circ}$ Fahrenheit. What is the difference between the temperature at noon and the temperature at midnight? Intuitively, we know the difference in the two temperatures is $35^{\circ}$. We also know that the word difference indicates subtraction. The difference between 20 and -15 is written

$$
20-(-15)
$$

It must be true that $20-(-15)=35$. In this section we will see how our definition for subtraction confirms that this last statement is in fact correct.

In the previous section we spent some time developing the rule for addition of real numbers. Because we want to make as few rules as possible, we can define subtraction in terms of addition. By doing so, we can then use the rule for addition to solve our subtraction problems.

Rule To subtract one real number from another, simply add its opposite.
Algebraically, the rule is written like this: If $a$ and $b$ represent two real numbers, then it is always true that

$$
\underbrace{a-b}_{\text {To subtract } b}=\underbrace{a+(-b)}_{\text {add the opposite of } b}
$$

This is how subtraction is defined in algebra. This definition of subtraction will not conflict with what you already know about subtraction, but it will allow you to do subtraction using negative numbers.

## EXAMPLE 1

Subtract all possible combinations of positive and nega-
tive 7 and 2.

## SOLUTION

$$
\left.\begin{array}{rl}
7-2 & =7+(-2)=5 \\
-7-2 & =-7+(-2)=-9
\end{array}\right\}
$$

Subtracting 2 is the same as adding - 2

Subtracting - 2 is the same as adding 2

Notice that each subtraction problem is first changed to an addition problem. The rule for addition is then used to arrive at the answer.

We have defined subtraction in terms of addition, and we still obtain answers consistent with the answers we are used to getting with subtraction. Moreover, we now can do subtraction problems involving both positive and negative numbers.

As you proceed through the following examples and the problem set, you will begin to notice shortcuts you can use in working the problems. You will not always have to change subtraction to addition of the opposite to be able to get answers quickly. Use all the shortcuts you wish as long as you consistently get the correct answers.

EXAMPLE 2 Subtract all combinations of positive and negative 8 and 13 .

## SOLUTION

$$
\begin{array}{rlrl}
8-13=8+(-13) & =-5 & & \text { Subtracting }+13 \text { is the } \\
-8-13=-8+(-13) & =-21 & & \text { same as adding }-13 \\
8-(-13)=8+13 & =21 & & \text { Subtracting }-13 \text { is the } \\
-8-(-13)=-8+13=5 & & \text { same as adding }+13
\end{array}
$$

## EXAMPLES Simplify each expression as much as possible.

3. $\left.\begin{array}{rl}7+(-3)-5=7+(-3)+ & (-5) \\ & =4+(-5) \\ & =-1\end{array}\right\}$
4. $8-(-2)-6=8+2+(-6)$

$$
=10+(-6) \quad\}
$$

$$
=4
$$

5. $-2-(-3+1)-5=-2-(-2)-5$

$$
\begin{aligned}
& =-2+2+(-5) \\
& =-5
\end{aligned}
$$

Begin by changing all subtractions to additions Then add left to right
Begin by changing all subtractions to additions Then add left to right
Do what is in the parentheses first

The next two examples involve multiplication and exponents as well as subtraction. Remember, according to the rule for order of operations, we evaluate the numbers containing exponents and multiply before we subtract.

## EXAMPLE 6 Simplify $2 \cdot 5-3 \cdot 8-4 \cdot 9$.

SOLUTION First, we multiply left to right, and then we subtract.

$$
\begin{aligned}
2 \cdot 5-3 \cdot 8-4 \cdot 9 & =10-24-36 \\
& =-14-36 \\
& =-50
\end{aligned}
$$

EXAMPLE 7 Simplify $3 \cdot 2^{3}-2 \cdot 4^{2}$.
SOLUTION We begin by evaluating each number that contains an exponent. Then we multiply before we subtract:

$$
\begin{aligned}
3 \cdot 2^{3}-2 \cdot 4^{2} & =3 \cdot 8-2 \cdot 16 \\
& =24-32 \\
& =-8
\end{aligned}
$$



EXAMPLE 8 Subtract 7 from -3 .
SOLUTION First, we write the problem in terms of subtraction. We then change to addition of the opposite:

$$
\begin{aligned}
-3-7 & =-3+(-7) \\
& =-10
\end{aligned}
$$

## EXAMPLE 9 Subtract -5 from 2.

SOLUTION Subtracting -5 is the same as adding +5 :

$$
\begin{aligned}
2-(-5) & =2+5 \\
& =7
\end{aligned}
$$

EXAMPLE 10 Find the difference of 9 and 2.
SOLUTION Written in symbols, the problem looks like this:

$$
9-2=7
$$

The difference of 9 and 2 is 7 .

## EXAMPLE 11 Find the difference of 3 and -5 .

SOLUTION Subtracting -5 from 3 we have

$$
\begin{aligned}
3-(-5) & =3+5 \\
& =8
\end{aligned}
$$

In the sport of drag racing, two cars at the starting line race to the finish line $\frac{1}{4}$ mile away. The car that crosses the finish line first wins the race.
Jim Rizzoli owns and races an alcohol dragster. On board the dragster is a computer that records data during each of Jim's races. Table 1 gives some of the data from a race Jim was in. Figure 1 gives the same information visually.

| TABLE 1 <br> Speed of a Race Car <br> Time in <br> Seconds | Speed in <br> Miles/Hour |
| :---: | :---: |
| 0 | 0 |
| 1 | 72.7 |
| 2 | 129.9 |
| 3 | 162.8 |
| 4 | 192.2 |
| 5 | 212.4 |
| 6 | 228.1 |



EXAMPLE 12 Use Table 1 to find the difference in speed after 5 seconds and after 2 seconds have elapsed during the race.

SOLUTION We know the word difference implies subtraction. The speed at 2 seconds is 129.9 miles per hour, whereas the speed at 5 seconds is 212.4 miles per hour. Therefore, the expression that represents the solution to our problem looks like this:
$212.4-129.9=82.5$ miles per hour


Complementary angles: $x+y=90^{\circ}$


Supplementary angles: $x+y=180^{\circ}$

## FACTS FROM GEOMETRY

## Complementary and Supplementary Angles

If you have studied geometry at all, you know that there are $360^{\circ}$ in a full rotation-the number of degrees swept out by the radius of a circle as it rotates once around the circle.


We can apply our knowledge of algebra to help solve some simple geometry problems. Before we do, however, we need to review some of the vocabulary associated with angles.

DEFINITION In geometry, two angles that add to $90^{\circ}$ are called complementary angles. In a similar manner, two angles that add to $180^{\circ}$ are called supplementary angles. The diagrams at the left illustrate the relationships between angles that are complementary and between angles that are supplementary.

EXAMPLE 13 Find $x$ in each of the following diagrams.


SOLUTION We use subtraction to find each angle.
a. Because the two angles are complementary, we can find $x$ by subtracting $30^{\circ}$ from $90^{\circ}$ :
$x=90^{\circ}-30^{\circ}=60^{\circ}$
We say $30^{\circ}$ and $60^{\circ}$ are complementary angles. The complement of $30^{\circ}$ is $60^{\circ}$.
b. The two angles in the diagram are supplementary. To find $x$, we subtract $45^{\circ}$ from $180^{\circ}$ :
$x=180^{\circ}-45^{\circ}=135^{\circ}$
We say $45^{\circ}$ and $135^{\circ}$ are supplementary angles. The supplement of $45^{\circ}$ is $135^{\circ}$.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,2
B 3-7
C 8 -12
D $\quad 13$

## Subtraction and Taking Away

For some people taking algebra for the first time, subtraction of positive and negative numbers can be a problem. These people may believe that $-5-9$ should be -4 or 4, not -14 . If this is happening to you, you probably are thinking of subtraction in terms of taking one number away from another. Thinking of subtraction in this way works well with positive numbers if you always subtract the smaller number from the larger. In algebra, however, we encounter many situations other than this. The definition of subtraction, that $a-b=a+(-b)$, clearly indicates the correct way to use subtraction; that is, when working subtraction problems, you should think "addition of the opposite," not "take one number away from another." To be successful in algebra, you need to apply properties and definitions exactly as they are presented here.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Why do we define subtraction in terms of addition?
2. Write the definition for $a-b$.
3. Explain in words how you would subtract 3 from -7 .
4. What are complementary angles?

## Problem Set 1.4

The following problems are intended to give you practice with subtraction of positive and negative numbers. Remember, in algebra subtraction is not taking one number away from another. Instead, subtracting a number is equivalent to adding its opposite.

Subtract.

1. $5-8$
2. $6-7$
3. $3-9$
4. $2-7$
5. $5-5$
6. $8-8$
7. $-8-2$
8. $-6-3$
9. $-4-12$
10. $-3-15$
11. $-6-6$
12. $-3-3$
13. $-8-(-1)$
14. $-6-(-2)$
15. $15-(-20)$
16. $20-(-5)$
17. $-4-(-4)$
18. $-5-(-5)$

Simplify each expression by applying the rule for order of operations.
19. $3-2-5$
21. $9-2-3$
23. $-6-8-10$
25. $-22+4-10$
27. $10-(-20)-5$
29. $8-(2-3)-5$
31. $7-(3-9)-6$
33. $5-(-8-6)-2$
35. $-(5-7)-(2-8)$
37. $-(3-10)-(6-3)$
39. $16-[(4-5)-1]$
41. $5-[(2-3)-4]$
43. $21-[-(3-4)-2]-5$
44. $30-[-(10-5)-15]-25$
= Videos available by instructor request
| = Online student support materials available at www.thomsonedu.com/login

The following problems involve multiplication and exponents. Use the rule for order of operations to simplify each expression as much as possible.
45. $2 \cdot 8-3 \cdot 5$
46. $3 \cdot 4-6 \cdot 7$
47. $3 \cdot 5-2 \cdot 7$
48. $6 \cdot 10-5 \cdot 20$
49. $5 \cdot 9-2 \cdot 3-6 \cdot 2$
50. $4 \cdot 3-7 \cdot 1-9 \cdot 4$
51. $3 \cdot 8-2 \cdot 4-6 \cdot 7$
52. $5 \cdot 9-3 \cdot 8-4 \cdot 5$
53. $2 \cdot 3^{2}-5 \cdot 2^{2}$
54. $3 \cdot 7^{2}-2 \cdot 8^{2}$
55. $4 \cdot 3^{3}-5 \cdot 2^{3}$
56. $3 \cdot 6^{2}-2 \cdot 3^{2}-8 \cdot 6^{2}$

Subtract.
57. $-3.4-7.9$
58. $-3.5-2.3$
59. $3.3-6.9$
60. $2.2-7.5$
61. Find the value of $x+y-4$ when
a. $x=-3$ and $y=-2$
b. $x=-9$ and $y=3$
c. $x=-\frac{3}{5}$ and $y=\frac{8}{5}$
62. Find the value of $x-y-3$ when
a. $x=-4$ and $y=1$
b. $x=2$ and $y=-1$
c. $x=-5$ and $y=-2$

Rewrite each of the following phrases as an equivalent expression in symbols, and then simplify.
63. Subtract 4 from -7 .
64. Subtract 5 from -19 .
65. Subtract -8 from 12 .
66. Subtract -2 from 10 .
67. Subtract -7 from -5 .
68. Subtract -9 from -3 .
69. Subtract 17 from the sum of 4 and -5 .
70. Subtract -6 from the sum of 6 and -3 .

Recall that the word difference indicates subtraction. The difference of $a$ and $b$ is $a-b$, in that order. Write a numerical expression that is equivalent to each of the following phrases, and then simplify.
71. The difference of 8 and 5
72. The difference of 5 and 8
73. The difference of -8 and 5
74. The difference of -5 and 8
75. The difference of 8 and -5
76. The difference of 5 and -8

Answer the following questions.
77. What number do you subtract from 8 to get -2 ?
78. What number do you subtract from 1 to get -5 ?
79. What number do you subtract from 8 to get 10 ?
80. What number do you subtract from 1 to get 5 ?

## Applying the Concepts

81. Savings Account Balance A man with $\$ 1,500$ in a savings account makes a withdrawal of $\$ 730$. Write an expression using subtraction that describes this situation.

| First Bank <br> Account No. 12345 |  |  |  |
| :--- | :---: | :---: | :---: |
| Date | Withdrawals | Deposits | Balance |
| $1 / 1 / 99$ |  |  | 1,500 |
| $2 / 2 / 99$ | 730 |  |  |

82. Temperature Change The temperature inside a Space Shuttle is $73^{\circ} \mathrm{F}$ before reentry. During reentry the temperature inside the craft increases $10^{\circ}$. On landing it drops $8^{\circ} \mathrm{F}$. Write an expression using the numbers 73,10 , and 8 to describe this situation. What is the temperature inside the shuttle on landing?
83. Gambling A man who has lost $\$ 35$ playing roulette in Las Vegas wins $\$ 15$ playing blackjack. He then loses $\$ 20$ playing the wheel of fortune. Write an expression using the numbers $-35,15$, and 20 to describe this situation and then simplify it.
84. Altitude Change An airplane flying at 10,000 feet lowers its altitude by 1,500 feet to avoid other air traffic. Then it increases its altitude by 3,000 feet to clear a mountain range. Write an expression that describes this situation and then simplify it.
85. Checkbook Balance Bob has $\$ 98$ in his checking account when he writes a check for $\$ 65$ and then another check for $\$ 53$. Write a subtraction problem that gives the new balance in Bob's checkbook. What is his new balance?
86. Temperature Change The temperature at noon is $23^{\circ} \mathrm{F}$. Six hours later it has dropped $19^{\circ} \mathrm{F}$, and by midnight it has dropped another $10^{\circ} \mathrm{F}$. Write a subtraction problem that gives the temperature at midnight. What is the temperature at midnight?
87. Depreciation Stacey buys a used car for $\$ 4,500$. With each year that passes, the car drops $\$ 550$ in value. Write a sequence of numbers that gives the value of the car at the beginning of each of the first 5 years she owns it. Can this sequence be considered an arithmetic sequence?
88. Depreciation Wade buys a computer system for $\$ 6,575$. Each year after that he finds that the system is worth $\$ 1,250$ less than it was the year before. Write a sequence of numbers that gives the value of the computer system at the beginning of each of the first four years he owns it. Can this sequence be considered an arithmetic sequence?

Drag Racing The table shown here extends the information given in Table 1 of this section. In addition to showing the time and speed of Jim Rizzoli's dragster during a race, it also shows the distance past the starting line that his dragster has traveled. Use the information in the table shown here to answer the following questions.

| Speed and Distance for a Race Car <br> Time in <br> Seconds |  |  |
| :---: | :---: | :---: |
| 0 | Speed in <br> Miles/Hour | Distance Traveled <br> in Feet |
| 1 | 0 | 0 |
| 2 | 72.7 | 69 |
| 3 | 129.9 | 231 |
| 4 | 162.8 | 439 |
| 5 | 212.4 | 728 |
| 6 | 228.1 | 1,000 |

89. Find the difference in the distance traveled by the dragster after 5 seconds and after 2 seconds.
90. How much faster is he traveling after 4 seconds than he is after 2 seconds?
91. How far from the starting line is he after 3 seconds?
92. How far from the starting line is he when his speed is 192.2 miles per hour?
93. How many seconds have gone by between the time his speed is 162.8 miles per hour and the time at which he has traveled 1,000 feet?
94. How many seconds have gone by between the time at which he has traveled 231 feet and the time at which his speed is 228.1 miles per hour?

Find $x$ in each of the following diagrams.
95.

96.

97.

98.

99. Grass Growth The bar chart below shows the growth of a certain species of grass over a period of 10 days.
a. Use the chart to fill in the missing entries in the table.
b. How much higher is the grass after 8 days than after 3 days?

| Day | Plant Height (inches) | Day | Plant Height (inches) |
| :---: | :---: | :---: | :---: |
| 0 | 0 | 6 |  |
| 1 | 0.5 | 7 |  |
| 2 |  |  | 13 |
|  | 1.5 | 9 | 18 |
| 4 |  | 10 |  |
| 5 | 4 |  |  |

## Overall Plant Height <br> 

100. Improving Your Quantitative Literacy The chart shown here appeared in USA Today during the first week of June 2004. Use the chart to answer the following questions.
a. Do you think the numbers in the chart have been rounded? If so, to which place were they rounded?
b. How many more participants were there in 2003 than in 2000?
c. If the trend shown in the table continues, estimate how many participants there will be in 2006.


By Ellen J. Horrow and Keith Simmons, USA TODAY

## Properties of Real Numbers

## OBJECTIVES

A Rewrite expressions using the commutative and associative properties.

B Multiply using the distributive property.

C Identify properties used to rewrite an expression.

In this section we will list all the facts (properties) that you know from past experience are true about numbers in general. We will give each property a name so we can refer to it later in this book. Mathematics is very much like a game. The game involves numbers. The rules of the game are the properties and rules we are developing in this chapter. The goal of the game is to extend the basic rules to as many situations as possible.
You know from past experience with numbers that it makes no difference in which order you add two numbers; that is, $3+5$ is the same as $5+3$. This fact about numbers is called the commutative property of addition. We say addition is a commutative operation. Changing the order of the numbers does not change the answer.

There is one other basic operation that is commutative. Because 3(5) is the same as 5(3), we say multiplication is a commutative operation. Changing the order of the two numbers you are multiplying does not change the answer.

For all properties listed in this section, $a, b$, and $c$ represent real numbers.

## Commutative Property of Addition

In symbols: $a+b=b+a$
In words: Changing the order of the numbers in a sum will not change the result.

## Note

At this point, some students are confused by the expression $x+8$; they feel that there is more to do, but they don't know what. At this point, there isn't any more that can be done with $x+8$ unless we know what $x$ is. So $x+8$ is as far as we can go with this problem.

## Commutative Property of Multiplication

In symbols: $a \cdot b=b \cdot a$
In words: Changing the order of the numbers in a product will not change the result.

## EXAMPLES

1. The statement $5+8=8+5$ is an example of the commutative property of addition.
2. The statement $2 \cdot y=y \cdot 2$ is an example of the commutative property of multiplication.
3. The expression $5+x+3$ can be simplified using the commutative property of addition:

$$
\begin{aligned}
5+x+3 & =x+5+3 & & \text { Commutative property of addition } \\
& =x+8 & & \text { Addition }
\end{aligned}
$$

The other two basic operations, subtraction and division, are not commutative. The order in which we subtract or divide two numbers makes a difference in the answer.
Another property of numbers that you have used many times has to do with grouping. You know that when we add three numbers it makes no difference which two we add first. When adding $3+5+7$, we can add the 3 and 5 first and then the 7 , or we can add the 5 and 7 first and then the 3 . Mathematically, it looks like this: $(3+5)+7=3+(5+7)$. This property is true of multiplication as well. Operations that behave in this manner are called associative operations. The answer will not change when we change the association (or grouping) of the numbers.

## Associative Property of Addition

In symbols: $a+(b+c)=(a+b)+c$
In words: Changing the grouping of the numbers in a sum will not change the result.

## Associative Property of Multiplication

In symbols: $a(b c)=(a b) c$
In words: Changing the grouping of the numbers in a product will not change the result.

The following examples illustrate how the associative properties can be used to simplify expressions that involve both numbers and variables.

## EXAMPLES Simplify.

4. $\begin{aligned} 4+(5+x) & =(4+5)+x \quad \text { Associative property of addition } \\ & =9+x\end{aligned}$ $=9+x \quad$ Addition
5. $5(2 x)=(5 \cdot 2) X \quad$ Associative property of multiplication
$=10 x \quad$ Multiplication

## Note

Because subtraction is defined in terms of addition, it is also true that the distributive property applies to subtraction as well as addition; that is, $a(b-c)=a b-a c$ for any three real numbers $a, b$, and $c$.


FIGURE 1
6. $\frac{1}{5}(5 x)=\left(\frac{1}{5} \cdot 5\right) x$

$$
=1 X
$$

$$
=x
$$

7. $3\left(\frac{1}{3} x\right)=\left(3 \cdot \frac{1}{3}\right) x$

$$
=1 x
$$

$$
=x
$$

8. $12\left(\frac{2}{3} x\right)=\left(12 \cdot \frac{2}{3}\right) x$

$$
=8 x
$$

Associative property of multiplication
Multiplication

Associative property of multiplication
Multiplication

Associative property of multiplication
Multiplication

The associative and commutative properties apply to problems that are either all multiplication or all addition. There is a third basic property that involves both addition and multiplication. It is called the distributive property and looks like this.

## Distributive Property

In symbols: $a(b+c)=a b+a c$
In words: Multiplication distributes over addition.

You will see as we progress through the book that the distributive property is used very frequently in algebra. We can give a visual justification to the distributive property by finding the areas of rectangles. Figure 1 shows a large rectangle that is made up of two smaller rectangles. We can find the area of the large rectangle two different ways.

Method 1 We can calculate the area of the large rectangle directly by finding its length and width. The width is 5 inches, and the length is $(3+4)$ inches.

$$
\begin{aligned}
\text { Area of large rectangle } & =5(3+4) \\
& =5(7) \\
& =35 \text { square inches }
\end{aligned}
$$

Method 2 Because the area of the large rectangle is the sum of the areas of the two smaller rectangles, we find the area of each small rectangle and then add to find the area of the large rectangle.

$$
\begin{array}{rlrl}
\text { Area of large rectangle } & =\text { Area of rectangle I }+ \text { Area of rectangle II } \\
& = & 5(3) & + \\
& = & 15 & +
\end{array}
$$

In both cases the result is 35 square inches. Because the results are the same, the two original expressions must be equal. Stated mathematically, $5(3+4)=$ $5(3)+5(4)$. We can either add the 3 and 4 first and then multiply that sum by 5 , or we can multiply the 3 and the 4 separately by 5 and then add the products. In either case we get the same answer.

Here are some examples that illustrate how we use the distributive property.

## EXAMPLES Apply the distributive property to each expression, and

 then simplify the result.9. $2(x+3)=2(x)+2(3)$
Distributive property
$=2 x+6$
Multiplication
10. $\begin{aligned} 5(2 x-8) & =5(2 x)-5(8) \\ & =10 x-40\end{aligned}$

$$
=10 x-40
$$

Distributive property
Multiplication

Notice in this example that multiplication distributes over subtraction as well as addition.
11. $4(x+y)=4 x+4 y$

Distributive property
12. $5(2 x+4 y)=5(2 x)+5(4 y)$

Distributive property
$=10 x+20 y$
Multiplication
13. $\frac{1}{2}(3 x+6)=\frac{1}{2}(3 x)+\frac{1}{2}(6)$

Distributive property
$=\frac{3}{2} x+3$
Multiplication
14. $4(2 a+3)+8=4(2 a)+4(3)+8 \quad$ Distributive property

$$
\begin{array}{ll}
=8 a+12+8 & \\
=8 a+20 & \\
\text { Multiplication } \\
\text { Addition }
\end{array}
$$

15. $a\left(1+\frac{1}{a}\right)=a \cdot 1+a \cdot \frac{1}{a}$

Distributive property

$$
=a+1
$$

## Multiplication

16. $3\left(\frac{1}{3} x+5\right)=3 \cdot \frac{1}{3} x+3 \cdot 5$

Distributive property

$$
=x+15
$$

Multiplication
17. $12\left(\frac{2}{3} x+\frac{1}{2} y\right)=12 \cdot \frac{2}{3} x+12 \cdot \frac{1}{2} y$

$$
=8 x+6 y \quad \text { Multiplication }
$$

## Special Numbers

In addition to the three properties mentioned so far, we want to include in our list two special numbers that have unique properties. They are the numbers zero and one.

Additive Identity Property There exists a unique number 0 such that In symbols: $\quad a+0=a$ and $0+a=a$
In words: Zero preserves identities under addition. (The identity of the number is unchanged after addition with 0 .)

Multiplicative Identity Property There exists a unique number 1 such that In symbols: $a(1)=a \quad$ and $\quad 1(a)=a$

In words: $\quad$ The number 1 preserves identities under multiplication. (The identity of the number is unchanged after multiplication by 1.)

Additive Inverse Property For each real number $a$, there exists a unique number - $a$ such that

In symbols: $\quad a+(-a)=0$
In words: Opposites add to 0.

Multiplicative Inverse Property For every real number $a$, except 0 , there exists a unique real number $\frac{1}{a}$ such that
In symbols: $\quad a\left(\frac{1}{a}\right)=1$
In words: Reciprocals multiply to 1 .
Of all the basic properties listed, the commutative, associative, and distributive properties are the ones we will use most often. They are important because they will be used as justifications or reasons for many of the things we will do.
The following examples illustrate how we use the preceding properties. Each one contains an algebraic expression that has been changed in some way. The property that justifies the change is written to the right.

## EXAMPLES

State the property that justifies the given statement.
18. $x+5=5+x$

Commutative property of addition
19. $(2+x)+y=2+(x+y)$

Associative property of addition
20. $6(x+3)=6 x+18$

Distributive property
21. $2+(-2)=0$

Additive inverse property
22. $3\left(\frac{1}{3}\right)=1$

Multiplicative inverse property
23. $(2+0)+3=2+3$

Additive identity property
24. $(2+3)+4=3+(2+4)$

Commutative and associative properties of addition
25. $(x+2)+y=(x+y)+2$

Commutative and associative properties of addition


As a final note on the properties of real numbers, we should mention that although some of the properties are stated for only two or three real numbers, they hold for as many numbers as needed. For example, the distributive property holds for expressions like $3(x+y+z+5+2)$; that is,

$$
3(x+y+z+5+2)=3 x+3 y+3 z+15+6
$$

It is not important how many numbers are contained in the sum, only that it is a sum. Multiplication, you see, distributes over addition, whether there are two numbers in the sum or 200.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-8

B 9-17
C $\quad 18-25$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the commutative property of addition?
2. Do you know from your experience with numbers that the commutative property of addition is true? Explain why.
3. Write the commutative property of multiplication in symbols and words.
4. How do you rewrite expressions using the distributive property?

State the property or properties that justify the following.

1. $3+2=2+3$
2. $5+0=5$
3. $4\left(\frac{1}{4}\right)=1$
4. $10(0.1)=1$
5. $4+x=x+4$
6. $3(x-10)=3 x-30$
7. $2(y+8)=2 y+16$
8. $3+(4+5)=(3+4)+5$
9. $(3+1)+2=1+(3+2)$
10. $(5+2)+9=(2+5)+9$
11. $(8+9)+10=(8+10)+9$
12. $(7+6)+5=(5+6)+7$
13. $3(x+2)=3(2+x)$
14. $2(7 y)=(7 \cdot 2) y$
15. $x(3 y)=3(x y)$
16. $a(5 b)=5(a b)$
17. $4(x y)=4(y x)$
18. $3[2+(-2)]=3(0)$
19. $8[7+(-7)]=8(0)$
20. $7(1)=7$

Each of the following problems has a mistake in it. Correct the right-hand side.
21. $3(x+2)=3 x+2$
22. $5(4+x)=4+5 x$
23. $9(a+b)=9 a+b$
24. $2(y+1)=2 y+1$
25. $3(0)=3$
26. $5\left(\frac{1}{5}\right)=5$
27. $3+(-3)=1$
29. $10(1)=0$
28. $8(0)=8$
30. $3 \cdot \frac{1}{3}=0$

Use the associative property to rewrite each of the following expressions, and then simplify the result. (See Examples 4, 5, and 6.)
31. $4+(2+x)$
32. $5+(6+x)$
33. $(x+2)+7$
34. $(x+8)+2$
35. $3(5 x)$
37. 9 (6y)
39. $\frac{1}{2}(3 a)$
40. $\frac{1}{3}(2 a)$
41. $\frac{1}{3}(3 x)$
42. $\frac{1}{4}(4 x)$
43. $\frac{1}{2}(2 y)$
44. $\frac{1}{7}(7 y)$
45. $\frac{3}{4}\left(\frac{4}{3} x\right)$
46. $\frac{3}{2}\left(\frac{2}{3} x\right)$
47. $\frac{6}{5}\left(\frac{5}{6} a\right)$
48. $\frac{2}{5}\left(\frac{5}{2} a\right)$

Apply the distributive property to each of the following expressions. Simplify when possible.
49. $8(x+2)$
50. $5(x+3)$
51. $8(x-2)$
52. $5(x-3)$
53. $4(y+1)$
54. $4(y-1)$
55. $3(6 x+5)$
56. $3(5 x+6)$
57. $2(3 a+7)$
58. $5(3 a+2)$
59. $9(6 y-8)$
60. $2(7 y-4)$

Apply the distributive property to each of the following expressions. Simplify when possible.
61. $\frac{1}{2}(3 x-6)$
62. $\frac{1}{3}(2 x-6)$
63. $\frac{1}{3}(3 x+6)$
64. $\frac{1}{2}(2 x+4)$
65. $3(x+y)$
66. $2(x-y)$
67. $8(a-b)$
68. $7(a+b)$
69. $6(2 x+3 y)$
70. $8(3 x+2 y)$
71. $4(3 a-2 b)$
72. $5(4 a-8 b)$
73. $\frac{1}{2}(6 x+4 y)$
74. $\frac{1}{3}(6 x+9 y)$
75. $4(a+4)+9$
76. $6(a+2)+8$
77. $2(3 x+5)+2$
78. $7(2 x+1)+3$
79. $7(2 x+4)+10$
80. $3(5 x+6)+20$

Here are some problems you will see later in the book. Apply the distributive property and simplify, if possible.
81. $\frac{1}{2}(4 x+2)$
82. $\frac{1}{3}(6 x+3)$
83. $\frac{3}{4}(8 x-4)$
84. $\frac{2}{5}(5 x+10)$
85. $\frac{5}{6}(6 x+12)$
86. $\frac{2}{3}(9 x-3)$
87. $10\left(\frac{3}{5} x+\frac{1}{2}\right)$
89. $15\left(\frac{1}{3} x+\frac{2}{5}\right)$
91. $12\left(\frac{1}{2} m-\frac{5}{12}\right)$
93. $21\left(\frac{1}{3}+\frac{1}{7} x\right)$
95. $6\left(\frac{1}{2} x-\frac{1}{3} y\right)$
88. $8\left(\frac{1}{4} x-\frac{5}{8}\right)$
90. $12\left(\frac{1}{12} m+\frac{1}{6}\right)$
92. $8\left(\frac{1}{8}+\frac{1}{2} m\right)$
94. $6\left(\frac{3}{2} y+\frac{1}{3}\right)$
96. $12\left(\frac{1}{4} x+\frac{2}{3} y\right)$
97. $0.09(x+2,000)$
98. $0.04(x+7,000)$
99. $0.12(x+500)$
100. $0.06(x+800)$
101. $a\left(1+\frac{1}{a}\right)$
102. $a\left(1-\frac{1}{a}\right)$
103. $a\left(\frac{1}{a}-1\right)$
104. $a\left(\frac{1}{a}+1\right)$

## Applying the Concepts

105. Getting Dressed While getting dressed for work, a man puts on his socks and puts on his shoes. Are the two statements "put on your socks" and "put on your shoes" commutative?
106. Getting Dressed Are the statements "put on your left shoe" and "put on your right shoe" commutative?
107. Skydiving A skydiver flying over the jump area is about to do two things: jump out of the plane and pull the rip
 cord. Are the two events "jump out of the plane" and "pull the rip cord" commutative? That is, will changing the order of the events always produce the same result?
108. Commutative Property Give an example of two events in your daily life that are commutative.
109. Division Give an example that shows that division is not a commutative operation; that is, find two numbers for which changing the order of division gives two different answers.
110. Subtraction Simplify the expression $10-(5-2)$ and the expression $(10-5)-2$ to show that subtraction is not an associative operation.
111. Take-Home Pay Jose works at a winery. His monthly salary is $\$ 2,400$. To cover his taxes and retirement, the winery withholds $\$ 480$ from each check. Calculate his yearly "take-home" pay using the numbers $2,400,480$, and 12 . Do the calculation two different ways so that the results give further justification for the distributive property.
112. Hours Worked Carlo works as a waiter. He works double shifts 4 days a week. The lunch shift is 2 hours and the dinner shift is 3 hours. Find the total number of hours he works per week using the numbers 2,3 , and 4 . Do the calculation two different ways so that the results give further justification for the distributive property.
113. College Expenses Maria is estimating her expenses for attending college for a year. Tuition is $\$ 650$ per academic quarter. She estimates she will spend $\$ 225$ on books each quarter. If she plans on attending 3 academic quarters during the year, how much can she expect to spend? Do the calculation two different ways so that the results give further justification for the distributive property.
114. Improving Your Quantitative Literacy Although everything you do in this course will help improve your quantitative literacy, these problems will extend the type of reasoning and thinking you are using in the classroom to situations you will find outside of the classroom. Here is what the Mathematical

Association of America has to say about quantitative literacy:
. . . . every college graduate should be able to apply simple mathematical methods to the solution of real-world problems. A quantitatively literate college graduate should be able to:

1. Interpret mathematical models such as formulas, graphs, tables, and schematics, and draw inferences from them.
2. Represent mathematical information symbolically, visually, numerically, and verbally.
3. Estimate and check answers to mathematical problems in order to determine reasonableness, identify alternatives, and select optimal results.
4. Recognize that mathematical and statistical methods have limits.

List two specific skills that you would like to acquire from this course that will improve your quantitative literacy. For example, you may want to better understand the financing options available for purchasing a new car.


## Multiplication of Real Numbers

## OBJECTIVES

A
Multiply any combination of positive and negative numbers.

B Simplify expressions using the rule for order of operations.

C Multiply fractions.
D Multiply using the associative property.

E Multiply using the distributive property.

F Extend a geometric sequence.

Suppose that you own 5 shares of a stock and the price per share drops $\$ 3$. How much money have you lost? Intuitively, we know the loss is $\$ 15$. Because it is a loss, we can express it as $-\$ 15$. To describe this situation with numbers, we would write


Reasoning in this manner, we conclude that the product of a positive number with a negative number is a negative number. Let's look at multiplication in more detail.

From our experience with counting numbers, we know that multiplication is simply repeated addition; that is, $3(5)=5+5+5$. We will use this fact, along with our knowledge of negative numbers, to develop the rule for multiplication of any two real numbers. The following examples illustrate multiplication with three of the possible combinations of positive and negative numbers.

## Note

You may have to read the explanation for Example 4 several times before you understand it completely. The purpose of the explanation in Example 4 is simply to justify the fact that the product of two negative numbers is a positive number. If you have no trouble believing that, then it is not so important that you understand everything in the explanation.

## EXAMPLES

1. Two positives:
2. One positive:
3. One negative:

$$
\begin{aligned}
3(-5) & =-5+(-5)+(-5) \\
& =-15 \quad \quad \text { Negative answer }
\end{aligned}
$$

$$
-3(5)=5(-3) \quad \text { Commutative property }
$$

$$
=-3+(-3)+(-3)+(-3)+(-3)
$$

$$
=-15 \quad \text { Negative answer }
$$

4. Two negatives: $-3(-5)=$ ?

With two negatives, $-3(-5)$, it is not possible to work the problem in terms of repeated addition. (It doesn't "make sense" to write -5 down a -3 number of times.) The answer is probably +15 (that's just a guess), but we need some justification for saying so. We will solve a different problem and in so doing get the answer to the problem $(-3)(-5)$.

## Note

Some students have trouble with the expression $-8(-3)$ because they want to subtract rather than multiply. Because we are very precise with the notation we use in algebra, the expression $-8(-3)$ has only one meaning-multiplication. A subtraction problem that uses the same numbers is $-8-3$. Compare the two following lists.

No
All Multiplication
Multiplication
$5(4)$
$-5(4)$
$5+4$
$-5(4) \quad-5+4$
$5(-4) \quad 5-4$
$-5(-4) \quad-5-4$

Here is a problem to which we know the answer. We will work it two different ways.

$$
-3[5+(-5)]=-3(0)=0
$$

The answer is zero. We also can work the problem using the distributive property.

$$
\begin{aligned}
-3[5+(-5)] & =-3(5)+(-3)(-5) \quad \text { Distributive property } \\
& =-15+?
\end{aligned}
$$

Because the answer to the problem is 0 , our ? must be +15 . (What else could we add to -15 to get 0 ? Only +15 .)

Here is a summary of the results we have obtained from the first four examples.

| Original Numbers Have |  | The Answer is |
| :--- | :--- | :--- |
| the same sign | $3(5)=15$ | positive |
| different signs | $3(-5)=-15$ | negative |
| different signs | $-3(5)=-15$ | negative |
| the same sign | $-3(-5)=15$ | positive |

By examining Examples 1 through 4 and the preceding table, we can use the information there to write the following rule. This rule tells us how to multiply any two real numbers.

Rule To multiply any two real numbers, simply multiply their absolute values. The sign of the answer is

1. Positive if both numbers have the same sign (both + or both - ).
2. Negative if the numbers have opposite signs (one + , the other - ).

The following examples illustrate how we use the preceding rule to multiply real numbers.

## EXAMPLES

5. $-8(-3)=24$
6. $-10(-5)=50$
7. $-4(-7)=28$
8. $5(-7)=-35$
9. $-4(8)=-32$
10. $-6(10)=-60$

Multiply.

If the two numbers in the product have the same sign, the answer is positive

If the two numbers in the product have different signs, the answer is negative

In the following examples, we combine the rule for order of operations with the rule for multiplication to simplify expressions. Remember, the rule for order of operations specifies that we are to work inside the parentheses first and then simplify numbers containing exponents. After this, we multiply and divide, left to right. The last step is to add and subtract, left to right.

## EXAMPLES Simplify as much as possible.

11. $-5(-3)(-4)=15(-4)$
$=-60$
12. $4(-3)+6(-5)-10=-12+(-30)-10 \quad$ Multiply
$=-42-10 \quad$ Add
$=-52$
13. $(-2)^{3}=(-2)(-2)(-2)$
$=-8$
Subtract
Definition of exponents
Multiply, left to right
14. $-3(-2)^{3}-5(-4)^{2}=-3(-8)-5(16)$

Exponents first
$=24-80$
$=-56$
15. $6-4(7-2)=6-4(5)$

$$
=6-20
$$

$$
=-14
$$

Multiply
Subtract
Inside parentheses first
Multiply
Subtract

## Multiplying Fractions

Previously, we mentioned that to multiply two fractions we multiply numerators and multiply denominators. We can apply the rule for multiplication of positive and negative numbers to fractions in the same way we apply it to other numbers. We multiply absolute values: The product is positive if both fractions have the same sign and negative if they have different signs. Here are some examples.

## EXAMPLES

Multiply.
16. $-\frac{3}{4}\left(\frac{5}{7}\right)=-\frac{3 \cdot 5}{4 \cdot 7}$

Different signs give a negative answer
$=-\frac{15}{28}$
17. $-6\left(\frac{1}{2}\right)=-\frac{6}{1}\left(\frac{1}{2}\right) \quad$ Different signs give a negative answer
$=-\frac{6}{2}$
$=-3$
18. $-\frac{2}{3}\left(-\frac{3}{2}\right)=\frac{2 \cdot 3}{3 \cdot 2} \quad$ Same signs give a positive answer
$=\frac{6}{6}$
$=1$

EXAMPLE 19
Figure 1 gives the calories that are burned in 1 hour for a variety of forms of exercise by a person weighing 150 pounds. Figure 2 gives the calories that are consumed by eating some popular fast foods. Find the net change in calories for a 150-pound person playing handball for 2 hours and then eating a Whopper.


SOLUTION The net change in calories will be the difference of the calories gained from eating and the calories lost from exercise.

$$
\text { Net change in calories }=630-2(680)=-730 \text { calories }
$$

We can use the rule for multiplication of real numbers, along with the associative property, to multiply expressions that contain numbers and variables.

## EXAMPLES

$$
\text { 20. }-3(2 x)=(-3 \cdot 2) x
$$

$$
=-6 x
$$

Apply the associative property, and then multiply.

$$
=-30 y
$$

21. $\begin{aligned} 6(-5 y) & =[6(-5)] y & & \text { Associative property } \\ & =-30 y & & \text { Multiplication }\end{aligned}$

## Associative property

 Multiplication22. $\begin{aligned}-2\left(-\frac{1}{2} x\right) & =\left[(-2)\left(-\frac{1}{2}\right)\right] x \\ & =1 x \\ & =x\end{aligned}$

Associative property
Multiplication
Multiplication
The following examples show how we can use both the distributive property and multiplication with real numbers.

## EXAMPLES

Apply the distributive property to each expression.
23. $-2(a+3)=-2 a+(-2)(3)$

$$
\begin{aligned}
& =-2 a+(-6) \\
& =-2 a-6
\end{aligned}
$$

24. $-3(2 x+1)=-3(2 x)+(-3)(1)$

## Distributive property

Multiplication

Distributive property

$$
=-6 x+(-3) \quad \text { Multiplication }
$$

$$
=-6 x-3
$$

25. $-\frac{1}{3}(2 x-6)=-\frac{1}{3}(2 x)-\left(-\frac{1}{3}\right)(6)$
$=-\frac{2}{3} x-(-2)$
Distributive property
$=-\frac{2}{3} x+2$
Multiplication

$$
\begin{aligned}
& =-12 x-(-20)-8 \\
& =-12 x+20-8 \\
& =-12 x+12
\end{aligned}
$$

Distributive property
26. $-4(3 x-5)-8=-4(3 x)-(-4)(5)-8$

## Multiplication

Definition of subtraction
Subtraction

The next examples continue the work we did previously with finding the value of an algebraic expression for given values of the variable or variables.

EXAMPLE 27 Find the value of $-\frac{2}{3} x-4$ when
a. $x=0$
b. $x=3$
c. $x=-\frac{9}{2}$

SOLUTION Substituting the values of $x$ into our expression one at a time we have
a. $-\frac{2}{3}(0)-4=0-4=-4$
b. $-\frac{2}{3}(3)-4=-2-4=-6$
c. $-\frac{2}{3}\left(-\frac{9}{2}\right)-4=3-4=-1$

## EXAMPLE 28

Find the value of $5 x-4 y$ when
a. $x=4$ and $y=0$
b. $x=0$ and $y=-8$
c. $x=-2$ and $y=3$

SOLUTION We substitute the given values for $x$ and $y$ and then simplify
a. $5(4)-4(0)=20-0=20$
b. $5(0)-4(-8)=0+32=32$
c. $5(-2)-4(3)=-10-12=-22$

## Geometric Sequences

A geometric sequence is a sequence of numbers in which each number (after the first number) comes from the number before it by multiplying by the same amount each time. For example, the sequence

$$
2,6,18,54, \ldots
$$

is a geometric sequence because each number is obtained by multiplying the number before it by 3 .

EXAMPLE 29 Each sequence below is a geometric sequence. Find the next number in each sequence.
a. $5,10,20, \ldots$
b. $3,-15,75, \ldots$
c. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$

SOLUTION Because each sequence is a geometric sequence, we know that each term is obtained from the previous term by multiplying by the same number each time.
a. $5,10,20, \ldots$ : Starting with 5 , each number is obtained from the previous number by multiplying by 2 each time. The next number will be $20 \cdot 2=40$.
b. $3,-15,75, \ldots$ : The sequence starts with 3 . After that, each number is obtained by multiplying by -5 each time. The next number will be $75(-5)=-375$.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $1-10$
B $\quad 11-15$
C $\quad 16-18$
D 20-22
E $\quad 23-26$
F 29
c. $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \ldots$ : This sequence starts with $\frac{1}{8}$. Multiplying each number in the sequence by 2 produces the next number in the sequence. To extend the sequence, we multiply $\frac{1}{2}$ by $2: \frac{1}{2} \cdot 2=1$ The next number in the sequence is 1 .

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How do you multiply two negative numbers?
2. How do you multiply two numbers with different signs?
3. Explain how some multiplication problems can be thought of as repeated addition.
4. What is a geometric sequence?

## Problem Set 1.6

Online support materials can be found at www.thomsonedu.com/login

Use the rule for multiplying two real numbers to find each of the following products.

1. $7(-6)$
2. $8(-4)$
3. $-8(2)$
4. $-16(3)$
5. $-3(-1)$
6. $-7(-1)$
7. $-11(-11)$
8. $-12(-12)$

Use the rule for order of operations to simplify each expression as much as possible.
9. $-3(2)(-1)$
10. $-2(3)(-4)$
11. $-3(-4)(-5)$
12. $-5(-6)(-7)$
13. $-2(-4)(-3)(-1)$
14. $-1(-3)(-2)(-1)$
15. $(-7)^{2}$
16. $(-8)^{2}$
17. $(-3)^{3}$
18. $(-2)^{4}$
19. $-2(2-5)$
20. $-3(3-7)$
21. $-5(8-10)$
22. $-4(6-12)$
23. $(4-7)(6-9)$
24. $(3-10)(2-6)$
25. $(-3-2)(-5-4)$
26. $(-3-6)(-2-8)$
27. $-3(-6)+4(-1)$
28. $-4(-5)+8(-2)$
29. 2(3) $-3(-4)+4(-5)$
31. $4(-3)^{2}+5(-6)^{2}$
33. $7(-2)^{3}-2(-3)^{3}$
35. $6-4(8-2)$
37. $9-4(3-8)$
39. $-4(3-8)-6(2-5)$
41. $7-2[-6-4(-3)]$
43. $7-3[2(-4-4)-3(-1-1)]$
44. $5-3[7(-2-2)-3(-3+1)]$
45. Simplify each expression.
a. $5(-4)(-3)$
b. $5(-4)-3$
c. $5-4(-3)$
d. $5-4-3$
46. Simplify each expression.
a. $-2(-3)(-5)$
b. $-2(-3)-5$
c. $-2-3(-5)$
d. $-2-3-5$

Multiply the following fractions.
47. $-\frac{2}{3} \cdot \frac{5}{7}$
48. $-\frac{6}{5} \cdot \frac{2}{7}$
49. $-8\left(\frac{1}{2}\right)$
50. $-12\left(\frac{1}{3}\right)$
51. $\left(-\frac{3}{4}\right)^{2}$
52. $\left(-\frac{2}{5}\right)^{2}$
53. Simplify each expression.
a. $\frac{5}{8}(24)+\frac{3}{7}(28)$
b. $\frac{5}{8}(24)-\frac{3}{7}(28)$
c. $\frac{5}{8}(-24)+\frac{3}{7}(-28)$
d. $-\frac{5}{8}(24)-\frac{3}{7}(28)$
54. Simplify each expression.
a. $\frac{5}{6}(18)+\frac{3}{5}(15)$
b. $\frac{5}{6}(18)-\frac{3}{5}(15)$
c. $\frac{5}{6}(-18)+\frac{3}{5}(-15)$
d. $-\frac{5}{6}(18)-\frac{3}{5}(15)$

Simplify.
55. $\left(\frac{1}{2} \cdot 6\right)^{2}$
56. $\left(\frac{1}{2} \cdot 10\right)^{2}$
57. $\left(\frac{1}{2} \cdot 5\right)^{2}$
58. $\left[\frac{1}{2}(0.8)\right]^{2}$
59. $\left[\frac{1}{2}(-4)\right]^{2}$
60. $\left[\frac{1}{2}(-12)\right]^{2}$
61. $\left[\frac{1}{2}(-3)\right]^{2}$
62. $\left[\frac{1}{2}(-0.8)\right]^{2}$

Find the following products.
63. $-2(4 x)$
64. $-8(7 x)$
65. $-7(-6 x)$
66. $-8(-9 x)$
67. $-\frac{1}{3}(-3 x)$
68. $-\frac{1}{5}(-5 x)$

Apply the distributive property to each expression, and then simplify the result.
69. $-\frac{1}{2}(3 x-6)$
70. $-\frac{1}{4}(2 x-4)$
71. $-3(2 x-5)-7$
72. $-4(3 x-1)-8$
73. $-5(3 x+4)-10$
74. $-3(4 x+5)-20$
75. $-4(3 x+5 y)$
76. $5(5 x+4 y)$
77. $-2(3 x+5 y)$
78. $-2(2 x-y)$
79. $\frac{1}{2}(-3 x+6)$
80. $\frac{1}{4}(5 x-20)$
81. $\frac{1}{3}(-2 x+6)$
82. $\frac{1}{5}(-4 x+20)$
83. $-\frac{1}{3}(-2 x+6)$
84. $-\frac{1}{2}(-2 x+6)$
85. $8\left(-\frac{1}{4} x+\frac{1}{8} y\right)$
86. $9\left(-\frac{1}{9} x+\frac{1}{3} y\right)$
87. Find the value of $-\frac{1}{3} x+2$ when
a. $x=0$
b. $x=3$
c. $x=-3$
88. Find the value of $-\frac{2}{3} x+1$ when
a. $x=0$
b. $x=3$
c. $x=-3$
89. Find the value of $2 x+y$ when
a. $x=2$ and $y=-1$
b. $x=0$ and $y=3$
c. $x=\frac{3}{2}$ and $y=-7$
90. Find the value of $2 x-5 y$ when
a. $x=2$ and $y=3$
b. $x=0$ and $y=-2$
c. $x=\frac{5}{2}$ and $y=1$
91. Find the value of $2 x^{2}-5 x$ when
a. $x=4$
b. $x=-\frac{3}{2}$
92. Find the value of $49 a^{2}-16$ when
a. $a=\frac{4}{7}$
b. $\quad a=-\frac{4}{7}$
93. Find the value of $y(2 y+3)$ when
a. $y=4$
b. $y=-\frac{11}{2}$
94. Find the value of $x(13-x)$ when
a. $x=5$
b. $x=8$
95. Five added to the product of 3 and -10 is what number?
96. If the product of -8 and -2 is decreased by 4 , what number results?
97. Write an expression for twice the product of -4 and $x$, and then simplify it.
98. Write an expression for twice the product of -2 and $3 x$, and then simplify it.
99. What number results if 8 is subtracted from the product of -9 and 2 ?
100. What number results if -8 is subtracted from the product of -9 and 2 ?

Each of the following is a geometric sequence. In each case, find the next number in the sequence.
101. $1,2,4, \ldots$
102. $1,5,25, \ldots$
103. $10,-20,40, \ldots$
105. $1, \frac{1}{2}, \frac{1}{4}, \ldots$
107. $3,-6,12, \ldots$
104. $10,-30,90, \ldots$
106. $1, \frac{1}{3}, \frac{1}{9}, \ldots$
108. $-3,6,-12, \ldots$

## Applying the Concepts

109. Stock Value Suppose you own 20 shares of a stock. If the price per share drops $\$ 3$, how much money have you lost?
110. Stock Value Imagine that you purchase 50 shares of a stock at a price of $\$ 18$ per share. If the stock is selling for $\$ 11$ a share a week after you purchased it, how much money have you lost?
111. Temperature Change The temperature is $25^{\circ} \mathrm{F}$ at $5: 00$ in the afternoon. If the temperature drops $6^{\circ} \mathrm{F}$ every hour after that, what is the temperature at 9:00 in the evening?
112. Temperature Change The temperature is $-5^{\circ} \mathrm{F}$ at $6: 00$ in the evening. If the temperature drops $3^{\circ} \mathrm{F}$ every hour after that, what is the temperature at midnight?
113. Improving Your Quantitative Literacy This table appeared as part of an article on tuition costs. Although tuition costs rose considerably between 1998 and 2003, the average cost per student declined.
a. Find the difference of student grants in 2003 and student grants in 1998.
b. Find the difference of actual cost in 2003 and actual cost in 1998.
c. How does the information in the table explain how students are paying less to go to college when tuition costs have increased?

## Actual tuition cost declines

The actual cost of tuition and fees at four-year public universities has fallen nearly one-third since 1998 because of new tax breaks and an increase in federal and state grants. Average per student:

| $\mathbf{1 9 9 8}$ |  |  |  |  |
| :--- | :---: | :--- | :--- | :--- |
| Actual cost |  | Student grants | Total cost |  |
| \$1,636 |  | $\mathbf{\$ 1 , 9 4 0}$ | $\mathbf{\$ 3 , 5 7 6}$ |  |
| $\mathbf{2 0 0 3}$ |  | Tax credits/deductions |  |  |
| Actual cost |  | Student grants | Total cost |  |
| $\mathbf{\$ 1 , 1 1 5}$ | $\mathbf{\$ 6 0 3}$ | $\mathbf{\$ 2 , 4 8 4}$ | $\mathbf{\$ 4 , 2 0 2}$ |  |

Sources: USA TODAY research: College Board, Office of Management and Budget

## Division of Real Numbers

## OBJECTIVES

A Divide any combination of positive and negative numbers.

B Divide fractions.
C Simplify expressions using the rule for order of operations.

## Note

We are defining division this way simply so that we can use what we already know about multiplication to do division problems. We actually want as few rules as possible. Defining division in terms of multiplication allows us to avoid writing a separate rule for division.

Suppose that you and four friends bought equal shares of an investment for a total of $\$ 15,000$ and then sold it later for only $\$ 13,000$. How much did each person lose? Because the total amount of money that was lost can be represented by $-\$ 2,000$, and there are 5 people with equal shares, we can represent each person's loss with division:

$$
\frac{-\$ 2,000}{5}=-\$ 400
$$

From this discussion it seems reasonable to say that a negative number divided by a positive number is a negative number. Here is a more detailed discussion of division with positive and negative numbers.

The last of the four basic operations is division. We will use the same approach to define division as we used for subtraction; that is, we will define division in terms of rules we already know.
Recall that we developed the rule for subtraction of real numbers by defining subtraction in terms of addition. We changed our subtraction problems to addition problems and then added to get our answers. Because we already have a rule for multiplication of real numbers, and division is the inverse operation of multiplication, we will simply define division in terms of multiplication.

We know that division by the number 2 is the same as multiplication by $\frac{1}{2}$; that is, 6 divided by 2 is 3 , which is the same as 6 times $\frac{1}{2}$. Similarly, dividing a number by 5 gives the same result as multiplying by $\frac{1}{5}$. We can extend this idea to all real numbers with the following rule.

Rule If $a$ and $b$ represent any two real numbers ( $b$ cannot be 0 ), then it is always true that

$$
a \div b=\frac{a}{b}=a\left(\frac{1}{b}\right)
$$

Division by a number is the same as multiplication by its reciprocal. Because every division problem can be written as a multiplication problem and because we already know the rule for multiplication of two real numbers, we do not have to write a new rule for division of real numbers. We will simply replace our division problem with multiplication and use the rule we already have.

EXAMPLES Write each division problem as an equivalent multiplication problem, and then multiply.

1. $\frac{6}{2}=6\left(\frac{1}{2}\right)=3 \quad$ The product of two positives is positive
2. $\frac{6}{-2}=6\left(-\frac{1}{2}\right)=-3$
3. $\frac{-6}{2}=-6\left(\frac{1}{2}\right)=-3$
4. $\frac{-6}{-2}=-6\left(-\frac{1}{2}\right)=3 \quad$ The product of two negatives is positive

## Note

What we are saying here is that the work shown in Examples 1 through 4 is shown simply to justify the answers we obtain. In the future we won't show the middle step in these kinds of problems. Even so, we need to know that division is defined to be multiplication by the reciprocal.

The second step in these examples is used only to show that we can write division in terms of multiplication. [In actual practice we wouldn't write $\frac{6}{2}$ as $6\left(\frac{1}{2}\right)$.] The answers, therefore, follow from the rule for multiplication; that is, like signs produce a positive answer, and unlike signs produce a negative answer.

Here are some examples. This time we will not show division as multiplication by the reciprocal. We will simply divide. If the original numbers have the same signs, the answer will be positive. If the original numbers have different signs, the answer will be negative.

## EXAMPLES Divide.

5. $\frac{12}{6}=2 \quad$ Like signs give a positive answer
6. $\frac{12}{-6}=-2 \quad$ Unlike signs give a negative answer
7. $\frac{-12}{6}=-2 \quad$ Unlike signs give a negative answer
8. $\frac{-12}{-6}=2 \quad$ Like signs give a positive answer
9. $\frac{15}{-3}=-5 \quad$ Unlike signs give a negative answer
10. $\frac{-40}{-5}=8 \quad$ Like signs give a positive answer
11. $\frac{-14}{2}=-7 \quad$ Unlike signs give a negative answer

## Division with Fractions

We can apply the definition of division to fractions. Because dividing by a fraction is equivalent to multiplying by its reciprocal, we can divide a number by the fraction $\frac{3}{4}$ by multiplying it by the reciprocal of $\frac{3}{4}$, which is $\frac{4}{3}$. For example,

$$
\frac{2}{5} \div \frac{3}{4}=\frac{2}{5} \cdot \frac{4}{3}=\frac{8}{15}
$$

You may have learned this rule in previous math classes. In some math classes, multiplication by the reciprocal is referred to as "inverting the divisor and multiplying." No matter how you say it, division by any number (except 0 ) is always equivalent to multiplication by its reciprocal. Here are additional examples that involve division by fractions.

## EXAMPLES Divide.

12. $\frac{2}{3} \div \frac{5}{7}=\frac{2}{3} \cdot \frac{7}{5} \quad$ Rewrite as multiplication by the reciprocal $=\frac{14}{15}$
13. $-\frac{3}{4} \div \frac{7}{9}=-\frac{3}{4} \cdot \frac{9}{7} \quad$ Rewrite as multiplication by the reciprocal

$$
=-\frac{27}{28} \quad \text { Multiply }
$$

14. $8 \div\left(-\frac{4}{5}\right)=\frac{8}{1}\left(-\frac{5}{4}\right) \quad$ Rewrite as multiplication by the reciprocal

$$
\begin{array}{ll}
=-\frac{40}{4} & \text { Multiply } \\
=-10 & \text { Divide } 40 \text { by } 4
\end{array}
$$

The last step in each of the following examples involves reducing a fraction to lowest terms. To reduce a fraction to lowest terms, we divide the numerator and denominator by the largest number that divides each of them exactly. For example, to reduce $\frac{15}{20}$ to lowest terms, we divide 15 and 20 by 5 to get $\frac{3}{4}$.

## EXAMPLES Simplify as much as possible.

In the examples that follow, the numerators and denominators contain expressions that are somewhat more complicated than those we have seen thus far. To apply the rule for order of operations to these examples, we treat fraction bars the same way we treat grouping symbols; that is, fraction bars separate numerators and denominators so that each will be simplified separately.

## EXAMPLES Simplify.

17. $\frac{-8-8}{-5-3}=\frac{-16}{-8}$

$$
=2
$$

## Simplify numerator and denominator separately <br> Divide

18. $\frac{2(-3)+4}{12}=\frac{-6+4}{12}$

$$
\begin{aligned}
& =\frac{-2}{12} \\
& =-\frac{1}{6}
\end{aligned}
$$

In the numerator, we multiply before we add

## Addition

Reduce to lowest terms by dividing numerator and denominator by 2
19. $\frac{5(-4)+6(-1)}{2(3)-4(1)}=\frac{-20+(-6)}{6-4}$

Multiplication before addition

$$
=\frac{-26}{2}
$$

$$
=-13 \quad \text { Divide }-\mathbf{2 6} \text { by } 2
$$

15. $\frac{-4(5)}{6}=\frac{-20}{6}$
16. $\frac{-4}{6}=\frac{-20}{6}$ Simplify numerator
$=-\frac{10}{3} \quad$ Reduce to lowest terms by dividing numerator and denominator by 2
17. $\frac{30}{-4-5}=\frac{30}{-9} \quad$ Simplify denominator
$=-\frac{10}{3}$ Reduce to lowest terms by dividing numerator and denominator by 3

We must be careful when we are working with expressions such as $(-5)^{2}$ and $-5^{2}$ that we include the negative sign with the base only when parentheses indicate we are to do so.
Unless there are parentheses to indicate otherwise, we consider the base to be only the number directly below and to the left of the exponent. If we want to include a negative sign with the base, we must use parentheses.
To simplify a more complicated expression, we follow the same rule. For example,

$$
7^{2}-3^{2}=49-9
$$

The bases are 7 and 3; the sign between the two terms is a subtraction sign

For another example,

$$
5^{3}-3^{4}=125-81 \quad \text { We simplify exponents first, then subtract }
$$

## EXAMPLES Simplify.

20. $\frac{5^{2}-3^{2}}{-5+3}=\frac{25-9}{-2} \quad$ Simplify numerator and denominator separately

$$
=\frac{16}{-2}
$$

$$
=-8
$$

21. $\frac{(3+2)^{2}}{-3^{2}-2^{2}}=\frac{5^{2}}{-9-4} \quad$ Simplify numerator and denominator separately

$$
\begin{aligned}
& =\frac{25}{-13} \\
& =-\frac{25}{13}
\end{aligned}
$$

| 11141
We can combine our knowledge of the properties of multiplication with our definition of division to simplify more expressions involving fractions. Here are two examples:

## EXAMPLES Simplify each expression.

22. $10\left(\frac{x}{2}\right)=10\left(\frac{1}{2} x\right) \quad$ Dividing by 2 is the same as multiplying by $\frac{1}{2}$

$$
\begin{array}{ll}
=\left(10 \cdot \frac{1}{2}\right) x & \text { Associative property } \\
=5 x & \text { Multiplication }
\end{array}
$$

23. $a\left(\frac{3}{a}-4\right)=a \cdot \frac{3}{a}-a \cdot 4 \quad$ Distributive property
$=3-4 a \quad$ Multiplication

## Division with the Number 0

For every division problem there is an associated multiplication problem involving the same numbers. For example, the following two problems say the same thing about the numbers 2,3 , and 6 :

$$
\begin{array}{cc}
\text { Division } & \text { Multiplication } \\
\frac{6}{3}=2 & 6=2(3)
\end{array}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-11
B $\quad 12-14$
C $15-21$

We can use this relationship between division and multiplication to clarify division involving the number 0 .

First, dividing 0 by a number other than 0 is allowed and always results in 0 . To see this, consider dividing 0 by 5 . We know the answer is 0 because of the relationship between multiplication and division. This is how we write it:

$$
\frac{0}{5}=0 \quad \text { because } \quad 0=0(5)
$$

However, dividing a nonzero number by 0 is not allowed in the real numbers. Suppose we were attempting to divide 5 by 0 . We don't know if there is an answer to this problem, but if there is, let's say the answer is a number that we can represent with the letter $n$. If 5 divided by 0 is a number $n$, then

$$
\frac{5}{0}=n \quad \text { and } \quad 5=n(0)
$$

This is impossible, however, because no matter what number $n$ is, when we multiply it by 0 the answer must be 0 . It can never be 5 . In algebra, we say expressions like $\frac{5}{0}$ are undefined because there is no answer to them; that is, division by 0 is not allowed in the real numbers.

The only other possibility for division involving the number 0 is 0 divided by 0 . We will treat problems like $\frac{0}{0}$ as if they were undefined also.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Why do we define division in terms of multiplication?
2. What is the reciprocal of a number?
3. How do we divide fractions?
4. Why is division by O not allowed with real numbers?

## Problem Set 1.7

Online support materials can be found at www.thomsonedu.com/login

Find the following quotients (divide).

1. $\frac{8}{-4}$
2. $\frac{10}{-5}$
3. $\frac{-48}{16}$
4. $\frac{-32}{4}$
5. $\frac{-7}{21}$
6. $\frac{-25}{100}$
7. $\frac{-39}{-13}$
8. $\frac{-18}{-6}$
9. $\frac{-6}{-42}$
10. $\frac{-4}{-28}$
11. $\frac{0}{-32}$
12. $\frac{0}{17}$

The following problems review all four operations with positive and negative numbers. Perform the indicated operations.
13. $-3+12$
14. $5+(-10)$
15. $-3-12$
16. $5-(-10)$

[^1]17. $-3(12)$
18. $5(-10)$
19. $-3 \div 12$
20. $5 \div(-10)$

Divide and reduce all answers to lowest terms.
21. $\frac{4}{5} \div \frac{3}{4}$
22. $\frac{6}{8} \div \frac{3}{4}$
23. $-\frac{5}{6} \div\left(-\frac{5}{8}\right)$
24. $-\frac{7}{9} \div\left(-\frac{1}{6}\right)$
25. $\frac{10}{13} \div\left(-\frac{5}{4}\right)$
26. $\frac{5}{12} \div\left(-\frac{10}{3}\right)$
27. $-\frac{5}{6} \div \frac{5}{6}$
28. $-\frac{8}{9} \div \frac{8}{9}$
29. $-\frac{3}{4} \div\left(-\frac{3}{4}\right)$
30. $-\frac{6}{7} \div\left(-\frac{6}{7}\right)$

The following problems involve more than one operation. Simplify as much as possible.
31. $\frac{3(-2)}{-10}$
32. $\frac{4(-3)}{24}$
33. $\frac{-5(-5)}{-15}$
34. $\frac{-7(-3)}{-35}$
35. $\frac{-8(-7)}{-28}$
36. $\frac{-3(-9)}{-6}$
37. $\frac{27}{4-13}$
39. $\frac{20-6}{5-5}$
41. $\frac{-3+9}{2 \cdot 5-10}$
38. $\frac{27}{13-4}$
40. $\frac{10-12}{3-3}$
43. $\frac{15(-5)-25}{2(-10)}$
42. $\frac{-4+8}{2 \cdot 4-8}$
45. $\frac{27-2(-4)}{-3(5)}$
47. $\frac{12-6(-2)}{12(-2)}$
44. $\frac{10(-3)-20}{5(-2)}$
46. $\frac{20-5(-3)}{10(-3)}$
48. $\frac{3(-4)+5(-6)}{10-6}$
49. $\frac{5^{2}-2^{2}}{-5+2}$
50. $\frac{7^{2}-4^{2}}{-7+4}$
51. $\frac{8^{2}-2^{2}}{8^{2}+2^{2}}$
52. $\frac{4^{2}-6^{2}}{4^{2}+6^{2}}$
53. $\frac{(5+3)^{2}}{-5^{2}-3^{2}}$
54. $\frac{(7+2)^{2}}{-7^{2}-2^{2}}$
55. $\frac{(8-4)^{2}}{8^{2}-4^{2}}$
56. $\frac{(6-2)^{2}}{6^{2}-2^{2}}$
57. $\frac{-4 \cdot 3^{2}-5 \cdot 2^{2}}{-8(7)}$
58. $\frac{-2 \cdot 5^{2}+3 \cdot 2^{3}}{-3(13)}$
59. $\frac{3 \cdot 10^{2}+4 \cdot 10+5}{345}$
60. $\frac{5 \cdot 10^{2}+6 \cdot 10+7}{567}$
61. $\frac{7-[(2-3)-4]}{-1-2-3}$
62. $\frac{2-[(3-5)-8]}{-3-4-5}$
63. $\frac{6(-4)-2(5-8)}{-6-3-5}$
64. $\frac{3(-4)-5(9-11)}{-9-2-3}$
65. $\frac{3(-5-3)+4(7-9)}{5(-2)+3(-4)}$
66. $\frac{-2(6-10)-3(8-5)}{6(-3)-6(-2)}$
67. $\frac{|3-9|}{3-9}$
68. $\frac{|4-7|}{4-7}$
69. $\frac{2+0.15(10)}{10}$
70. $\frac{5(5)+250}{640(5)}$
71. $\frac{1-3}{3-1}$
72. $\frac{25-16}{16-25}$
73. Simplify.
a. $\frac{5-2}{3-1}$
b. $\frac{2-5}{1-3}$
74. Simplify.
a. $\frac{6-2}{3-5}$
b. $\frac{2-6}{5-3}$
75. Simplify.
a. $\frac{-4-1}{5-(-2)}$
b. $\frac{1-(-4)}{-2-5}$
76. Simplify.
a. $\frac{-6-1}{4-(-5)}$
b. $\frac{1-(-6)}{-5-4}$
77. Simplify each expression.
a. $\frac{3+2.236}{2}$
b. $\frac{3-2.236}{2}$
c. $\frac{3+2.236}{2}+\frac{3-2.236}{2}$
78. Simplify each expression.
a. $\frac{1+1.732}{2}$
b. $\frac{1-1.732}{2}$
c. $\frac{1+1.732}{2}+\frac{1-1.732}{2}$
79. Simplify each expression.
a. $20 \div 4 \cdot 5$
b. $-20 \div 4 \cdot 5$
c. $20 \div(-4) \cdot 5$
d. $20 \div 4(-5)$
e. $-20 \div 4(-5)$
80. Simplify each expression.
a. $32 \div 8 \cdot 4$
b. $-32 \div 8 \cdot 4$
c. $32 \div(-8) \cdot 4$
d. $32 \div 8(-4)$
e. $-32 \div 8(-4)$
81. Simplify each expression.
a. $8 \div \frac{4}{5}$
b. $8 \div \frac{4}{5}-10$
c. $8 \div \frac{4}{5}(-10)$
d. $8 \div\left(-\frac{4}{5}\right)-10$
82. Simplify each expression.
a. $10 \div \frac{5}{6}$
b. $10 \div \frac{5}{6}-12$
c. $10 \div \frac{5}{6}(-12)$
d. $10 \div\left(-\frac{5}{6}\right)-12$

Apply the distributive property.
83. $10\left(\frac{x}{2}+\frac{3}{5}\right)$
84. $6\left(\frac{x}{3}+\frac{5}{2}\right)$
85. $15\left(\frac{x}{5}+\frac{4}{3}\right)$
86. $6\left(\frac{x}{3}+\frac{1}{2}\right)$
87. $x\left(\frac{3}{x}+1\right)$
88. $x\left(\frac{4}{x}+3\right)$
89. $21\left(\frac{x}{7}-\frac{y}{3}\right)$
90. $36\left(\frac{x}{4}-\frac{y}{9}\right)$
91. $a\left(\frac{3}{a}-\frac{2}{a}\right)$
92. $a\left(\frac{7}{a}+\frac{1}{a}\right)$
93. $2 y\left(\frac{1}{y}-\frac{1}{2}\right)$
94. $5 y\left(\frac{3}{y}-\frac{4}{5}\right)$

Answer the following questions.
95. What is the quotient of -12 and -4 ?
96. The quotient of -4 and -12 is what number?
97. What number do we divide by -5 to get 2 ?
98. What number do we divide by -3 to get 4 ?
99. Twenty-seven divided by what number is -9 ?
100. Fifteen divided by what number is -3 ?
101. If the quotient of -20 and 4 is decreased by 3 , what number results?
102. If -4 is added to the quotient of 24 and -8 , what number results?

## Applying the Concepts

103. Investment Suppose that you and 3 friends bought equal shares of an investment for a total of $\$ 15,000$ and then sold it later for only $\$ 13,600$. How much did each person lose?
104. Investment If 8 people invest $\$ 500$ each in a stamp collection and after a year the collection is worth $\$ 3,800$, how much did each person lose?
105. Temperature Change Suppose that the temperature outside is dropping at a constant rate. If the temperature is $75^{\circ} \mathrm{F}$ at noon and drops to $61^{\circ} \mathrm{F}$ by 4:00 in the afternoon, by how much did the temperature change each hour?
106. Temperature Change In a chemistry class, a thermometer is placed in a beaker of hot water. The initial temperature of the water is $165^{\circ} \mathrm{F}$. After 10 minutes the water has cooled to $72^{\circ} \mathrm{F}$. If the water temperature drops at a constant rate, by how much does the water temperature change each minute?
107. Internet Mailing Lists A company sells products on the Internet through an email list. They predict that they sell one $\$ 50$ product for every 25 people on their mailing list.
a. What is their projected revenue if their list contains 10,000 email addresses?
b. What is their projected revenue if their list contains 25,000 email addresses?
c. They can purchase a list of 5,000 email addresses for $\$ 5,000$. Is this a wise purchase?
108. Internet Mailing Lists A new band has a following on the Internet. They sell their CDs through an email list. They predict that they sell one \$15 CD for every 10 people on their mailing list.
a. What is their projected revenue if their list contains 5,000 email addresses?
b. What is their projected revenue if their list contains 20,000 email addresses?
c. If they need to make $\$ 45,000$, how many people do they need on their email list?

## Subsets of the Real Numbers

## OBJECTIVES

A Perform operations with sets.

B Associate numbers with subsets of the real numbers. Factor whole numbers into
C the product of prime factors.

D Reduce fractions to lowest terms.

In Section 1.2 we introduced the real numbers and defined them as the numbers associated with points on the real number line. At that time, we said the real numbers include whole numbers, fractions, and decimals, as well as other numbers that are not as familiar to us as these numbers. In this section we take a more detailed look at the kinds of numbers that make up the set of real numbers.

Let's begin with the definition of a set, the starting point for all branches of mathematics.

DEFINITION A set is a collection of objects or things. The objects in the set are called elements, or members, of the set.

Sets are usually denoted by capital letters, and elements of sets are denoted by lowercase letters. We use braces, \{ \}, to enclose the elements of a set.

To show that an element is contained in a set we use the symbol $\in$. That is,

$$
x \in A \text { is read " } x \text { is an element (member) of set } A \text { " }
$$

For example, if $A$ is the set $\{1,2,3\}$, then $2 \in A$. However, $5 \notin A$ means 5 is not an element of set $A$.

DEFINITION Set $A$ is a subset of set $B$, written $A \subset B$, if every element in $A$ is also an element of $B$. That is:

$$
A \subset B \quad \text { if and only if } \quad A \text { is contained in } B
$$

## EXAMPLES

1. The set of numbers used to count things is $\{1,2,3, \ldots\}$. The dots mean the set continues indefinitely in the same manner. This is an example of an infinite set.
2. The set of all numbers represented by the dots on the faces of a regular die is $\{1,2,3,4,5,6\}$. This set is a subset of the set in Example 1. It is an example of a finite set because it has a limited number of elements.

DEFINITION The set with no members is called the empty, or null, set. It is denoted by the symbol $\varnothing$. The empty set is considered a subset of every set.

The diagrams shown here are called Venn diagrams after John Venn (1834-1923). They can be used to visualize operations with sets. The region inside the circle labeled $A$ is set $A$; the region inside the circle labeled $B$ is set $B$.

$A \cup B$
FIGURE 1 The union of two sets

$A \cap B$
FIGURE 2 The intersection of two sets

## Operations with Sets

Two basic operations are used to combine sets: union and intersection.

DEFINITION The union of two sets $A$ and $B$, written $A \cup B$, is the set of all elements that are either in $A$ or in $B$, or in both $A$ and $B$. The key word here is or. For an element to be in $A \cup B$ it must be in $A$ or $B$. In symbols, the definition looks like this:

$$
x \in A \cup B \quad \text { if and only if } \quad x \in A \text { or } x \in B
$$

DEFINITION The intersection of two sets $A$ and $B$, written $A \cap B$, is the set of elements in both $A$ and $B$. The key word in this definition is the word and. For an element to be in $A \cap B$ it must be in both $A$ and $B$, or

$$
x \in A \cap B \quad \text { if and only if } \quad x \in A \text { and } x \in B
$$

## EXAMPLES <br> Let $A=\{1,3,5\}, B=\{0,2,4\}$, and $C=\{1,2,3, \ldots\}$. Then

3. $A \cup B=\{0,1,2,3,4,5\}$
4. $A \cap B=\varnothing \quad$ ( $A$ and $B$ have no elements in common.)
5. $A \cap C=\{1,3,5\}=A$
6. $B \cup C=\{0,1,2,3, \ldots\}$

Another notation we can use to describe sets is called set-builder notation. Here is how we write our definition for the union of two sets $A$ and $B$ using setbuilder notation:

$$
A \cup B=\{x \mid x \in A \text { or } x \in B\}
$$

The right side of this statement is read "the set of all $x$ such that $x$ is a member of $A$ or $x$ is a member of $B . "$ As you can see, the vertical line after the first $x$ is read "such that."

## EXAMPLE 7 If $A=\{1,2,3,4,5,6\}$, find $C=\{x \mid x \in A$ and $x \geq 4\}$.

SOLUTION We are looking for all the elements of $A$ that are also greater than or equal to 4 . They are 4,5 , and 6 . Using set notation, we have

$$
C=\{4,5,6\}
$$

The numbers that make up the set of real numbers can be classified as counting numbers, whole numbers, integers, rational numbers, and irrational numbers; each is a subset of the real numbers. Here is a detailed description of the major subsets of the real numbers.
The counting numbers are the numbers with which we count. They are the numbers $1,2,3$, and so on. The notation we use to specify a group of numbers like this is set notation. We use the symbols $\{$ and \} to enclose the members of the set.

$$
\text { Counting numbers }=\{1,2,3, \ldots\}
$$

EXAMPLE 8 Which of the numbers in the following set are not count-
ing numbers?

$$
\left\{-3,0, \frac{1}{2}, 1,1.5,3\right\}
$$

SOLUTION The numbers $-3,0, \frac{1}{2}$, and 1.5 are not counting numbers.
The whole numbers include the counting numbers and the number 0 .

$$
\text { Whole numbers }=\{0,1,2, \ldots\}
$$

The set of integers includes the whole numbers and the opposites of all the counting numbers.

$$
\text { Integers }=\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}
$$

When we refer to positive integers, we are referring to the numbers $1,2,3, \ldots$ Likewise, the negative integers are $-1,-2,-3, \ldots$. The number 0 is neither positive nor negative.

EXAMPLE 9 Which of the numbers in the following set are not integers?

$$
\left\{-5,-1.75,0, \frac{2}{3}, 1, \pi, 3\right\}
$$

SOLUTION The only numbers in the set that are not integers are $-1.75, \frac{2}{3}$, and $\pi$.

The set of rational numbers is the set of numbers commonly called "fractions" together with the integers. The set of rational numbers is difficult to list in the same way we have listed the other sets, so we will use a different kind of notation:

$$
\text { Rational numbers }=\left\{\left.\frac{a}{b} \right\rvert\, a \text { and } b \text { are integers }(b \neq 0)\right\}
$$

This notation is read "The set of elements $\frac{a}{b}$ such that $a$ and $b$ are integers (and $b$ is not 0 )." If a number can be put in the form $\frac{a}{b}$, where $a$ and $b$ are both from the set of integers, then it is called a rational number.

Rational numbers include any number that can be written as the ratio of two integers; that is, rational numbers are numbers that can be put in the form

$$
\frac{\text { integer }}{\text { integer }}
$$

EXAMPLE 10 Show why each of the numbers in the following set is a
rational number.

$$
\left\{-3,-\frac{2}{3}, 0,0.333 \ldots, 0.75\right\}
$$

SOLUTION The number -3 is a rational number because it can be written as the ratio of -3 to 1 ; that is,

$$
-3=\frac{-3}{1}
$$

Similarly, the number $-\frac{2}{3}$ can be thought of as the ratio of -2 to 3 , whereas the number 0 can be thought of as the ratio of 0 to 1 .
Any repeating decimal, such as $0.333 \ldots$ (the dots indicate that the 3 's repeat forever), can be written as the ratio of two integers. In this case $0.333 \ldots$ is the same as the fraction $\frac{1}{3}$.

Finally, any decimal that terminates after a certain number of digits can be written as the ratio of two integers. The number 0.75 is equal to the fraction $\frac{3}{4}$ and is therefore a rational number.

Still other numbers exist, each of which is associated with a point on the real number line, that cannot be written as the ratio of two integers. In decimal form they never terminate and never repeat a sequence of digits indefinitely. They are called irrational numbers (because they are not rational):

$$
\begin{aligned}
\text { Irrational numbers }= & \text { nonrational numbers; nonrepeating, } \\
& \text { nonterminating decimals }\}
\end{aligned}
$$

We cannot write any irrational number in a form that is familiar to us because they are all nonterminating, nonrepeating decimals. Because they are not rational, they cannot be written as the ratio of two integers. They have to be represented in other ways. One irrational number you have probably seen before is $\pi$. It is not 3.14 . Rather, 3.14 is an approximation to $\pi$. It cannot be written as a terminating decimal number. Other representations for irrational numbers are $\sqrt{2}$, $\sqrt{3}, \sqrt{5}, \sqrt{6}$, and, in general, the square root of any number that is not itself a perfect square. (If you are not familiar with square roots, you will be after Chapter 9.) Right now it is enough to know that some numbers on the number line cannot be written as the ratio of two integers or in decimal form. We call them irrational numbers.

The set of real numbers is the set of numbers that are either rational or irrational; that is, a real number is either rational or irrational.

Real numbers $=$ \{all rational numbers and all irrational numbers $\}$

## Prime Numbers and Factoring

The following diagram shows the relationship between multiplication and factoring:

$$
\begin{gathered}
\text { Multiplication } \\
\text { Factors } \rightarrow 3 \cdot 4=12 \leftarrow \text { Product } \\
\text { Factoring }
\end{gathered}
$$

When we read the problem from left to right, we say the product of 3 and 4 is 12. Or we multiply 3 and 4 to get 12 . When we read the problem in the other direction, from right to left, we say we have factored 12 into 3 times 4 , or 3 and 4 are factors of 12 .

The number 12 can be factored still further:

$$
\begin{aligned}
12 & =4 \cdot 3 \\
& =2 \cdot 2 \cdot 3 \\
& =2^{2} \cdot 3
\end{aligned}
$$

## Note

The number 15 is not a prime number because it has factors of 3 and 5 ; that is, $15=3 \cdot 5$. When a whole number larger than 1 is not prime, it is said to be composite.

## Note

It is customary to write the prime factors in order from smallest to largest.

## Note

There are some "tricks" to finding the divisors of a number. For instance, if a number ends in 0 or 5 , then it is divisible by 5 . If a number ends in an even number ( $0,2,4,6$, or 8 ), then it is divisible by 2 . A number is divisible by 3 if the sum of its digits is divisible by 3. For example, 921 is divisible by 3 because the sum of its digits is $9+2+1=12$, which is divisible by 3 .

The numbers 2 and 3 are called prime factors of 12 because neither of them can be factored any further.

DEFINITION If $a$ and $b$ represent integers, then $a$ is said to be a factor (or divisor) of $b$ if $a$ divides $b$ evenly; that is, if $a$ divides $b$ with no remainder.

DEFINITION A prime number is any positive integer larger than 1 whose only positive factors (divisors) are itself and 1.

Here is a list of the first few prime numbers.

$$
\text { Prime numbers }=\{2,3,5,7,11,13,17,19,23,29,31,37,41, \ldots\}
$$

When a number is not prime, we can factor it into the product of prime numbers. To factor a number into the product of primes, we simply factor it until it cannot be factored further.

## EXAMPLE 11 Factor the number 60 into the product of prime numbers.

SOLUTION We begin by writing 60 as the product of any two positive integers whose product is 60 , like 6 and 10 :

$$
60=6 \cdot 10
$$

We then factor these numbers:

$$
\begin{aligned}
60 & =6 \cdot 10 \\
& =(2 \cdot 3) \cdot(2 \cdot 5) \\
& =2 \cdot 2 \cdot 3 \cdot 5 \\
& =2^{2} \cdot 3 \cdot 5
\end{aligned}
$$

EXAMPLE 12 Factor the number 630 into the product of primes.
SOLUTION Let's begin by writing 630 as the product of 63 and 10:


It makes no difference which two numbers we start with, as long as their product is 630. We always will get the same result because a number has only one set of prime factors.

$$
\begin{aligned}
630 & =18 \cdot 35 \\
& =3 \cdot 6 \cdot 5 \cdot 7 \\
& =3 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \\
& =2 \cdot 3^{2} \cdot 5 \cdot 7
\end{aligned}
$$

## Note

The small lines we have drawn through the factors that are common to the numerator and denominator are used to indicate that we have divided the numerator and denominator by those factors.

When we have factored a number into the product of its prime factors, we not only know what prime numbers divide the original number, but we also know all of the other numbers that divide it as well. For instance, if we were to factor 210 into its prime factors, we would have $210=2 \cdot 3 \cdot 5 \cdot 7$, which means that 2, 3,5 , and 7 divide 210 , as well as any combination of products of $2,3,5$, and 7 ; that is, because 3 and 7 divide 210 , then so does their product 21 . Because 3,5 , and 7 each divide 210 , then so does their product 105 :


Although there are many ways in which factoring is used in arithmetic and algebra, one simple application is in reducing fractions to lowest terms.

Recall that we reduce fractions to lowest terms by dividing the numerator and denominator by the same number. We can use the prime factorization of numbers to help us reduce fractions with large numerators and denominators.

EXAMPLE 13 Reduce $\frac{210}{231}$ to lowest terms.
SOLUTION First we factor 210 and 231 into the product of prime factors. Then we reduce to lowest terms by dividing the numerator and denominator by any ztors they have in common.

$$
\begin{aligned}
\frac{210}{231} & =\frac{2 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 7 \cdot 11} \text { Factor the numerator and denominator completely } \\
& =\frac{2 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 7 \cdot 11} \text { Divide the numerator and denominator by } 3 \cdot 7 \\
& =\frac{2 \cdot 5}{11} \\
& =\frac{10}{11}
\end{aligned}
$$

When we are working with fractions or with division, some of the instructions we use are equivalent; they mean the same thing. For example, each of the problems below will yield the same result:

$$
\begin{aligned}
& \text { Reduce to lowest terms: } \frac{50}{-80} . \\
& \text { Divide: } \frac{50}{-80} . \\
& \text { Simplify: } \frac{50}{-80} .
\end{aligned}
$$

Whether you think of the problem as a division problem, a simplification, or reducing a fraction to lowest terms, the answer will be $-\frac{5}{8}$, or -0.625 , if a decimal is more appropriate for the situation. Sometimes you will see the instruction simplify, and sometimes you will see the instruction reduce. In either case, you will work the problem in the same way.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 3-7

B $\quad 8-10$
C 11,12
D 13

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a whole number?
2. How are factoring and multiplication related?
3. Is every integer also a rational number? Explain.
4. What is a prime number?

## Problem Set 1.8

For the following problems, let $A=\{0,2,4,6\}$,
$B=\{1,2,3,4,5\}$, and $C=\{1,3,5,7\}$.

1. $A \cup B$
2. $A \cup C$
3. $A \cap B$
4. $A \cap C$
5. $B \cap C$
6. $B \cup C$
7. $A \cup(B \cap C)$
8. $C \cup(A \cap B)$
9. $\{x \mid x \in A$ and $x<4\}$
10. $\{x \mid x \in B$ and $x>3\}$
11. $\{x \mid x \in A$ and $x \notin B\}$
12. $\{x \mid x \in B$ and $x \notin C\}$
13. $\{x \mid x \in A$ or $x \in C\}$
14. $\{x \mid x \in A$ or $x \in B\}$
15. $\{x \mid x \in B$ and $x \neq 3\}$
16. $\{x \mid x \in C$ and $x \neq 5\}$

Given the numbers in the set $\left\{-3,-2.5,0,1, \frac{3}{2}, \sqrt{15}\right\}$ :
17. List all the whole numbers.
18. List all the integers.
19. List all the rational numbers.
20. List all the irrational numbers.
21. List all the real numbers.

Given the numbers in the set
$\left\{-10,-8,-0.333 \ldots,-2,9, \frac{25}{3}, \pi\right\}$ :
22. List all the whole numbers.
23. List all the integers.
24. List all the rational numbers.
25. List all the irrational numbers.
26. List all the real numbers.

Identify the following statements as either true or false.
27. Every whole number is also an integer.
28. The set of whole numbers is a subset of the set of integers.
29. A number can be both rational and irrational.
30. The set of rational numbers and the set of irrational numbers have some elements in common.
31. Some whole numbers are also negative integers.
32. Every rational number is also a real number.
33. All integers are also rational numbers.
34. The set of integers is a subset of the set of rational numbers.
= Videos available by instructor request
> = Online student support materials available at www.thomsonedu.com/login

Label each of the following numbers as prime or composite. If a number is composite, then factor it completely.
35. 48
36. 72
37. 37
38. 23
39. 1,023
40. 543

Factor the following into the product of primes. When the number has been factored completely, write its prime factors from smallest to largest.
41. 144
42. 288
43. 38
44. 63
45. 105
46. 210
47. 180
48. 900
49. 385
50. 1,925
51. 121
52. 546
53. 420
54. 598
55. 620
56. 2,310

Reduce each fraction to lowest terms by first factoring the numerator and denominator into the product of prime factors and then dividing out any factors they have in common.
57. $\frac{105}{165}$
58. $\frac{165}{385}$
59. $\frac{525}{735}$
60. $\frac{550}{735}$
61. $\frac{385}{455}$
62. $\frac{385}{735}$
63. $\frac{322}{345}$
64. $\frac{266}{285}$
65. $\frac{205}{369}$
66. $\frac{111}{185}$
67. $\frac{215}{344}$
68. $\frac{279}{310}$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working. You will see a number of problems like this throughout the book. Working these problems is an excellent way to get ready for a test or a quiz.
69. Work each problem according to the instructions given. (Note that each of these instructions could be replaced with the instruction Simplify.)
a. Add: $50+(-80)$
b. Subtract: $50-(-80)$
c. Multiply: 50(-80)
d. Divide: $\frac{50}{-80}$
70. Work each problem according to the instructions given.
a. Add: $-2.5+7.5$
b. Subtract: $-2.5-7.5$
c. Multiply: $-2.5(7.5)$
d. Divide: $\frac{-2.5}{7.5}$

Simplify each expression without using a calculator.
71. $\frac{6.28}{9(3.14)}$
72. $\frac{12.56}{4(3.14)}$
73. $\frac{9.42}{2(3.14)}$
74. $\frac{12.56}{2(3.14)}$
75. $\frac{32}{0.5}$
76. $\frac{16}{0.5}$
77. $\frac{5,599}{11}$
78. $\frac{840}{80}$
79. Find the value of $\frac{2+0.15 x}{x}$ for each of the values of $x$ given below. Write your answers as decimals, to the nearest hundredth.
a. $x=10$
b. $x=15$
c. $x=20$
80. Find the value of $\frac{5 x+250}{640 x}$ for each of the values of $x$ given below. Write your answers as decimals, to the nearest thousandth.
a. $x=10$
b. $x=15$
c. $x=20$
81. Factor $6^{3}$ into the product of prime factors by first factoring 6 and then raising each of its factors to the third power.
82. Factor $12^{2}$ into the product of prime factors by first factoring 12 and then raising each of its factors to the second power.
83. Factor $9^{4} \cdot 16^{2}$ into the product of prime factors by first factoring 9 and 16 completely.
84. Factor $10^{2} \cdot 12^{3}$ into the product of prime factors by first factoring 10 and 12 completely.
85. Simplify the expression $3 \cdot 8+3 \cdot 7+3 \cdot 5$, and then factor the result into the product of primes. (Notice one of the factors of the answer is 3 .)
86. Simplify the expression $5 \cdot 4+5 \cdot 9+5 \cdot 3$, and then factor the result into the product of primes.

Recall the Fibonacci sequence we introduced earlier in this chapter.

$$
\text { Fibonacci sequence }=1,1,2,3,5,8, \ldots
$$

Any number in the Fibonacci sequence is a Fibonacci number.
87. The Fibonacci numbers are not a subset of which of the following sets: real numbers, rational numbers, irrational numbers, whole numbers?
88. Name three Fibonacci numbers that are prime numbers.
89. Name three Fibonacci numbers that are composite numbers.
90. Is the sequence of odd numbers a subset of the Fibonacci numbers?

## N 1.9

## OBJECTIVES

A Add or subtract two or more fractions with the same denominator.

B Find the least common denominator for a set of fractions.

C Add or subtract fractions with different denominators.

D Extend a sequence of numbers containing fractions.

## Addition and Subtraction with Fractions

You may recall from previous math classes that to add two fractions with the same denominator, you simply add their numerators and put the result over the common denominator:

$$
\frac{3}{4}+\frac{2}{4}=\frac{3+2}{4}=\frac{5}{4}
$$

The reason we add numerators but do not add denominators is that we must follow the distributive property. To see this, you first have to recall that $\frac{3}{4}$ can be written as $3 \cdot \frac{1}{4}$, and $\frac{2}{4}$ can be written as $2 \cdot \frac{1}{4}$ (dividing by 4 is equivalent to multiplying by $\frac{1}{4}$ ). Here is the addition problem again, this time showing the use of the distributive property:

$$
\begin{aligned}
\frac{3}{4}+\frac{2}{4} & =3 \cdot \frac{1}{4}+2 \cdot \frac{1}{4} \\
& =(3+2) \cdot \frac{1}{4} \quad \text { Distributive property } \\
& =5 \cdot \frac{1}{4} \\
& =\frac{5}{4}
\end{aligned}
$$

What we have here is the sum of the numerators placed over the common denominator. In symbols we have the following.

Addition and Subtraction of Fractions If $a, b$, and $c$ are integers and $c$ is not equal to 0 , then

$$
\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}
$$

This rule holds for subtraction as well; that is,

$$
\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}
$$

In Examples 1-4, find the sum or difference. (Add or subtract as indicated.) Reduce all answers to lowest terms. (Assume all variables represent nonzero numbers.)

## EXAMPLES

1. $\frac{3}{8}+\frac{1}{8}=\frac{3+1}{8} \quad$ Add numerators; keep the same denominator

$$
\begin{array}{ll}
=\frac{4}{8} & \text { The sum of } 3 \text { and } 1 \text { is } 4 \\
=\frac{1}{2} & \text { Reduce to lowest terms }
\end{array}
$$

2. $\frac{a+5}{8}-\frac{3}{8}=\frac{a+5-3}{8} \quad$ Combine numerators; keep the same denominator

$$
=\frac{a+2}{8}
$$

3. $\frac{9}{x}-\frac{3}{x}=\frac{9-3}{x}$

Subtract numerators; keep the same denominator

$$
=\frac{6}{x} \quad \text { The difference of } 9 \text { and } 3 \text { is } 6
$$

4. $\frac{3}{7}+\frac{2}{7}-\frac{9}{7}=\frac{3+2-9}{7}$

$$
=\frac{-4}{7}
$$

$$
=-\frac{4}{7} \quad \text { Unlike signs give a negative answer }
$$

As Examples 1-4 indicate, addition and subtraction are simple, straightforward processes when all the fractions have the same denominator. We will now turn our attention to the process of adding fractions that have different denominators. To get started, we need the following definition.

DEFINITION The least common denominator (LCD) for a set of denominators is the smallest number that is exactly divisible by each denominator. (Note that in some books the least common denominator is also called the least common multiple.)

In other words, all the denominators of the fractions involved in a problem must divide into the least common denominator exactly; that is, they divide it without giving a remainder.

## Note

The ability to find least common denominators is very important in mathematics. The discussion here is a detailed explanation of how to do it.

EXAMPLE 5 Find the LCD for the fractions $\frac{5}{12}$ and $\frac{7}{18}$.
SOLUTION The least common denominator for the denominators 12 and 18 must be the smallest number divisible by both 12 and 18 . We can factor 12 and 18 completely and then build the LCD from these factors. Factoring 12 and 18 completely gives us

$$
12=2 \cdot 2 \cdot 3 \quad 18=2 \cdot 3 \cdot 3
$$

Now, if 12 is going to divide the LCD exactly, then the LCD must have factors of $2 \cdot 2 \cdot 3$. If 18 is to divide it exactly, it must have factors of $2 \cdot 3 \cdot 3$. We don't need to repeat the factors that 12 and 18 have in common:


In other words, first we write down the factors of 12 , then we attach the factors of 18 that do not already appear as factors of 12 . We start with $2 \cdot 2 \cdot 3$ because those are the factors of 12 . Then we look at the first factor of 18 . It is 2 . Because 2 already appears in the expression $2 \cdot 2 \cdot 3$, we don't need to attach another one. Next, we look at the factors $3 \cdot 3$. The expression $2 \cdot 2 \cdot 3$ has one 3 . For it to contain the expression $3 \cdot 3$, we attach another 3. The final expression, our LCD, is $2 \cdot 2 \cdot 3 \cdot 3$.
The LCD for 12 and 18 is 36 . It is the smallest number that is divisible by both 12 and $18 ; 12$ divides it exactly three times, and 18 divides it exactly two times.

We can use the results of Example 5 to find the sum of the fractions $\frac{5}{12}$ and $\frac{7}{18}$.

## EXAMPLE 6 Add $\frac{5}{12}+\frac{7}{18}$.

SOLUTION We can add fractions only when they have the same denominators. In Example 5 we found the LCD for $\frac{5}{12}$ and $\frac{7}{18}$ to be 36 . We change $\frac{5}{12}$ and $\frac{7}{18}$ to equivalent fractions that each have 36 for a denominator by applying Property 1 , on page 14 , for fractions:

$$
\begin{aligned}
& \frac{5}{12}=\frac{5 \cdot \mathbf{3}}{12 \cdot \mathbf{3}}=\frac{15}{36} \\
& \frac{7}{18}=\frac{7 \cdot \mathbf{2}}{18 \cdot \mathbf{2}}=\frac{14}{36}
\end{aligned}
$$

The fraction $\frac{15}{36}$ is equivalent to $\frac{5}{12}$, because it was obtained by multiplying both the numerator and denominator by 3 . Likewise, $\frac{14}{36}$ is equivalent to $\frac{7}{18}$ because it was obtained by multiplying the numerator and denominator by 2. All we have left to do is to add numerators:

$$
\frac{15}{36}+\frac{14}{36}=\frac{29}{36}
$$

The sum of $\frac{5}{12}$ and $\frac{7}{18}$ is the fraction $\frac{29}{36}$. Let's write the complete solution again step-by-step.

$$
\begin{aligned}
\frac{5}{12}+\frac{7}{18} & =\frac{5 \cdot \mathbf{3}}{12 \cdot \mathbf{3}}+\frac{7 \cdot \mathbf{2}}{18 \cdot \mathbf{2}} \\
& =\frac{15}{36}+\frac{14}{36}
\end{aligned}
$$

Rewrite each fraction as an equivalent fraction with denominator 36

$$
=\frac{29}{36} \quad \begin{aligned}
& \text { Add numerators; keep the } \\
& \text { common denominator }
\end{aligned}
$$

EXAMPLE $7 \quad$ Find the LCD for $\frac{3}{4}$ and $\frac{1}{6}$.
SOLUTION We factor 4 and 6 into products of prime factors and build the LCD from these factors:

$$
\left.\begin{array}{l}
4=2 \cdot 2 \\
6=2 \cdot 3
\end{array}\right\} \quad \mathrm{LCD}=2 \cdot 2 \cdot 3=12
$$

The LCD is 12 . Both denominators divide it exactly; 4 divides 12 exactly three times, and 6 divides 12 exactly two times.

## EXAMPLE $8 \quad$ Add $\frac{3}{4}+\frac{1}{6}$.

SOLUTION In Example 7 we found that the LCD for these two fractions is 12 . We begin by changing $\frac{3}{4}$ and $\frac{1}{6}$ to equivalent fractions with denominator 12 :

$$
\begin{aligned}
& \frac{3}{4}=\frac{3 \cdot \mathbf{3}}{4 \cdot \mathbf{3}}=\frac{9}{12} \\
& \frac{1}{6}=\frac{1 \cdot \mathbf{2}}{6 \cdot \mathbf{2}}=\frac{2}{12}
\end{aligned}
$$

The fraction $\frac{9}{12}$ is equal to the fraction $\frac{3}{4}$ because it was obtained by multiplying the numerator and denominator of $\frac{3}{4}$ by 3 . Likewise, $\frac{2}{12}$ is equivalent to $\frac{1}{6}$ because it was obtained by multiplying the numerator and denominator of $\frac{1}{6}$ by 2 . To complete the problem, we add numerators:

$$
\frac{9}{12}+\frac{2}{12}=\frac{11}{12}
$$

The sum of $\frac{3}{4}$ and $\frac{1}{6}$ is $\frac{11}{12}$. Here is how the complete problem looks:

$$
\begin{aligned}
\frac{3}{4}+\frac{1}{6} & =\frac{3 \cdot \mathbf{3}}{4 \cdot 3}+\frac{1 \cdot \mathbf{2}}{6 \cdot 2} \quad \begin{array}{l}
\text { Rewrite each fraction as an equivalent } \\
\text { fraction with denominator } \mathbf{1 2}
\end{array} \\
& =\frac{9}{12}+\frac{2}{12} \\
& =\frac{11}{12} \quad \text { Add numerators; keep the same denominator } \||\|
\end{aligned}
$$

## EXAMPLE 9 Subtract $\frac{7}{15}-\frac{3}{10}$.

SOLUTION Let's factor 15 and 10 completely and use these factors to build the LCD:

$$
\left.\begin{array}{l}
15=3 \cdot 5 \\
10=2 \cdot 5
\end{array}\right\} \mathrm{LCD}=2 \cdot 3 \cdot 5=30
$$

Changing to equivalent fractions and subtracting, we have

$$
\begin{aligned}
\frac{7}{15}-\frac{3}{10} & =\frac{7 \cdot \mathbf{2}}{15 \cdot \mathbf{2}}-\frac{3 \cdot \mathbf{3}}{10 \cdot \mathbf{3}} & & \begin{array}{l}
\text { Rewrite as equivalent fractions with } \\
\text { the LCD for denominator }
\end{array} \\
& =\frac{14}{30}-\frac{9}{30} & & \\
& =\frac{5}{30} & & \text { Subtract numerators; keep the LCD } \\
& =\frac{1}{6} & & \text { Reduce to lowest terms }
\end{aligned}
$$

As a summary of what we have done so far and as a guide to working other problems, we will now list the steps involved in adding and subtracting fractions with different denominators.

## To Add or Subtract Any Two Fractions

Step 1: Factor each denominator completely and use the factors to build the LCD. (Remember, the LCD is the smallest number divisible by each of the denominators in the problem.)

Step 2: Rewrite each fraction as an equivalent fraction that has the LCD for its denominator. This is done by multiplying both the numerator and denominator of the fraction in question by the appropriate whole number.

Step 3: Add or subtract the numerators of the fractions produced in step 2. This is the numerator of the sum or difference. The denominator of the sum or difference is the LCD.
Step 4: Reduce the fraction produced in step 3 to lowest terms if it is not already in lowest terms.

The idea behind adding or subtracting fractions is really very simple. We can add or subtract only fractions that have the same denominators. If the fractions we are trying to add or subtract do not have the same denominators, we rewrite each of them as an equivalent fraction with the LCD for a denominator.
Here are some further examples of sums and differences of fractions.
EXAMPLE 10 Add $\frac{1}{6}+\frac{1}{8}+\frac{1}{4}$.
SOLUTION We begin by factoring the denominators completely and building the LCD from the factors that result:


We then change to equivalent fractions and add as usual:

$$
\begin{aligned}
\frac{1}{6}+\frac{1}{8}+\frac{1}{4} & =\frac{1 \cdot 4}{6 \cdot 4}+\frac{1 \cdot 3}{8 \cdot 3}+\frac{1 \cdot 6}{4 \cdot 6} \\
& =\frac{4}{24}+\frac{3}{24}+\frac{6}{24} \\
& =\frac{13}{24}
\end{aligned}
$$

## EXAMPLE 11 Subtract $3-\frac{5}{6}$.

SOLUTION The denominators are 1 (because $3=\frac{3}{1}$ ) and 6 . The smallest number divisible by both 1 and 6 is 6 .

$$
\begin{aligned}
3-\frac{5}{6} & =\frac{3}{1}-\frac{5}{6} \\
& =\frac{3 \cdot 6}{1 \cdot 6}-\frac{5}{6} \\
& =\frac{18}{6}-\frac{5}{6} \\
& =\frac{13}{6}
\end{aligned}
$$

EXAMPLE 12 Find the next number in each sequence.
a. $\frac{1}{2}, 0,-\frac{1}{2}, \ldots$
b. $\frac{1}{2}, 1, \frac{3}{2}, \ldots$
c. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

SOLUTION a. $\frac{1}{2}, 0,-\frac{1}{2}, \ldots$ : Adding $-\frac{1}{2}$ to each term produces the next term. The fourth term will be $-\frac{1}{2}+\left(-\frac{1}{2}\right)=-1$. This is an arithmetic sequence.
b. $\frac{1}{2}, 1, \frac{3}{2}, \ldots$ : Each term comes from the term before it by adding $\frac{1}{2}$. The fourth term will be $\frac{3}{2}+\frac{1}{2}=2$. This sequence is also an arithmetic sequence.
c. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ : This is a geometric sequence in which each term comes from the term before it by multiplying by $\frac{1}{2}$ each time. The next term will be $\frac{1}{8} \cdot \frac{1}{2}=\frac{1}{16}$.

## EXAMPLE 13 Subtract: $\frac{x}{5}-\frac{1}{6}$

SOLUTION The LCD for 5 and 6 is their product, 30 . We begin by rewriting each fraction with this common denominator:

$$
\begin{aligned}
\frac{x}{5}-\frac{1}{6} & =\frac{x \cdot 6}{5 \cdot 6}-\frac{1 \cdot 5}{6 \cdot 5} \\
& =\frac{6 x}{30}-\frac{5}{30} \\
& =\frac{6 x-5}{30}
\end{aligned}
$$

## Note

In Example 14, it is understood that $x$ cannot be 0 . Do you know why?

EXAMPLE 14 Add: $\frac{4}{x}+\frac{2}{3}$
SOLUTION The LCD for $x$ and 3 is $3 x$. We multiply the numerator and the denominator of the first fraction by 3 and the numerator and the denominator of the second fraction by $x$ to get two fractions with the same denominator. We then add the numerators:

$$
\begin{array}{rlr}
\frac{4}{x}+\frac{2}{3} & =\frac{4 \cdot 3}{x \cdot 3}+\frac{2 \cdot x}{3 \cdot x} & \\
& \text { Change to equivalent fractions } \\
& =\frac{12}{3 x}+\frac{2 x}{3 x} & \\
& =\frac{12+2 x}{3 x} & \text { Add the numerators }
\end{array}
$$

When we are working with fractions, we can change the form of a fraction without changing its value. There will be times when one form is easier to work with than another form. Look over the material below and be sure you see that the pairs of expressions are equal.

The expressions $\frac{x}{2}$ and $\frac{1}{2} x$ are equal.
The expressions $\frac{3 a}{4}$ and $\frac{3}{4} a$ are equal.
The expressions $\frac{7 y}{3}$ and $\frac{7}{3} y$ are equal.

## EXAMPLE 15 Add: $\frac{x}{3}+\frac{5 x}{6}$.

SOLUTION We can do the problem two ways. One way probably seems easier, but both ways are valid methods of finding this sum. You should understand both of them.
Method 1: $\quad \frac{x}{3}+\frac{5 x}{6}=\frac{\mathbf{2} \cdot x}{\mathbf{2} \cdot 3}+\frac{5 x}{6} \quad$ LCD

$$
\begin{aligned}
& =\frac{2 x+5 x}{6} \quad \text { Add numerators } \\
& =\frac{7 x}{6}
\end{aligned}
$$

Method 2: $\quad \frac{x}{3}+\frac{5 x}{6}=\frac{1}{3} x+\frac{5}{6} x$

$$
\begin{aligned}
& =\left(\frac{1}{3}+\frac{5}{6}\right) x \quad \text { Distributive property } \\
& =\left(\frac{\mathbf{2} \cdot 1}{\mathbf{2} \cdot 3}+\frac{5}{6}\right) x \\
& =\left(\frac{2}{6}+\frac{5}{6}\right) x \\
& =\frac{7}{6} x
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $\quad 1-4$
B 5,7
C $6,8-11,13-15$
D 12

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How do we add two fractions that have the same denominators?
2. What is a least common denominator?
3. What is the first step in adding two fractions that have different denominators?
4. What is the last thing you do when adding two fractions?

## Problem Set 1.9

Find the following sums and differences, and reduce to lowest terms. Add and subtract as indicated. Assume all variables represent nonzero numbers.

1. $\frac{3}{6}+\frac{1}{6}$
2. $\frac{2}{5}+\frac{3}{5}$
3. $\frac{3}{8}-\frac{5}{8}$
4. $\frac{1}{7}-\frac{6}{7}$
5. $-\frac{1}{4}+\frac{3}{4}$
6. $-\frac{4}{9}+\frac{7}{9}$
7. $\frac{x}{3}-\frac{1}{3}$
8. $\frac{x}{8}-\frac{1}{8}$
9. $\frac{1}{4}+\frac{2}{4}+\frac{3}{4}$
10. $\frac{2}{5}+\frac{3}{5}+\frac{4}{5}$
11. $\frac{x+7}{2}-\frac{1}{2}$
12. $\frac{x+5}{4}-\frac{3}{4}$
13. $\frac{1}{10}-\frac{3}{10}-\frac{4}{10}$
14. $\frac{3}{20}-\frac{1}{20}-\frac{4}{20}$
15. $\frac{1}{a}+\frac{4}{a}+\frac{5}{a}$
16. $\frac{5}{a}+\frac{4}{a}+\frac{3}{a}$

Find the LCD for each of the following; then use the methods developed in this section to add and subtract as indicated.
17. $\frac{1}{8}+\frac{3}{4}$
18. $\frac{1}{6}+\frac{2}{3}$
19. $\frac{3}{10}-\frac{1}{5}$
20. $\frac{5}{6}-\frac{1}{12}$
21. $\frac{4}{9}+\frac{1}{3}$
22. $\frac{1}{2}+\frac{1}{4}$
23. $2+\frac{1}{3}$
25. $-\frac{3}{4}+1$
27. $\frac{1}{2}+\frac{2}{3}$
29. $\frac{5}{12}-\left(-\frac{3}{8}\right)$
31. $-\frac{1}{20}+\frac{8}{30}$
33. $\frac{17}{30}+\frac{11}{42}$
35. $\frac{25}{84}+\frac{41}{90}$
37. $\frac{13}{126}-\frac{13}{180}$
39. $\frac{3}{4}+\frac{1}{8}+\frac{5}{6}$
41. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}$
43. $1-\frac{5}{2}$
24. $3+\frac{1}{2}$
26. $-\frac{3}{4}+2$
28. $\frac{2}{3}+\frac{1}{4}$
30. $\frac{9}{16}-\left(-\frac{7}{12}\right)$
32. $-\frac{1}{30}+\frac{9}{40}$
34. $\frac{19}{42}+\frac{13}{70}$
36. $\frac{23}{70}+\frac{29}{84}$
38. $\frac{17}{84}-\frac{17}{90}$
40. $\frac{3}{8}+\frac{2}{5}+\frac{1}{4}$
42. $\frac{1}{8}+\frac{1}{4}+\frac{1}{5}+\frac{1}{10}$
44. $1-\frac{5}{3}$
= Videos available by instructor request
> = Online student support materials available at www.thomsonedu.com/login
45. $1+\frac{1}{2}$
46. $1+\frac{2}{3}$

Find the fourth term in each sequence.
47. $\frac{1}{3}, 0,-\frac{1}{3}, \ldots$
48. $\frac{2}{3}, 0,-\frac{2}{3}, \ldots$
49. $\frac{1}{3}, 1, \frac{5}{3}, \ldots$
50. $1, \frac{3}{2}, 2, \ldots$
51. $1, \frac{1}{5}, \frac{1}{25}, \ldots$
52. $1,-\frac{1}{2}, \frac{1}{4}, \ldots$

Use the rule for order of operations to simplify.
53. $9-3\left(\frac{5}{3}\right)$
54. $6-4\left(\frac{7}{2}\right)$
55. $-\frac{1}{2}+2\left(-\frac{3}{4}\right)$
56. $\frac{5}{4}-3\left(\frac{7}{12}\right)$
57. $\frac{3}{5}(-10)+\frac{4}{7}(-21)$
58. $-\frac{3}{5}(10)-\frac{4}{7}(21)$
59. $16\left(-\frac{1}{2}\right)^{2}-125\left(-\frac{2}{5}\right)^{2}$
60. $16\left(-\frac{1}{2}\right)^{3}-125\left(-\frac{2}{5}\right)^{3}$
61. $-\frac{4}{3} \div 2 \cdot 3$
62. $-\frac{8}{7} \div 4 \cdot 2$
63. $-\frac{4}{3} \div 2(-3)$
64. $-\frac{8}{7} \div 4(-2)$
65. $-6 \div \frac{1}{2} \cdot 12$
66. $-6 \div\left(-\frac{1}{2}\right) \cdot 12$
67. $-15 \div \frac{5}{3} \cdot 18$
68. $-15 \div\left(-\frac{5}{3}\right) \cdot 18$

Add or subtract the following fractions. (Assume all variables represent nonzero numbers. See Examples 13-15.)
69. $\frac{x}{4}+\frac{1}{5}$
70. $\frac{x}{3}+\frac{1}{5}$
71. $\frac{1}{3}+\frac{a}{12}$
72. $\frac{1}{8}+\frac{a}{32}$
73. $\frac{x}{2}+\frac{1}{3}+\frac{x}{4}$
74. $\frac{x}{3}+\frac{1}{4}+\frac{x}{5}$
75. $\frac{2}{x}+\frac{3}{5}$
76. $\frac{3}{x}-\frac{2}{5}$
77. $\frac{3}{7}+\frac{4}{y}$
78. $\frac{2}{9}+\frac{5}{y}$
79. $\frac{3}{a}+\frac{3}{4}+\frac{1}{5}$
80. $\frac{4}{a}+\frac{2}{3}+\frac{1}{2}$
81. $\frac{1}{2} x+\frac{1}{6} x$
82. $\frac{2}{3} x+\frac{5}{6} x$
83. $\frac{1}{2} x-\frac{3}{4} x$
84. $\frac{2}{3} x-\frac{5}{6} x$
85. $\frac{1}{3} x+\frac{3}{5} x$
86. $\frac{2}{3} x-\frac{3}{5} x$
87. $\frac{3 x}{4}+\frac{x}{6}$
88. $\frac{3 x}{4}-\frac{2 x}{3}$
89. $\frac{2 x}{5}+\frac{5 x}{8}$
90. $\frac{3 x}{5}-\frac{3 x}{8}$
91. $1-\frac{1}{X}$
92. $1+\frac{1}{X}$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working. As we mentioned previously, working these problems is an excellent way to get ready for a test or a quiz.
93. Work each problem according to the instructions given. (Note that each of these instructions could be replaced with the instruction Simplify.)
a. Add: $\frac{3}{4}+\left(-\frac{1}{2}\right)$
b. Subtract: $\frac{3}{4}-\left(-\frac{1}{2}\right)$
c. Multiply: $\frac{3}{4}\left(-\frac{1}{2}\right)$
d. Divide: $\frac{3}{4} \div\left(-\frac{1}{2}\right)$
94. Work each problem according to the instructions given.
a. Add: $-\frac{5}{8}+\left(-\frac{1}{2}\right)$
b. Subtract: $-\frac{5}{8}-\left(-\frac{1}{2}\right)$
c. Multiply: $-\frac{5}{8}\left(-\frac{1}{2}\right)$
d. Divide: $-\frac{5}{8} \div\left(-\frac{1}{2}\right)$

Simplify.
95. $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)$
96. $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)$
97. $\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)$
98. $\left(1+\frac{1}{3}\right)\left(1-\frac{1}{3}\right)$
99. Find the value of $1+\frac{1}{x}$ when $x$ is
a. 2
b. 3
c. 4
100. Find the value of $1-\frac{1}{x}$ when $x$ is
a. 2
b. 3
c. 4
101. Find the value of $2 x+\frac{6}{x}$ when $x$ is
a. 1
b. 2
c. 3
102. Find the value of $x+\frac{4}{x}$ when $x$ is
a. 1
b. 2
c. 3

## Chapter 1 SUMMARY

## NOTE

We will use the margins in the chapter summaries to give examples that correspond to the topic being reviewed whenever it is appropriate

The number(s) in brackets next to each heading indicates the section(s) in which that topic is discussed.

## Symbols [1.1]

$$
\begin{array}{ll}
a=b & a \text { is equal to } b \\
a \neq b & a \text { is not equal to } b \\
a<b & a \text { is less than } b \\
a \nless b & a \text { is not less than } b \\
a>b & a \text { is greater than } b \\
a \ngtr b & a \text { is not greater than } b \\
a \geq b & a \text { is greater than or equal to } b \\
a \leq b & a \text { is less than or equal to } b
\end{array}
$$

## Exponents [1.1]

Exponents are notation used to indicate repeated multiplication. In the expression $3^{4}, 3$ is the base and 4 is the exponent.

$$
3^{4}=3 \cdot 3 \cdot 3 \cdot 3=81
$$

## Order of Operations [1.1]

When evaluating a mathematical expression, we will perform the operations in the following order, beginning with the expression in the innermost parentheses or brackets and working our way out.

1. Simplify all numbers with exponents, working from left to right if more than one of these numbers is present.
2. Then do all multiplications and divisions left to right.
3. Finally, perform all additions and subtractions left to right.

## Addition of Real Numbers [1.3]

To add two real numbers with

1. The same sign: Simply add their absolute values and use the common sign.
2. Different signs: Subtract the smaller absolute value from the larger absolute value. The answer has the same sign as the number with the larger absolute value.

## Subtraction of Real Numbers [1.4]

4. Subtracting 2 is the same as adding -2 : $7-2=7+(-2)=5$

To subtract one number from another, simply add the opposite of the number you are subtracting; that is, if $a$ and $b$ represent real numbers, then

$$
a-b=a+(-b)
$$

## Multiplication of Real Numbers [1.6]

To multiply two real numbers, simply multiply their absolute values. Like signs give a positive answer. Unlike signs give a negative answer.
6. $\frac{-6}{2}=-6\left(\frac{1}{2}\right)=-3$
$\frac{-6}{-2}=-6\left(-\frac{1}{2}\right)=3$
7. $|5|=5$
$|-5|=5$
8. The numbers 3 and -3 are opposites; their sum is 0 : $3+(-3)=0$

## Division of Real Numbers [1.7]

Division by a number is the same as multiplication by its reciprocal. Like signs give a positive answer. Unlike signs give a negative answer.

## Absolute Value [1.2]

The absolute value of a real number is its distance from zero on the real number line. Absolute value is never negative.

## Opposites [1.2, 1.5]

Any two real numbers the same distance from zero on the number line but in opposite directions from zero are called opposites. Opposites always add to zero.

## Reciprocals [1.2, 1.5]

Any two real numbers whose product is 1 are called reciprocals. Every real number has a reciprocal except 0 .

## Properties of Real Numbers [1.5]

|  | For Addition | For Multiplication |
| :--- | :--- | :--- |
| Commutative: | $a+b=b+a$ | $a \cdot b=b \cdot a$ |
| Associative: | $a+(b+c)=(a+b)+c$ | $a \cdot(b \cdot c)=(a \cdot b) \cdot c$ |
| Identity: | $a+0=a$ | $a \cdot 1=a$ |
| Inverse: | $a+(-a)=0$ | $a\left(\frac{1}{a}\right)=1$ | Distributive: $\quad a(b+c)=a b+a c$

## Subsets of the Real Numbers [1.8]

10. a. 7 and 100 are counting numbers, but 0 and -2 are not.
b. 0 and 241 are whole numbers, but -4 and $\frac{1}{2}$ are not.
c. $-15,0$, and 20 are integers.
d. $-4,-\frac{1}{2}, 0.75$, and 0.666 are rational numbers.
e. $-\pi, \sqrt{3}$, and $\pi$ are irrational numbers.
f. All the numbers listed above are real numbers.

Counting numbers: Whole numbers:
Integers:
Rational numbers:

Irrational numbers:

## Real numbers:

$\{1,2,3, \ldots\}$
$\{0,1,2,3, \ldots\}$
$\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}$
\{all numbers that can be expressed as the ratio of two integers\}
\{all numbers on the number line that cannot be expressed as the ratio of two integers\}
\{all numbers that are either rational or irrational\}

## Factoring [1.8]

Factoring is the reverse of multiplication.


## Least Common Denominator (LCD) [1.9]

The least common denominator (LCD) for a set of denominators is the smallest number that is exactly divisible by each denominator.

## Addition and Subtraction of Fractions [1.9]

To add (or subtract) two fractions with a common denominator, add (or subtract) numerators and use the common denominator.

$$
\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \quad \text { and } \quad \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}
$$

## COMMON MISTAKES

1. Interpreting absolute value as changing the sign of the number inside the absolute value symbols. $|-5|=+5,|+5|=-5$. (The first expression is correct; the second one is not.) To avoid this mistake, remember: Absolute value is a distance and distance is always measured in positive units.
2. Using the phrase "two negatives make a positive." This works only with multiplication and division. With addition, two negative numbers produce a negative answer. It is best not to use the phrase "two negatives make a positive" at all.

## Chapter 1 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

The numbers in brackets refer to the sections of the text in which similar problems can be found.

Write the numerical expression that is equivalent to each phrase, and then simplify. [1.3, 1.4, 1.6, 1.7]

1. The sum of -7 and -10
2. Five added to the sum of -7 and 4
3. The sum of -3 and 12 increased by 5
4. The difference of 4 and 9
5. The difference of 9 and -3
6. The difference of -7 and -9
7. The product of -3 and -7 decreased by 6
8. Ten added to the product of 5 and -6
9. Twice the product of -8 and $3 x$
10. The quotient of -25 and -5

Simplify. [1.2]
11. $|-1.8|$
12. $-|-10|$

For each number, give the opposite and the reciprocal. [1.2]
13. 6
14. $-\frac{12}{5}$

Multiply. [1.2, 1.6]
15. $\frac{1}{2}(-10)$
16. $\left(-\frac{4}{5}\right)\left(\frac{25}{16}\right)$

Add. [1.3]
17. $-9+12$
18. $-18+(-20)$
19. $-2+(-8)+[-9+(-6)]$
20. $(-21)+40+(-23)+5$

Subtract. [1.4]
21. $6-9$
22. $14-(-8)$
23. $-12-(-8)$
24. $4-9-15$

Find the products. [1.6]
25. $(-5)(6)$
26. $4(-3)$
27. $-2(3)(4)$
28. $(-1)(-3)(-1)(-4)$

Find the following quotients. [1.7]
29. $\frac{12}{-3}$
30. $-\frac{8}{9} \div \frac{4}{3}$

Simplify. [1.1, 1.6, 1.7]
31. $4 \cdot 5+3$
32. $9 \cdot 3+4 \cdot 5$
33. $2^{3}-4 \cdot 3^{2}+5^{2}$
34. $12-3(2 \cdot 5+7)+4$
35. $20+8 \div 4+2 \cdot 5$
36. $2(3-5)-(2-8)$
37. $30 \div 3 \cdot 2$
38. $(-2)(3)-(4)(-3)-9$
39. $3(4-7)^{2}-5(3-8)^{2}$
40. $(-5-2)(-3-7)$
41. $\frac{4(-3)}{-6}$
42. $\frac{3^{2}+5^{2}}{(3-5)^{2}}$
43. $\frac{15-10}{6-6}$
44. $\frac{2(-7)+(-11)(-4)}{7-(-3)}$

State the property or properties that justify the following. [1.5]
45. $9(3 y)=(9 \cdot 3) y$
46. $\quad 8(1)=8$
47. $(4+y)+2=(y+4)+2$
48. $5+(-5)=0$
49. $(4+2)+y=(4+y)+2$
50. $5(w-6)=5 w-30$

Use the associative property to rewrite each expression, and then simplify the result. [1.5]
51. $7+(5+x)$
52. $4(7 a)$
53. $\frac{1}{9}(9 x)$
54. $\frac{4}{5}\left(\frac{5}{4} y\right)$

Apply the distributive property to each of the following expressions. Simplify when possible. [1.5, 1.6]
55. $7(2 x+3)$
56. $3(2 a-4)$
57. $\frac{1}{2}(5 x-6)$
58. $-\frac{1}{2}(3 x-6)$

For the set $\left\{\sqrt{7},-\frac{1}{3}, 0,5,-4.5, \frac{2}{5}, \pi,-3\right\}$ list all the [1.8]
59. rational numbers
60. whole numbers
61. irrational numbers
62. integers

Factor into the product of primes. [1.8]
63. 90
64. 840

Combine. [1.9]
65. $\frac{18}{35}+\frac{13}{42}$
66. $\frac{x}{6}+\frac{7}{12}$

Find the next number in each sequence. [1.1, 1.2, 1.3, 1.6, 1.9]
67. $10,7,4,1, \ldots$
68. $10,-30,90,-270, \ldots$
69. $1,1,2,3,5, \ldots$
71. $1, \frac{1}{2}, 0,-\frac{1}{2}, \ldots$
70. $4,6,8,10, \ldots$
72. $1,-\frac{1}{2}, \frac{1}{4},-\frac{1}{8}, \ldots$

## Chapter 1 Projects

## Basic Properties and Definitions

## GROUP PROJECT Binary Numbers

Students and Instructors: The end of each chapter in this book will contain a section like this one containing two projects. The group project is intended to be done in class. The research projects are to be completed outside of class. They can be done in groups or individually. In my classes, I use the research projects for extra credit. I require all research projects to be done on a word processor and to be free of spelling errors.

Number of People 2 or 3
Time Needed 10 minutes
Equipment Paper and pencil
Background Our decimal number system is a base 10 number system. We have 10 digits- $0,1,2,3,4,5,6,7$, 8 , and 9-which we use to write all the numbers in our number system. The number 10 is the first number that is written with a combination of digits. Although our number system is very useful, there are other number systems that are more appropriate for some disciplines. For example, computers and computer programmers use both the binary number system, which is base 2 , and the hexadecimal number system, which is base 16 . The binary number system has only digits 0 and 1 , which are used to write all the other numbers. Every number in our base 10 number system can be written in the base 2 number system as well.

Procedure To become familiar with the binary number system, we first learn to count in base 2. Imagine that the odometer on your car had only 0's and 1's. Here is what the odometer would look like for the first 6 miles the car was driven.

ODOMETER READING


MILEAGE

0

1

2

3

4

5

6

Continue the table at left to show the odometer reading for the first 32 miles the car is driven. At 32 miles, the odometer should read


## RESEARCH PROJECT

## Sophie Germain



T.he photograph at the left shows the street sign in Paris named for the French mathematician Sophie Germain (1776-1831). Among her contributions to mathematics is her work with prime numbers. In this chapter we had an introductory look at some of the classifications for numbers, including the prime numbers. Within the prime numbers themselves, there are still further classifications. In fact, a Sophie Germain prime is a prime number $P$, for which both $P$ and $2 P+1$ are primes. For example, the prime number 2 is the first Sophie Germain prime because both 2 and $2 \cdot 2+1=5$ are prime numbers. The next Germain prime is 3 because both 3 and $2 \cdot 3+1=7$ are primes.

Sophie Germain was born on April 1, 1776, in Paris, France. She taught herself mathematics by reading the books in her father's library at home. Today she is recognized most for her work in number theory, which includes her work with prime numbers. Research the life of Sophie Germain. Write a short essay that includes information on her work with prime numbers and how her results contributed to solving Fermat's Last Theorem almost 200 years later.

## Linear Equations and Inequalities


ust before starting work on this edition of your text, I flew to Europe for vacation. From time to time the television screens on the plane displayed statistics about the flight. At one point during the flight the temperature outside the plane was $-60^{\circ} \mathrm{F}$. When I returned home, I did some research and found that the relationship between temperature $T$ and altitude $A$ can be described with the formula

$$
T=-0.0035 A+70
$$

when the temperature on the ground is $70^{\circ} \mathrm{F}$. The table and the line graph also describe this relationship.

| Air Temperature and Altitude <br> Altitude <br> (feet) | Temperature <br> $\left({ }^{\circ}\right.$ F) |
| :---: | :---: |
| 0 | 70 |
| 10,000 | 35 |
| 20,000 | 0 |
| 30,000 | -35 |
| 40,000 | -70 |



In this chapter we will start our work with formulas, and you will see how we use formulas to produce tables and line graphs like the ones above.

## Simplifying Expressions

## OBJECTIVES

A Simplify expressions by combining similar terms.

B Simplify expressions by applying the distributive property and then combining similar terms.

C Calculate the value of an expression for a given value of the variable.

If a cellular phone company charges $\$ 35$ per month plus $\$ 0.25$ for each minute, or fraction of a minute, that you use one of their cellular phones, then the amount of your monthly bill is given by the expression $35+0.25 t$. To find the amount you will pay for using that phone 30 minutes in one month, you substitute 30 for $t$ and simplify the resulting expression. This process is one of the topics we will study in this section.
As you will see in the next few sections, the first step in solving an equation is to simplify both sides as much as possible. In the first part of this section, we will practice simplifying expressions by combining what are called similar (or like) terms.
For our immediate purposes, a term is a number or a number and one or more variables multiplied together. For example, the number 5 is a term, as are the expressions $3 x,-7 y$, and $15 x y$.

DEFINITION Two or more terms with the same variable part are called similar (or like) terms.

The terms $3 x$ and $4 x$ are similar because their variable parts are identical. Likewise, the terms $18 y,-10 y$, and $6 y$ are similar terms.
To simplify an algebraic expression, we simply reduce the number of terms in the expression. We accomplish this by applying the distributive property along with our knowledge of addition and subtraction of positive and negative real numbers. The following examples illustrate the procedure.

## EXAMPLES Simplify by combining similar terms.

| 1. $3 x+4 x$ | $=(3+4) X$ |  |  |
| ---: | :--- | ---: | :--- |
|  | $=7 x$ |  | Distributive property |
| 2. $7 a-10$ | $=(7-10) a$ |  | Addition of 3 and 4 |
|  | $=-3 a$ |  | Distributive property |
| 3. $18 y-10 y+6 y=(18-10+6) y$ |  | Addition of 7 and $-\mathbf{1 0}$ |  |
|  | $=14 y$ |  | Distributive property |

Addition of 18, -10 , and 6


When the expression we intend to simplify is more complicated, we use the commutative and associative properties first.

EXAMPLES Simplify each expression.

4. | $3 x+5+2 x-3$ | $=3 x+2 x+5-3$ |  | Commutative property |
| ---: | :--- | ---: | :--- |
|  | $=(3 x+2 x)+(5-3)$ |  | Associative property |
|  | $=(3+2) x+(5-3)$ |  | Distributive property |
|  | $=5 x+2$ |  | Addition |
| 5. $4 a-7-2 a+3$ | $=(4 a-2 a)+(-7+3)$ |  | Commutative and <br> associative properties |
|  | $=(4-2) a+(-7+3)$ |  |  |
|  | $=2 a-4$ |  | Distributive property |
| Addition |  |  |  |

$$
\text { 6. } \begin{aligned}
5 x+8-x-6 & =(5 x-x)+(8-6) \\
& =(5-1) x+(8-6) \\
& =4 x+2
\end{aligned}
$$

## Commutative and associative properties Distributive property Addition

Notice that in each case the result has fewer terms than the original expression. Because there are fewer terms, the resulting expression is said to be simpler than the original expression.

## Simplifying Expressions Containing Parentheses

If an expression contains parentheses, it is often necessary to apply the distributive property to remove the parentheses before combining similar terms.

## EXAMPLE 7 Simplify the expression $5(2 x-8)-3$.

SOLUTION We begin by distributing the 5 across $2 x-8$. We then combine similar terms:

$$
\begin{align*}
5(2 x-8)-3 & =10 x-40-3 \quad \text { Distributive property } \\
& =10 x-43 \tag{111419}
\end{align*}
$$

## EXAMPLE 8 Simplify $7-3(2 y+1)$.

SOLUTION By the rule for order of operations, we must multiply before we add or subtract. For that reason, it would be incorrect to subtract 3 from 7 first. Instead, we multiply -3 and $2 y+1$ to remove the parentheses and then combine similar terms:

$$
\begin{align*}
7-3(2 y+1) & =7-6 y-3 \quad \text { Distributive property } \\
& =-6 y+4 \tag{1114}
\end{align*}
$$

## EXAMPLE 9 Simplify $5(x-2)-(3 x+4)$.

SOLUTION We begin by applying the distributive property to remove the parentheses. The expression $-(3 x+4)$ can be thought of as $-1(3 x+4)$. Thinking of it in this way allows us to apply the distributive property:

$$
\begin{aligned}
-1(3 x+4) & =-1(3 x)+(-1)(4) \\
& =-3 x-4
\end{aligned}
$$

The complete solution looks like this:

$$
\begin{aligned}
5(x-2)-(3 x+4) & =5 x-10-3 x-4 & & \text { Distributive property } \\
& =2 x-14 & & \text { Combine similar terms }
\end{aligned}
$$

As you can see from the explanation in Example 9, we use the distributive property to simplify expressions in which parentheses are preceded by a negative sign. In general we can write

$$
\begin{aligned}
-(a+b) & =-1(a+b) \\
& =-a+(-b) \\
& =-a-b
\end{aligned}
$$

The negative sign outside the parentheses ends up changing the sign of each term within the parentheses. In words, we say "the opposite of a sum is the sum of the opposites."

## The Value of an Expression

Recall from Chapter 1, an expression like $3 x+2$ has a certain value depending on what number we assign to $x$. For instance, when $x$ is $4,3 x+2$ becomes $3(4)+2$, or 14 . When $x$ is $-8,3 x+2$ becomes $3(-8)+2$, or -22 . The value of an expression is found by replacing the variable with a given number.

## EXAMPLES

Find the value of the following expressions by replacing the variable with the given number.

| Expression | Value of the Variable | Value of the Expression |
| :---: | :---: | :---: |
| 10. $3 x-1$ | $x=2$ | $3(2)-1=6-1=5$ |
| 11. $7 a+4$ | $a=-3$ | $7(-3)+4=-21+4=-17$ |
| 12. $2 x-3+4 x$ | $x=-1$ | $\begin{aligned} & 2(-1)-3+4(-1) \\ & \quad=-2-3+(-4)=-9 \end{aligned}$ |
| 13. $2 x-5-8 x$ | $x=5$ | $\begin{aligned} & 2(5)-5-8(5) \\ & \quad=10-5-40=-35 \end{aligned}$ |
| 14. $y^{2}-6 y+9$ | $y=4$ | $4^{2}-6(4)+9=16-24+9=1$ |

Simplifying an expression should not change its value; that is, if an expression has a certain value when $x$ is 5 , then it will always have that value no matter how much it has been simplified as long as $x$ is 5 . If we were to simplify the expression in Example 13 first, it would look like

$$
2 x-5-8 x=-6 x-5
$$

When $x$ is 5 , the simplified expression $-6 x-5$ is

$$
-6(5)-5=-30-5=-35
$$

It has the same value as the original expression when $x$ is 5 .
We also can find the value of an expression that contains two variables if we know the values for both variables.

EXAMPLE 15 Find the value of the expression $2 x-3 y+4$ when $x$ is -5
and $y$ is 6 .
SOLUTION Substituting -5 for $x$ and 6 for $y$, the expression becomes

$$
\begin{aligned}
2(-5)-3(6)+4 & =-10-18+4 \\
& =-28+4 \\
& =-24
\end{aligned}
$$

EXAMPLE 16 Find the value of the expression $x^{2}-2 x y+y^{2}$ when $x$ is 3
and $y$ is -4 .
SOLUTION Replacing each $x$ in the expression with the number 3 and each $y$ in the expression with the number -4 gives us

$$
\begin{aligned}
3^{2}-2(3)(-4)+(-4)^{2} & =9-2(3)(-4)+16 \\
& =9-(-24)+16 \\
& =33+16 \\
& =49
\end{aligned}
$$

## More About Sequences

As the next example indicates, when we substitute the counting numbers, in order, into algebraic expressions, we form some of the sequences of numbers that we studied in Chapter 1. To review, recall that the sequence of counting numbers (also called the sequence of positive integers) is

Counting numbers $=1,2,3, \ldots$

## EXAMPLE 17 Substitute $1,2,3$, and 4 for $n$ in the expression $2 n-1$.

SOLUTION Substituting as indicated, we have

$$
\begin{aligned}
& \text { When } n=1,2 n-1=2 \cdot 1-1=1 \\
& \text { When } n=2,2 n-1=2 \cdot 2-1=3 \\
& \text { When } n=3,2 n-1=2 \cdot 3-1=5 \\
& \text { When } n=4,2 n-1=2 \cdot 4-1=7
\end{aligned}
$$

As you can see, substituting the first four counting numbers into the formula $2 n-1$ produces the first four numbers in the sequence of odd numbers.

The next example is similar to Example 17 but uses tables to display the information. substituting the first four counting numbers into the expressions $2 n$ and $n^{2}$.
a.

| $n$ | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- |
|  |  |  |  |  |

$2 n$
b.

| $n$ | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| $n^{2}$ |  |  |  |  |

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $\quad 1-6$
B 7-9
C 10-18

SOLUTION Proceeding as we did in the previous example, we substitute the numbers 1, 2, 3, and 4 into the given expressions.
a. When $n=1,2 n=2 \cdot 1=2$

When $n=2,2 n=2 \cdot 2=4$
When $n=3,2 n=2 \cdot 3=6$
When $n=4,2 n=2 \cdot 4=8$
As you can see, the expression $2 n$ produces the sequence of even numbers when $n$ is replaced by the counting numbers. Placing these results into our first
 table gives us
b. The expression $n^{2}$ produces the sequence of squares when $n$ is replaced by $1,2,3$, and 4 . In table form we have

| $n$ | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| $n^{2}$ | 1 | 4 | 9 | 16 |

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What are similar terms?
2. Explain how the distributive property is used to combine similar terms.
3. What is wrong with writing $3 x+4 x=7 x^{2}$ ?
4. Explain how you would find the value of $5 x+3$ when $x$ is 6 .

## Problem Set 2.1

Simplify the following expressions.

1. $3 x-6 x$
2. $7 x-5 x$
3. $-2 a+a$
4. $3 a-a$
5. $7 x+3 x+2 x$
6. $8 x-2 x-x$
7. $3 a-2 a+5 a$
8. $7 a-a+2 a$
9. $4 x-3+2 x$
10. $5 x+6-3 x$
11. $3 a+4 a+5$
12. $6 a+7 a+8$
13. $2 x-3+3 x-2$
14. $6 x+5-2 x+3$
15. $3 a-1+a+3$
16. $-a+2+8 a-7$
17. $-4 x+8-5 x-10$
18. $-9 x-1+x-4$
19. $7 a+3+2 a+3 a$
20. $5(2 x-1)+4$
21. $7(3 y+2)-8$
22. $-3(2 x-1)+5$
23. $5-2(a+1)$
24. $6-4(x-5)$
25. $12-3(4 x-2)$
26. $-9-4(2-y)+1$
27. $-10-3(2-y)+3$
28. $-6+2(2-3 x)+1$
29. $-7-4(3-x)+1$
30. $(4 x-7)-(2 x+5)$
31. $(7 x-3)-(4 x+2)$
32. $8(2 a+4)-(6 a-1)$
33. $9(3 a+5)-(8 a-7)$
34. $3(x-2)+(x-3)$
35. $2(2 x+1)-(x+4)$
36. $4(2 y-8)-(y+7)$
37. $5(y-3)-(y-4)$
38. $-9(2 x+1)-(x+5)$
39. $-3(3 x-2)-(2 x+3)$

Evaluate the following expressions when $x$ is 2 .
45. $3 x-1$
46. $4 x+3$
47. $-2 x-5$
48. $-3 x+6$
49. $x^{2}-8 x+16$
50. $x^{2}-10 x+25$
51. $(x-4)^{2}$
52. $(x-5)^{2}$

Find the value of each expression when $x$ is -5 . Then simplify the expression, and check to see that it has the same value for $x=-5$.
53. $7 x-4-x-3$
54. $3 x+4+7 x-6$
55. $5(2 x+1)+4$
56. $2(3 x-10)+5$

Find the value of each expression when $x$ is -3 and $y$ is 5 .
57. $x^{2}-2 x y+y^{2}$
58. $x^{2}+2 x y+y^{2}$
59. $(x-y)^{2}$
60. $(x+y)^{2}$
61. $x^{2}+6 x y+9 y^{2}$
62. $x^{2}+10 x y+25 y^{2}$
63. $(x+3 y)^{2}$
64. $(x+5 y)^{2}$

Find the value of $12 x-3$ for each of the following values of $x$.
65. $\frac{1}{2}$
66. $\frac{1}{3}$
67. $\frac{1}{4}$
68. $\frac{1}{6}$
69. $\frac{3}{2}$
70. $\frac{2}{3}$
71. $\frac{3}{4}$
72. $\frac{5}{6}$
73. Fill in the tables below to find the sequences formed by substituting the first four counting numbers into the expressions $3 n$ and $n^{3}$.
a. $n$
$3 n$
b
$n^{3}$
74. Fill in the tables below to find the sequences formed by substituting the first four counting numbers into the expressions $2 n-1$ and $2 n+1$
a.

b.

$$
\begin{array}{lllll}
n & 1 & 2 & 3 & 4 \\
2 n+1 & & & &
\end{array}
$$

Find the sequences formed by substituting the counting numbers, in order, into the following expressions.
75. $3 n-2$
76. $2 n-3$
77. $n^{2}-2 n+1$
78. $(n-1)^{2}$

Here are some problems you will see later in the book. Simplify.
79. $7-3(2 y+1)$
80. $4(3 x-2)-(6 x-5)$
81. $0.08 x+0.09 x$
82. $0.04 x+0.05 x$
83. $(x+y)+(x-y)$
84. $(-12 x-20 y)+(25 x+20 y)$

Simplify.
85. $3 x+2(x-2)$
86. $2(x-2)+3(5 x)$
87. $4(x+1)+3(x-3)$
88. $5(x+2)+3(x-1)$
89. $x+(x+3)(-3)$
90. $x-2(x+2)$
91. $3(4 x-2)-(5 x-8)$
92. $2(5 x-3)-(2 x-4)$
93. $-(3 x+1)-(4 x-7)$
94. $-(6 x+2)-(8 x-3)$
95. $(x+3 y)+3(2 x-y)$
96. $(2 x-y)-2(x+3 y)$
97. $3(2 x+3 y)-2(3 x+5 y)$
98. $5(2 x+3 y)-3(3 x+5 y)$
99. $-6\left(\frac{1}{2} x-\frac{1}{3} y\right)+12\left(\frac{1}{4} x+\frac{2}{3} y\right)$
100. $6\left(\frac{1}{3} x+\frac{1}{2} y\right)-4\left(x+\frac{3}{4} y\right)$
101. $0.08 x+0.09(x+2,000)$
102. $0.06 x+0.04(x+7,000)$
103. $0.10 x+0.12(x+500)$
104. $0.08 x+0.06(x+800)$
105. Find $a$ so the expression $(5 x+4 y)+a(2 x-y)$ simplifies to an expression that does not contain $y$. Using that value of $a$, simplify the expression.
106. Find $a$ so the expression $(5 x+4 y)-a(x-2 y)$ simplifies to an expression that does not contain $x$. Using that value of $a$, simplify the expression.

Find the value of $b^{2}-4 a c$ for the given values of $a, b$, and $c$. (You will see these problems later in the book.)
107. $a=1, b=-5, c=-6$
108. $a=1, b=-6, c=7$
109. $a=2, b=4, c=-3$
110. $a=3, b=4, c=-2$

## Applying the Concepts

111. Temperature and Altitude If the temperature on the ground is $70^{\circ} \mathrm{F}$, then the temperature at $A$ feet above the ground can be found from the expression $-0.0035 A+70$. Find the temperature at the following altitudes.
a. 8,000 feet
b. 12,000 feet
c. 24,000 feet

112. Perimeter of a Rectangle The expression $2 l+2 w$ gives the perimeter of a rectangle with length $l$ and width $w$. Find the perimeter of the rectangles with the following lengths and widths.
a. Length $=8$ meters, width $=5$ meters

b. Length $=10$ feet, width $=3$ feet

113. Cellular Phone Rates A cellular phone company charges $\$ 35$ per month plus $\$ 0.25$ for each minute, or fraction of a minute, that you use one of their cellular phones. The expression $35+0.25 t$ gives the amount of money you will pay for using one of their phones for $t$ minutes a month. Find the monthly bill for using one of their phones.
a. 10 minutes in a month
b. 20 minutes in a month
c. 30 minutes in a month
114. Cost of Bottled Water A water bottling company charges $\$ 7.00$ per month for their water dispenser and $\$ 1.10$ for each gallon of water delivered. If you have $g$ gallons of
 water delivered in a month, then the expression $7+1.1 \mathrm{~g}$ gives the amount of your bill for that month. Find the monthly bill for each of the following deliveries.
a. 10 gallons
b. 20 gallons
c. 30 gallons
115. Taxes We all have to pay taxes. Suppose that $21 \%$ of your monthly pay is withheld for federal income taxes and another $8 \%$ is withheld for Social Security, state income tax, and other miscellaneous items. If $G$ is your monthly pay before any money is deducted (your gross pay), then the amount of money that you take home each month is given by the expression $G-0.21 G-0.08 G$. Simplify this expression and then find your takehome pay if your gross pay is $\$ 1,250$ per month.
116. Improving Your Quantitative Literacy The chart shows how the average cost per minute for using a cellular phone declined over a specific period of years. For each of the following years, use the chart to write an expression for the average cost to talk on a cell phone for $x$ minutes, and then evaluate the expression to find the average cost to talk on a cell phone for an hour.
a. 1997
b. 2001
c. 2004

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## Maintaining Your Skills

From this point on, each problem set will contain a number of problems under the heading Maintaining Your Skills. These problems cover the most important skills you have learned in previous sections and chapters. Hopefully, by working these problems on a regular basis, you will keep yourself current on all the topics we have covered and possibly need less time to study for tests and quizzes.
117. $\frac{1}{8}-\frac{1}{6}$
118. $\frac{x}{8}-\frac{x}{6}$
119. $\frac{5}{9}-\frac{4}{3}$
120. $\frac{x}{9}-\frac{x}{3}$
121. $-\frac{7}{30}+\frac{5}{28}$
122. $-\frac{11}{105}+\frac{11}{30}$

## Getting Ready for the Next Section

Problems under this heading, Getting Ready for the Next Section, are problems that you must be able to work in order to understand the material in the next section. The problems below are exactly the types of problems you will see in the explanations and examples in the next section.

Simplify.
123. $17-5$
125. $2-5$
127. $-2.4+(-7.3)$
129. $-\frac{1}{2}+\left(-\frac{3}{4}\right)$
124. $12+(-2)$
126. $25-20$
128. $8.1+2.7$
130. $-\frac{1}{6}+\left(-\frac{2}{3}\right)$
131. $4(2 \cdot 9-3)-7 \cdot 9$
132. $5(3 \cdot 45-4)-14 \cdot 45$
133. $4(2 a-3)-7 a$
134. $5(3 a-4)-14 a$
135. $-3-\frac{1}{2}$
136. $-5-\frac{1}{3}$
137. $\frac{4}{5}+\frac{1}{10}+\frac{3}{8}$
138. $\frac{3}{10}+\frac{7}{25}+\frac{3}{4}$
139. Find the value of $2 x-3$ when $x$ is 5 .
140. Find the value of $3 x+4$ when $x$ is -2 .

## Addition Property of Equality

## OBJECTIVES

A Check the solution to an equation by substitution.

B Use the addition property of equality to solve an equation.

## Note

We can use a question mark over the equal signs to show that we don't know yet whether the two sides of the equation are equal.

When light comes into contact with any object, it is reflected, absorbed, and transmitted, as shown in Figure 1.
For a certain type of glass, $88 \%$ of the light hitting the glass is transmitted through to the other side, whereas $6 \%$ of the light is absorbed into the glass. To find the percent of light that is reflected by the glass, we can solve the equation

$$
88+R+6=100
$$



FIGURE 1

Solving equations of this type is what we study in this section. To solve an equation we must find all replacements for the variable that make the equation a true statement.

DEFINITION The solution set for an equation is the set of all numbers that when used in place of the variable make the equation a true statement.

For example, the equation $x+2=5$ has solution set $\{3\}$ because when $x$ is 3 the equation becomes the true statement $3+2=5$, or $5=5$.

## EXAMPLE $1 \quad$ Is 5 a solution to $2 \mathrm{x}-3=7$ ?

SOLUTION We substitute 5 for $x$ in the equation, and then simplify to see if a true statement results. A true statement means we have a solution; a false statement indicates the number we are using is not a solution.

| When | $x$ |
| ---: | :--- |
| the equation | $=5$ |
| $2 x-3$ | $=7$ |
| becomes | $2(5)-3$ |
| $=\stackrel{?}{=} 7$ |  |
| $10-3$ | $\stackrel{?}{=} 7$ |
| 7 | $=7$ |

A true statement
Because $x=5$ turns the equation into the true statement $7=7$, we know 5 is a solution to the equation.

EXAMPLE 2 Is -2 a solution to $8=3 x+4$ ?
SOLUTION Substituting -2 for $x$ in the equation, we have

$$
\begin{array}{ll}
8 \stackrel{?}{=} 3(-2)+4 \\
8 \stackrel{?}{=}-6+4 \\
8 & =-2
\end{array} \quad \text { A false statement }
$$

Substituting -2 for $x$ in the equation produces a false statement. Therefore, $x=-2$ is not a solution to the equation.

The important thing about an equation is its solution set. We therefore make the following definition to classify together all equations with the same solution set.

DEFINITION Two or more equations with the same solution set are said to be equivalent equations.

Equivalent equations may look different but must have the same solution set.

## EXAMPLE 3

a. $x+2=5$ and $x=3$ are equivalent equations because both have solution set $\{3\}$.
b. $a-4=3, a-2=5$, and $a=7$ are equivalent equations because they all have solution set $\{7\}$.
c. $y+3=4, y-8=-7$, and $y=1$ are equivalent equations because they all have solution set $\{1\}$.

If two numbers are equal and we increase (or decrease) both of them by the same amount, the resulting quantities are also equal. We can apply this concept to equations. Adding the same amount to both sides of an equation always produces an equivalent equation-one with the same solution set. This fact about equations is called the addition property of equality and can be stated more formally as follows.

Addition Property of Equality For any three algebraic expressions $A, B$, and $C$,

$$
\left.\begin{array}{llrl} 
& \text { if } & & A
\end{array}\right)=B
$$

In words: Adding the same quantity to both sides of an equation will not change the solution set.

This property is just as simple as it seems. We can add any amount to both sides of an equation and always be sure we have not changed the solution set.

Consider the equation $x+6=5$. We want to solve this equation for the value of $x$ that makes it a true statement. We want to end up with $x$ on one side of the equal sign and a number on the other side. Because we want $x$ by itself, we will add -6 to both sides:

$$
\begin{aligned}
x+6+(-\mathbf{6}) & =5+(-\mathbf{6}) & & \text { Addition property of equality } \\
x+0 & =-1 & & \text { Addition } \\
x & =-1 & &
\end{aligned}
$$

All three equations say the same thing about $x$. They all say that $x$ is -1 . All three equations are equivalent. The last one is just easier to read.

Here are some further examples of how the addition property of equality can be used to solve equations.

## EXAMPLE 4 Solve the equation $x-5=12$ for $x$.

SOLUTION Because we want $x$ alone on the left side, we choose to add +5 to both sides:

$$
\begin{aligned}
x-5+\mathbf{5} & =12+\mathbf{5} \quad \text { Addition property of equality } \\
x+0 & =17 \\
x & =17
\end{aligned}
$$

To check our solution, we substitute 17 for $x$ in the original equation:

| When | $x=17$ |  |
| :---: | :---: | :---: |
| the equation | $x-5=12$ |  |
| becomes | $17-5 \stackrel{?}{\underline{=}} 12$ |  |
|  | $12=12$ | A true statement |

As you can see, our solution checks. The purpose for checking a solution to an equation is to catch any mistakes we may have made in the process of solving the equation.

## EXAMPLE 5 Solve for $a: a+\frac{3}{4}=-\frac{1}{2}$.

SOLUTION Because we want $a$ by itself on the left side of the equal sign, we add the opposite of $\frac{3}{4}$ to each side of the equation.

$$
\begin{aligned}
a+\frac{3}{4}+\left(-\frac{\mathbf{3}}{\mathbf{4}}\right) & =-\frac{1}{2}+\left(-\frac{\mathbf{3}}{\mathbf{4}}\right) & & \text { Addition property of equality } \\
a+0 & =-\frac{1}{2} \cdot \frac{\mathbf{2}}{\mathbf{2}}+\left(-\frac{3}{4}\right) & & \text { LCD on the right side is } 4 \\
a & =-\frac{2}{4}+\left(-\frac{3}{4}\right) & & \frac{2}{4} \text { is equivalent to } \frac{1}{2} \\
a & =-\frac{5}{4} & & \text { Add fractions }
\end{aligned}
$$

The solution is $a=-\frac{5}{4}$. To check our result, we replace $a$ with $-\frac{5}{4}$ in the original equation. The left side then becomes $-\frac{5}{4}+\frac{3}{4}$, which reduces to $-\frac{1}{2}$, so our solution checks.

## EXAMPLE 6 Solve for $x: 7.3+x=-2.4$.

SOLUTION Again, we want to isolate $x$, so we add the opposite of 7.3 to both sides:

$$
\begin{aligned}
7.3+(-\mathbf{7 . 3})+x & =-2.4+(-\mathbf{7 . 3}) \quad \text { Addition property of equality } \\
0+x & =-9.7 \\
x & =-9.7
\end{aligned}
$$

The addition property of equality also allows us to add variable expressions to each side of an equation.

EXAMPLE 7 Solve for $x: 3 x-5=4 x$.
SOLUTION Adding $-3 x$ to each side of the equation gives us our solution.

$$
\begin{aligned}
3 x-5 & =4 x \\
3 x+(-\mathbf{3 x})-5 & =4 x+(-\mathbf{3 x}) \quad \text { Distributive property } \\
-5 & =x
\end{aligned}
$$

## Note

Again, we place a question mark over the equal sign because we don't know yet whether the expressions on the left and right side of the equal sign will be equal

Sometimes it is necessary to simplify each side of an equation before using the addition property of equality. The reason we simplify both sides first is that we want as few terms as possible on each side of the equation before we use the addition property of equality. The following examples illustrate this procedure.

## EXAMPLE 8 Solve $4(2 a-3)-7 a=2-5$.

SOLUTION We must begin by applying the distributive property to separate terms on the left side of the equation. Following that, we combine similar terms and then apply the addition property of equality.

$$
\begin{aligned}
4(2 a-3)-7 a & =2-5 & & \text { Original equation } \\
8 a-12-7 a & =2-5 & & \text { Distributive property } \\
a-12 & =-3 & & \text { Simplify each side } \\
a-12+\mathbf{1 2} & =-3+\mathbf{1 2} & & \text { Add 12 to each side } \\
a & =9 & & \text { Addition }
\end{aligned}
$$

To check our solution, we replace $a$ with 9 in the original equation.

$$
\begin{aligned}
4(2 \cdot 9-3)-7 \cdot 9 & \stackrel{?}{=} 2-5 \\
4(15)-63 & \stackrel{?}{=}-3 \\
60-63 & \stackrel{?}{=}-3 \\
-3 & =-3 \quad \text { A true statement }
\end{aligned}
$$

We can also add a term involving a variable to both sides of an equation.

## EXAMPLE 9 Solve $3 x-5=2 x+7$.

SOLUTION We can solve this equation in two steps. First, we add $-2 x$ to both sides of the equation. When this has been done, $x$ appears on the left side only. Second, we add 5 to both sides:

$$
\begin{array}{rlrl}
3 x+(-\mathbf{2 x})-5 & =2 x+(-\mathbf{2} \boldsymbol{x})+7 \\
x-5 & =7 & & \text { Add }-\mathbf{2} \boldsymbol{x} \text { to both sides } \\
x-5+\mathbf{5} & =7+\mathbf{5} & & \text { Simplify each side } \\
x & =12 & & \text { Add } 5 \text { to both sides } \\
x & & \text { Simplify each side }
\end{array}
$$



A Note on Subtraction Although the addition property of equality is stated for addition only, we can subtract the same number from both sides of an equation as well. Because subtraction is defined as addition of the opposite, subtracting the same quantity from both sides of an equation does not change the solution.

$$
\begin{aligned}
x+2 & =12 & & \text { Original equation } \\
x+2-\mathbf{2} & =12-\mathbf{2} & & \text { Subtract } \mathbf{2} \text { from each side } \\
x & =10 & & \text { Subtraction }
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-3
B 4-9

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a solution to an equation?
2. What are equivalent equations?
3. Explain in words the addition property of equality.
4. How do you check a solution to an equation?

## Problem Set 2.2

Online support materials can be found at www.thomsonedu.com/login

Find the solution for the following equations. Be sure to show when you have used the addition property of equality.

1. $x-3=8$
2. $x-2=7$
3. $x+2=6$
4. $x+5=4$
5. $a+\frac{1}{2}=-\frac{1}{4}$
6. $a+\frac{1}{3}=-\frac{5}{6}$
7. $x+2.3=-3.5$
8. $x+7.9=-3.4$
9. $y+11=-6$
10. $y-3=-1$
11. $x-\frac{5}{8}=-\frac{3}{4}$
12. $x-\frac{2}{5}=-\frac{1}{10}$
13. $m-6=2 m$
14. $3 m-10=4 m$
15. $6.9+x=3.3$
16. $7.5+x=2.2$
17. $5 a=4 a-7$
18. $12 a=-3+11 a$
19. $-\frac{5}{9}=x-\frac{2}{5}$
20. $-\frac{7}{8}=x-\frac{4}{5}$

Simplify both sides of the following equations as much as possible, and then solve.
21. $4 x+2-3 x=4+1$
22. $5 x+2-4 x=7-3$
23. $8 a-\frac{1}{2}-7 a=\frac{3}{4}+\frac{1}{8}$
24. $9 a-\frac{4}{5}-8 a=\frac{3}{10}-\frac{1}{5}$
25. $-3-4 x+5 x=18$
26. $10-3 x+4 x=20$
27. $-11 x+2+10 x+2 x=9$
28. $-10 x+5-4 x+15 x=0$
29. $-2.5+4.8=8 x-1.2-7 x$
30. $-4.8+6.3=7 x-2.7-6 x$
31. $2 y-10+3 y-4 y=18-6$
32. $15-21=8 x+3 x-10 x$

The following equations contain parentheses. Apply the distributive property to remove the parentheses, then simplify each side before using the addition property of equality.
33. $2(x+3)-x=4 \quad$ 34. $5(x+1)-4 x=2$
35. $-3(x-4)+4 x=3-7$
36. $-2(x-5)+3 x=4-9$
37. $5(2 a+1)-9 a=8-6$
38. $4(2 a-1)-7 a=9-5$
39. $-(x+3)+2 x-1=6$
40. $-(x-7)+2 x-8=4$
41. $4 y-3(y-6)+2=8$
42. $7 y-6(y-1)+3=9$
43. $-3(2 m-9)+7(m-4)=12-9$
44. $-5(m-3)+2(3 m+1)=15-8$

Solve the following equations by the method used in Example 9 in this section. Check each solution in the original equation.
45. $4 x=3 x+2$
47. $8 a=7 a-5$
49. $2 x=3 x+1$
51. $2 y+1=3 y+4$
53. $2 m-3=m+5$
55. $4 x-7=5 x+1$
57. $4 x+\frac{4}{3}=5 x-\frac{2}{3}$
59. $8 a-7.1=7 a+3.9$
61. Solve each equation.
a. $2 x=3$
b. $2+x=3$
c. $2 x+3=0$
d. $2 x+3=-5$
e. $2 x+3=7 x-5$
46. $6 x=5 x-4$
48. $9 a=8 a-3$
50. $4 x=3 x+5$
52. $4 y+2=5 y+6$
54. $8 m-1=7 m-3$
56. $3 x-7=4 x-6$
58. $2 x+\frac{1}{4}=3 x-\frac{5}{4}$
60. $10 a-4.3=9 a+4.7$
62. Solve each equation.
a. $5 t=10$
b. $5+t=10$
c. $5 t+10=0$
d. $5 t+10=12$
e. $5 t+10=8 t+12$

## Applying the Concepts

63. Light When light comes into contact with any object, it is reflected, absorbed, and transmitted, as shown in the following figure. If $T$ represents the percent of light transmitted, $R$ the percent of light reflected, and $A$ the percent of light
 absorbed by a surface, then the equation $T+R+A=$ 100 shows one way these quantities are related.
a. For glass, $T=88$ and $A=6$, meaning that $88 \%$ of the light hitting the glass is transmitted and $6 \%$ is absorbed. Substitute $T=88$ and $A=6$ into the equation $T+R+A=100$ and solve for $R$ to find the percent of light that is reflected.
b. For flat black paint, $A=95$ and no light is transmitted, meaning that $T=0$. What percent of light is reflected by flat black paint?
c. A pure white surface can reflect $98 \%$ of light, so $R=98$. If no light is transmitted, what percent of light is absorbed by the pure white surface?
d. Typically, shiny gray metals reflect $70-80 \%$ of light. Suppose a thick sheet of aluminum absorbs $25 \%$ of light. What percent of light is reflected by this shiny gray metal? (Assume no light is transmitted.)
64. Improving Your Quantitative Literacy According to a survey done by Seventeen magazine in 2005, the average spending amount for girls going to the prom was $\$ 338$.

USA TODAY Snapshots ${ }^{\circledR}$


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a. Suppose Kendra spent $\$ 210$ for a prom dress, then bought new shoes to go with the dress. If the total bill was $\$ 289$, does the equation

$$
210+x=289
$$

describe this situation? If so, what does $x$ represent?
b. Suppose Ava buys a prom dress on sale and then spends $\$ 120$ on her makeup, nails, and hair. If the total bill is $\$ 219$, and $x$ represents a positive number, does the equation

$$
x+219=120
$$

describe the situation?
65. Geometry The three angles shown in the triangle at the front of the tent in the following figure add up to $180^{\circ}$. Use this fact to write an equation containing $x$, and then solve the equation to find the number of degrees in the angle at the top of the triangle.

66. Geometry The figure shows part of a room. From a point on the floor, the angle of elevation to the top of the door is $47^{\circ}$, whereas the angle of elevation to the ceiling above the door is $59^{\circ}$. Use this diagram to write an equation involving $x$, and then solve the equa-
 tion to find the number of degrees in the angle that extends from the top of the door to the ceiling.

## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in previous sections and chapters. You can consider the time you spend working these problems as time spent studying for exams.
67. $3(6 x)$
68. $5(4 x)$
69. $\frac{1}{5}(5 x)$
70. $\frac{1}{3}(3 x)$
71. $8\left(\frac{1}{8} y\right)$
72. $6\left(\frac{1}{6} y\right)$
73. $-2\left(-\frac{1}{2} x\right)$
74. $-4\left(-\frac{1}{4} x\right)$
75. $-\frac{4}{3}\left(-\frac{3}{4} a\right)$
76. $-\frac{5}{2}\left(-\frac{2}{5} a\right)$

## Getting Ready for the Next Section

To understand all of the explanations and examples in the next section you must be able to work the problems below.

Simplify.
77. $\frac{3}{2}\left(\frac{2}{3} y\right)$
78. $\frac{5}{2}\left(\frac{2}{5} y\right)$
79. $\frac{1}{5}(5 x)$
81. $\frac{1}{5}(30)$
83. $\frac{3}{2}(4)$
85. $12\left(-\frac{3}{4}\right)$
87. $\frac{3}{2}\left(-\frac{5}{4}\right)$
89. $-13+(-5)$
91. $-\frac{3}{4}+\left(-\frac{1}{2}\right)$
93. $7 x+(-4 x)$
80. $-\frac{1}{4}(-4 a)$
82. $-\frac{1}{4}(24)$
84. $\frac{1}{26}(13)$
86. $12\left(\frac{1}{2}\right)$
88. $\frac{5}{3}\left(-\frac{6}{5}\right)$
90. $-14+(-3)$
92. $-\frac{7}{10}+\left(-\frac{1}{2}\right)$
94. $5 x+(-2 x)$

## 20

## OBJECTIVES

A Use the multiplication property of equality to solve an equation.

B Use the addition and multiplication properties of equality together to solve an equation.

## Multiplication Property of Equality

As we have mentioned before, we all have to pay taxes. According to Figure 1, people have been paying taxes for quite a long time.

If $21 \%$ of your monthly pay is withheld for federal income taxes and another $8 \%$ is withheld for Social Security, state income tax, and other miscellaneous items, leaving you with $\$ 987.50$ a month in take-home pay, then the amount you earned before the deductions were removed from your check is given by the equation

$$
G-0.21 G-0.08 G=987.5
$$

In this section we will learn how to solve equations of this type.

## Note

This property is also used many times throughout the book. Make every effort to understand it completely.

In the previous section, we found that adding the same number to both sides of an equation never changed the solution set. The same idea holds for multiplication by numbers other than zero. We can multiply both sides of an equation by the same nonzero number and always be sure we have not changed the solution set. (The reason we cannot multiply both sides by zero will become apparent later.) This fact about equations is called the multiplication property of equality, which can be stated formally as follows.

Multiplication Property of Equality For any three algebraic expressions
$A, B$, and $C$, where $C \neq 0$,

$$
\begin{aligned}
& \text { if } & A & =B \\
& \text { then } & A C & =B C
\end{aligned}
$$

In words: Multiplying both sides of an equation by the same nonzero number will not change the solution set.

Suppose we want to solve the equation $5 x=30$. We have $5 x$ on the left side but would like to have just $x$. We choose to multiply both sides by $\frac{1}{5}$ because $\left(\frac{1}{5}\right)(5)=1$. Here is the solution:

$$
\begin{aligned}
5 x & =30 \\
\frac{\mathbf{1}}{5}(5 x) & =\frac{\mathbf{1}}{5}(30) \quad \text { Multiplication property of equality } \\
\left(\frac{1}{5} \cdot 5\right) X & =\frac{1}{5}(30) \quad \text { Associative property of multiplication } \\
1 X & =6 \\
x & =6
\end{aligned}
$$

We chose to multiply by $\frac{1}{5}$ because it is the reciprocal of 5 . We can see that multiplication by any number except zero will not change the solution set. If, however, we were to multiply both sides by zero, the result would always be $0=0$ because multiplication by zero always results in zero. Although the statement $0=0$ is true, we have lost our variable and cannot solve the equation. This is the only restriction of the multiplication property of equality. We are free to multiply both sides of an equation by any number except zero.

Here are some more examples that use the multiplication property of equality.

## EXAMPLE 1 Solve for $a:-4 a=24$.

SOLUTION Because we want $a$ alone on the left side, we choose to multiply both sides by $-\frac{1}{4}$ :

$$
\begin{aligned}
-\frac{\mathbf{1}}{\mathbf{4}}(-4 a) & =-\frac{\mathbf{1}}{\mathbf{4}}(24) & \text { Multiplication property of equality } \\
{\left[-\frac{1}{4}(-4)\right] a } & =-\frac{1}{4}(24) & \text { Associative property } \\
a & =-6 &
\end{aligned}
$$

## EXAMPLE 2 Solve for $t:-\frac{t}{3}=5$.

SOLUTION Because division by 3 is the same as multiplication by $\frac{1}{3}$, we can write $-\frac{t}{3}$ as $-\frac{1}{3} t$. To solve the equation, we multiply each side by the reciprocal of $-\frac{1}{3}$, which is -3 .

$$
\begin{aligned}
-\frac{t}{3} & =5 & & \text { Original equation } \\
-\frac{1}{3} t & =5 & & \text { Dividing by } 3 \text { is equivalent to multiplying by } \frac{1}{3} \\
-\mathbf{3}\left(-\frac{1}{3} t\right) & =-\mathbf{3}(5) & & \text { Multiply each side by }-3 \\
t & =-15 & & \text { Multiplication }
\end{aligned}
$$

## EXAMPLE 3 Solve $\frac{2}{3} y=4$.

## Note

Notice in Examples 1 through 3 that if the variable is being multiplied by a number like -4 or $\frac{2}{3}$, we always multiply by the number's reciprocal, $-\frac{1}{4}$ or $\frac{3}{2}$, to end up with just the variable on one side of the equation.

SOLUTION We can multiply both sides by $\frac{3}{2}$ and have $1 y$ on the left side:

$$
\begin{aligned}
\frac{3}{2}\left(\frac{2}{3} y\right) & =\frac{3}{2}(4) & & \text { Multiplication property of equality } \\
\left(\frac{3}{2} \cdot \frac{2}{3}\right) y & =\frac{3}{2}(4) & & \text { Associative property } \\
y & =6 & & \text { Simplify } \frac{3}{2}(4)=\frac{3}{2}\left(\frac{4}{1}\right)=\frac{12}{2}=6
\end{aligned}
$$

EXAMPLE $4 \quad$ Solve $5+8=10 x+20 x-4 x$.
SOLUTION Our first step will be to simplify each side of the equation:

$$
\begin{aligned}
13 & =26 x & & \text { Simplify both sides first } \\
\frac{\mathbf{1}}{\mathbf{2 6}}(13) & =\frac{\mathbf{1}}{\mathbf{2 6}}(26 x) & & \text { Multiplication property of equality } \\
\frac{13}{26} & =X & & \text { Multiplication } \\
\frac{1}{2} & =x & & \text { Reduce to lowest terms }
\end{aligned}
$$

In the next three examples, we will use both the addition property of equality and the multiplication property of equality.

## EXAMPLE 5 Solve for $x: 6 x+5=-13$.

SOLUTION We begin by adding -5 to both sides of the equation:

$$
\begin{aligned}
6 x+5+(-\mathbf{5}) & =-13+(-\mathbf{5}) & & \text { Add }-5 \text { to both sides } \\
6 x & =-18 & & \text { Simplify } \\
\frac{\mathbf{1}}{\mathbf{6}}(6 x) & =\frac{\mathbf{1}}{\mathbf{6}}(-18) & & \text { Multiply both sides by } \frac{1}{6} \\
x & =-3 & &
\end{aligned}
$$

## Note

Notice that in Example 6 we used the addition property of equality first to combine all the terms containing $x$ on the left side of the equation. Once this had been done, we used the multiplication property to isolate $x$ on the left side.

## EXAMPLE 6 Solve for $x: 5 x=2 x+12$.

SOLUTION We begin by adding $-2 x$ to both sides of the equation:

$$
\begin{aligned}
5 x+(-\mathbf{2 x}) & =2 x+(-\mathbf{2 x})+12 \\
3 x & =12 \\
\frac{\mathbf{1}}{\mathbf{3}}(3 x) & =\frac{\mathbf{1}}{\mathbf{3}}(12) \\
x & =4
\end{aligned}
$$

EXAMPLE 7 Solve for $x: 3 x-4=-2 x+6$.
SOLUTION We begin by adding $2 x$ to both sides:

$$
\begin{aligned}
3 x+\mathbf{2 x}-4 & =-2 x+\mathbf{2 x}+6 & & \text { Add } 2 x \text { to both sides } \\
5 x-4 & =6 & & \text { Simplify }
\end{aligned}
$$

Now we add 4 to both sides:

$$
\begin{aligned}
5 x-4+\mathbf{4} & =6+\mathbf{4} & & \text { Add } 4 \text { to both sides } \\
5 x & =10 & & \text { Simplify } \\
\frac{\mathbf{1}}{\mathbf{5}}(5 x) & =\frac{\mathbf{1}}{\mathbf{5}}(10) & & \text { Multiply by } \frac{1}{5} \\
x & =2 & & \text { Simplify }
\end{aligned}
$$

The next example involves fractions. You will see that the properties we use to solve equations containing fractions are the same as the properties we used to solve the previous equations. Also, the LCD that we used previously to add fractions can be used with the multiplication property of equality to simplify equations containing fractions.

## EXAMPLE 8 Solve $\frac{2}{3} x+\frac{1}{2}=-\frac{3}{4}$.

SOLUTION We can solve this equation by applying our properties and working with the fractions, or we can begin by eliminating the fractions.

Method 1 Working with the fractions.

$$
\begin{aligned}
\frac{2}{3} x+\frac{1}{2}+\left(-\frac{\mathbf{1}}{\mathbf{2}}\right) & =-\frac{3}{4}+\left(-\frac{\mathbf{1}}{\mathbf{2}}\right) & & \text { Add }-\frac{1}{2} \text { to each side } \\
\frac{2}{3} x & =-\frac{5}{4} & & \text { Note that }-\frac{3}{4}+\left(-\frac{1}{2}\right)=-\frac{3}{4}+\left(-\frac{2}{4}\right) \\
\frac{\mathbf{3}}{\mathbf{2}}\left(\frac{2}{3} x\right) & =\frac{\mathbf{3}}{\mathbf{2}}\left(-\frac{5}{4}\right) & & \text { Multiply each side by } \frac{3}{2} \\
x & =-\frac{15}{8} & &
\end{aligned}
$$

Method 2 Eliminating the fractions in the beginning.

$$
\begin{aligned}
12\left(\frac{2}{3} x+\frac{1}{2}\right) & =\mathbf{1 2}\left(-\frac{3}{4}\right) & \text { Multiply each side by the LCD } 12 \\
\mathbf{1 2}\left(\frac{2}{3} x\right)+\mathbf{1 2}\left(\frac{1}{2}\right) & =\mathbf{1 2}\left(-\frac{3}{4}\right) & \text { Distributive property on the left side }
\end{aligned}
$$

$$
\begin{aligned}
8 x+6 & =-9 & & \text { Multiply } \\
8 x & =-15 & & \text { Add }-6 \text { to each side } \\
X & =-\frac{15}{8} & & \text { Multiply each side by } \frac{1}{8}
\end{aligned}
$$

As the third line in Method 2 indicates, multiplying each side of the equation by the LCD eliminates all the fractions from the equation.
As you can see, both methods yield the same solution.

A Note on Division Because division is defined as multiplication by the reciprocal, multiplying both sides of an equation by the same number is equivalent to dividing both sides of the equation by the reciprocal of that number; that is, multiplying each side of an equation by $\frac{1}{3}$ and dividing each side of the equation by 3 are equivalent operations. If we were to solve the equation $3 x=18$ using division instead of multiplication, the steps would look like this:

$$
\begin{aligned}
3 x & =18 & & \text { Original equation } \\
\frac{3 x}{\mathbf{3}} & =\frac{18}{\mathbf{3}} & & \text { Divide each side by } 3 \\
x & =6 & & \text { Division }
\end{aligned}
$$

Using division instead of multiplication on a problem like this may save you some writing. However, with multiplication, it is easier to explain "why" we end up with just one $x$ on the left side of the equation. (The "why" has to do with the associative property of multiplication.) My suggestion is that you continue to use multiplication to solve equations like this one until you understand the process completely. Then, if you find it more convenient, you can use division instead of multiplication.

## GETTING READY FOR CLASS

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

| A | $1-4$ |
| :--- | :--- |
| B | $5-8$ |

After reading through the preceding section, respond in your own words and in complete sentences.

1. Explain in words the multiplication property of equality.
2. If an equation contains fractions, how do you use the multiplication property of equality to clear the equation of fractions?
3. Why is it okay to divide both sides of an equation by the same nonzero number?
4. Explain in words how you would solve the equation $3 x=7$ using the multiplication property of equality.

## Problem Set 2.3

Solve the following equations. Be sure to show your work.

1. $5 x=10$
2. $6 x=12$
3. $7 a=28$
4. $4 a=36$
5. $-8 x=4$
6. $-6 x=2$
7. $8 m=-16$
8. $5 m=-25$
9. $-3 x=-9$
10. $-9 x=-36$
11. $-7 y=-28$
12. $-15 y=-30$
13. $2 x=0$
14. $7 x=0$
15. $-5 x=0$
16. $-3 x=0$
17. $\frac{x}{3}=2$
18. $\frac{x}{4}=3$
19. $-\frac{m}{5}=10$
20. $-\frac{m}{7}=1$
21. $-\frac{x}{2}=-\frac{3}{4}$
22. $-\frac{x}{3}=\frac{5}{6}$
23. $\frac{2}{3} a=8$
24. $\frac{3}{4} a=6$
25. $-\frac{3}{5} x=\frac{9}{5}$
26. $-\frac{2}{5} x=\frac{6}{15}$
27. $-\frac{5}{8} y=-20$
28. $-\frac{7}{2} y=-14$

Simplify both sides as much as possible, and then solve.
29. $-4 x-2 x+3 x=24$
30. $7 x-5 x+8 x=20$
31. $4 x+8 x-2 x=15-10$
32. $5 x+4 x+3 x=4+8$
33. $-3-5=3 x+5 x-10 x$
34. $10-16=12 x-6 x-3 x$
35. $18-13=\frac{1}{2} a+\frac{3}{4} a-\frac{5}{8} a$
36. $20-14=\frac{1}{3} a+\frac{5}{6} a-\frac{2}{3} a$

Solve the following equations by multiplying both sides by -1 .
37. $-x=4$
38. $-x=-3$
39. $-x=-4$
40. $-x=3$
41. $15=-a$
42. $-15=-a$
43. $-y=\frac{1}{2}$
44. $-y=-\frac{3}{4}$

Solve each of the following equations.
45. $3 x-2=7$
46. $2 x-3=9$
47. $2 a+1=3$
48. $5 a-3=7$
49. $\frac{1}{8}+\frac{1}{2} x=\frac{1}{4}$
50. $\frac{1}{3}+\frac{1}{7} x=-\frac{8}{21}$
51. $6 x=2 x-12$
52. $8 x=3 x-10$
53. $2 y=-4 y+18$
54. $3 y=-2 y-15$
55. $-7 x=-3 x-8$
56. $-5 x=-2 x-12$
57. $2 x-5=8 x+4$
58. $3 x-6=5 x+6$
59. $x+\frac{1}{2}=\frac{1}{4} x-\frac{5}{8}$
60. $\frac{1}{3} x+\frac{2}{5}=\frac{1}{5} x-\frac{2}{5}$
61. $m+2=6 m-3$
62. $m+5=6 m-5$
63. $\frac{1}{2} m-\frac{1}{4}=\frac{1}{12} m+\frac{1}{6}$
64. $\frac{1}{2} m-\frac{5}{12}=\frac{1}{12} m+\frac{5}{12}$
65. $6 y-4=9 y+2$
66. $2 y-2=6 y+14$
67. $\frac{3}{2} y+\frac{1}{3}=y-\frac{2}{3}$
68. $\frac{3}{2} y+\frac{7}{2}=\frac{1}{2} y-\frac{1}{2}$
69. $5 x+6=2$
70. $2 x+15=3$
71. $\frac{x}{2}=\frac{6}{12}$
72. $\frac{x}{4}=\frac{6}{8}$
73. $\frac{3}{x}=\frac{6}{7}$
74. $\frac{2}{9}=\frac{8}{x}$
75. $\frac{a}{3}=\frac{5}{12}$
76. $\frac{a}{2}=\frac{7}{20}$
77. $\frac{10}{20}=\frac{20}{x}$
78. $\frac{15}{60}=\frac{60}{x}$
79. $\frac{2}{x}=\frac{6}{7}$
80. $\frac{4}{x}=\frac{6}{7}$

## Applying the Concepts

81. Break-Even Point Movie theaters pay a certain price for the movies that you and I see. Suppose a theater pays $\$ 1,500$ for each showing of a popular movie. If they charge $\$ 7.50$ for each ticket they sell,
then the equation $7.5 x=1,500$ gives the number of tickets they must sell to equal the $\$ 1,500$ cost of showing the movie. This number is called the break-even point. Solve the equation for $x$ to find the break-even point.
82. Basketball Laura plays basketball for her community college. In one game she scored 13 points total, with a combination of free throws, field goals, and three-pointers. Each free throw is worth 1 point, each field goal is 2 points, and each three-pointer is worth 3 points. If she made 1 free throw and 3 field goals, then solving the equation

$$
1+3(2)+3 x=13
$$

will give us the number of three-pointers she made. Solve the equation to find the number of threepoint shots Laura made.
83. Taxes If $21 \%$ of your monthly pay is withheld for federal income taxes and another $8 \%$ is withheld for Social Security, state income tax, and other miscellaneous items, leaving you with $\$ 987.50$ a month in take-home pay, then the amount you earned before the deductions were removed from your check is given by the equation

$$
G-0.21 G-0.08 G=987.5
$$

Solve this equation to find your gross income.
84. Rhind Papyrus The Rhind Papyrus is an ancient document that contains mathematical riddles. One problem asks the reader to find a quantity such that when it is added to one-fourth of itself the sum is 15 . The equation that describes this situation is

$$
x+\frac{1}{4} x=15
$$

Solve this equation.


## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in previous sections and chapters. You can consider the time you spend working these problems as time spent studying for exams.

Apply the distributive property, and then simplify each expression as much as possible.
85. $2(3 x-5)$
86. $4(2 x-6)$
87. $\frac{1}{2}(3 x+6)$
88. $\frac{1}{4}(2 x+8)$
89. $\frac{1}{3}(-3 x+6)$
90. $\frac{1}{2}(-2 x+6)$

Simplify each expression.
91. $5(2 x-8)-3$
92. $4(3 x-1)+7$
93. $-2(3 x+5)+3(x-1)$
94. $6(x+3)-2(2 x+4)$
95. $7-3(2 y+1)$
96. $8-5(3 y-4)$

## Getting Ready for the Next Section

To understand all of the explanations and examples in the next section you must be able to work the problems below.

Solve each equation.
97. $2 x=4$
98. $3 x=24$
99. $30=5 x$
100. $0=5 x$
101. $0.17 x=510$
102. $0.1 x=400$

Apply the distributive property, then simplify if possible.
103. $3(x-5)+4$
104. $5(x-3)+2$
105. $0.09(x+2,000)$
106. $0.04(x+7,000)$
107. $7-3(2 y+1)$
108. $4-2(3 y+1)$
109. $3(2 x-5)-(2 x-4)$
110. $4(3 x-2)-(6 x-5)$

Simplify.
111. $10 x+(-5 x)$
112. $12 x+(-7 x)$
113. $0.08 x+0.09 x$
114. $0.06 x+0.04 x$

## Solving Linear Equations

## OBJECTIVES

A Solve a linear equation in one variable.

## Note

You may have some previous experience solving equations. Even so, you should solve the equations in this section using the method developed here. Your work should look like the examples in the text. If you have learned shortcuts or a different method of solving equations somewhere else, you always can go back to them later. What is important now is that you are able to solve equations by the methods shown here.

We will now use the material we have developed in the first three sections of this chapter to build a method for solving any linear equation.

DEFINITION A linear equation in one variable is any equation that can be put in the form $a x+b=0$, where $a$ and $b$ are real numbers and $a$ is not zero.

Each of the equations we will solve in this section is a linear equation in one variable. The steps we use to solve a linear equation in one variable are listed here.

## Strategy for Solving Linear Equations in One Variable

Step 1a: Use the distributive property to separate terms, if necessary.
$\mathbf{1 b}$ : If fractions are present, consider multiplying both sides by the LCD to eliminate the fractions. If decimals are present, consider multiplying both sides by a power of 10 to clear the equation of decimals.

1c: Combine similar terms on each side of the equation.
Step 2: Use the addition property of equality to get all variable terms on one side of the equation and all constant terms on the other side. A variable term is a term that contains the variable (for example, $5 x$ ). A constant term is a term that does not contain the variable (the number 3 , for example).

Step 3: Use the multiplication property of equality to get $x$ (that is, $1 x$ ) by itself on one side of the equation.
Step 4: Check your solution in the original equation to be sure that you have not made a mistake in the solution process.

As you will see as you work through the examples in this section, it is not always necessary to use all four steps when solving equations. The number of steps used depends on the equation. In Example 1 there are no fractions or decimals in the original equation, so step 1 b will not be used. Likewise, after applying the distributive property to the left side of the equation in Example 1, there are no similar terms to combine on either side of the equation, making step 1c also unnecessary.

## EXAMPLE $1 \quad$ Solve $2(x+3)=10$.

SOLUTION To begin, we apply the distributive property to the left side of the equation to separate terms:

Step 1a:

$$
2 x+6=10
$$

Step 2: $\quad\left\{\begin{aligned} 2 x+6+(-\mathbf{6}) & =10+(-\mathbf{6}) \\ 2 x & =4\end{aligned}\right.$
Step 3: $\quad\left\{\begin{array}{l}\frac{\mathbf{1}}{\mathbf{2}}(2 x)=\frac{\mathbf{1}}{\mathbf{2}}(4) \quad \text { Multiply each side by } \frac{1}{2}\end{array}\right.$

## Distributive property

Addition property of equality

The solution is 2

The solution to our equation is 2 . We check our work (to be sure we have not made either a mistake in applying the properties or an arithmetic mistake) by substituting 2 into our original equation and simplifying each side of the result separately.

Check: $\begin{aligned} & \text { When }=2 \\ & \text { the equation } 2(x+3) \\ &=10 \\ & \text { becomes } 2(2+3) \\ & \stackrel{?}{=} 10 \\ & 2(5) \stackrel{?}{=} 10 \\ & 10=10 \quad \text { A true statement }\end{aligned}$
Our solution checks.

The general method of solving linear equations is actually very simple. It is based on the properties we developed in Chapter 1 and on two very simple new properties. We can add any number to both sides of the equation and multiply both sides by any nonzero number. The equation may change in form, but the solution set will not. If we look back to Example 1, each equation looks a little different from each preceding equation. What is interesting and useful is that each equation says the same thing about $x$. They all say $x$ is 2 . The last equation, of course, is the easiest to read, and that is why our goal is to end up with $x$ by itself.

## EXAMPLE 2 Solve for $x: 3(x-5)+4=13$.

SOLUTION Our first step will be to apply the distributive property to the left side of the equation:

Step 1a: $3 x-15+4=13 \quad$ Distributive property
Step 1c: $\quad 3 x-11=13 \quad$ Simplify the left side
Step 2: $\left\{\begin{aligned} 3 x-11+\mathbf{1 1} & =13+\mathbf{1 1} \quad \text { Add } 11 \text { to both sides } \\ 3 x & =24\end{aligned}\right.$
Step 3: $\quad\left\{\begin{aligned} \frac{\mathbf{1}}{\mathbf{3}}(3 x) & =\frac{\mathbf{1}}{\mathbf{3}}(24) \\ x & =8\end{aligned}\right.$
Check: $\left\{\begin{aligned} & x=8 \\ & \text { When }=13 \\ & \text { the equation } 3(x-5)+4 \\ & \text { becomes } 3(8-5)+4 \\ & \stackrel{?}{=} 13 \\ & 3(3)+4 \stackrel{?}{=} 13 \\ & 9+4 \stackrel{?}{=} 13 \\ & 13=13 \quad \text { A true statement }\end{aligned}\right.$

## EXAMPLE 3 Solve $5(x-3)+2=5(2 x-8)-3$.

SOLUTION In this case we apply the distributive property on each side of the equation:

$$
\text { Step } 1 \text { a: }
$$

| $5 x-15+2$ | $=10 x-40-3$ |  | Distributive property |
| ---: | :--- | ---: | :--- |
| $5 x-13$ | $=10 x-43$ |  | Simplify each side |

$$
\left\{\begin{aligned}
5 x+(-\mathbf{5 x})-13 & =10 x+(-\mathbf{5 x})-43 & & \text { Add }-\mathbf{5 x} \text { to both sides } \\
-13 & =5 x-43 & & \\
-13+\mathbf{4 3} & =5 x-43+\mathbf{4 3} & & \text { Add } 43 \text { to both sides } \\
30 & =5 x & & \\
\frac{\mathbf{1}}{5}(30) & =\frac{\mathbf{1}}{5}(5 x) & & \text { Multiply both sides by } \frac{1}{5} \\
6 & =x & & \text { The solution is } 6
\end{aligned}\right.
$$

Step 3:
Check: $\quad$ Replacing $x$ with 6 in the original equation, we have $x$. Both expressions, however, $x$. Both expressions, however,
have exactly the same meaning.

Step 1 c:

## Note

It makes no difference on which side of the equal sign $x$ ends up. Most people prefer to have $x$ on the left side because we read from left to right, and it seems to sound better to say $x$ is 6 rather than 6 is

Method 2 Eliminating the decimals in the beginning.

|  | $0.08 x+0.09(x+2,000)=690$ | Original equation |
| :---: | :---: | :---: |
| Step 1a: | $0.08 x+0.09 x+180=690$ | Distributive property |
|  | $\int \mathbf{1 0 0}(0.08 x+0.09 x+180)=\mathbf{1 0 0}(690)$ | Multiply both sides by 100 |
| Step 1b: | $8 x+9 x+18,000=69,000$ |  |
| Step 1c: | $17 x+18,000=69,000$ | Simplify the left side |
| Step 2: | $17 x=51,000$ | Add - 18,000 to each side |
| Step 3: | $\left\{\frac{17 x}{\mathbf{1 7}}=\frac{51,000}{\mathbf{1 7}}\right.$ | Divide each side by 17 |
|  | $x=3,000$ |  |

Check: Substituting 3,000 for $x$ in the original equation, we have
Step 4:

$$
\begin{aligned}
0.08(3,000)+0.09(3,000+2,000) & \stackrel{?}{=} 690 \\
0.08(3,000)+0.09(5,000) & \stackrel{?}{=} 690 \\
240+450 & \stackrel{?}{=} 690 \\
690 & =690 \quad \text { A true statement }
\end{aligned}
$$

## EXAMPLE $5 \quad$ Solve $7-3(2 y+1)=16$.

SOLUTION We begin by multiplying -3 times the sum of $2 y$ and 1 :
Step 1a: $\quad 7-6 y-3=16 \quad$ Distributive property
Step 1c: $\quad-6 y+4=16 \quad$ Simplify the left side
Step 2: $\left\{\begin{aligned}-6 y+4+(-4) & =16+(-4) \quad \text { Add }-4 \text { to both sides } \\ -6 y & =12\end{aligned}\right.$
Step 3: $\quad\left\{\begin{aligned}-\frac{\mathbf{1}}{\mathbf{6}}(-6 y) & =-\frac{\mathbf{1}}{\mathbf{6}}(12) \quad \text { Multiply both sides by }-\frac{1}{6} \\ y & =-2\end{aligned}\right.$
Step 4: Replacing $y$ with -2 in the original equation yields a true statement.

There are two things to notice about the example that follows: first, the distributive property is used to remove parentheses that are preceded by a negative sign, and, second, the addition property and the multiplication property are not shown in as much detail as in the previous examples.

## EXAMPLE 6 Solve $3(2 x-5)-(2 x-4)=6-(4 x+5)$.

SOLUTION When we apply the distributive property to remove the grouping symbols and separate terms, we have to be careful with the signs. Remember, we can think of $-(2 x-4)$ as $-1(2 x-4)$, so that

$$
-(2 x-4)=-1(2 x-4)=-2 x+4
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

$$
\text { A } \quad 1-6
$$

It is not uncommon for students to make a mistake with this type of simplification and write the result as $-2 x-4$, which is incorrect. Here is the complete solution to our equation:

$$
\begin{aligned}
3(2 x-5)-(2 x-4) & =6-(4 x+5) & & \text { Original equation } \\
6 x-15-2 x+4 & =6-4 x-5 & & \text { Distributive property } \\
4 x-11 & =-4 x+1 & & \text { Simplify each side } \\
8 x-11 & =1 & & \text { Add } 4 x \text { to each side } \\
8 x & =12 & & \text { Add } 11 \text { to each side } \\
x & =\frac{12}{8} & & \text { Multiply each side by } \frac{1}{8} \\
x & =\frac{3}{2} & & \text { Reduce to lowest terms }
\end{aligned}
$$

The solution, $\frac{3}{2}$, checks when replacing $x$ in the original equation.

## GETTING READY FOR CLASS

## After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the first step in solving a linear equation containing parentheses?
2. What is the last step in solving a linear equation?
3. Explain in words how you would solve the equation $2 x-3=8$.
4. If an equation contains decimals, what can you do to eliminate the decimals?

## Problem Set 2.4

Online support materials can be found at www.thomsonedu.com/login

Solve each of the following equations using the four steps shown in this section.

1. $2(x+3)=12$
2. $3(x-2)=6$
3. $6(x-1)=-18$
4. $4(x+5)=16$
5. $2(4 a+1)=-6$
6. $3(2 a-4)=12$
7. $14=2(5 x-3)$
8. $-25=5(3 x+4)$
9. $-2(3 y+5)=14$
10. $-3(2 y-4)=-6$
11. $-5(2 a+4)=0$
12. $-3(3 a-6)=0$
13. $1=\frac{1}{2}(4 x+2)$
14. $1=\frac{1}{3}(6 x+3)$
15. $3(t-4)+5=-4$
16. $5(t-1)+6=-9$
17. $4(2 y+1)-7=1$
18. $6(3 y+2)-8=-2$
19. $\frac{1}{2}(x-3)=\frac{1}{4}(x+1)$
20. $\frac{1}{3}(x-4)=\frac{1}{2}(x-6)$
21. $-0.7(2 x-7)=0.3(11-4 x)$
22. $-0.3(2 x-5)=0.7(3-x)$
23. $-2(3 y+1)=3(1-6 y)-9$
24. $-5(4 y-3)=2(1-8 y)+11$
25. $\frac{3}{4}(8 x-4)+3=\frac{2}{5}(5 x+10)-1$
26. $\frac{5}{6}(6 x+12)+1=\frac{2}{3}(9 x-3)+5$
= Videos available by instructor request

- = Online student support materials available at www.thomsonedu.com/login

27. $0.06 x+0.08(100-x)=6.5$
28. $0.05 x+0.07(100-x)=6.2$
29. $6-5(2 a-3)=1$
30. $-8-2(3-a)=0$
31. $0.2 x-0.5=0.5-0.2(2 x-13)$
32. $0.4 x-0.1=0.7-0.3(6-2 x)$
33. $2(t-3)+3(t-2)=28$
34. $-3(t-5)-2(2 t+1)=-8$
35. $5(x-2)-(3 x+4)=3(6 x-8)+10$
36. $3(x-1)-(4 x-5)=2(5 x-1)-7$
37. $2(5 x-3)-(2 x-4)=5-(6 x+1)$
38. $3(4 x-2)-(5 x-8)=8-(2 x+3)$
39. $-(3 x+1)-(4 x-7)=4-(3 x+2)$
40. $-(6 x+2)-(8 x-3)=8-(5 x+1)$
41. $x+(2 x-1)=2$
42. $x+(5 x+2)=20$
43. $x-(3 x+5)=-3$
44. $15=3(x-1)$
45. $4 x-(-4 x+1)=5$
46. $5 x-8(2 x-5)=7$
47. $3 x+4(8 x-15)=10$
48. $7(2 y-1)-6 y=-1$
49. $4(4 y-3)+2 y=3$
50. $0.2 x+0.5(12-x)=3.6$
51. $0.3 x+0.6(25-x)=12$
52. $0.5 x+0.2(18-x)=5.4$
53. $0.1 x+0.5(40-x)=32$
54. $x+(x+3)(-3)=x-3$
55. $x-2(x+2)=x-2$
56. $5(x+2)+3(x-1)=-9$
57. $4(x+1)+3(x-3)=2$
58. $3(x-3)+2(2 x)=5$
59. $2(x-2)+3(5 x)=30$
60. $5(y+2)=4(y+1)$
61. $3(y-3)=2(y-2)$
62. $3 x+2(x-2)=6$
63. $5 x-(x-5)=25$
64. $50(x-5)=30(x+5)$
65. $34(x-2)=26(x+2)$
66. $0.08 x+0.09(x+2,000)=860$
67. $0.11 x+0.12(x+4,000)=940$
68. $0.10 x+0.12(x+500)=214$
69. $0.08 x+0.06(x+800)=104$
70. $5 x+10(x+8)=245$
71. $5 x+10(x+7)=175$
72. $5 x+10(x+3)+25(x+5)=435$
73. $5(x+3)+10 x+25(x+7)=390$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working.
Working these problems is an excellent way to get ready for a test or a quiz.
77. Work each problem according to the instructions given.
a. Solve: $4 x-5=0$
b. Solve: $4 x-5=25$
c. Add: $(4 x-5)+(2 x+25)$
d. Solve: $4 x-5=2 x+25$
e. Multiply: $4(x-5)$
f. Solve: $4(x-5)=2 x+25$
78. Work each problem according to the instructions given.
a. Solve: $3 x+6=0$
b. Solve: $3 x+6=4$
c. Add: $(3 x+6)+(7 x+4)$
d. Solve: $3 x+6=7 x+4$
e. Multiply: $3(x+6)$
f. Solve: $3(x+6)=7 x+4$

## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in previous sections and chapters. You can consider the time you spend working these problems as time spent studying for exams.

Multiply.
79. $\frac{1}{2}(3)$
80. $\frac{1}{3}(2)$
81. $\frac{2}{3}(6)$
82. $\frac{3}{2}(4)$
83. $\frac{5}{9} \cdot \frac{9}{5}$
84. $\frac{3}{7} \cdot \frac{7}{3}$

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Fill in the tables by finding the value of each expression for the given values of the variables.
85.

| $x$ | $3(x+2)$ | $3 x+2$ | $3 x+6$ |
| :--- | :--- | :--- | :--- |
| 0 |  |  |  |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |

86. 

| $\boldsymbol{x}$ | $7(x-5)$ | $7 x-5$ | $7 x-35$ |
| :--- | :--- | :--- | :--- |
| -3 |  |  |  |
| -2 |  |  |  |
| -1 |  |  |  |
| 0 |  |  |  |

87. 

| $a$ | $(2 a+1)^{2}$ | $4 a^{2}+4 a+1$ |
| :--- | :--- | :--- |
| 1 |  |  |
| 2 |  |  |

88. 

| $a$ | $(a+1)^{3}$ | $a^{3}+3 a^{2}+3 a+1$ |
| :--- | :--- | :--- |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |

## Getting Ready for the Next Section

To understand all of the explanations and examples in the next section you must be able to work the problems below.

Solve each equation.
89. $40=2 x+12$
90. $80=2 x+12$
91. $12+2 y=6$
92. $3 x+18=6$
93. $24 x=6$
94. $45=0.75 x$
95. $70=x \cdot 210$
96. $15=x \cdot 80$

Apply the distributive property.
97. $\frac{1}{2}(-3 x+6)$
98. $-\frac{1}{4}(-5 x+20)$

### 2.5 Formulas

## OBJECTIVES

A Find the value of a variable in a formula given replacements for the other variables.

B Solve a formula for one of its variables.

C Find the complement and supplement of an angle.
D Solve simple percent problems.

In this section we continue solving equations by working with formulas. To begin, here is the definition of a formula.

DEFINITION In mathematics, a formula is an equation that contains more than one variable.

The equation $P=2 l+2 w$, which tells us how to find the perimeter of a rectangle, is an example of a formula.

To begin our work with formulas, we will consider some examples in which we are given numerical replacements for all but one of the


## EXAMPLE 1 The perime-

ter $P$ of a rectangular livestock pen is 40 feet. If the width $w$ is 6 feet, find the length.


SOLUTION First we substitute 40 for $P$ and 6 for $w$ in the formula $P=2 l+2 w$. Then we solve for 1 :

| When | $P=40$ and $w=6$ |  |
| :---: | :---: | :---: |
| the formula | $P=2 l+2 w$ |  |
| becomes | $40=2 l+2(6)$ |  |
| or | $40=2 l+12$ | Multiply 2 and 6 |
|  | $28=21$ | Add - 12 to each side |
|  | $14=1$ | Multiply each side by $\frac{1}{2}$ |

To summarize our results, if a rectangular pen has a perimeter of 40 feet and a width of 6 feet, then the length must be 14 feet.

EXAMPLE 2 Find $y$ when $x=4$ in the formula $3 x+2 y=6$.
SOLUTION We substitute 4 for $x$ in the formula and then solve for $y$ :

| When | $x=4$ |  |
| :---: | :---: | :---: |
| the formula | $3 x+2 y=6$ |  |
| becomes | $3(4)+2 y=6$ |  |
| or | $12+2 y=6$ | Multiply 3 and 4 |
|  | $2 y=-6$ | Add - 12 to each side |
|  | $y=-3$ | Multiply each side by $\frac{1}{2}$ |

In the next examples we will solve a formula for one of its variables without
 being given numerical replacements for the other variables.
Consider the formula for the area of a triangle:
where $A=$ area, $b=$ length of the base, and $h=$ height of the triangle.
Suppose we want to solve this formula for $h$. What we must do is isolate the variable $h$ on one side of the equal sign. We begin by multiplying both sides by 2 :

$$
\begin{aligned}
2 \cdot A & =2 \cdot \frac{1}{2} b h \\
2 A & =b h
\end{aligned}
$$

Then we divide both sides by $b$ :

$$
\begin{aligned}
\frac{2 A}{b} & =\frac{b h}{b} \\
h & =\frac{2 A}{b}
\end{aligned}
$$

The original formula $A=\frac{1}{2} b h$ and the final formula $h=\frac{2 A}{b}$ both give the same relationship among $A, b$, and $h$. The first one has been solved for $A$ and the second one has been solved for $h$.

Rule To solve a formula for one of its variables, we must isolate that variable on either side of the equal sign. All other variables and constants will appear on the other side.

## EXAMPLE 3 Solve $3 x+2 y=6$ for $y$.

SOLUTION To solve for $y$, we must isolate $y$ on the left side of the equation. To begin, we use the addition property of equality to add $-3 x$ to each side:

$$
\begin{aligned}
3 x+2 y & =6 & & \text { Original formula } \\
3 x+(-\mathbf{3 x})+2 y & =(-\mathbf{3 x})+6 & & \text { Add }-\mathbf{3 x} \text { to each side } \\
2 y & =-3 x+6 & & \text { Simplify the left side } \\
\frac{\mathbf{1}}{\mathbf{2}}(2 y) & =\frac{\mathbf{1}}{\mathbf{2}}(-3 x+6) & & \text { Multiply each side by } \frac{1}{2} \\
y & =-\frac{3}{2} x+3 & & \text { Multiplication }
\end{aligned}
$$

## EXAMPLE $4 \quad$ Solve $h=v t-16 t^{2}$ for $v$.

SOLUTION Let's begin by interchanging the left and right sides of the equation. That way, the variable we are solving for, $v$, will be on the left side.

$$
\begin{aligned}
v t-16 t^{2} & =h & & \text { Exchange sides } \\
v t-16 t^{2}+\mathbf{1 6} \boldsymbol{t}^{\mathbf{2}} & =h+\mathbf{1 6} \boldsymbol{t}^{\mathbf{2}} & & \text { Add } \mathbf{1 6 t ^ { 2 }} \text { to each side } \\
v t & =h+16 t^{2} & & \\
\frac{v t}{\boldsymbol{t}} & =\frac{h+16 t^{2}}{\boldsymbol{t}} & & \text { Divide each side by } t \\
v & =\frac{h+16 t^{2}}{t} & &
\end{aligned}
$$

We know we are finished because we have isolated the variable we are solving for on the left side of the equation and it does not appear on the other side.

## EXAMPLE 5 Solve for $y: \frac{y-1}{x}=\frac{3}{2}$.

SOLUTION Although we will do more extensive work with formulas of this form later in the book, we need to know how to solve this particular formula for $y$ in order to understand some things in the next chapter. We begin by multiplying each side of the formula by $x$. Doing so will simplify the left side of the equation and make the rest of the solution process simple.

$$
\frac{y-1}{x}=\frac{3}{2} \quad \text { Original formula }
$$

$$
\begin{aligned}
x \cdot \frac{y-1}{x} & =\frac{3}{2} \cdot x & & \text { Multiply each side by } x \\
y-1 & =\frac{3}{2} x & & \text { Simplify each side } \\
y & =\frac{3}{2} x+1 & & \text { Add } 1 \text { to each side }
\end{aligned}
$$

This is our solution. If we look back to the first step, we can justify our result on the left side of the equation this way: Dividing by $x$ is equivalent to multiplying by its reciprocal $\frac{1}{x}$. Here is what it looks like when written out completely:

$$
x \cdot \frac{y-1}{x}=x \cdot \frac{1}{x} \cdot(y-1)=1(y-1)=y-1
$$

## FACTS FROM GEOMETRY

More on Complementary and Supplementary Angles
In Chapter 1 we defined complementary angles as angles that add to $90^{\circ}$; that is, if $x$ and $y$ are complementary angles, then

$$
x+y=90^{\circ}
$$

If we solve this formula for $y$, we obtain a formula equivalent to our original formula:

$$
y=90^{\circ}-x
$$

Because $y$ is the complement of $x$, we can generalize by saying that the complement of angle $x$ is the angle $90^{\circ}-x$. By a similar reasoning process, we can say that the supplement of angle $x$ is the angle $180^{\circ}-x$. To summarize, if $x$ is an angle, then

The complement of $x$ is $90^{\circ}-x$, and
The supplement of $x$ is $180^{\circ}-x$
If you go on to take a trigonometry class, you will see this formula again.

## EXAMPLE 6 Find the complement and the supplement of $25^{\circ}$.

SOLUTION We can use the formulas above with $x=25^{\circ}$.
The complement of $25^{\circ}$ is $90^{\circ}-25^{\circ}=65^{\circ}$.
The supplement of $25^{\circ}$ is $180^{\circ}-25^{\circ}=155^{\circ}$.

## Basic Percent Problems

The last examples in this section show how basic percent problems can be translated directly into equations. To understand these examples, you must recall that percent means "per hundred" that is, $75 \%$ is the same as $\frac{75}{100}, 0.75$, and, in reduced fraction form, $\frac{3}{4}$. Likewise, the decimal 0.25 is equivalent to $25 \%$. To change a decimal to a percent, we move the decimal point two places to the right and write the \% symbol. To change from a percent to a decimal, we drop the \% symbol and
move the decimal point two places to the left. The table that follows gives some of the most commonly used fractions and decimals and their equivalent percents.

| Fraction | Decimal | Percent |
| :---: | :---: | :---: |
| $\frac{1}{2}$ | 0.5 | $50 \%$ |
| $\frac{1}{4}$ | 0.25 | $25 \%$ |
| $\frac{3}{4}$ | 0.75 | $75 \%$ |
| $\frac{1}{3}$ | $0.333 \ldots$ | $33 \frac{1}{3} \%$ |
| $\frac{2}{3}$ | $0.666 \ldots$ | $66 \frac{2}{3} \%$ |
| $\frac{1}{5}$ | 0.2 | $20 \%$ |
| $\frac{2}{5}$ | 0.4 | $40 \%$ |
| $\frac{3}{5}$ | 0.6 | $60 \%$ |
| $\frac{4}{5}$ | 0.8 | $80 \%$ |
|  |  |  |

## EXAMPLE 7 What number is $25 \%$ of 60 ?

SOLUTION To solve a problem like this, we let $x=$ the number in question (that is, the number we are looking for). Then, we translate the sentence directly into an equation by using an equal sign for the word "is" and multiplication for the word "of." Here is how it is done:


Notice that we must write $25 \%$ as a decimal in order to do the arithmetic in the problem.

The number 15 is $25 \%$ of 60 .

## EXAMPLE $8 \quad$ What percent of 24 is 6 ?

SOLUTION Translating this sentence into an equation, as we did in Example 7, we have:


Next, we multiply each side by $\frac{1}{24}$. (This is the same as dividing each side by 24.)

$$
\begin{aligned}
\frac{\mathbf{1}}{\mathbf{2 4}}(24 x) & =\frac{\mathbf{1}}{\mathbf{2 4}}(6) \\
x & =\frac{6}{24} \\
& =\frac{1}{4} \\
& =0.25, \text { or } 25 \%
\end{aligned}
$$

The number 6 is $25 \%$ of 24 .

## EXAMPLE $9 \quad 45$ is $75 \%$ of what number?

SOLUTION Again, we translate the sentence directly:


Next, we multiply each side by $\frac{1}{0.75}$ (which is the same as dividing each side by 0.75):

$$
\begin{aligned}
\frac{\mathbf{1}}{\mathbf{0 . 7 5}}(45) & =\frac{\mathbf{1}}{\mathbf{0 . 7 5}}(0.75 x) \\
\frac{45}{0.75} & =x \\
60 & =x
\end{aligned}
$$

The number 45 is $75 \%$ of 60 .

We can solve application problems involving percent by translating each problem into one of the three basic percent problems shown in Examples 7, 8, and 9.

## EXAMPLE 10

 The American Dietetic Association (ADA) recommends eating foods in which the calories from fat are less than $30 \%$ of the total calories. The nutrition labels from two kinds of granola bars are shown in Figure 1. For each bar, what percent of the total calories come from fat?SOLUTION The information needed to solve this problem is located toward the top of each label. Each serving of Bar I contains 210 calories, of which 70 calories come from fat. To find the percent of total calories that come from fat, we must answer this question:

70 is what percent of 210 ?

| Nutrition Facts |  |  |
| :--- | ---: | :---: |
| Serving Size 2 bars (47g) <br> Servings Per Container 6 |  |  |
| Amount Per Serving |  |  |
| Calories | 210 |  |
| Calories from Fat | $\mathbf{7 0}$ |  |
|  |  |  |
| Total Fat 8g Daily Value* |  |  |

$\left.\begin{array}{|lr|}\hline \text { Nutrition Facts } \\ \text { Serving Size 1 bar (21g) } \\ \text { Servings Per Container 8 }\end{array}\right)$

FIGURE 1

For Bar II, one serving contains 80 calories, of which 15 calories come from fat. To find the percent of total calories that come from fat, we must answer this question: 15 is what percent of 80 ?

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,2
B ${ }^{3-5}$
C 6
D $\quad 7-10$

Translating each equation into symbols, we have

70 is what percent of 210

$$
\begin{aligned}
70 & =x \cdot 210 \\
x & =\frac{70}{210} \\
x & =0.33 \text { to the nearest hundredth } \\
x & =33 \%
\end{aligned}
$$

15 is what percent of 80

$$
15=x \cdot 80
$$

$$
x=\frac{15}{80}
$$

$x=0.19$ to the nearest hundredth
$x=19 \%$

Comparing the two bars, $33 \%$ of the calories in Bar I are fat calories, whereas $19 \%$ of the calories in Bar II are fat calories. According to the ADA, Bar II is the healthier choice.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a formula?
2. How do you solve a formula for one of its variables?
3. What are complementary angles?
4. What does percent mean?

## Problem Set 2.5

Online support materials can be found at www.thomsonedu.com/login

Use the formula $P=2 l+2 w$ to find the length $l$ of a rectangular lot if

1. The width $w$ is 50 feet and the perimeter $P$ is 300 feet
2. The width $w$ is 75 feet and the perimeter $P$ is 300 feet
3. For the equation $2 x+3 y=6$,
a. Find $y$ when $x$ is 0 .
b. Find $x$ when $y$ is 1 .
c. Find $y$ when $x$ is 3 .
4. For the equation $2 x-5 y=20$,
a. Find $y$ when $x$ is 0 .
b. Find $x$ when $y$ is 0 .
c. Find $x$ when $y$ is 2 .
5. For the equation $y=-\frac{1}{3} x+2$,
a. Find $y$ when $x$ is 0 .
b. Find $x$ when $y$ is 3 .
c. Find $y$ when $x$ is 3 .
6. For the equation $y=-\frac{2}{3} x+1$,
a. Find $y$ when $x$ is 0 .
b. Find $x$ when $y$ is -1 .
c. Find $y$ when $x$ is -3 .

Use the equation $y=(x+1)^{2}-3$ to find the value of $y$ when
7. $x=-2$
8. $x=-1$
9. $x=1$
10. $x=2$
11. Use the formula $y=\frac{20}{x}$ to find $y$ when
a. $x=10$
b. $x=5$
12. Use the formula $y=2 x^{2}$ to find $y$ when
a. $x=5$
b. $x=-6$
$=$ Videos available by instructor request
$\rangle$ = Online student support materials available at www.thomsonedu.com/login
13. Use the formula $y=K x$ to find $K$ when
a. $y=15$ and $x=3$
b. $y=72$ and $x=4$
14. Use the formula $y=K x^{2}$ to find $K$ when
a. $y=32$ and $x=4$
b. $y=45$ and $x=3$
15. If $y=\frac{K}{x}$, find $K$ if
a. $x$ is 5 and $y$ is 4 .
b. $x$ is 5 and $y$ is 15 .
16. If $I=\frac{K}{d^{2}}$, find $K$ if
a. $I=200$ and $d=10$.
b. $I=200$ and $d=5$.

Solve each of the following for the indicated variable.
17. $A=l w$ for $l$
19. $V=l w h$ for $h$
21. $P=a+b+c$ for $a$
22. $P=a+b+c$ for $b$
23. $x-3 y=-1$ for $x$
24. $x+3 y=2$ for $x$
25. $-3 x+y=6$ for $y$
26. $2 x+y=-17$ for $y$
27. $2 x+3 y=6$ for $y$
28. $4 x+5 y=20$ for $y$
29. $P=2 l+2 w$ for $w$
30. $P=2 l+2 w$ for $l$
31. $h=v t+16 t^{2}$ for $v$
32. $h=v t-16 t^{2}$ for $v$
33. $A=\pi r^{2}+2 \pi r h$ for $h$
34. $A=2 \pi r^{2}+2 \pi r h$ for $h$
35. Solve for $y$.
a. $y-3=-2(x+4)$
b. $y-5=4(x-3)$
36. Solve for $y$.
a. $y+1=-\frac{2}{3}(x-3)$
b. $y-3=-\frac{2}{3}(x+3)$
37. Solve for $y$.
a. $y-1=\frac{3}{4}(x-1)$
b. $y+2=\frac{3}{4}(x-4)$
38. Solve for $y$.
a. $y+3=\frac{3}{2}(x-2)$
b. $y+4=\frac{4}{3}(x-3)$
39. Solve for $y$.
a. $\frac{y-1}{x}=\frac{3}{5}$
b. $\frac{y-2}{x}=\frac{1}{2}$
c. $\frac{y-3}{x}=4$
40. Solve for $y$.
a. $\frac{y+1}{x}=-\frac{3}{5}$
b. $\frac{y+2}{x}=-\frac{1}{2}$
c. $\frac{y+3}{x}=-4$

Solve each formula for $y$.
41. $\frac{x}{7}-\frac{y}{3}=1$
42. $\frac{x}{4}-\frac{y}{9}=1$
43. $-\frac{1}{4} x+\frac{1}{8} y=1$
44. $-\frac{1}{9} x+\frac{1}{3} y=1$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working. As we have mentioned previously, working these problems is an excellent way to get ready for a test or a quiz.
45. Work each problem according to the instructions given.
a. Solve: $4 x+5=20$
b. Find the value of $4 x+5$ when $x$ is 3 .
c. Solve for $y: 4 x+5 y=20$
d. Solve for $x: 4 x+5 y=20$
46. Work each problem according to the instructions given.
a. Solve: $-2 x+1=4$
b. Find the value of $-2 x+1$ when $x$ is 8 .
c. Solve for $y:-2 x+y=20$
d. Solve for $x:-2 x+y=20$

Find the complement and the supplement of each angle.
47. $30^{\circ}$
48. $60^{\circ}$
49. $45^{\circ}$
50. $15^{\circ}$

Translate each of the following into an equation, and then solve that equation.
51. What number is $25 \%$ of 40 ?
52. What number is $75 \%$ of 40 ?
53. What number is $12 \%$ of 2,000 ?
54. What number is $9 \%$ of 3,000 ?
55. What percent of 28 is 7 ?
56. What percent of 28 is 21 ?
57. What percent of 40 is 14 ?
58. What percent of 20 is 14 ?
59. 32 is $50 \%$ of what number?
60. 16 is $50 \%$ of what number?
61. 240 is $12 \%$ of what number?
62. 360 is $12 \%$ of what number?

## Applying the Concepts

More About Temperatures As we mentioned in Chapter 1, in the U.S. system, temperature is measured on the Fahrenheit scale. In the metric system, temperature is measured on the Celsius scale. On the Celsius scale, water boils at 100 degrees and freezes at 0 degrees. To denote a temperature of 100 degrees on the Celsius scale, we write
$100^{\circ} \mathrm{C}$, which is read " 100 degrees Celsius"
Table 1 is intended to give you an intuitive idea of the relationship between the two temperature scales. Table 2 gives the formulas, in both symbols and words, that are used to convert between the two scales.
63. Let $\mathrm{F}=212$ in the formula $\mathrm{C}=\frac{5}{9}(\mathrm{~F}-32)$, and solve for $C$. Does the value of $C$ agree with the information in Table 1?

TABLE 1

|  | Temperature |  |
| :--- | :---: | ---: |
| Situation | Fahrenheit | Celsius |
| Water freezes | $32^{\circ} \mathrm{F}$ | $0^{\circ} \mathrm{C}$ |
| Room temperature | $68^{\circ} \mathrm{F}$ | $20^{\circ} \mathrm{C}$ |
| Normal body temperature | $98.6^{\circ} \mathrm{F}$ | $37^{\circ} \mathrm{C}$ |
| Water boils | $212^{\circ} \mathrm{F}$ | $100^{\circ} \mathrm{C}$ |
| Bake cookies | $365^{\circ} \mathrm{F}$ | $185^{\circ} \mathrm{C}$ |

TABLE 2

| To Convert from | Formula in Symbols | Formula in Words |
| :---: | :---: | :--- |
| Fahrenheit to Celsius | $\mathrm{C}=\frac{5}{9}(\mathrm{~F}-32)$ | Subtract 32, multiply <br> by 5, then divide by 9. |
| Celsius to Fahrenheit | $\mathrm{F}=\frac{9}{5} \mathrm{C}+32$ | Multiply by $\frac{9}{5}$, then <br> add 32. |

64. Let $\mathrm{C}=100$ in the formula $\mathrm{F}=\frac{9}{5} \mathrm{C}+32$, and solve for $F$. Does the value of $F$ agree with the information in Table 1?
65. Let $\mathrm{F}=68$ in the formula $\mathrm{C}=\frac{5}{9}(\mathrm{~F}-32)$, and solve for $C$. Does the value of C agree with the information in Table 1?
66. Let $\mathrm{C}=37$ in the formula $\mathrm{F}=\frac{9}{5} \mathrm{C}+32$, and solve for F. Does the value of F agree with the information in Table 1?
67. Solve the formula $\mathrm{F}=\frac{9}{5} \mathrm{C}+32$ for C .
68. Solve the formula $C=\frac{5}{9}(F-32)$ for $F$.

Nutrition Labels The nutrition label in Figure 2 is from a quart of vanilla ice cream. The label in Figure 3 is from a pint of vanilla frozen yogurt. Use the information on these labels for problems 69-72. Round your answers to the nearest tenth of a percent.

| Nutrition Facts |  |
| :---: | :---: |
| Serving Size $1 / 2$ cup (65g) Servings 8 |  |
|  |  |
| Amount/Serving |  |
| Calories 150 | Calories from Fat 90 |
|  | \% Daily Value* |
| Total Fat 10 g | 16\% |
| Saturated Fat 6g | 32\% |
| Cholesterol 35mg | 12\% |
| Sodium 30mg | 1\% |
| Total Carbohydrate 14g | 5\% |
| Dietary Fiber Og | 0\% |
| Sugars 11g |  |
| Protein 2 g |  |
| Vitamin A 6\% | Vitamin C 0\% |
| Calcium 6\% | Iron 0\% |
| * Percent Daily Values are based on a 2,000 calorie diet. |  |

FIGURE 2 Vanilla ice cream


FIGURE 3 Vanilla frozen yogurt
69. What percent of the calories in one serving of the vanilla ice cream are fat calories?
70. What percent of the calories in one serving of the frozen yogurt are fat calories?
71. One serving of frozen yogurt is 98 grams, of which 26 grams are carbohydrates. What percent of one serving are carbohydrates?
72. One serving of vanilla ice cream is 65 grams. What percent of one serving is sugar?

Circumference The circumference of a circle is given by the formula $C=2 \pi r$. Find $r$ if
73. The circumference $C$ is 44 meters and $\pi$ is $\frac{22}{7}$
74. The circumference $C$ is 176 meters and $\pi$ is $\frac{22}{7}$
75. The circumference is 9.42 inches and $\pi$ is 3.14
76. The circumference is 12.56 inches and $\pi$ is 3.14

Volume The volume of a cylinder is given by the formula $V=\pi r^{2} h$. Find the height $h$ if
77. The volume $V$ is 42 cubic feet, the radius is $\frac{7}{22}$ feet, and $\pi$ is $\frac{22}{7}$
78. The volume $V$ is 84 cubic inches, the radius is $\frac{7}{11}$ inches, and $\pi$ is $\frac{22}{7}$
79. The volume is 6.28 cubic centimeters, the radius is 3 centimeters, and $\pi$ is 3.14
80. The volume is 12.56 cubic centimeters, the radius is 2 centimeters, and $\pi$ is 3.14

## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in previous sections and chapters. You can consider the time you spend working these problems as time spent studying for exams.
81. a. $27-(-68)$
82. a. $55-(-29)$
b. $27+(-68)$
b. $55+(-29)$
c. $-27-68$
c. $-55-29$
d. $-27+68$
d. $-55+29$
83. a. $-32-(-41)$

84 a. $-56-(-35)$
b. $-32+(-41)$
b. $-56+(-35)$
c. $-32+41$
c. $-56+35$
d. $-32-41$
d. $-56-35$

## Getting Ready for the Next Section

To understand all of the explanations and examples in the next section you must be able to work the problems below.

Write an equivalent expression in English. Include the words sum and difference when possible.
85. $4+1=5$
86. $7+3=10$
87. $6-2=4$
88. $8-1=7$
89. $x-5=-12$
90. $2 x+3=7$
91. $x+3=4(x-3)$
92. $2(2 x-5)=2 x-34$

For each of the following expressions, write an equivalent equation.
93. Twice the sum of 6 and 3 is 18 .
94. Four added to the product of 3 and -1 is 1 .
95. The sum of twice 5 and 3 is 13 .
96. Twice the difference of 8 and 2 is 12 .
97. The sum of a number and five is thirteen.
98. The difference of ten and a number is negative eight.
99. Five times the sum of a number and seven is thirty.
100. Five times the difference of twice a number and six is negative twenty.


## Applications

As you begin reading through the examples in this section, you may find yourself asking why some of these problems seem so contrived. The title of the section is "Applications," but many of the problems here don't seem to have much to do with "real life." You are right about that. Example 3 is what we refer to as an "age problem." But imagine a conversation in which you ask someone how old her children are and she replies, "Bill is 6 years older than Tom. Three years ago the sum of their ages was 21 . You figure it out." Although many of the "application" problems in this section are contrived, they are also good for practicing the strategy we will use to solve all application problems.

To begin this section, we list the steps used in solving application problems. We call this strategy the Blueprint for Problem Solving. It is an outline that will overlay the solution process we use on all application problems.

## BLUEPRINT FOR PROBLEM SOLVING

STEP 1: Read the problem, and then mentally list the items that are known and the items that are unknown.

STEP 2: Assign a variable to one of the unknown items. (In most cases this will amount to letting $x=$ the item that is asked for in the problem.) Then translate the other information in the problem to expressions involving the variable.

STEP 3: Reread the problem, and then write an equation, using the items and variables listed in steps 1 and 2, that describes the situation.
STEP 4: Solve the equation found in step 3.
STEP 5: Write your answer using a complete sentence.
STEP 6: Reread the problem, and check your solution with the original words in the problem.

There are a number of substeps within each of the steps in our blueprint. For instance, with steps 1 and 2 it is always a good idea to draw a diagram or picture if it helps visualize the relationship between the items in the problem. In other cases a table helps organize the information. As you gain more experience using the blueprint to solve application problems, you will find additional techniques that expand the blueprint.

To help with problems of the type shown next in Example 1, in the margin are some common English words and phrases and their mathematical translations.

## Number Problems

## EXAMPLE 1

The sum of twice a number and three is seven. Find the number.

SOLUTION Using the Blueprint for Problem Solving as an outline, we solve the problem as follows:

Step 1: Read the problem, and then mentally list the items that are known and the items that are unknown.

Known items: $\quad$ The numbers 3 and 7
Unknown items: The number in question
Step 2: Assign a variable to one of the unknown items. Then translate the other information in the problem to expressions involving the variable.

Let $x=$ the number asked for in the problem,
then "The sum of twice a number and three" translates to $2 x+3$.

Step 3: Reread the problem, and then write an equation, using the items and variables listed in steps 1 and 2 , that describes the situation. With all word problems, the word is translates to $=$.


Step 4: Solve the equation found in step 3.

$$
\begin{aligned}
2 x+3 & =7 \\
2 x+3+(-\mathbf{3}) & =7+(-\mathbf{3}) \\
2 x & =4 \\
\frac{\mathbf{1}}{\mathbf{2}}(2 x) & =\frac{\mathbf{1}}{\mathbf{2}}(4) \\
x & =2
\end{aligned}
$$

Step 5: Write your answer using a complete sentence.
The number is 2 .
Step 6: Reread the problem, and check your solution with the original words in the problem.

The sum of twice 2 and 3 is 7 ; a true statement.

You may find some examples and problems in this section that you can solve without using algebra or our blueprint. It is very important that you solve these problems using the methods we are showing here. The purpose behind these problems is to give you experience using the blueprint as a guide to
solving problems written in words. Your answers are much less important than the work that you show to obtain your answer. You will be able to condense the steps in the blueprint later in the course. For now, though, you need to show your work in the same detail that we are showing in the examples in this section.

## EXAMPLE 2 One number is three more than twice another; their sum

 is eighteen. Find the numbers.
## SOLUTION

Step 1: Read and list.
Known items: Two numbers that add to 18 . One is 3 more than twice the other.

Unknown items: The numbers in question.

## Step 2: Assign a variable, and translate information.

Let $x=$ the first number. The other is $2 x+3$.
Step 3: Reread, and write an equation.

$$
\underbrace{\text { Their sum }}_{x+(2 x+3)} \text { is } 18
$$

## Step 4: Solve the equation.

$$
\begin{aligned}
x+(2 x+3) & =18 \\
3 x+3 & =18 \\
3 x+3+(-\mathbf{3}) & =18+(-\mathbf{3}) \\
3 x & =15 \\
x & =5
\end{aligned}
$$

## Step 5: Write the answer.

The first number is 5 . The other is $2 \cdot 5+3=13$.

## Step 6: Reread, and check.

The sum of 5 and 13 is 18 , and 13 is 3 more than twice 5 .

## Age Problem

Remember as you read through the steps in the solutions to the examples in this section that step 1 is done mentally. Read the problem, and then mentally list the items that you know and the items that you don't know. The purpose of step 1 is to give you direction as you begin to work application problems. Finding the solution to an application problem is a process; it doesn't happen all at once. The first step is to read the problem with a purpose in mind. That purpose is to mentally note the items that are known and the items that are unknown.

## EXAMPLE 3 <br> Bill is 6 years older than Tom. Three years ago Bill's age

 was four times Tom's age. Find the age of each boy now.SOLUTION Applying the Blueprint for Problem Solving, we have
Step 1: Read and list.
Known items: Bill is 6 years older than Tom. Three years ago Bill's age was four times Tom's age.

Unknown items: Bill's age and Tom's age
Step 2: Assign a variable, and translate information.
Let $x=$ Tom's age now. That makes Bill $x+6$ years old now. A table like the one shown here can help organize the information in an age problem. Notice how we placed the $x$ in the box that corre-

|  | Three Years <br> Ago | Now |
| :--- | :---: | :---: |
| Bill |  | $x+6$ |
| Tom | $x$ |  | sponds to Tom's age now.

If Tom is $x$ years old now, 3 years ago he was $x-3$ years old. If Bill is $x+6$ years old now, 3 years ago he was $x+6-3=x+3$ years old. We use this information to fill in

|  | Three Years <br> Ago | Now |
| :--- | :---: | :---: |
| Bill | $x+3$ | $x+6$ |
| Tom | $x-3$ | $x$ | the remaining squares in the table.

Step 3: Reread, and write an equation.
Reading the problem again, we see that 3 years ago Bill's age was four times Tom's age. Writing this as an equation, we have Bill's age 3 years ago $=4 \cdot$ (Tom's age 3 years ago):

$$
x+3=4(x-3)
$$

Step 4: Solve the equation.

$$
\begin{aligned}
x+3 & =4(x-3) \\
x+3 & =4 x-12 \\
x+(-\boldsymbol{x})+3 & =4 x+(-\boldsymbol{x})-12 \\
3 & =3 x-12 \\
3+\mathbf{1 2} & =3 x-12+\mathbf{1 2} \\
15 & =3 x \\
x & =5
\end{aligned}
$$

## Step 5: Write the answer.

Tom is 5 years old. Bill is 11 years old.

## Step 6: Reread, and check.

If Tom is 5 and Bill is 11 , then Bill is 6 years older than Tom. Three years ago Tom was 2 and Bill was 8. At that time, Bill's age was four times Tom's age. As you can see, the answers check with the original problem.

## Geometry Problem

To understand Example 4 completely, you need to recall from Chapter 1 that the perimeter of a rectangle is the sum of the lengths of the sides. The formula for the perimeter is $P=2 l+2 w$.

EXAMPLE 4 The length of a rectangle is 5 inches more than twice the width. The perimeter is 34 inches. Find the length and width.
SOLUTION When working problems that involve geometric figures, a sketch of the figure helps organize and visualize the problem.
Step 1: Read and list.
Known items: The figure is a rectangle. The length is 5 inches more than twice the width. The perimeter is 34 inches.
Unknown items: The length and the width
Step 2: Assign a variable, and translate information.
Because the length is given in terms of the width (the length is 5 more than twice the width), we let $x=$ the width of the rectangle. The length is 5 more than twice the width, so it must be
 $2 x+5$. The diagram is a visual description of the relationships we have listed so far.
Step 3: Reread, and write an equation.
The equation that describes the situation is
Twice the length + twice the width is the perimeter

$$
2(2 x+5)+2 x=34
$$

Step 4: Solve the equation.

$$
\begin{aligned}
2(2 x+5)+2 x & =34 & & \text { Original equation } \\
4 x+10+2 x & =34 & & \text { Distributive property } \\
6 x+10 & =34 & & \text { Add } 4 x \text { and } 2 \boldsymbol{x} \\
6 x & =24 & & \text { Add }-\mathbf{1 0} \text { to each side } \\
x & =4 & & \text { Divide each side by } \mathbf{6}
\end{aligned}
$$

## Step 5: Write the answer.

The width $x$ is 4 inches. The length is $2 x+5=2(4)+5=13$ inches.

## Step 6: Reread, and check.

If the length is 13 and the width is 4 , then the perimeter must be $2(13)+2(4)=26+8=34$, which checks with the original problem.

## Coin Problem

EXAMPLE 5 Jennifer has $\$ 2.45$ in dimes and nickels. If she has 8 more dimes than nickels, how many of each coin does she have?

## SOLUTION

## Step 1: Read and list.

Known items: The type of coins, the total value of the coins, and that there are 8 more dimes than nickels.

Unknown items: The number of nickels and the number of dimes
Step 2: Assign a variable, and translate information.
If we let $x=$ the number of nickels, then $x+8=$ the number of dimes. Because the value of each nickel is 5 cents, the amount of money

|  | Nickels | Dimes |
| :--- | :---: | :---: |
| Number | $x$ | $x+8$ |
| Value <br> (in cents) | $5 x$ | $10(x+8)$ | in nickels is $5 x$. Similarly, because each dime is worth 10 cents, the amount of money in dimes is $10(x+8)$. The table summarizes the information we have so far.

Step 3: Reread, and write an equation.
Because the total value of all the coins is 245 cents, the equation that describes this situation is

| Amount of money | Amount of money | Total amount |  |  |
| :---: | :---: | :---: | :---: | :---: |
| in nickels | + | in dimes | $=$ | of money |
| $5 x$ | + | $10(x+8)$ | $=$ | 245 |

Step 4: Solve the equation.
To solve the equation, we apply the distributive property first.

$$
\begin{aligned}
5 x+10 x+80 & =245 & & \text { Distributive property } \\
15 x+80 & =245 & & \text { Add } 5 x \text { and } 10 x \\
15 x & =165 & & \text { Add }-\mathbf{8 0} \text { to each side } \\
x & =11 & & \text { Divide each side by } \mathbf{1 5}
\end{aligned}
$$

## Step 5: Write the answer.

The number of nickels is $x=11$.
The number of dimes is $x+8=11+8=19$.
Step 6: Reread, and check.
To check our results

$$
\begin{aligned}
11 \text { nickels are worth } 5(11) & =55 \text { cents } \\
19 \text { dimes are worth } 10(19) & =190 \text { cents }
\end{aligned}
$$

When you begin working the problems in the problem set that follows, there are a couple of things to remember. The first is that you may have to read the problems a number of times before you begin to see how to solve them. The second thing to remember is that word problems are not always solved correctly the first time you try them. Sometimes it takes a couple of attempts and some wrong answers before you can set up and solve these problems correctly.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-5

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the first step in the Blueprint for Problem Solving?
2. What is the last thing you do when solving an application problem?
3. What good does it do you to solve application problems even when they don't have much to do with real life?
4. Write an application problem whose solution depends on solving the equation $2 x+3=7$.

## Problem Set 2.6

Online support materials can be found at www.thomsonedu.com/login

Solve the following word problems. Follow the steps given in the Blueprint for Problem Solving.

## Number Problems

1. The sum of a number and five is thirteen. Find the number.
2. The difference of ten and a number is negative eight. Find the number.
3. The sum of twice a number and four is fourteen. Find the number.
4. The difference of four times a number and eight is sixteen. Find the number.
5. Five times the sum of a number and seven is thirty. Find the number.
6. Five times the difference of twice a number and six is negative twenty. Find the number.
7. One number is two more than another. Their sum is eight. Find both numbers.
8. One number is three less than another. Their sum is fifteen. Find the numbers.
9. One number is four less than three times another. If their sum is increased by five, the result is twentyfive. Find the numbers.
10. One number is five more than twice another. If their sum is decreased by ten, the result is twentytwo. Find the numbers.

## Age Problems

11. Shelly is 3 years older than Michele. Four years ago the sum of their ages was 67 . Find the age of each person now.

|  | Four Years Ago | Now |
| :--- | :---: | :---: |
| Shelly | $x-1$ | $x+3$ |
| Michele | $x-4$ | $x$ |

12. Cary is 9 years older than Dan. In 7 years the sum of their ages will be 93. Find the age of each man now.

|  | Now | In Seven Years |
| :---: | :---: | :---: |
| Cary | $x+9$ |  |
| Dan | $x$ | $x+7$ |

13. Cody is twice as old as Evan. Three years ago the sum of their ages was 27 . Find the age of each boy now.

|  | Three Years Ago | Now |
| :--- | :---: | :---: |
| Cody |  |  |
| Evan | $x-3$ | $x$ |

14. Justin is 2 years older than Ethan. In 9 years the sum of their ages will be 30. Find the age of each boy now.

|  | Now | In Nine Years |
| :--- | :---: | :---: |
| Justin |  |  |
| Ethan | $x$ |  |

15. Fred is 4 years older than Barney. Five years ago the sum of their ages was 48 . How old are they now?

|  | Five Years Ago | Now |
| :--- | :---: | :---: |
| Fred |  |  |
| Barney | $x$ |  |

16. Tim is 5 years older than JoAnn. Six years from now the sum of their ages will be 79. How old are they now?

|  | Now | Six Years From Now |
| :--- | :---: | :---: |
| Tim |  |  |
| JoAnn | $X$ |  |

17. Jack is twice as old as Lacy. In 3 years the sum of their ages will be 54 . How old are they now?
18. John is 4 times as old as Martha. Five years ago the sum of their ages was 50 . How old are they now?
19. Pat is 20 years older than his son Patrick. In 2 years Pat will be twice as old as Patrick. How old are they now?
20. Diane is 23 years older than her daughter Amy. In 6 years Diane will be twice as old as Amy. How old are they now?

## Perimeter Problems

21. The perimeter of a square is 36 inches. Find the length of one side.
22. The perimeter of a square is 44 centimeters. Find the length of one side.
23. The perimeter of a square is 60 feet. Find the length of one side.
24. The perimeter of a square is 84 meters. Find the length of one side.
25. One side of a triangle is three times the shortest side. The third side is 7 feet more than the shortest side. The perimeter is 62 feet. Find all three sides.
26. One side of a triangle is half the longest side. The third side is 10 meters less than the longest side. The perimeter is 45 meters. Find all three sides.
27. One side of a triangle is half the longest side. The third side is 12 feet less than the longest side. The perimeter is 53 feet. Find all three sides.
28. One side of a triangle is 6 meters more than twice the shortest side. The third side is 9 meters more than the shortest side. The perimeter is 75 meters. Find all three sides.
29. The length of a rectangle is 5 inches more than the width. The perimeter is 34 inches. Find the length and width.

30. The width of a rectangle is 3 feet less than the length. The perimeter is 10 feet. Find the length and width.
31. The length of a rectangle is 7 inches more than twice the width. The perimeter is 68 inches. Find the length and width.
32. The length of a rectangle is 4 inches more than three times the width. The perimeter is 72 inches. Find the length and width.
33. The length of a rectangle is 6 feet more than three times the width. The perimeter is 36 feet. Find the length and width.
34. The length of a rectangle is 3 feet less than twice the width. The perimeter is 54 feet. Find the length and width.

## Coin Problems

35. Marissa has $\$ 4.40$ in quarters and dimes. If she has 5 more quarters than dimes, how many of each coin does she have?

|  | Dimes | Quarters |
| :--- | :---: | :---: |
| Number | $x$ | $x+5$ |
| Value (cents) | $10(x)$ | $25(x+5)$ |

36. Kendra has $\$ 2.75$ in dimes and nickels. If she has twice as many dimes as nickels, how many of each coin does she have?

|  | Nickels | Dimes |
| :--- | :---: | :---: |
| Number | $x$ | $2 x$ |
| Value (cents) | $5(x)$ |  |

37. Tanner has $\$ 4.35$ in nickels and quarters. If he has 15 more nickels than quarters, how many of each coin does he have?

|  | Nickels | Quarters |
| :--- | :---: | :---: |
| Number | $x+15$ | $x$ |
| Value (cents) |  |  |

38. Connor has $\$ 9.00$ in dimes and quarters. If he has twice as many quarters as dimes, how many of each coin does he have?

|  | Dimes | Quarters |
| :---: | :---: | :---: |
| Number | $x$ | $2 x$ |

Value (cents)
39. Sue has $\$ 2.10$ in dimes and nickels. If she has 9 more dimes than nickels, how many of each coin does she have?
40. Mike has $\$ 1.55$ in dimes and nickels. If he has 7 more nickels than dimes, how many of each coin does he have?
41. Katie has a collection of nickels, dimes, and quarters with a total value of $\$ 4.35$. There are 3 more dimes than nickels and 5 more quarters than nickels. How many of each coin is in her collection?

|  | Nickels | Dimes | Quarters |
| :--- | :---: | :---: | :---: |
| Number | $x$ |  |  |
| Value |  |  |  |

42. Mary Jo has $\$ 3.90$ worth of nickels, dimes, and quarters. The number of nickels is 3 more than the number of dimes. The number of quarters is 7 more than the number of dimes. How many of each coin is in her collection?

|  | Nickels | Dimes | Quarters |
| :--- | :--- | :--- | :--- |
| Number |  |  |  |
| Value |  |  |  |

43. Cory has a collection of nickels, dimes, and quarters with a total value of $\$ 2.55$. There are 6 more dimes than nickels and twice as many quarters as nickels. How many of each coin is in her collection?

|  | Nickels | Dimes | Quarters |
| :--- | :---: | :---: | :---: |
| Number | $x$ |  |  |
| Value |  |  |  |

44. Kelly has a collection of nickels, dimes, and quarters with a total value of $\$ 7.40$. There are four more nickels than dimes and twice as many quarters as nickels. How many of each coin is in her collection?

|  | Nickels | Dimes |
| :--- | :--- | :--- |
| Number |  | Quarters |
| Value |  |  |

## Maintaining Your Skills

Write an equivalent statement in English.
45. $4<10$
46. $4 \leq 10$
47. $9 \geq-5$
48. $x-2>4$

Place the symbol < or the symbol > between the quantities in each expression.
49. 1220
50. $-12 \quad 20$
51. $-8-6$
52. $-10-20$

Simplify.
53. $|8-3|-|5-2|$
54. $|9-2|-|10-8|$
55. $15-|9-3(7-5)|$
56. $10-|7-2(5-3)|$

## Getting Ready for the Next Section

To understand all of the explanations and examples in the next section you must be able to work the problems below.

Simplify the following expressions.
57. $x+2 x+2 x$
58. $x+2 x+3 x$
59. $x+0.075 x$
60. $x+0.065 x$
61. $0.09(x+2,000)$
62. $0.06(x+1,500)$

Solve each of the following equations.
63. $0.05 x+0.06(x-1,500)=570$
64. $0.08 x+0.09(x+2,000)=690$
65. $x+2 x+3 x=180$
66. $2 x+3 x+5 x=180$

## 2.7

## OBJECTIVES

A Apply the Blueprint for Problem Solving to a variety of application problems.

## More Applications

Now that you have worked through a number of application problems using our blueprint, you probably have noticed that step 3, in which we write an equation that describes the situation, is the key step. Anyone with experience solving application problems will tell you that there will be times when your first attempt at step 3 results in the wrong equation. Remember, mistakes are part of the process of learning to do things correctly. Many times the correct equation will become obvious after you have written an equation that is partially wrong. In any case it is better to write an equation that is partially wrong and be actively involved with the problem than to write nothing at all. Application problems, like other problems in algebra, are not always solved correctly the first time.

## Consecutive Integers

Our first example involves consecutive integers. When we ask for consecutive integers, we mean integers that are next to each other on the number line, like 5 and 6 , or 13 and 14 , or -4 and -3 . In the dictionary, consecutive is defined as following one another in uninterrupted order. If we ask for consecutive odd integers, then we mean odd integers that follow one another on the number line. For example, 3 and 5, 11 and 13, and -9 and -7 are consecutive odd integers. As you can see, to get from one odd integer to the next consecutive odd integer we add 2.

If we are asked to find two consecutive integers and we let $x$ equal the first integer, the next one must be $x+1$, because consecutive integers always differ by 1. Likewise, if we are asked to find two consecutive odd or even integers, and
we let $x$ equal the first integer, then the next one will be $x+2$ because consecutive even or odd integers always differ by 2 . Here is a table that summarizes this information.

| In Words | Using Algebra | Example |
| :---: | :---: | :---: |
| Two consecutive integers | $x, x+1$ | The sum of two consecutive integers is 15 . $\begin{aligned} x+(x+1) & =15 \text { or } \\ 7+8 & =15 \end{aligned}$ |
| Three consecutive integers | $x, x+1, x+2$ | The sum of three consecutive integers is 24 . $\begin{aligned} x+(x+1)+(x+2) & =24 \text { or } \\ 7+8+9 & =24 \end{aligned}$ |
| Two consecutive odd integers | $x, x+2$ | The sum of two consecutive odd integers is 16 . $\begin{aligned} x+(x+2) & =16 \text { or } \\ 7+9 & =16 \end{aligned}$ |
| Two consecutive even integers | $x, x+2$ | The sum of two consecutive even integers is 18 . $\begin{aligned} x+(x+2) & =18 \text { or } \\ 8+10 & =18 \end{aligned}$ |

## EXAMPLE 1

The sum of two consecutive odd integers is 28 . Find the
two integers.

## SOLUTION

## Step 1: Read and list.

Known items: Two consecutive odd integers. Their sum is equal to 28. Unknown items: The numbers in question.

## Step 2: Assign a variable, and translate information.

If we let $x=$ the first of the two consecutive odd integers, then $x+2$ is the next consecutive one.

## Step 3: Reread, and write an equation.

Their sum is 28 .

$$
x+(x+2)=28
$$

Step 4: Solve the equation.

$$
\begin{aligned}
2 x+2 & =28 & & \text { Simplify the left side } \\
2 x & =26 & & \text { Add }-\mathbf{2} \text { to each side } \\
x & =13 & & \text { Multiply each side by } \frac{1}{2}
\end{aligned}
$$

## Step 5: Write the answer.

The first of the two integers is 13 . The second of the two integers will be two more than the first, which is 15 .

## Step 6: Reread, and check.

Suppose the first integer is 13 . The next consecutive odd integer is 15 . The sum of 15 and 13 is 28 .

## Interest

EXAMPLE 2 Suppose you invest a certain amount of money in an account that earns $8 \%$ in annual interest. At the same time, you invest $\$ 2,000$ more than that in an account that pays $9 \%$ in annual interest. If the total interest from both accounts at the end of the year is $\$ 690$, how much is invested in each account?

## SOLUTION

## Step 1: Read and list.

Known items: The interest rates, the total interest earned, and how much more is invested at 9\%
Unknown items: The amounts invested in each account
Step 2: Assign a variable, and translate information.
Let $x=$ the amount of money invested at $8 \%$. From this, $x+2,000=$ the amount of money invested at $9 \%$. The interest earned on $x$ dollars invested at $8 \%$ is $0.08 x$. The interest earned on $x+2,000$ dollars invested at $9 \%$ is $0.09(x+2,000)$.
Here is a table that summarizes this information:

|  | Dollars Invested <br> at 8\% | Dollars Invested <br> at 9\% |
| :--- | :---: | :---: |
| Number of | $X$ | $x+2,000$ |
| Interest on | $0.08 x$ | $0.09(x+2,000)$ |

Step 3: Reread, and write an equation.
Because the total amount of interest earned from both accounts is $\$ 690$, the equation that describes the situation is

| Interest earned | Interest earned |  | Total interest |  |
| :---: | :---: | :---: | :---: | :---: |
| at $8 \%$ | + | at $9 \%$ | $=$ | earned |
| $0.08 x$ | $+0.09(x+2,000)$ | $=$ | 690 |  |

Step 4: Solve the equation.

$$
\begin{aligned}
0.08 x+0.09(x+2,000) & =690 & & \\
0.08 x+0.09 x+180 & =690 & & \text { Distributive property } \\
0.17 x+180 & =690 & & \text { Add } 0.08 x \text { and } 0.09 x \\
0.17 x & =510 & & \text { Add }-\mathbf{1 8 0} \text { to each side } \\
x & =3,000 & & \text { Divide each side by } \mathbf{0 . 1 7}
\end{aligned}
$$

## Step 5: Write the answer.

The amount of money invested at $8 \%$ is $\$ 3,000$, whereas the amount of money invested at $9 \%$ is $x+2,000=3,000+2,000=\$ 5,000$.

Step 6: Reread, and check.
The interest at $8 \%$ is $8 \%$ of $3,000=0.08(3,000)=\$ 240$
The interest at $9 \%$ is $9 \%$ of $5,000=0.09(5,000)=\$ 450$
The total interest is $\$ 690$

## FACTS FROM GEOMETRY

## Labeling Triangles and the Sum of the Angles in a Triangle

One way to label the important parts of a triangle is to label the vertices with capital letters and the sides with small letters, as shown in Figure 1.


In Figure 1, notice that side $a$ is opposite vertex $A$, side $b$ is opposite vertex $B$, and side $c$ is opposite vertex $C$. Also, because each vertex is the vertex of one of the angles of the triangle, we refer to the three interior angles as $A, B$, and $C$.

In any triangle, the sum of the interior angles is $180^{\circ}$. For the triangle shown in Figure 1, the relationship is written

$$
A+B+C=180^{\circ}
$$

## EXAMPLE 3

The angles in a triangle are such that one angle is twice the smallest angle, whereas the third angle is three times as large as the smallest angle. Find the measure of all three angles.

## SOLUTION

## Step 1: Read and list.

Known items: The sum of all three angles is $180^{\circ}$, one angle is twice the smallest angle, the largest angle is three times the smallest angle. Unknown items: The measure of each angle

Step 2: Assign a variable, and translate information.
Let $x$ be the smallest angle, then $2 x$ will be the measure of another angle and $3 x$ will be the measure of the largest angle.

Step 3: Reread, and write an equation.
When working with geometric objects, drawing a generic diagram sometimes will help us visualize what it is that we are asked to find. In Figure 2, we draw a triangle with angles $A, B$, and $C$.


FIGURE 2

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-3

We can let the value of $A=x$, the value of $B=2 x$, and the value of $C=$ $3 x$. We know that the sum of angles $A, B$, and $C$ will be $180^{\circ}$, so our equation becomes

$$
x+2 x+3 x=180^{\circ}
$$

Step 4: Solve the equation.

$$
\begin{aligned}
x+2 x+3 x & =180^{\circ} \\
6 x & =180^{\circ} \\
x & =30^{\circ}
\end{aligned}
$$

## Step 5: Write the answer.

The smallest angle $A$ measures $30^{\circ}$
Angle $B$ measures $2 x$, or $2\left(30^{\circ}\right)=60^{\circ}$
Angle $C$ measures $3 x$, or $3\left(30^{\circ}\right)=90^{\circ}$

## Step 6: Reread, and check.

The angles must add to $180^{\circ}$ :

$$
\begin{array}{r}
A+B+C=180^{\circ} \\
30^{\circ}+60^{\circ}+90^{\circ} \stackrel{?}{=} 180^{\circ}
\end{array}
$$

$$
180^{\circ}=180^{\circ} \quad \text { Our answers check }
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How do we label triangles?
2. What rule is always true about the three angles in a triangle?
3. Write an application problem whose solution depends on solving the equation $x+0.075 x=500$.
4. Write an application problem whose solution depends on solving the equation $0.05 x+0.06(x+200)=67$.

## Problem Set 2.7

## Consecutive Integer Problems

1. The sum of two consecutive integers is 11 . Find the numbers.
2. The sum of two consecutive integers is 15 . Find the numbers.
3. The sum of two consecutive integers is -9 . Find the numbers.
= Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
4. The sum of two consecutive integers is -21 . Find the numbers.
5. The sum of two consecutive odd integers is 28 . Find the numbers.
6. The sum of two consecutive odd integers is 44 . Find the numbers.
7. The sum of two consecutive even integers is 106 . Find the numbers.
8. The sum of two consecutive even integers is 66 Find the numbers.
9. The sum of two consecutive even integers is -30 . Find the numbers.
10. The sum of two consecutive odd integers is -76 . Find the numbers.
11. The sum of three consecutive odd integers is 57 . Find the numbers.
12. The sum of three consecutive odd integers is -51 . Find the numbers.
13. The sum of three consecutive even integers is 132 . Find the numbers.
14. The sum of three consecutive even integers is -108 . Find the numbers.

## Interest Problems

15. Suppose you invest money in two accounts. One of the accounts pays $8 \%$ annual interest, whereas the other pays $9 \%$ annual interest. If you have \$2,000 more invested at $9 \%$ than you have invested at $8 \%$, how much do you have invested in each account if the total amount of interest you earn in a year is $\$ 860$ ? (Begin by completing the following table.)

|  | Dollars <br> Invested <br> at 8\% | Dollars <br> Invested <br> at 9\% |
| :--- | :---: | :---: |
| Number of | $x$ |  |
| Interest on |  |  |

16. Suppose you invest a certain amount of money in an account that pays $11 \%$ interest annually, and $\$ 4,000$ more than that in an account that pays $12 \%$ annually. How much money do you have in each account if the total interest for a year is \$940?

|  | Dollars <br> Invested <br> at $\mathbf{1 1 \%}$ | Dollars <br> Invested <br> at $\mathbf{1 2 \%}$ |
| :--- | :---: | :---: |
| Number of | $x$ |  |
| Interest on |  |  |

17. Tyler has two savings accounts that his grandparents opened for him. The two accounts pay $10 \%$
and $12 \%$ in annual interest; there is $\$ 500$ more in the account that pays $12 \%$ than there is in the other account. If the total interest for a year is $\$ 214$, how much money does he have in each account?
18. Travis has a savings account that his parents opened for him. It pays $6 \%$ annual interest. His uncle also opened an account for him, but it pays $8 \%$ annual interest. If there is $\$ 800$ more in the account that pays $6 \%$, and the total interest from both accounts is $\$ 104$, how much money is in each of the accounts?
19. A stockbroker has money in three accounts. The interest rates on the three accounts are $8 \%, 9 \%$, and $10 \%$. If she has twice as much money invested at $9 \%$ as she has invested at $8 \%$, three times as much at $10 \%$ as she has at $8 \%$, and the total interest for the year is $\$ 280$, how much is invested at each rate? (Hint: Let $x=$ the amount invested at 8\%.)
20. An accountant has money in three accounts that pay $9 \%, 10 \%$, and $11 \%$ in annual interest. He has twice as much invested at $9 \%$ as he does at $10 \%$ and three times as much invested at $11 \%$ as he does at $10 \%$. If the total interest from the three accounts is $\$ 610$ for the year, how much is invested at each rate? (Hint: Let $x=$ the amount invested at 10\%.)

## Triangle Problems

21. Two angles in a triangle are equal and their sum is equal to the third angle in the triangle. What are the measures of each of the three interior angles?
22. One angle in a triangle measures twice the smallest angle, whereas the largest angle is six times the smallest angle. Find the measures of all three angles.
23. The smallest angle in a triangle is $\frac{1}{5}$ as large as the largest angle. The third angle is twice the smallest angle. Find the three angles.
24. One angle in a triangle is half the largest angle but three times the smallest. Find all three angles.
25. A right triangle has one $37^{\circ}$ angle. Find the other two angles.
26. In a right triangle, one of the acute angles is twice as large as the other acute angle. Find the measure of the two acute angles.
27. One angle of a triangle measures $20^{\circ}$ more than the smallest, while a third angle is twice the smallest. Find the measure of each angle.
28. One angle of a triangle measures $50^{\circ}$ more than the smallest, while a third angle is three times the smallest. Find the measure of each angle.

## Miscellaneous Problems

29. Ticket Prices Miguel is selling tickets to a barbecue. Adult tickets cost $\$ 6.00$ and children's tickets cost $\$ 4.00$. He sells six more children's tickets than adult tickets. The total amount of money he collects is $\$ 184$. How many adult tickets and how many children's tickets did he sell?

|  | Adult | Child |
| :--- | :---: | :---: |
| Number | $x$ |  |
| Income |  |  |

30. Working Two Jobs Maggie has a job working in an office for $\$ 10$ an hour and another job driving a tractor for $\$ 12$ an hour. One week she works in the office twice as long as she drives the tractor. Her total income for that week is $\$ 416$. How many hours did she spend at each job?

| Job | Office | Tractor |
| :---: | :---: | :---: |
| Hours Worked | $X$ |  |
| Wages Earned |  |  |

31. Phone Bill The cost of a long-distance phone call is $\$ 0.41$ for the first minute and $\$ 0.32$ for each additional minute. If the total charge for a long-distance call is $\$ 5.21$, how many minutes was the call?
32. Phone Bill Danny, who is 1 year old, is playing with the telephone when he accidentally presses one of the buttons his mother has programmed to dial her friend Sue's number. Sue answers the phone and realizes Danny is on the other end. She talks to

Danny, trying to get him to hang up. The cost for a call is $\$ 0.23$ for the first minute and $\$ 0.14$ for every minute after that. If the total charge for the call is $\$ 3.73$, how long did it take Sue to convince Danny to hang up the phone?
33. Hourly Wages JoAnn works in the publicity office at the state university. She is paid $\$ 12$ an hour for the first 35 hours she works each week and $\$ 18$ an hour for every hour after that. If she makes \$492 one week, how many hours did she work?
34. Hourly Wages Diane has a part-time job that pays her $\$ 6.50$ an hour. During one week she works 26 hours and is paid $\$ 178.10$. She realizes when she sees her check that she has been given a raise. How much per hour is that raise?
35. Office Numbers Professors Wong and Gil have offices in the mathematics building at Miami Dade College. Their office numbers are consecutive odd integers with a sum of 14,660 . What are the office numbers of these two professors?
36. Cell Phone Numbers Diana and Tom buy two cell phones. The phone numbers assigned to each are consecutive integers with a sum of $11,109,295$. If the smaller number is Diana's, what are their phone numbers?
37. Age Marissa and Kendra are 2 years apart in age. Their ages are two consecutive even integers. Kendra is the younger of the two. If Marissa's age is added to twice Kendra's age, the result is 26 . How old is each girl?
38. Age Justin's and Ethan's ages form two consecutive odd integers. What is the difference of their ages?
39. Arrival Time Jeff and Carla Cole are driving separately from San Luis Obispo, California, to the north shore of Lake Tahoe, a distance of 425 miles. Jeff leaves San Luis Obispo at 11:00 Am and averages 55 miles per hour on the drive, Carla leaves later, at 1:00 Pm but averages 65 miles per hour. Which person arrives in Lake Tahoe first?
40. Piano Lessons Tyler is taking piano lessons. Because he doesn't practice as often as his parents would like him to, he has to pay for part of the lessons himself. His parents pay him $\$ 0.50$ to do the laun-
dry and $\$ 1.25$ to mow the lawn. In one month, he does the laundry 6 more times than he mows the lawn. If his parents pay him $\$ 13.50$ that month, how many times did he mow the lawn?

At one time, the Texas Junior College Teachers Association annual conference was held in Austin. At that time a taxi ride in Austin was $\$ 1.25$ for the first $\frac{1}{5}$ of a mile and $\$ 0.25$ for each additional $\frac{1}{5}$ of a mile. Use this information for Problems 41 and 42.
41. Cost of a Taxi Ride If the distance from one of the convention hotels to the airport is 7.5 miles, how much will it cost to take at taxi from that hotel to the airport?
42. Cost of a Taxi Ride Suppose the distance from one of the hotels to one of the western dance clubs in Austin is 12.4 miles. If the fare meter in the taxi gives the charge for that trip as $\$ 16.50$, is the meter working correctly?
43. Geometry The length and width of a rectangle are consecutive even integers. The perimeter is 44 meters. Find the length and width.
44. Geometry The length and width of a rectangle are consecutive odd integers. The perimeter is 128 meters. Find the length and width.
45. Geometry The angles of a triangle are three consecutive integers. Find the measure of each angle.
46. Geometry The angles of a triangle are three consecutive even integers. Find the measure of each angle.

Ike and Nancy Lara give western dance lessons at the Elks Lodge on Sunday nights. The lessons cost $\$ 3.00$ for members of the lodge and $\$ 5.00$ for nonmembers. Half of the money collected for the lesson is paid to Ike and Nancy. The Elks Lodge keeps the other half. One Sunday night Ike counts 36 people in the dance lesson. Use this information to work Problems 47 through 50.
47. Dance Lessons What is the least amount of money Ike and Nancy will make?
48. Dance Lessons What is the largest amount of money Ike and Nancy will make?
49. Dance Lessons At the end of the evening, the Elks Lodge gives Ike and Nancy a check for $\$ 80$ to cover half of the receipts. Can this amount be correct?
50. Dance Lessons Besides the number of people in the dance lesson, what additional information does Ike need to know to always be sure he is being paid the correct amount?

## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in previous sections and chapters. You can consider the time you spend working these problems as time spent studying for exams.

Simplify the expression $36 x-12$ for each of the following values of $x$.
51. $\frac{1}{4}$
52. $\frac{1}{6}$
53. $\frac{1}{9}$
54. $\frac{3}{2}$
55. $\frac{1}{3}$
56. $\frac{5}{12}$
57. $\frac{5}{9}$
58. $\frac{2}{3}$

Find the value of each expression when $x=-4$.
59. $3(x-4)$
60. $-3(x-4)$
61. $-5 x+8$
62. $5 x+8$
63. $\frac{x-14}{36}$
64. $\frac{x-12}{36}$
65. $\frac{16}{x}+3 x$
66. $\frac{16}{x}-3 x$
67. $7 x-\frac{12}{x}$
68. $7 x+\frac{12}{x}$
69. $8\left(\frac{x}{2}+5\right)$
70. $-8\left(\frac{x}{2}+5\right)$

## Getting Ready for the Next Section

To understand all of the explanations and examples in the next section you must be able to work the problems below.

Solve the following equations.
71. a. $x-3=6$
b. $x+3=6$
c. $-x-3=6$
d. $-x+3=6$
72. a. $x-7=16$
b. $x+7=16$
c. $-x-7=16$
d. $-x+7=16$
73. a. $\frac{x}{4}=-2$
b. $-\frac{x}{4}=-2$
c. $\frac{x}{4}=2$
d. $-\frac{x}{4}=2$
74. a. $3 a=15$
b. $3 a=-15$
c. $-3 a=15$
d. $-3 a=-15$
75. $2.5 x-3.48=4.9 x+2.07$
76. $2(1-3 x)+4=4 x-14$
77. $3(x-4)=-2$
78. Solve $2 x-3 y=6$ for $y$.

## 2.8

## OBJECTIVES

A Use the addition property for inequalities to solve an inequality.

B Use the multiplication property for inequalities to solve an inequality.

C Use both the addition and multiplication properties to solve an inequality.

D Graph the solution set for an inequality.

E Translate and solve application problems involving inequalities.

## Linear Inequalities

Linear inequalities are solved by a method similar to the one used in solving linear equations. The only real differences between the methods are in the multiplication property for inequalities and in graphing the solution set.

An inequality differs from an equation only with respect to the comparison symbol between the two quantities being compared. In place of the equal sign, we use $<$ (less than), $\leq$ (less than or equal to), $>$ (greater than), or $\geq$ (greater than or equal to). The addition property for inequalities is almost identical to the addition property for equality.

Addition Property for Inequalities For any three algebraic expressions A, $B$, and $C$,

| if |  | $A$ | $<B$ |
| :--- | :--- | ---: | :--- |
| then | $A+C$ | $<B+C$ |  |

In words: Adding the same quantity to both sides of an inequality will not change the solution set.

It makes no difference which inequality symbol we use to state the property. Adding the same amount to both sides always produces an inequality equivalent to the original inequality. Also, because subtraction can be thought of as addition of the opposite, this property holds for subtraction as well as addition.

## EXAMPLE 1

Solve the inequality $x+5<7$.
SOLUTION To isolate $x$, we add -5 to both sides of the inequality:

$$
\begin{aligned}
x+5 & <7 \\
x+5+(-\mathbf{5}) & <7+(-\mathbf{5}) \quad \text { Addition property for inequalities } \\
x & <2
\end{aligned}
$$

## Note

This discussion is intended to show why the multiplication property for inequalities is written the way it is. You may want to look ahead to the property itself and then come back to this discussion if you are having trouble making sense out of it.

We can go one step further here and graph the solution set. The solution set is all real numbers less than 2. To graph this set, we simply draw a straight line and label the center 0 (zero) for reference. Then we label the 2 on the right side of zero and extend an arrow beginning at 2 and pointing to the left. We use an open circle at 2 because it is not included in the solution set. Here is the graph.


EXAMPLE 2 Solve $x-6 \leq-3$.
SOLUTION Adding 6 to each side will isolate $x$ on the left side:

$$
\begin{aligned}
x-6 & \leq-3 \\
x-6+\mathbf{6} & \leq-3+\mathbf{6} \quad \text { Add } 6 \text { to both sides } \\
x & \leq 3
\end{aligned}
$$



The graph of the solution set is


Notice that the dot at the 3 is darkened because 3 is included in the solution set. We always will use open circles on the graphs of solution sets with $<$ or $>$ and closed (darkened) circles on the graphs of solution sets with $\leq$ or $\geq$.

To see the idea behind the multiplication property for inequalities, we will consider three true inequality statements and explore what happens when we multiply both sides by a positive number and then what happens when we multiply by a negative number.

Consider the following three true statements:

$$
3<5 \quad-3<5 \quad-5<-3
$$

Now multiply both sides by the positive number 4:

$$
\begin{aligned}
4(3) & <4(5) & 4(-3) & <4(5) \\
12 & <20 & -12 & <20
\end{aligned}
$$

In each case, the inequality symbol in the result points in the same direction it did in the original inequality. We say the "sense" of the inequality doesn't change when we multiply both sides by a positive quantity.
Notice what happens when we go through the same process but multiply both sides by -4 instead of 4 :

$$
\begin{gathered}
3<5 \\
-4(3) \stackrel{y}{>}
\end{gathered}
$$




In each case, we have to change the direction in which the inequality symbol points to keep each statement true. Multiplying both sides of an inequality by a negative quantity always reverses the sense of the inequality. Our results are summarized in the multiplication property for inequalities.

## Note

Because division is defined in terms of multiplication, this property is also true for division. We can divide both sides of an inequality by any nonzero number we choose. If that number happens to be negative, we must also reverse the direction of the inequality symbol.

Multiplication Property for Inequalities For any three algebraic expressions $A, B$, and $C$,

| if | $A$ | $<B$ |
| :--- | :--- | :--- |
|  |  |  |
| then | $A C$ | $<B C$ |
| and | $A C$ | when $C$ is positive |
|  |  | when $C$ is negative |

In words: Multiplying both sides of an inequality by a positive number does not change the solution set. When multiplying both sides of an inequality by a negative number, it is necessary to reverse the inequality symbol to produce an equivalent inequality.

We can multiply both sides of an inequality by any nonzero number we choose. If that number happens to be negative, we must also reverse the sense of the inequality.

## EXAMPLE 3 Solve $3 a<15$ and graph the solution.

SOLUTION We begin by multiplying each side by $\frac{1}{3}$. Because $\frac{1}{3}$ is a positive number, we do not reverse the direction of the inequality symbol:

$$
\begin{aligned}
3 a & <15 \\
\frac{\mathbf{1}}{\mathbf{3}}(3 a) & <\frac{\mathbf{1}}{\mathbf{3}}(15) \quad \text { Multiply each side by } \frac{1}{3} \\
a & <5 \\
& \\
&
\end{aligned}
$$

## EXAMPLE 4 Solve $-3 a \leq 18$, and graph the solution.

SOLUTION We begin by multiplying both sides by $-\frac{1}{3}$. Because $-\frac{1}{3}$ is a negative number, we must reverse the direction of the inequality symbol at the same time that we multiply by $-\frac{1}{3}$.

$$
\begin{aligned}
-3 a & \leq 18 \\
-\frac{1}{3}(-3 a) & \geq-\frac{1}{3}(18) \quad \\
a \geq-6 & \begin{array}{l}
\text { Multiply both sides by }-\frac{1}{3} \text { and reverse } \\
\text { the direction of the inequality symbol }
\end{array} \\
\hline-6 &
\end{aligned}
$$

EXAMPLE 5 Solve $-\frac{X}{4}>2$ and graph the solution.
SOLUTION To isolate $x$, we multiply each side by -4 . Because -4 is a negative number, we also must reverse the direction of the inequality symbol:

$$
\begin{aligned}
-\frac{x}{4} & >2 \\
6 & \\
-\mathbf{4}\left(-\frac{x}{4}\right) & <-\mathbf{4}(2) \quad \begin{array}{l}
\text { Multiply each side by }-4, \text { and reverse } \\
\text { the direction of the inequality symbol }
\end{array}
\end{aligned}
$$

To solve more complicated inequalities, we use the following steps.

Strategy for Solving Linear Inequalities in One Variable
Step 1a: Use the distributive property to separate terms, if necessary.
1b: If fractions are present, consider multiplying both sides by the LCD to eliminate the fractions. If decimals are present, consider multiplying both sides by a power of 10 to clear the inequality of decimals.

1c: Combine similar terms on each side of the inequality.
Step 2: Use the addition property for inequalities to get all variable terms on one side of the inequality and all constant terms on the other side.

Step 3: Use the multiplication property for inequalities to get $x$ by itself on one side of the inequality.

Step 4: Graph the solution set.

## EXAMPLE 6 Solve $2.5 x-3.48<-4.9 x+2.07$.

SOLUTION We have two methods we can use to solve this inequality. We can simply apply our properties to the inequality the way it is currently written and work with the decimal numbers, or we can eliminate the decimals to begin with and solve the resulting inequality.

Method 1 Working with the decimals.

$$
\begin{aligned}
2.5 x-3.48 & <-4.9 x+2.07 & & \text { Original inequality } \\
2.5 x+\mathbf{4 . 9 x}-3.48 & <-4.9 x+\mathbf{4 . 9 x}+2.07 & & \text { Add } 4.9 \boldsymbol{x} \text { to each side } \\
7.4 x-3.48 & <2.07 & & \\
7.4 x-3.48+\mathbf{3 . 4 8} & <2.07+\mathbf{3 . 4 8} & & \text { Add } 3.48 \text { to each side } \\
7.4 x & <5.55 & & \\
\frac{7.4 x}{\mathbf{7 . 4}} & <\frac{5.55}{\mathbf{7 . 4}} & & \text { Divide each side by } 7.4 \\
x & <0.75 & &
\end{aligned}
$$

Method 2 Eliminating the decimals in the beginning.
Because the greatest number of places to the right of the decimal point in any of the numbers is 2 , we can multiply each side of the inequality by 100 and we will be left with an equivalent inequality that contains only whole numbers.

$$
\begin{aligned}
2.5 x-3.48 & <-4.9 x+2.07 & & \text { Original inequality } \\
\mathbf{1 0 0}(2.5 x-3.48) & <\mathbf{1 0 0}(-4.9 x+2.07) & & \text { Multiply each side by } 100 \\
\mathbf{1 0 0}(2.5 x)-\mathbf{1 0 0}(3.48) & <\mathbf{1 0 0}(-4.9 x)+\mathbf{1 0 0}(2.07) & & \text { Distributive property } \\
250 x-348 & <-490 x+207 & & \text { Multiplication } \\
740 x-348 & <207 & & \text { Add 490x to each side }
\end{aligned}
$$

$$
\begin{aligned}
740 x & <555 & & \text { Add } 348 \text { to each side } \\
\frac{740 x}{\mathbf{7 4 0}} & <\frac{555}{\mathbf{7 4 0}} & & \text { Divide each side by } 740 \\
x & <0.75 & &
\end{aligned}
$$

The solution by either method is $x<0.75$. Here is the graph:


## EXAMPLE 7 Solve $3(x-4) \geq-2$.

## SOLUTION

$$
\begin{array}{rlrl}
3 x-12 & \geq-2 & & \text { Distributive property } \\
3 x-12+\mathbf{1 2} & \geq-2+\mathbf{1 2} & & \text { Add } \mathbf{1 2} \text { to both sides } \\
3 x & \geq 10 & & \\
\frac{\mathbf{1}}{\mathbf{3}(3 x)} & \geq \frac{\mathbf{1}}{\mathbf{3}}(10) & & \text { Multiply both sides by } \frac{\mathbf{1}}{3} \\
x & \geq \frac{10}{3} &
\end{array}
$$

## EXAMPLE $8 \quad$ Solve and graph $2(1-3 x)+4<4 x-14$.

## SOLUTION

$$
\begin{array}{rll}
2-6 x+4 & <4 x-14 & \text { Distributive property } \\
-6 x+6 & <4 x-14 & \text { Simplify } \\
-6 x+6+(-\mathbf{6})<4 x-14+(-\mathbf{6}) & \text { Add }-\mathbf{6} \text { to both sides } \\
-6 x<4 x-20 & \\
-6 x+(-\mathbf{4 x})<4 x+(-\mathbf{4 x})-20 & \text { Add }-\mathbf{4 x} \text { to both sides } \\
-10 x<-20 & \begin{array}{l}
\text { Multiply by }-\frac{1}{10}, \text { reverse } \\
\text { the sense of the inequality }
\end{array} \\
\left(-\frac{\mathbf{1}}{\mathbf{1 0}}\right)(-10 x) \geqslant\left(-\frac{\mathbf{1}}{\mathbf{1 0}}\right)(-20) \\
x>2 & \xrightarrow[2]{\longrightarrow}
\end{array}
$$

EXAMPLE 9 Solve $2 x-3 y<6$ for $y$.
SOLUTION We can solve this inequality for $y$ by first adding $-2 x$ to each side and then multiplying each side by $-\frac{1}{3}$. When we multiply by $-\frac{1}{3}$ we must reverse
the direction of the inequality symbol. Because this is an inequality in two variables, we will not graph the solution.

$$
\begin{array}{rlrl}
2 x-3 y & <6 & & \text { Original inequality } \\
2 x+(-\mathbf{2 x})-3 y & <(-\mathbf{2 x})+6 & & \text { Add }-\mathbf{2 x} \text { to each side } \\
-3 y & <-2 x+6 & & \\
\zeta & & \\
-\frac{\mathbf{1}}{\mathbf{3}}(-3 y) & >-\frac{\mathbf{1}}{\mathbf{3}}(-2 x+6) & & \text { Multiply each side by }-\frac{\mathbf{1}}{3} \\
y & >\frac{2}{3} x-2 & & \text { Distributive property }
\end{array}
$$

When working application problems that involve inequalities, the phrases "at least" and "at most" translate as follows:

| In Words | In Symbols |
| :---: | :---: |
| $x$ is at least 30 | $x \geq 30$ |
| $x$ is at most 20 | $x \leq 20$ |

## Applying the Concepts

Our next example is similar to an example done earlier in this chapter. This time it involves an inequality instead of an equation.

We can modify our Blueprint for Problem Solving to solve application problems whose solutions depend on writing and then solving inequalities.

EXAMPLE 10 The sum of two consecutive odd integers is at most 28. What are the possibilities for the first of the two integers?
SOLUTION When we use the phrase "their sum is at most 28," we mean that their sum is less than or equal to 28 .
Step 1: Read and list.
Known items: Two consecutive odd integers. Their sum is less than or equal to 28.
Unknown items: The numbers in question.
Step 2: Assign a variable, and translate information.
If we let $x=$ the first of the two consecutive odd integers, then $x+2$ is the next consecutive one.

Step 3: Reread, and write an inequality.
Their sum is at most 28.

$$
x+(x+2) \leq 28
$$

Step 4: Solve the inequality.

$$
\begin{aligned}
& 2 x+2 \leq 28 \quad \text { Simplify the left side } \\
& 2 x \leq 26 \quad \text { Add }-2 \text { to each side } \\
& x \leq 13 \quad \text { Multiply each side by } \frac{1}{2}
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,2
B 3-5
C ${ }^{6-9}$
D 1-8
E $\quad 10$

## Step 5: Write the answer.

The first of the two integers must be an odd integer that is less than or equal to 13 . The second of the two integers will be two more than whatever the first one is.

## Step 6: Reread, and check.

Suppose the first integer is 13 . The next consecutive odd integer is 15 . The sum of 15 and 13 is 28 . If the first odd integer is less than 13 , the sum of it and the next consecutive odd integer will be less than 28.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. State the addition property for inequalities.
2. How is the multiplication property for inequalities different from the multiplication property of equality?
3. When do we reverse the direction of an inequality symbol?
4. Under what conditions do we not change the direction of the inequality symbol when we multiply both sides of an inequality by a number?

## Problem Set 2.8

Solve the following inequalities using the addition property of inequalities. Graph each solution set.

1. $x-5<7$
2. $x+3<-5$
3. $a-4 \leq 8$
4. $a+3 \leq 10$
5. $x-4.3>8.7$
6. $x-2.6>10.4$
7. $y+6 \geq 10$
8. $y+3 \geq 12$
9. $2<x-7$
10. $3<x+8$

Solve the following inequalities using the multiplication property of inequalities. If you multiply both sides by a negative number, be sure to reverse the direction of the inequality symbol. Graph the solution set.
11. $3 x<6$
12. $2 x<14$
13. $5 a \leq 25$
14. $4 a \leq 16$
15. $\frac{x}{3}>5$
16. $\frac{x}{7}>1$
17. $-2 x>6$
18. $-3 x \geq 9$
19. $-3 x \geq-18$
20. $-8 x \geq-24$
21. $-\frac{x}{5} \leq 10$
22. $-\frac{x}{9} \geq-1$
23. $-\frac{2}{3} y>4$
24. $-\frac{3}{4} y>6$

Solve the following inequalities. Graph the solution set in each case.
25. $2 x-3<9$
26. $3 x-4<17$
27. $-\frac{1}{5} y-\frac{1}{3} \leq \frac{2}{3}$
28. $-\frac{1}{6} y-\frac{1}{2} \leq \frac{2}{3}$
29. $-7.2 x+1.8>-19.8$
= Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
30. $-7.8 x-1.3>22.1$
31. $\frac{2}{3} x-5 \leq 7$
32. $\frac{3}{4} x-8 \leq 1$
33. $-\frac{2}{5} a-3>5$
34. $-\frac{4}{5} a-2>10$
35. $5-\frac{3}{5} y>-10$
36. $4-\frac{5}{6} y>-11$
37. $0.3(a+1) \leq 1.2$
38. $0.4(a-2) \leq 0.4$
39. $2(5-2 x) \leq-20$
40. $7(8-2 x)>28$
41. $3 x-5>8 x$
42. $8 x-4>6 x$
43. $\frac{1}{3} y-\frac{1}{2} \leq \frac{5}{6} y+\frac{1}{2}$
44. $\frac{7}{6} y+\frac{4}{3} \leq \frac{11}{6} y-\frac{7}{6}$
45. $-2.8 x+8.4<-14 x-2.8$
46. $-7.2 x-2.4<-2.4 x+12$
47. $3(m-2)-4 \geq 7 m+14$
48. $2(3 m-1)+5 \geq 8 m-7$
49. $3-4(x-2) \leq-5 x+6$
50. $8-6(x-3) \leq-4 x+12$

Solve each of the following inequalities for $y$.
51. $3 x+2 y<6$
52. $-3 x+2 y<6$
53. $2 x-5 y>10$
54. $-2 x-5 y>5$
55. $-3 x+7 y \leq 21$
56. $-7 x+3 y \leq 21$
57. $2 x-4 y \geq-4$
58. $4 x-2 y \geq-8$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working.
59. Work each problem according to the instructions given.
a. Evaluate when $x=0:-5 x+3$
b. Solve: $-5 x+3=-7$
c. Is 0 a solution to $-5 x+3<-7$
d. Solve: $-5 x+3<-7$
60. Work each problem according to the instructions given.
a. Evaluate when $x=0:-2 x-5$
b. Solve: $-2 x-5=1$
c. Is 0 a solution to $-2 x-5>1$
d. Solve: $-2 x-5>1$

For each graph below, write an inequality whose solution is the graph.
61.

62.

63.

64.


## Applying the Concepts

65. Consecutive Integers The sum of two consecutive integers is at least 583. What are the possibilities for the first of the two integers?
66. Consecutive Integers The sum of two consecutive integers is at most 583. What are the possibilities for the first of the two integers?
67. Number Problems The sum of twice a number and six is less than ten. Find all solutions.
68. Number Problems Twice the difference of a number and three is greater than or equal to the number increased by five. Find all solutions.
69. Number Problems The product of a number and four is greater than the number minus eight. Find the solution set.
70. Number Problems The quotient of a number and five is less than the sum of seven and two. Find the solution set.
71. Geometry Problems The length of a rectangle is 3 times the width. If the perimeter is to be at least 48 meters, what are the possible values for the width? (If the perimeter is at least 48 meters, then it is greater than or equal to 48 meters.)
72. Geometry Problems The length of a rectangle is 3 more than twice the width. If the perimeter is to be at least 51 meters, what are the possible values for the width? (If the perimeter is at least 51 meters, then it is greater than or equal to 51 meters.)
73. Geometry Problems The numerical values of the three sides of a triangle are given by three consecutive even integers. If the perimeter is greater than 24 inches, what are the possibilities for the shortest side?
74. Geometry Problems The numerical values of the three sides of a triangle are given by three consecutive odd integers. If the perimeter is greater than 27 inches, what are the possibilities for the shortest side?
75. Car Heaters If you have ever gotten in a cold car early in the morning you know that the heater does not work until the engine warms up. This is because the heater relies on the heat coming off the engine. Write an equation using an inequality sign to express when the heater will work if the heater works only after the engine is $100^{\circ} \mathrm{F}$.
76. Exercise When Kate exercises, she either swims or runs. She wants to spend a minimum of 8 hours a week exercising, and she wants to swim 3 times the amount she runs. What is the minimum amount of time she must spend doing each exercise?
77. Profit and Loss Movie theaters pay a certain price for the movies that you and I see. Suppose a theater pays $\$ 1,500$ for each showing of a popular movie. If they charge $\$ 7.50$ for each ticket they sell, then they will lose money if ticket sales are less than $\$ 1,500$. However, they will make a profit if ticket sales are greater than $\$ 1,500$. What is the range of tickets they can sell and still lose money? What is the range of tickets they can sell and make a profit?
78. Stock Sales Suppose you purchase $x$ shares of a stock at $\$ 12$ per share. After 6 months you decide to sell all your shares at $\$ 20$ per share. Your broker charges you $\$ 15$ for the trade. If your profit is at least $\$ 3,985$, how many shares did you purchase in the first place?

## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in previous sections and chapters. You can consider the time you spend working these problems as time spent studying for exams.

Apply the distributive property, then simplify.
79. $\frac{1}{6}(12 x+6)$
80. $\frac{3}{5}(15 x-10)$
81. $\frac{2}{3}(-3 x-6)$
82. $\frac{3}{4}(-4 x-12)$
83. $3\left(\frac{5}{6} a+\frac{4}{9}\right)$
84. $2\left(\frac{3}{4} a-\frac{5}{6}\right)$
85. $-3\left(\frac{2}{3} a+\frac{5}{6}\right)$
86. $-4\left(\frac{5}{6} a+\frac{4}{9}\right)$

Apply the distributive property, then find the LCD and simplify.
87. $\frac{1}{2} x+\frac{1}{6} x$
88. $\frac{1}{2} x-\frac{3}{4} x$
89. $\frac{2}{3} x-\frac{5}{6} x$
90. $\frac{1}{3} x+\frac{3}{5} x$
91. $\frac{3}{4} x+\frac{1}{6} x$
92. $\frac{3}{2} x-\frac{2}{3} x$
93. $\frac{2}{5} x+\frac{5}{8} x$
94. $\frac{3}{5} x-\frac{3}{8} x$

## Getting Ready for the Next Section

Solve each inequality. Do not graph.
95. $2 x-1 \geq 3$
96. $3 x+1 \geq 7$
97. $-2 x>-8$
98. $-3 x>-12$
99. $-3 \leq 4 x+1$
100. $4 x+1 \leq 9$

## Chapter 2 <br> SUMMARY

## Similar Terms [2.1]

A term is a number or a number and one or more variables multiplied together. Similar terms are terms with the same variable part.

## Simplifying Expressions [2.1]

In this chapter we simplified expressions that contained variables by using the distributive property to combine similar terms.

## Solution Set [2.2]

The solution set for an equation (or inequality) is all the numbers that, when used in place of the variable, make the equation (or inequality) a true statement.

## Equivalent Equations [2.2]

4. The equation $a-4=3$ and $a-2=5$ are equivalent because both have solution set \{7\}.

Two equations are called equivalent if they have the same solution set.

## Addition Property of Equality [2.2]

5. Solve $x-5=12$

$$
\begin{aligned}
x-5+\mathbf{5} & =12+\mathbf{5} \\
x+0 & =17 \\
x & =17
\end{aligned}
$$

When the same quantity is added to both sides of an equation, the solution set for the equation is unchanged. Adding the same amount to both sides of an equation produces an equivalent equation.

## Multiplication Property of Equality [2.3]

If both sides of an equation are multiplied by the same nonzero number, the solution set is unchanged. Multiplying both sides of an equation by a nonzero quantity produces an equivalent equation.

## Strategy for Solving Linear Equations in One Variable [2.4]

7. Solve $2(x+3)=10$

$$
2 x+6=10
$$

$$
2 x+6+(-\mathbf{6})=10+(-\mathbf{6})
$$

$$
2 x=4
$$

$$
\frac{1}{2}(2 x)=\frac{1}{2}(4)
$$

Step 1a: Use the distributive property to separate terms, if necessary.
1b: If fractions are present, consider multiplying both sides by the LCD to eliminate the fractions. If decimals are present, consider multiplying both sides by a power of 10 to clear the equation of decimals.
1c: Combine similar terms on each side of the equation.

$$
x=2
$$

Step 2: Use the addition property of equality to get all variable terms on one side of the equation and all constant terms on the other side. A variable term is a term that contains the variable (for example, $5 x$ ). A constant term is a term that does not contain the variable (the number 3, for example.)
Step 3: Use the multiplication property of equality to get $x$ (that is, $1 x$ ) by itself on one side of the equation.
Step 4: Check your solution in the original equation to be sure that you have not made a mistake in the solution process.

## Formulas [2.5]

8. Solving $P=2 l+2 w$ for $l$, we have

$$
\begin{aligned}
P-2 w & =2 l \\
\frac{P-2 w}{2} & =1
\end{aligned}
$$

A formula is an equation with more than one variable. To solve a formula for one of its variables, we use the addition and multiplication properties of equality to move everything except the variable in question to one side of the equal sign so the variable in question is alone on the other side.

## Blueprint for Problem Solving [2.6, 2.7]

Step 1: Read the problem, and then mentally list the items that are known and the items that are unknown.
Step 2: Assign a variable to one of the unknown items. (In most cases this will amount to letting $x=$ the item that is asked for in the problem.) Then translate the other information in the problem to expressions involving the variable.
Step 3: Reread the problem, and then write an equation, using the items and variables listed in steps 1 and 2, that describes the situation.
Step 4: Solve the equation found in step 3.
Step 5: Write your answer using a complete sentence.
Step 6: Reread the problem, and check your solution with the original words in the problem.

## Addition Property for Inequalities [2.8]

9. Solve $x+5<7$.
$x+5+(-5)<7+(-5)$

$$
x<2
$$

10. Solve $-3 a \leq 18$.

$$
\begin{aligned}
-\frac{1}{3}(-3 a) & \geq-\frac{1}{3}(18) \\
a & \geq-6
\end{aligned}
$$

## Multiplication Property for Inequalities [2.8]

Multiplying both sides of an inequality by a positive number never changes the solution set. If both sides are multiplied by a negative number, the sense of the inequality must be reversed to produce an equivalent inequality.
11. Solve $3(x-4) \geq-2$.
$3 x-12 \geq-2$
$3 x-12+\mathbf{1 2} \geq-2+\mathbf{1 2}$ $3 x \geq 10$ $\frac{\mathbf{1}}{\mathbf{3}}(3 x) \geq \frac{\mathbf{1}}{\mathbf{3}}(10)$ $x \geq \frac{10}{3}$


## Strategy for Solving Linear Inequalities in One Variable [2.8]

Step 1a: Use the distributive property to separate terms, if necessary.
$\mathbf{1 b}$ : If fractions are present, consider multiplying both sides by the LCD to eliminate the fractions. If decimals are present, consider multiplying both sides by a power of 10 to clear the inequality of decimals.
1c: Combine similar terms on each side of the inequality.
Step 2: Use the addition property for inequalities to get all variable terms on one side of the inequality and all constant terms on the other side.
Step 3: Use the multiplication property for inequalities to get $x$ by itself on one side of the inequality.
Step 4: Graph the solution set.

## COMMON MISTAKES

1. Trying to subtract away coefficients (the number in front of variables) when solving equations. For example:

$$
\begin{aligned}
4 x & =12 \\
4 x-\mathbf{4} & =12-\mathbf{4} \\
x & =8 \leftarrow \text { Mistake }
\end{aligned}
$$

It is not incorrect to add ( -4 ) to both sides; it's just that $4 x-4$ is not equal to $x$. Both sides should be multiplied by $\frac{1}{4}$ to solve for $x$.
2. Forgetting to reverse the direction of the inequality symbol when multiplying both sides of an inequality by a negative number. For instance:

$$
\begin{aligned}
-3 x & <12 \\
-\frac{\mathbf{1}}{\mathbf{3}}(-3 x) & <-\frac{\mathbf{1}}{\mathbf{3}}(12) \leftarrow \text { Mistake } \\
x & <-4
\end{aligned}
$$

It is not incorrect to multiply both sides by $-\frac{1}{3}$. But if we do, we must also reverse the sense of the inequality.

## Chapter 2 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

The numbers in brackets refer to the sections of the text in which similar problems can be found.

Simplify each expression as much as possible. [2.1]

1. $5 x-8 x$
2. $6 x-3-8 x$
3. $-a+2+5 a-9$
4. $5(2 a-1)-4(3 a-2)$
5. $6-2(3 y+1)-4$
6. $4-2(3 x-1)-5$

Find the value of each expression when $x$ is 3. [2.1]
7. $7 x-2$
8. $-4 x-5+2 x$
9. $-x-2 x-3 x$

Find the value of each expression when $x$ is -2 . [2.1]
10. $5 x-3$
11. $-3 x+2$
12. $7-x-3$

Solve each equation. [2.2, 2.3]
13. $x+2=-6$
14. $x-\frac{1}{2}=\frac{4}{7}$
15. $10-3 y+4 y=12$
16. $-3-4=-y-2+2 y$
17. $2 x=-10$
18. $3 x=0$
19. $\frac{x}{3}=4$
20. $-\frac{x}{4}=2$
21. $3 a-2=5 a$
22. $\frac{7}{10} a=\frac{1}{5} a+\frac{1}{2}$
23. $3 x+2=5 x-8$
24. $6 x-3=x+7$
25. $0.7 x-0.1=0.5 x-0.1$
26. $0.2 x-0.3=0.8 x-0.3$

Solve each equation. Be sure to simplify each side first. [2.4]
27. $2(x-5)=10$
29. $\frac{1}{2}(3 t-2)+\frac{1}{2}=\frac{5}{2}$
28. $12=2(5 x-4)$
30. $\frac{3}{5}(5 x-10)=\frac{2}{3}(9 x+3)$
31. $2(3 x+7)=4(5 x-1)+18$
32. $7-3(y+4)=10$

Use the formula $4 x-5 y=20$ to find $y$ if [2.5]
33. $x$ is 5
34. $x$ is 0
35. $x$ is -5
36. $x$ is 10

Solve each of the following formulas for the indicated variable. [2.5]
37. $2 x-5 y=10$ for $y$
38. $5 x-2 y=10$ for $y$
39. $V=\pi r^{2} h$ for $h$
40. $P=2 l+2 w$ for $w$
41. What number is $86 \%$ of 240 ? [2.5]
42. What percent of 2,000 is 180 ? [2.5]

Solve each of the following word problems. In each case, be sure to show the equation that describes the situation. [2.6, 2.7]
43. Number Problem The sum of twice a number and 6 is 28. Find the number.
44. Geometry The length of a rectangle is 5 times as long as the width. If the perimeter is 60 meters, find the length and the width.
45. Investing A man invests a certain amount of money in an account that pays 9\% annual interest. He invests \$300 more than that in an account that pays $10 \%$ annual interest. If his total interest after a year is $\$ 125$, how much does he have invested in each account?
46. Coin Problem A collection of 15 coins is worth $\$ 1.00$. If the coins are dimes and nickels, how many of each coin are there?

Solve each inequality. [2.8]
47. $-2 x<4$
48. $-5 x>-10$
49. $-\frac{a}{2} \leq-3$
50. $-\frac{a}{3}>5$

Solve each inequality, and graph the solution. [2.8]
51. $-4 x+5>37$
52. $2 x+10<5 x-11$
53. $2(3 t+1)+6 \geq 5(2 t+4)$

## Chapter 2 Projects



## Linear Equations and Inequalities

## GROUP PROJECT Tables and Graphs

Number of People 2-3

Time Needed 5-10 minutes

Equipment Pencil and graph paper
Background Building tables is a method of visualizing information. We can build a table from a situation (as below) or from an equation. In this project, we will first build a table and then write an equation from the information in the table.

Procedure A parking meter, which accepts only dimes and quarters, is emptied at the end of each day. The amount of money in the meter at the end of one particular day is $\$ 3.15$.

1. Complete the following table so that all possible combinations of dimes and quarters, along with the total number of coins, is shown. Remember, although the number of coins will vary, the value of the dimes and quarters must total $\$ 3.15$.
2. From the information in the table, answer the following questions.
a. What is the maximum possible number of coins taken from the meter?
b. What is the minimum possible number of coins taken from the meter?
c. When is the number of dimes equal to the number of quarters?
3. Let $x=$ the number of dimes and $y=$ the number of quarters. Write an equation in two variables such that the value of the dimes added to the value of the quarters is $\$ 3.15$.

## RESEARCH PROJECT

## Stand and Deliver



T.
he 1988 film Stand and Deliver starring Edward James Olmos and Lou Diamond Phillips is based on a true story. Olmos, in his portrayal of high-school math teacher Jaime Escalante, earned an Academy Award nomination for best actor.

Watch the movie Stand and Deliver. After briefly describing the movie, explain how Escalante's students became successful in math. Make a list of specific things you observe that the students had to do to become successful. Indicate which items on this list you think will also help you become successful.

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## Graphing and Systems of Equations

## 3



When light comes into contact with a surface that does not transmit light, then all the light that contacts the surface is either reflected off the surface or absorbed into the surface. If we let $R$ represent the percentage of light reflected and $A$ represent the percentage of light absorbed, then the relationship between these two variables can be written as

$$
R+A=100
$$

which is a linear equation in two variables. The following table and graph show the same relationship as that described by the equation. The table is a numerical description; the graph is a visual description.

| Reflected and Absorbed Light <br> Percent <br> Reflected | Percent <br> Absorbed |
| :---: | :---: |
| 0 | 100 |
| 20 | 80 |
| 40 | 60 |
| 60 | 40 |
| 80 | 20 |
| 100 | 0 |



In this chapter we learn how to build tables and draw graphs from linear equations in two variables.

## CHAPTER OUTLINE

3.1 Paired Data and Graphing Ordered Pairs
3.2 Solutions to Linear Equations in Two Variables
3.3 Graphing Linear Equations in Two Variables
3.4 More on Graphing: Intercepts
3.5 Slope and the Equation of a Line
3.6 Solving Linear Systems by Graphing
3.7 The Elimination Method
3.8 The Substitution Method
3.9 Applications

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Try to arrange your daily study habits so that you have very little studying to do the night before your next exam. The next two goals will help you achieve this.

## 1 Review with the Exam in Mind

Each day you should review material that will be covered on the next exam. Your review should consist of working problems. Preferably, the problems you work should be problems from your list of difficult problems.

## 2 Pay Attention to Instructions

Each of the following is a valid instruction with respect to the equation $y=3 x-2$, and the result of applying the instructions will be different in each case:

$$
\begin{array}{ll}
\text { Find } x \text { when } y \text { is } 10 . & \text { (Section 2.5) } \\
\text { Solve for } x . & \text { (Section 2.5) } \\
\text { Graph the equation. } & \text { (Section 3.3) } \\
\text { Find the intercepts. } & \text { (Section 3.4) }
\end{array}
$$

There are many things to do with the equation $y=3 x-2$. If you train yourself to pay attention to the instructions that accompany a problem as you work through the assigned problems, you will not find yourself confused about what to do with a problem when you see it on a test.

## Paired Data and Graphing Ordered Pairs

## OBJECTIVES

A Create a bar chart, scatter diagram, or line graph from a table of data.

```
B Graph ordered pairs on a rectangular coordinate system.
```

TABLE 1
Speed of a Race Car

| Time in <br> Seconds | Speed in <br> Miles per Hour |
| :---: | :---: |
| 0 | 0 |
| 1 | 72.7 |
| 2 | 129.9 |
| 3 | 162.8 |
| 4 | 192.2 |
| 5 | 212.4 |
| 6 | 228.1 |



FIGURE 1

In Chapter 1 we showed the relationship between the table of values for the speed of a race car and the corresponding bar chart. Table 1 and Figure 1 from the introduction of Chapter 1 are reproduced here for reference. In Figure 1, the horizontal line that shows the elapsed time in seconds is called the horizontal axis, and the vertical line that shows the speed in miles per hour is called the vertical axis.

The data in Table 1 are called paired data because the information is organized so that each number in the first column is paired with a specific number in the second column. Each pair of numbers is associated with one of the solid bars in Figure 1. For example, the third bar in the bar chart is associated with the pair of numbers 3 seconds and 162.8 miles per hour. The first number, 3 seconds, is associated with the horizontal axis, and the second number, 162.8 miles per hour, is associated with the vertical axis.

## Scatter Diagrams and Line Graphs

The information in Table 1 can be visualized with a scatter diagram and line graph as well. Figure 2 is a scatter diagram of the information in Table 1. We use dots instead of the bars shown in Figure 1 to show the speed of the race car at each second during the race. Figure 3 is called a line graph. It is constructed by taking the dots in Figure 2 and connecting each one to the next with a straight line. Notice that we have labeled the axes in these two figures a little differently than we did with the bar chart by making the axes intersect at the number 0 .


FIGURE 2


FIGURE 3

The number sequences we have worked with in the past can also be written as paired data by associating each number in the sequence with its position in the sequence. For instance, in the sequence of odd numbers

$$
1,3,5,7,9, \ldots
$$

the number 7 is the fourth number in the sequence. Its position is 4 , and its value is 7 . Here is the sequence of odd numbers written so that the position of each term is noted:

Position $\quad 1,2,3,4,5, \ldots$
Value $\quad 1,3,5,7,9, \ldots$

EXAMPLE 1 Tables 2 and 3 give the first five terms of the sequence of odd numbers and the sequence of squares as paired data. In each case construct a scatter diagram.

TABLE 2

| Odd Numbers <br> Position |  |
| :---: | :---: |
| 1 | Value |
| 2 | 1 |
| 3 | 3 |
| 4 | 5 |
| 5 | 7 |
|  | 9 |

TABLE 3

| Squares <br> Position | Value |
| :---: | :---: |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| 4 | 16 |
| 5 | 25 |

SOLUTION The two scatter diagrams are based on the data from Tables 2 and 3 shown here. Notice how the dots in Figure 4 seem to line up in a straight line, whereas the dots in Figure 5 give the impression of a curve. We say the points in Figure 4 suggest a linear relationship between the two sets of data, whereas the points in Figure 5 suggest a nonlinear relationship.


As you know, each dot in Figures 4 and 5 corresponds to a pair of numbers, one of which is associated with the horizontal axis and the other with the vertical axis. Paired data play a very important role in the equations we will solve in the next section. To prepare ourselves for those equations, we need to expand the concept of paired data to include negative numbers. At the same time, we want to standardize the position of the axes in the diagrams that we use to visualize paired data.

DEFINITION A pair of numbers enclosed in parentheses and separated by a comma, such as $(-2,1)$, is called an ordered pair of numbers. The first number in the pair is called the $x$-coordinate of the ordered pair; the second number is called the $y$-coordinate. For the ordered pair $(-2,1)$, the $x$-coordinate is -2 and the $y$-coordinate is 1 .

Ordered pairs of numbers are important in the study of mathematics because they give us a way to visualize solutions to equations. To see the visual compo-
nent of ordered pairs, we need the diagram shown in Figure 6. It is called the rectangular coordinate system.


FIGURE 6
The rectangular coordinate system is built from two number lines oriented perpendicular to each other. The horizontal number line is exactly the same as our real number line and is called the $x$-axis. The vertical number line is also the same as our real number line with the positive direction up and the negative direction down. It is called the $y$-axis. The point where the two axes intersect is called the origin. As you can see from Figure 6, the axes divide the plane into four quadrants, which are numbered I through IV in a counterclockwise direction.

## Graphing Ordered Pairs

To graph the ordered pair $(a, b)$, we start at the origin and move $a$ units forward or back (forward if $a$ is positive and back if $a$ is negative). Then we move $b$ units up or down (up if $b$ is positive, down if $b$ is negative). The point where we end up is the graph of the ordered pair $(a, b)$.

EXAMPLE 2 Graph the ordered pairs $(3,4),(3,-4),(-3,4)$, and $(-3,-4)$.

## SOLUTION

## Note

It is very important that you graph ordered pairs quickly and accurately. Remember, the first coordinate goes with the horizontal axis and the second coordinate goes with the vertical axis.


FIGURE 7

## Note

If we do not label the axes of a coordinate system, we assume that each square is one unit long and one unit wide.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1
B 2,3

We can see in Figure 7 that when we graph ordered pairs, the $x$-coordinate corresponds to movement parallel to the $x$-axis (horizontal) and the $y$-coordinate corresponds to movement parallel to the $y$-axis (vertical).

## EXAMPLE 3

$(4,0)$.
SOLUTION See Figure 8.


FIGURE 8

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is an ordered pair of numbers?
2. Explain in words how you would graph the ordered pair $(3,4)$.
3. How do you construct a rectangular coordinate system?
4. Where is the origin on a rectangular coordinate system?

## Problem Set 3.1

Graph the following ordered pairs.

1. $(3,2)$
2. $(-3,2)$
3. $(5,1)$
4. $(5,-1)$
5. $(1,5)$
6. $(1,-5)$
7. $(-1,5)$
8. $\left(2, \frac{1}{2}\right)$
9. $\left(-4,-\frac{5}{2}\right)$
10. $(3,0)$
11. $(0,5)$
12. $(-1,-5)$
13. $\left(3, \frac{3}{2}\right)$
14. $\left(-5,-\frac{3}{2}\right)$
15. $(-2,0)$
16. $(0,0)$

Give the coordinates of each numbered point in the figure.

19.
21.
23.
25.
27.
20.
22.
.
24.
26.
28.

Graph the points $(4,3)$ and $(-4,-1)$, and draw a straight line that passes through both of them. Then answer the following questions.
29. Does the graph of $(2,2)$ lie on the line?
30. Does the graph of $(-2,0)$ lie on the line?
31. Does the graph of $(0,-2)$ lie on the line?
32. Does the graph of $(-6,2)$ lie on the line?

Graph the points $(-2,4)$ and $(2,-4)$, and draw a straight line that passes through both of them. Then answer the following questions.
33. Does the graph of $(0,0)$ lie on the line?
34. Does the graph of $(-1,2)$ lie on the line?
35. Does the graph of $(2,-1)$ lie on the line?
36. Does the graph of $(1,-2)$ lie on the line?

Draw a straight line that passes through the points $(3,4)$ and $(3,-4)$. Then answer the following questions.
37. Is the graph of $(3,0)$ on this line?
38. Is the graph of $(0,3)$ on this line?
39. Is there any point on this line with an $x$-coordinate other than 3 ?
40. If you extended the line, would it pass through a point with a $y$-coordinate of 10 ?

Draw a straight line that passes through the points $(3,4)$ and $(-3,4)$. Then answer the following questions.
41. Is the graph of $(4,0)$ on this line?
42. Is the graph of $(0,4)$ on this line?
43. Is there any point on this line with a $y$-coordinate other than 4 ?
44. If you extended the line, would it pass through a point with an $x$-coordinate of 10 ?

## Applying the Concepts

45. Hourly Wages Jane takes a job at the local Marcy's department store. Her job pays $\$ 8.00$ per hour. The graph shows how much Jane earns for working from 0 to 40 hours in a week.

a. List three ordered pairs that lie on the line graph.
b. How much will she earn for working 40 hours?
c. If her check for one week is $\$ 240$, how many hours did she work?
d. She works 35 hours one week, but her paycheck before deductions are subtracted out is for $\$ 260$. Is this correct? Explain.
46. Hourly Wages Judy takes a job at Gigi's boutique. Her job pays $\$ 6.00$ per hour plus $\$ 50$ per week in commission. The graph shows how much Judy earns for working from 0 to 40 hours in a week.

a. List three ordered pairs that lie on the line graph.
b. How much will she earn for working 40 hours?
c. If her check for one week is $\$ 230$, how many hours did she work?
d. She works 35 hours one week, but her paycheck before deductions are subtracted out is for $\$ 260$. Is this correct? Explain.
47. Garbage Production The table and bar chart shown here give the annual production of garbage in the United States for some specific years. Use the information from the table and bar chart to construct a line graph.

48. Grass Height The table and bar chart from Problem 99 in Problem Set 1.4 are shown here. Each gives the growth of a certain species of grass over time. Use the information from the table and chart to construct a line graph.

49. Wireless Phone Costs The table and bar chart shown here provide the projected cost of wireless phone use. Use the information from the table and chart to construct a line graph.

| Year | Cents per <br> Minute |
| :---: | :---: |
| 1998 | 33 |
| 1999 | 28 |
| 2000 | 25 |
| 2001 | 23 |
| 2002 | 22 |
| 2003 | 20 |


50. Digital Camera Sales The table and bar chart shown here show the sales of digital cameras from 1996 to 1999. Use the information from the table and chart to construct a line graph.

51. Kentucky Derby The bar chart gives the monetary bets placed at the Kentucky Derby for specific years. If $x$ represents the year in question and $y$ represents the total wagering for that year (in millions), write six ordered pairs that describe the information in the table.

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52. Improving Your Quantitative Literacy Use the chart to answer the following questions.
a. Are the expenditures for 2003 more or less than \$6,000?
b. Are the projected expenditures for 2009 more or less than $\$ 8,000$ ?

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53. Right triangle $A B C$ has legs of length 5. Point $C$ is the ordered pair $(6,2)$. Find the coordinates of $A$ and $B$.

54. Right triangle $A B C$ has legs of length 7. Point $C$ is the ordered pair $(-8,-3)$. Find the coordinates of $A$ and $B$.

55. Rectangle $A B C D$ has a length of 5 and a width of 3 . Point $D$ is the ordered pair $(7,2)$. Find points $A, B$, and $C$.

56. Rectangle $A B C D$ has a length of 5 and a width of 3 . Point $D$ is the ordered pair $(-1,1)$. Find points $A, B$, and $C$.


## Maintaining Your Skills

Add or subtract as indicated.
57. $\frac{x}{5}+\frac{3}{5}$
58. $\frac{x}{5}+\frac{3}{4}$
59. $\frac{2}{7}-\frac{a}{7}$
60. $\frac{2}{7}-\frac{a}{5}$
61. $\frac{1}{14}-\frac{y}{7}$
62. $\frac{3}{4}+\frac{x}{5}$
63. $\frac{1}{2}+\frac{3}{x}$
64. $\frac{2}{3}-\frac{6}{y}$
65. $\frac{5+x}{6}-\frac{5}{6}$
66. $\frac{3-x}{3}+\frac{2}{3}$
67. $\frac{4}{x}+\frac{1}{2}$
68. $\frac{3}{y}+\frac{2}{3}$

## Getting Ready for the Next Section

69. Let $2 x+3 y=6$.
a. Find $x$ if $y=4$.
b. Find $x$ if $y=-2$.
c. Find $y$ if $x=3$.
d. Find $y$ if $x=9$.
70. Let $2 x-5 y=20$.
a. Find $x$ if $y=0$.
b. Find $x$ if $y=-6$.
c. Find $y$ if $x=0$.
d. Find $y$ if $x=5$.
71. Let $y=2 x-1$.
a. Find $x$ if $y=7$.
b. Find $x$ if $y=3$.
c. Find $y$ if $x=0$.
d. Find $y$ if $x=5$.
72. Let $y=3 x-2$.
a. Find $x$ if $y=4$.
b. Find $x$ if $y=7$.
c. Find $y$ if $x=0$.
d. Find $y$ if $x=-3$.


## OBJECTIVES

A
Find solutions to linear equations in two variables.
B Decide if an ordered pair is a solution to a linear equation in two variables.

## Note

If this discussion seems a little long and confusing, you may want to look over some of the examples first and then come back and read this. Remember, it isn't always easy to read material in mathematics. What is important is that you understand what you are doing when you work problems. The reading is intended to assist you in understanding what you are doing. It is important to read everything in the book, but you don't always have to read it in the order it is written.

## Solutions to Linear Equations in Two Variables

In this section we will begin to investigate equations in two variables. As you will see, equations in two variables have pairs of numbers for solutions. Because we know how to use paired data to construct tables, histograms, and other charts, we can take our work with paired data further by using equations in two variables to construct tables of paired data. Let's begin this section by reviewing the relationship between equations in one variable and their solutions.

If we solve the equation $3 x-2=10$, the solution is $x=4$. If we graph this solution, we simply draw the real number line and place a dot at the point whose coordinate is 4 . The relationship between linear equations in one variable, their solutions, and the graphs of those solutions look like this:


When the equation has one variable, the solution is a single number whose graph is a point on a line.

Now, consider the equation $2 x+y=3$. The first thing we notice is that there are two variables instead of one. Therefore, a solution to the equation $2 x+y=3$ will be not a single number but a pair of numbers, one for $x$ and one for $y$, that makes the equation a true statement. One pair of numbers that works is $x=2$, $y=-1$ because when we substitute them for $x$ and $y$ in the equation, we get a true statement.

$$
\begin{aligned}
2(2)+(-1) & \stackrel{?}{=} 3 \\
4-1 & =3 \\
3 & =3 \quad \text { A true statement }
\end{aligned}
$$

The pair of numbers $x=2, y=-1$ is written as $(2,-1)$. As you know from Section $3.1,(2,-1)$ is called an ordered pair because it is a pair of numbers written in a specific order. The first number is always associated with the variable $x$,
and the second number is always associated with the variable $y$. We call the first number in the ordered pair the $x$-coordinate (or $x$ component) and the second number the $y$-coordinate (or $y$ component) of the ordered pair.

Let's look back to the equation $2 x+y=3$. The ordered pair $(2,-1)$ is not the only solution. Another solution is $(0,3)$ because when we substitute 0 for $x$ and 3 for $y$ we get

$$
\begin{aligned}
2(0)+3 & \stackrel{?}{=} 3 \\
0+3 & =3 \\
3 & =3 \quad \text { A true statement }
\end{aligned}
$$

Still another solution is the ordered pair $(5,-7)$ because

$$
\begin{aligned}
2(5)+(-7) & \stackrel{?}{=} 3 \\
10-7 & =3 \\
3 & =3 \quad \text { A true statement }
\end{aligned}
$$

As a matter of fact, for any number we want to use for $x$, there is another number we can use for $y$ that will make the equation a true statement. There is an infinite number of ordered pairs that satisfy (are solutions to) the equation $2 x+$ $y=3$; we have listed just a few of them.

EXAMPLE 1 Given the equation $2 x+3 y=6$, complete the following ordered pairs so they will be solutions to the equation: $(0),,(, 1),(3$,$) .$
SOLUTION To complete the ordered pair ( 0 , ), we substitute 0 for $x$ in the equation and then solve for $y$ :

$$
\begin{aligned}
2(0)+3 y & =6 \\
3 y & =6 \\
y & =2
\end{aligned}
$$

The ordered pair is $(0,2)$.
To complete the ordered pair $(, 1)$, we substitute 1 for $y$ in the equation and solve for $x$ :

$$
\begin{aligned}
2 x+3(1) & =6 \\
2 x+3 & =6 \\
2 x & =3 \\
x & =\frac{3}{2}
\end{aligned}
$$

The ordered pair is $\left(\frac{3}{2}, 1\right)$.
To complete the ordered pair (3, ), we substitute 3 for $x$ in the equation and solve for $y$ :

$$
\begin{aligned}
2(3)+3 y & =6 \\
6+3 y & =6 \\
3 y & =0 \\
y & =0
\end{aligned}
$$

The ordered pair is $(3,0)$.

Notice in each case that once we have used a number in place of one of the variables, the equation becomes a linear equation in one variable. We then use the method explained in Chapter 2 to solve for that variable.

EXAMPLE 2 Complete the following table for the equation $2 x-5 y=20$.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | :---: |
| 0 |  |
|  | 2 |
|  | 0 |
| -5 |  |

SOLUTION Filling in the table is equivalent to completing the following ordered pairs: $(0),,(, 2),(, 0),(-5$,$) . So we proceed as in Example 1$.

| When $x=0$, we have | When $y=2$, we have |  |  |
| ---: | :--- | ---: | :--- |
| $2(0)-5 y$ | $=20$ | $2 x-5(2)$ | $=20$ |
| $0-5 y$ | $=20$ | $2 x-10$ | $=20$ |
| $-5 y$ | $=20$ | $2 x$ | $=30$ |
| $y$ | $=-4$ | $x$ | $=15$ |

When $y=0$, we have
When $x=-5$, we have

$$
\begin{aligned}
2 x-5(0) & =20 \\
2 x-0 & =20 \\
2 x & =20 \\
x & =10
\end{aligned}
$$

$$
2(-5)-5 y=20
$$

$$
-10-5 y=20
$$

$$
-5 y=30
$$

$$
y=-6
$$

The completed table looks like this:

| $x$ | $y$ |
| :---: | ---: |
| 0 | -4 |
| 15 | 2 |
| 10 | 0 |
| -5 | -6 |

which is equivalent to the ordered pairs $(0,-4),(15,2),(10,0)$, and $(-5,-6)$.

EXAMPLE 3 Complete the following table for the equation $y=2 x-1$.

| $x$ | $y$ |
| :--- | :--- |
| 0 |  |
| 5 |  |
|  | 7 |
|  | 3 |

SOLUTION When $x=0$, we have

$$
\begin{aligned}
& y=2(0)-1 \\
& y=0-1 \\
& y=-1
\end{aligned}
$$

When $y=7$, we have
$7=2 x-1$
$8=2 x$
$4=x$

When $x=5$, we have
$y=2(5)-1$
$y=10-1$
$y=9$
When $y=3$, we have
$3=2 x-1$
$4=2 x$
$2=x$

The completed table is

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :--- | ---: |
| 0 | -1 |
| 5 | 9 |
| 4 | 7 |
| 2 | 3 |

which means the ordered pairs $(0,-1),(5,9),(4,7)$, and $(2,3)$ are among the solutions to the equation $y=2 x-1$.

## EXAMPLE 4 Which of the ordered pairs $(2,3),(1,5)$, and $(-2,-4)$ are

 solutions to the equation $y=3 x+2$ ?SOLUTION If an ordered pair is a solution to the equation, then it must satisfy the equation; that is, when the coordinates are used in place of the variables in the equation, the equation becomes a true statement.

$$
\begin{aligned}
& \text { Try }(2,3) \text { in } y=3 x+2 \text { : } \\
& 3 \stackrel{?}{=} 3(2)+2 \\
& 3=6+2 \\
& 3=8 \quad \text { A false statement } \\
& \operatorname{Try}(1,5) \text { in } y=3 x+2 \text { : } \\
& 5 \stackrel{?}{=} 3(1)+2 \\
& 5=3+2 \\
& 5=5 \quad \text { A true statement } \\
& \text { Try }(-2,-4) \text { in } y=3 x+2 \text { : } \\
& -4 \stackrel{?}{=} 3(-2)+2 \\
& -4=-6+2 \\
& -4=-4 \quad \text { A true statement }
\end{aligned}
$$

The ordered pairs $(1,5)$ and $(-2,-4)$ are solutions to the equation $y=3 x+2$, and $(2,3)$ is not.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## B 4

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How can you tell if an ordered pair is a solution to an equation?
2. How would you find a solution to $y=3 x-5$ ?
3. Why is $(3,2)$ not a solution to $y=3 x-5$ ?
4. How many solutions are there to an equation that contains two variables?

## Problem Set 3.2

For each equation, complete the given ordered pairs.

1. $2 x+y=6$
$(0),,(, 0),(,-6$
2. $3 x-y=5$
$(0, \quad),(1, \quad),(, 5)$
3. $3 x+4 y=12$
$(0),,(, 0),(-4$,
4. $5 x-5 y=20$
$(0, \quad),(,-2),(1, \quad)$
5. $y=4 x-3$
$(1),,(, 0),(5, \quad)$
6. $y=3 x-5$
$(, 13),(0, \quad),(-2, \quad)$
7. $y=7 x-1$
$(2, \quad),(, 6),(0, \quad)$
8. $y=8 x+2$
$(3),,(\quad, 0),(\quad,-6)$
9. $x=-5 \quad(\quad, 4),(,-3),(\quad, 0)$
10. $y=2$
$(5),,(-8),,\left(\frac{1}{2},\right)$

For each of the following equations, complete the given table.

## 11. $y=3 x$

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | :---: |
| 1 | 3 |
| -3 |  |
|  | 12 |
|  | 18 |

12. $y=-2 x$

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | :---: |
| -4 |  |
| 0 |  |
|  | 10 |
|  | 12 |

13. $y=4 x$

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | :---: |
| 0 |  |
|  | -2 |
| -3 |  |
|  | 12 |

15. $x+y=5$

| $x$ | $y$ |
| :---: | ---: |
| 2 |  |
| 3 |  |
|  | 0 |
|  | -4 |

17. $2 x-y=4$

18. $y=6 x-1$

| $\boldsymbol{x}$ | $y$ |
| :---: | :---: |
| 0 |  |
|  | -7 |
| -3 |  |
|  | 8 |

For the following equations, tell which of the given ordered pairs are solutions.
21. $2 x-5 y=10$
$(2,3),(0,-2),\left(\frac{5}{2}, 1\right)$
22. $3 x+7 y=21$
$(0,3),(7,0),(1,2)$
23. $y=7 x-2$
$(1,5),(0,-2),(-2,-16)$
24. $y=8 x-3$
$(0,3),(5,16),(1,5)$
25. $y=6 x$
$(1,6),(-2,-12),(0,0)$
26. $y=-4 x$
$(0,0),(2,4),(-3,12)$
27. $x+y=0$
$(1,1),(2,-2),(3,3)$
28. $x-y=1$
$(0,1),(0,-1),(1,2)$
29. $x=3$
$(3,0),(3,-3),(5,3)$
30. $y=-4$
$(3,-4),(-4,4),(0,-4)$

## Applying the Concepts

31. Perimeter If the perimeter of a rectangle is 30 inches, then the relationship between the length 1 and the width $w$ is given by the equation

$$
2 l+2 w=30
$$

What is the length when the width is 3 inches?
32. Perimeter The relationship between the perimeter $P$ of a square and the length of its side $s$ is given by the formula $P=4 s$. If each side of a square is 5 inches, what is the perimeter? If the perimeter of a square is 28 inches, how long is a side?
33. Janai earns $\$ 12$ per hour working as a math tutor. We can express the amount she earns each week, $y$, for working $x$ hours with the equation $y=12 x$. Indicate with a yes or no which of the following could be one of Janai's paychecks. If you answer no, explain your answer.
a. $\$ 60$ for working 5 hours
b. $\$ 100$ for working nine hours
c. $\$ 80$ for working 7 hours
d. $\$ 168$ for working 14 hours
34. Erin earns $\$ 15$ per hour working as a graphic designer. We can express the amount she earns each week, $y$, for working $x$ hours with the equation $y=15 x$. Indicate with a yes or no which of the following could be one of Erin's paychecks. If you answer no, explain your answer.
a. $\$ 75$ for working 5 hours
b. $\$ 125$ for working 9 hours.
c. $\$ 90$ for working 6 hours
d. $\$ 500$ for working 35 hours.
35. The equation $V=-45,000 t+600,000$ can be used to find the value, $V$, of a small crane at the end of $t$ years.
a. What is the value of the crane at the end of 5 years?
b. When is the crane worth $\$ 330,000$ ?
c. Is it true that the crane will be worth $\$ 150,000$ after 9 years?
d. How much did the crane cost?
36. The equation $P=-400 t+2,500$, can be used to find the price, $P$, of a notebook computer at the end of $t$ years.
a. What is the value of the notebook computer at the end of 4 years?
b. When is the notebook computer worth $\$ 1,700$ ?
c. Is it true that the notebook computer will be worth $\$ 100$ after 5 years?
d. How much did the notebook computer cost?

## Maintaining Your Skills

37. $\frac{11(-5)-17}{2(-6)}$
38. $\frac{12(-4)+15}{3(-11)}$
39. $\frac{13(-6)+18}{4(-5)}$
40. $\frac{9^{2}-6^{2}}{-9-6}$
41. $\frac{7^{2}-5^{2}}{(7-5)^{2}}$
42. $\frac{7^{2}-2^{2}}{-7-2}$
43. $\frac{-3 \cdot 4^{2}-3 \cdot 2^{4}}{-3(8)}$
44. $\frac{-4(8-13)-2(6-11)}{-5(3)+5}$

## Getting Ready for the Next Section

45. Find $y$ when $x$ is 4 in the formula $3 x+2 y=6$.
46. Find $y$ when $x$ is 0 in the formula $3 x+2 y=6$.
47. Find $y$ when $x$ is 0 in $y=-\frac{1}{3} x+2$.
48. Find $y$ when $x$ is 3 in $y=-\frac{1}{3} x+2$.
49. Find $y$ when $x$ is 2 in $y=\frac{3}{2} x-3$.
50. Find $y$ when $x$ is 4 in $y=\frac{3}{2} x-3$.
51. Solve $5 x+y=4$ for $y$.
52. Solve $-3 x+y=5$ for $y$.
53. Solve $3 x-2 y=6$ for $y$.
54. Solve $2 x-3 y=6$ for $y$.

## 3.3 <br> Graphing Linear Equations in Two Variables

## OBJECTIVES

A Graph a linear equation in two variables.

B Graph horizontal lines, vertical lines, and lines through the origin.

In this section we will use the rectangular coordinate system introduced in Section 3.1 to obtain a visual picture of all solutions to a linear equation in two variables. The process we use to obtain a visual picture of all solutions to an equation is called graphing. The picture itself is called the graph of the equation. Graphing equations is an important part of algebra. It is a powerful tool that we can use to help apply algebra to the world around us.

## EXAMPLE 1 Graph the solution set for $x+y=5$.

SOLUTION We know from the previous section that an infinite number of ordered pairs are solutions to the equation $x+y=5$. We can't possibly list them all. What we can do is list a few of them and see if there is any pattern to their graphs.

Some ordered pairs that are solutions to $x+y=5$ are $(0,5),(2,3),(3,2),(5,0)$. The graph of each is shown in Figure 1.


Now, by passing a straight line through these points we can graph the solution set for the equation $x+y=5$. Linear equations in two variables always have
graphs that are straight lines. The graph of the solution set for $x+y=5$ is shown in Figure 2.


FIGURE 2
Every ordered pair that satisfies $x+y=5$ has its graph on the line, and any point on the line has coordinates that satisfy the equation. So, there is a one-toone correspondence between points on the line and solutions to the equation.

Our ability to graph an equation as we have done in Example 1 is due to the invention of the rectangular coordinate system. The French philosopher René Descartes (1596-1650) is the person usually credited with the invention of the rectangular coordinate system. As a philosopher, Descartes is responsible for the statement "I think, therefore I am." Until Descartes invented his coordinate system in 1637, algebra and geometry were treated as separate subjects. The rectangular coordinate system allows us to connect algebra and geometry by associating geometric shapes with algebraic equations.

Here is the precise definition for a linear equation in two variables.

DEFINITION Any equation that can be put in the form $a x+b y=c$, where $a$, $b$, and $c$ are real numbers and $a$ and $b$ are not both 0 , is called a linear equation in two variables. The graph of any equation of this form is a straight line (that is why these equations are called "linear"). The form $a x+b y=c$ is called standard form.

To graph a linear equation in two variables, we simply graph its solution set; that is, we draw a line through all the points whose coordinates satisfy the equation. Here are the steps to follow.

## To Graph a Linear Equation in Two Variables

Step 1: Find any three ordered pairs that satisfy the equation. This can be done by using a convenient number for one variable and solving for the other variable.

Step 2: Graph the three ordered pairs found in step 1. Actually, we need only two points to graph a straight line. The third point serves as a check. If all three points do not line up, there is a mistake in our work.

Step 3: Draw a straight line through the three points graphed in step 2.

## Note

It may seem that we have simply picked the numbers 0,2 , and -1 out of the air and used them for $x$. In fact we have done just that. Could we have used numbers other than these? The answer is yes, we can substitute any number for $x$; there will always be a value of $y$ to go with it.

## EXAMPLE $2 \quad$ Graph the equation $y=3 x-1$.

SOLUTION Because $y=3 x-1$ can be put in the form $a x+b y=c$, it is a linear equation in two variables. Hence, the graph of its solution set is a straight line. We can find some specific solutions by substituting numbers for $x$ and then solving for the corresponding values of $y$. We are free to choose any numbers for $x$, so let's use 0,2 , and -1 .

$$
\text { Let } x=0: \quad \begin{aligned}
& y=3(0)-1 \\
& y=0-1 \\
& y=-1
\end{aligned}
$$

The ordered pair $(0,-1)$ is one solution.

$$
\text { Let } x=2: \quad \begin{aligned}
& y=3(2)-1 \\
& y=6-1 \\
& y=5
\end{aligned}
$$

In table form

| $x$ | $y$ |
| ---: | ---: |
| 0 | -1 |
| 2 | 5 |
| -1 | -4 |

The ordered pair $(2,5)$ is a second solution.

$$
\text { Let } x=-1: \quad \begin{array}{ll} 
& y=3(-1)-1 \\
& y=-3-1 \\
& y=-4
\end{array}
$$

The ordered pair $(-1,-4)$ is a third solution.
Next, we graph the ordered pairs $(0,-1),(2,5),(-1,-4)$ and draw a straight line through them.

The line we have drawn in Figure 3 is the graph of $y=3 x-1$.


FIGURE 3
Example 2 again illustrates the connection between algebra and geometry that we mentioned previously. Descartes's rectangular coordinate system allows us to associate the equation $y=3 x-1$ (an algebraic concept) with a specific straight line (a geometric concept). The study of the relationship between equations in algebra and their associated geometric figures is called analytic geometry. The rectangular coordinate system often is referred to as the Cartesian coordinate system in honor of Descartes.

## EXAMPLE $3 \quad$ Graph the equation $y=-\frac{1}{3} x+2$.

SOLUTION We need to find three ordered pairs that satisfy the equation. To do so, we can let $x$ equal any numbers we choose and find corresponding values of

## Note

In Example 3 the values of $x$ we used, $-3,0$, and 3 , are referred to as convenient values of $x$ because they are easier to work with than some other numbers. For instance, if we let $x=2$ in our original equation, we would have to add $-\frac{2}{3}$ and 2 to find the corresponding value of $y$. Not only would the arithmetic be more difficult but also the ordered pair we obtained would have a fraction for its $y$ coordinate, making it more difficult to graph accurately
$y$. But, every value of $x$ we substitute into the equation is going to be multiplied by $-\frac{1}{3}$. Let's use numbers for $x$ that are divisible by 3 , like $-3,0$, and 3 . That way, when we multiply them by $-\frac{1}{3}$, the result will be an integer.

$$
\text { Let } x=-3: \quad \begin{aligned}
& y=-\frac{1}{3}(-3)+2 \\
y & =1+2 \\
& y=3
\end{aligned}
$$

The ordered pair $(-3,3)$ is one solution.
In table form

| $x$ | $y$ |
| :---: | :---: |
| -3 | 3 |
| 0 | 2 |
| 3 | 1 |

$$
\begin{aligned}
\text { Let } x=0: & y \\
\quad & =-\frac{1}{3}(0)+2 \\
y & =0+2 \\
y & =2
\end{aligned}
$$

The ordered pair $(0,2)$ is a second solution.

$$
\begin{aligned}
& \text { Let } x=3: \quad y=-\frac{1}{3}(3)+2 \\
& y=-1+2 \\
& y=1
\end{aligned}
$$

The ordered pair $(3,1)$ is a third solution.
Graphing the ordered pairs $(-3,3),(0,2)$, and $(3,1)$ and drawing a straight line through their graphs, we have the graph of the equation $y=-\frac{1}{3} x+2$, as shown in Figure 4.


FIGURE 4

## EXAMPLE $4 \quad$ Graph the solution set for $3 x-2 y=6$.

SOLUTION It will be easier to find convenient values of $x$ to use in the equation if we first solve the equation for $y$. To do so, we add $-3 x$ to each side, and then we multiply each side by $-\frac{1}{2}$.

$$
\begin{aligned}
3 x-2 y & =6 & & \text { Original equation } \\
-2 y & =-3 x+6 & & \text { Add }-3 x \text { to each side }
\end{aligned}
$$

$$
\begin{aligned}
-\frac{\mathbf{1}}{\mathbf{2}}(-2 y) & =-\frac{\mathbf{1}}{\mathbf{2}}(-3 x+6) & & \text { Multiply each side by }-\frac{1}{2} \\
y & =\frac{3}{2} x-3 & & \text { Simplify each side }
\end{aligned}
$$

Now, because each value of $x$ will be multiplied by $\frac{3}{2}$, it will be to our advantage to choose values of $x$ that are divisible by 2 . That way, we will obtain values of $y$ that do not contain fractions. This time, let's use 0,2 , and 4 for $x$.

$$
\begin{aligned}
& \text { When } x=0: \quad \begin{aligned}
& y=\frac{3}{2}(0)-3 \\
& y=0-3 \\
& y=-3 \quad(0,-3) \text { is one solution } \\
& \text { When } x=2: \quad y \\
&=\frac{3}{2}(2)-3 \\
& y=3-3 \\
& y=0 \quad(2,0) \text { is a second solution } \\
& \text { When } x=4: \\
& y=\frac{3}{2}(4)-3 \\
& y=6-3 \\
& y=3 \quad(4,3) \text { is a third solution }
\end{aligned}
\end{aligned}
$$

Graphing the ordered pairs $(0,-3),(2,0)$, and $(4,3)$ and drawing a line through them, we have the graph shown in Figure 5.

## Note

After reading through Example 4, many students ask why we didn't use -2 for $x$ when we were finding ordered pairs that were solutions to the original equation. The answer is, we could have. If we were to let $x=-2$, the corresponding value of $y$ would have been -6 . As you can see by looking at the graph in Figure 5, the ordered pair $(-2,-6)$ is on the graph.


FIGURE 5

EXAMPLE 5 Graph each of the following lines.
a. $y=\frac{1}{2} x$
b. $x=3$
c. $y=-2$

## SOLUTION

a. The line $y=\frac{1}{2} x$ passes through the origin because $(0,0)$ satisfies the equation. To sketch the graph we need at least one more point on the line. When $x$ is 2 , we obtain the
point $(2,1)$, and when $x$ is -4 , we obtain the point $(-4,-2)$. The graph of $y=\frac{1}{2} x$ is shown in Figure 6a.
b. The line $x=3$ is the set of all points whose $x$-coordinate is 3. The variable $y$ does not appear in the equation, so the $y$-coordinate can be any number. Note that we can write our equation as a linear equation in two variables by writing it as $x+0 y=3$. Because the product of 0 and $y$ will always be $0, y$ can be any number. The graph of $x=3$ is the vertical line shown in Figure 6b.
c. The line $y=-2$ is the set of all points whose $y$-coordinate is -2 . The variable $x$ does not appear in the equation, so the $x$-coordinate can be any number. Again, we can write our equation as a linear equation in two variables by writing it as $0 x+y=-2$. Because the product of 0 and $x$ will always be $0, x$ can be any number. The graph of $y=-2$ is the horizontal line shown in Figure 6c.


## FACTS FROM GEOMETRY

## Special Equations and Their Graphs

For the equations below, $m, a$, and $b$ are real numbers.

Through the Origin


FIGURE 7A Any equation of the form $y=m x$ has a graph that passes through the origin.

Vertical Line


FIGURE 7B Any equation of the form $x=a$ has a vertical line for its graph.

Horizontal Line


FIGURE 7C Any equation of the form $y=b$ has a horizontal line for its graph.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-4

B 5

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Explain how you would go about graphing the line $x+y=5$.
2. When graphing straight lines, why is it a good idea to find three points, when every straight line is determined by only two points?
3. What kind of equations have vertical lines for graphs?
4. What kind of equations have horizontal lines for graphs?

## Problem Set 3.3

For the following equations, complete the given ordered pairs, and use the results to graph the solution set for the equation.

1. $x+y=4$
$(0),,(2),,(, 0)$
2. $x-y=3$
$(0, \quad),(2, \quad),(, 0)$
3. $x+y=3$
$(0),,(2),,(,-1)$
4. $x-y=4$
$(1, \quad),(-1, \quad),(, 0)$
5. $y=2 x$
$(0),,(-2, \quad),(2, \quad)$
6. $y=\frac{1}{2} x$
$(0),,(-2, \quad),(2$,
7. $y=\frac{1}{3} x$
$(-3, \quad),(0),,(3$,
8. $y=3 x$
9. $y=2 x+1$
$(0),,(-1, \quad),(1$,
10. $y=-2 x+1$
$(0),,(-1),,(1$,
11. $y=4$
12. $x=3$
$(,-2),(, 0),(, 5)$
13. $y=\frac{1}{2} x+3 \quad(-2),,(0),,(2$,
14. $y=\frac{1}{2} x-3 \quad(-2, \quad),(0, \quad),(2, \quad)$
15. $y=-\frac{2}{3} x+1 \quad(-3),,(0),,(3, \quad)$
16. $y=-\frac{2}{3} x-1 \quad(-3),,(0, \quad),(3, \quad)$

Solve each equation for $y$. Then, complete the given ordered pairs, and use them to draw the graph.
17. $2 x+y=3 \quad(-1),,(0),,(1$,
18. $3 x+y=2 \quad(-1),,(0),,(1$,
19. $3 x+2 y=6$
$(0),,(2),,(4, \quad)$
20. $2 x+3 y=6$
$(0),,(3),,(6, \quad)$
21. $-x+2 y=6$
$(-2),,(0),,(2$,
22. $-x+3 y=6$
$(-3),,(0),,(3$,
Find three solutions to each of the following equations, and then graph the solution set.
23. $y=-\frac{1}{2} x$
24. $y=-2 x$
25. $y=3 x-1$
26. $y=-3 x-1$
27. $-2 x+y=1$
28. $-3 x+y=1$
29. $3 x+4 y=8$
30. $3 x-4 y=8$
31. $x=-2$
32. $y=3$
33. $y=2$
34. $x=-3$

Graph each equation.
35. $y=\frac{3}{4} x+1$
36. $y=\frac{2}{3} x+1$
37. $y=\frac{1}{3} x+\frac{2}{3}$
38. $y=\frac{1}{2} x+\frac{1}{2}$
39. $y=\frac{2}{3} x+\frac{2}{3}$
40. $y=-\frac{3}{4} x+\frac{3}{2}$
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For each equation in each table below, indicate whether the graph is horizontal (H), or vertical (V), or whether it passes through the origin (O).
41.

| Equation $\quad \mathbf{H}, \mathbf{V}$, and/or $\mathbf{O}$ |  |
| :--- | :--- |
| $x=3$ |  |
| $y=3$ |  |
| $y=3 x$ |  |
| $y=0$ |  |

42. 

| Equation | $\mathbf{H}, \mathbf{V}$, and/or $\mathbf{O}$ |
| :--- | :--- |
| $x=\frac{1}{2}$ |  |
| $y=\frac{1}{2}$ |  |
| $y=\frac{1}{2} X$ |  |
| $x=0$ |  |

43. 

Equation $\quad \mathrm{H}, \mathrm{V}$, and/or O

$$
\begin{aligned}
& x=-\frac{3}{5} \\
& y=-\frac{3}{5} \\
& y=-\frac{3}{5} x \\
& x=0
\end{aligned}
$$

44. 

Equation $\quad \mathrm{H}, \mathrm{V}$, and/or $\mathbf{O}$

$$
\begin{aligned}
& x=-4 \\
& y=-4 \\
& y=-4 x \\
& y=0
\end{aligned}
$$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working. Working these problems is an excellent way to get ready for a test or a quiz.
45. Work each problem according to the instructions given.
a. Solve: $2 x+5=10$
b. Find $x$ when $y$ is $0: 2 x+5 y=10$
c. Find $y$ when $x$ is $0: 2 x+5 y=10$
d. Graph: $2 x+5 y=10$
e. Solve for $y: 2 x+5 y=10$
46. Work each problem according to the instructions given.
a. Solve: $x-2=6$
b. Find $x$ when $y$ is 0: $x-2 y=6$
c. Find $y$ when $x$ is 0 : $x-2 y=6$
d. Graph: $x-2 y=6$
e. Solve for $y: x-2 y=6$

## Maintaining Your Skills

Apply the distributive property.
47. $\frac{1}{2}(4 x+10)$
48. $\frac{1}{2}(6 x-12)$
49. $\frac{2}{3}(3 x-9)$
50. $\frac{1}{3}(2 x+12)$
51. $\frac{3}{4}(4 x+10)$
52. $\frac{3}{4}(8 x-6)$
53. $\frac{3}{5}(10 x+15)$
54. $\frac{2}{5}(5 x-10)$
55. $5\left(\frac{2}{5} x+10\right)$
56. $3\left(\frac{2}{3} x+5\right)$
57. $4\left(\frac{3}{2} x-7\right)$
58. $4\left(\frac{3}{4} x+5\right)$
59. $\frac{3}{4}(2 x+12 y)$
60. $\frac{3}{4}(8 x-16 y)$
61. $\frac{1}{2}(5 x-10 y)+6$
62. $\frac{1}{3}(5 x-15 y)-5$

## Getting Ready for the Next Section

63. Let $3 x-2 y=6$.
a. Find $x$ when $y=0$.
b. Find $y$ when $x=0$.
64. Let $2 x-5 y=10$.
a. Find $x$ when $y=0$.
b. Find $y$ when $x=0$.
65. Let $-x+2 y=4$.
a. Find $x$ when $y=0$.
b. Find $y$ when $x=0$.
66. Let $3 x-y=6$.
a. Find $x$ when $y=0$.
b. Find $y$ when $x=0$.
67. Let $y=-\frac{1}{3} x+2$.
a. Find $x$ when $y=0$.
b. Find $y$ when $x=0$.
68. Let $y=\frac{3}{2} x-3$.
a. Find $x$ when $y=0$.
b. Find $y$ when $x=0$.

## 3.4

## OBJECTIVES

A Find the intercepts of a line from the equation of the line.

B Use intercepts to graph a line.

## More on Graphing: Intercepts

In this section we continue our work with graphing lines by finding the points where a line crosses the axes of our coordinate system. To do so, we use the fact that any point on the $x$-axis has a $y$-coordinate of 0 and any point on the $y$-axis has an $x$-coordinate of 0 . We begin with the following definition.

DEFINITION The $x$-intercept of a straight line is the $x$-coordinate of the point where the graph crosses the $x$-axis. The $y$-intercept is defined similarly. It is the $y$-coordinate of the point where the graph crosses the $y$-axis.

If the $x$-intercept is $a$, then the point ( $a$, 0 ) lies on the graph. (This is true because any point on the $x$-axis has a $y$-coordinate of 0 .)

If the $y$-intercept is $b$, then the point $(0$, b) lies on the graph. (This is true because any point on the $y$-axis has an $x$-coordinate of 0 .)

Graphically, the relationship is shown in Figure 1.


FIGURE 1

EXAMPLE 1 Find the $x$ - and $y$-intercepts for $3 x-2 y=6$, and then use them to draw the graph.
SOLUTION To find where the graph crosses the $x$-axis, we let $y=0$. (The $y$-coordinate of any point on the $x$-axis is 0 .)

## $x$-intercept:

$$
\begin{array}{lr}
\text { When } & y=0 \\
\text { the equation } & 3 x-2 y \\
\text { becomes } & 6 \\
& 3 x-2(0) \\
\text { b } & 6 \\
3 x-0 & =6 \\
x & =2 \quad \text { Multiply each side by } \frac{1}{3}
\end{array}
$$

The graph crosses the $x$-axis at $(2,0)$, which means the $x$-intercept is 2 .

## $y$-intercept:

$$
\begin{array}{lr}
\text { When } & x=0 \\
\text { the equation } & 3 x-2 y=6 \\
\text { becomes } & 3(0)-2 y \\
=6 \\
0-2 y & =6 \\
-2 y & =6 \\
y & =-3
\end{array}
$$

$$
\text { Multiply each side by }-\frac{1}{2}
$$

The graph crosses the $y$-axis at $(0,-3)$, which means the $y$-intercept is -3 .

Plotting the $x$ - and $y$-intercepts and then drawing a line through them, we have the graph of $3 x-2 y=6$, as shown in Figure 2.


FIGURE 2

EXAMPLE 2 Graph $-x+2 y=4$ by finding the intercepts and using them to draw the graph.

SOLUTION Again, we find the $x$-intercept by letting $y=0$ in the equation and solving for $x$. Similarly, we find the $y$-intercept by letting $x=0$ and solving for $y$.

## $x$-intercept:

When | $y$ | $=0$ |
| ---: | :--- |
| the equation | $-x+2 y$ |
| becomes | $=4$ |
| $-x+2(0)$ | $=4$ |
| $-x+0$ | $=4$ |
| $-x$ | $=4$ |
| $x$ | $=-4$ |

Multiply each side by - 1
The $x$-intercept is -4 , indicating that the point $(-4,0)$, is on the graph of $-x+2 y=4$.

## $y$-intercept:

When $\quad$| $x$ | $=0$ |
| ---: | :--- |
| the equation | $-x+2 y$ |
| becomes | $=4$ |
| $-0+2 y$ | $=4$ |
| $2 y$ | $=4$ |
| $y$ | $=2 \quad$ Multiply each side by $\frac{1}{2}$ |

The $y$-intercept is 2 , indicating that the point $(0,2)$ is on the graph of $-x+2 y=4$.
Plotting the intercepts and drawing a line through them, we have the graph of $-x+2 y=4$, as shown in Figure 3.


FIGURE 3

Graphing a line by finding the intercepts, as we have done in Examples 1 and 2 , is an easy method of graphing if the equation has the form $a x+b y=c$ and both the numbers $a$ and $b$ divide the number $c$ evenly.

In our next example we use the intercepts to graph a line in which $y$ is given in terms of $x$.

## EXAMPLE 3 Use the intercepts for $y=-\frac{1}{3} x+2$ to draw its graph.

SOLUTION We graphed this line previously in Example 3 of Section 3.3 by substituting three different values of $x$ into the equation and solving for $y$. This time we will graph the line by finding the intercepts.

## $x$-intercept:

$$
\begin{array}{rlrl} 
& \text { When } & y & =0 \\
& \text { the equation } & y & =-\frac{1}{3} x+2 \\
& \text { becomes } & 0 & =-\frac{1}{3} x+2 \\
& & \\
-2 & =-\frac{1}{3} x & & \\
& & \text { Add }-\mathbf{2} \text { to each side } \\
& & =x & \\
\text { Multiply each side by }-\mathbf{3}
\end{array}
$$

The $x$-intercept is 6 , which means the graph passes through the point $(6,0)$.

## $y$-intercept:

$$
\begin{array}{ll}
\text { When } & x=0 \\
\text { the equation } & y=-\frac{1}{3} x+2 \\
\text { becomes } & y=-\frac{1}{3}(0)+2 \\
& y=2
\end{array}
$$

The $y$-intercept is 2 , which means the graph passes through the point $(0,2)$.
The graph of $y=-\frac{1}{3} x+2$ is shown in Figure 4. Compare this graph, and the method used to obtain it, with Example 3 in Section 3.3.


FIGURE 4

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A ${ }^{1-3}$
B ${ }^{1-3}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the $x$-intercept for a graph?
2. What is the $y$-intercept for a graph?
3. How do we find the $y$-intercept for a line from the equation?
4. How do we graph a line using its intercepts?

Find the $x$ - and $y$-intercepts for the following equations. Then use the intercepts to graph each equation.

1. $2 x+y=4$
2. $2 x+y=2$
3. $-x+y=3$
4. $-x+y=4$
5. $-x+2 y=2$
6. $-x+2 y=4$
7. $5 x+2 y=10$
8. $2 x+5 y=10$
9. $-4 x+5 y=20$
10. $-5 x+4 y=20$
11. $3 x-4 y=-4$
12. $-2 x+3 y=3$
13. $x-3 y=2$
14. $x-2 y=1$
15. $2 x-3 y=-2$
16. $3 x+4 y=6$
17. $y=2 x-6$
18. $y=2 x+6$
19. $y=2 x-1$
20. $y=\frac{1}{2} x+3$
21. $y=-\frac{1}{3} x-2$

For each of the following lines, the $x$-intercept and the $y$-intercept are both 0 , which means the graph of each will go through the origin, $(0,0)$. Graph each line by finding a point on each, other than the origin, and then drawing a line through that point and the origin.
25. $y=-2 x$
26. $y=\frac{1}{2} x$
= Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
27. $y=-\frac{1}{3} x$
28. $y=-3 x$
29. $y=\frac{2}{3} x$
30. $y=\frac{3}{2} x$

Complete each table.
31.

## Equation $\quad x$-intercept $y$-intercept

$$
\begin{aligned}
& 3 x+4 y=12 \\
& 3 x+4 y=4 \\
& 3 x+4 y=3 \\
& 3 x+4 y=2
\end{aligned}
$$

32. 

| Equation | $x$-intercept | $y$-intercept |
| :---: | :---: | :---: |
| $-2 x+3 y=6$ |  |  |
| $-2 x+3 y=3$ |  |  |
| $-2 x+3 y=2$ |  |  |
| $-2 x+3 y=1$ |  |  |

33. 

| Equation | $\boldsymbol{x}$-intercept | $y$-intercept |
| ---: | :--- | ---: |
| $x-3 y$ | $=2$ |  |
| $y$ | $=\frac{1}{3} x-\frac{2}{3}$ |  |
| $x-3 y$ | $=0$ |  |
| $y$ | $=\frac{1}{3} x$ |  |

34. 

| Equation | $x$-intercept | $y$-intercept |
| ---: | :--- | ---: |
| $x-2 y$ | $=1$ |  |
| $y$ | $=\frac{1}{2} x-\frac{1}{2}$ |  |
| $x-2 y$ | $=0$ |  |
| $y$ | $=\frac{1}{2} x$ |  |

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working. Working these problems is an excellent way to get ready for a test or a quiz.
35. Work each problem according to the instructions given.
a. Solve: $2 x-3=-3$
b. Find the $x$-intercept: $2 x-3 y=-3$
c. Find $y$ when $x$ is $0: 2 x-3 y=-3$
d. Graph: $2 x-3 y=-3$
e. Solve for $y: 2 x-3 y=-3$
36. Work each problem according to the instructions given.
a. Solve: $3 x-4=-4$
b. Find the $y$-intercept: $3 x-4 y=-4$
c. Find $x$ when $y$ is $0: 3 x-4 y=-4$
d. Graph: $3 x-4 y=-4$
e. Solve for $y$ : $3 x-4 y=-4$
37. Graph the line that passes through the points $(-2,5)$ and $(5,-2)$. What are the $x$ - and $y$-intercepts for this line?
38. Graph the line that passes through the points $(5,3)$ and $(-3,-5)$. What are the $x$ - and $y$-intercepts for this line?

From the graphs below, find the $x$ - and $y$-intercepts for each line.
39.

40.

41. Use the graph at the right to complete the following table.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | ---: |
| -2 |  |
| 0 |  |
|  | 0 |
|  | -2 |


42. Use the graph at the right to complete the following table.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | :---: |
| -2 |  |
| 0 |  |
|  | 0 |
|  | 6 |


43. The vertical line $x=3$ has only one intercept. Graph $x=3$, and name its intercept. [Remember, ordered pairs $(x, y)$ that are solutions to the equation $x=3$ are ordered pairs with an $x$-coordinate of 3 and any $y$-coordinate.]
44. Graph the vertical line $x=-2$. Then name its intercept.
45. The horizontal line $y=4$ has only one intercept. Graph $y=4$, and name its intercept. [Ordered pairs $(x, y)$ that are solutions to the equation $y=4$ are ordered pairs with a $y$-coordinate of 4 and any $x$-coordinate.]
46. Graph the horizontal line $y=-3$. Then name its intercept.

## Applying the Concepts

47. Working Two Jobs Maggie has a job working in an office for $\$ 10$ an hour and another job driving a
tractor for $\$ 12$ an hour. Maggie works at both jobs and earns $\$ 480$ in one week.
a. Write an equation that gives the number of hours $x$, she worked for $\$ 10$ per hour and the number of hours, $y$, she worked for $\$ 12$ per hour.
b. Find the $x$ - and $y$-intercepts for this equation.
c. Graph this equation from the intercepts, using only the first quadrant.
d. From the graph, find how many hours she worked at $\$ 12$ if she worked 36 hours at $\$ 10$ per hour.
e. From the graph, find how many hours she worked at \$10 if she worked 25 hours at \$12 per hour.
48. Improving Your Quantitative Literacy Give the $y$-intercept of the graph of poverty rates and tell what it represents. Why is there not an $x$-intercept?


## Maintaining Your Skills

Solve each equation.
49. $-12 y-4=-148$
50. $-2 x-18=4$
51. $-5 y-4=51$
52. $-2 y+18=-14$
53. $11 x-12=-78$
54. $21+9 y=-24$
55. $9 x+3=66$
56. $-11-15 a=-71$
57. $-9 c-6=12$
58. $-7 a+28=-84$
59. $4+13 c=-9$
60. $-3 x+15=-24$
61. $3 y-12=30$
62. $9 x+11=-16$
63. $-11 y+9=75$
64. $9 x-18=-72$

## Getting Ready for the Next Section

65. Evaluate:
a. $\frac{5-2}{3-1}$
b. $\frac{2-5}{1-3}$
66. Evaluate.
a. $\frac{-4-1}{5-(-2)}$
b. $\frac{1+4}{-2-5}$
67. Evaluate the following expressions when $x=3$ and $y=5$.
a. $\frac{y-2}{x-1}$
b. $\frac{2-y}{1-x}$
68. Evaluate the following expressions when $x=3$ and $y=2$.
a. $\frac{-4-y}{5-x}$
b. $\frac{y+4}{x-5}$

## 3.5

## Slope and the Equation of a Line

## OBJECTIVES

A Find the slope of a line from two points on the line.

B Graph a line given the slope and $y$-intercept.

C Find the slope-intercept form of a linear equation.

## Note

The 2 in $x_{2}$ is called a subscript. It is notation that allows us to distinguish between the variables $x_{1}$ and $x_{2}$, while still showing that they are both $x$-coordinates.

In defining the slope of a straight line, we are looking for a number to associate with a straight line that does two things. First of all, we want the slope of a line to measure the "steepness" of the line; that is, in comparing two lines, the slope of the steeper line should have the larger numerical value. Second, we want a line that rises going from left to right to have a positive slope. We want a line that falls going from left to right to have a negative slope. (A line that neither rises nor falls going from left to right must, therefore, have 0 slope.) These are illustrated in Figure 1.


Negative slope


Zero slope


Positive slope

FIGURE 1

Suppose we know the coordinates of two points on a line. Because we are trying to develop a general formula for the slope of a line, we will use general points-call the two points $P_{1}\left(x_{1}, y_{1}\right)$ and $P_{2}\left(x_{2}, y_{2}\right)$. They represent the coordinates of any two different points on our line. We define the slope of our line to be the ratio of the vertical change to the horizontal change as we move from point $\left(x_{1}, y_{1}\right)$ to point $\left(x_{2}, y_{2}\right)$ on the line. (See Figure 2.)


FIGURE 2

We call the vertical change the rise in the graph and the horizontal change the run in the graph. The slope, then, is

$$
\text { Slope }=\frac{\text { vertical change }}{\text { horizontal change }}=\frac{\text { rise }}{\text { run }}
$$

We would like to have a numerical value to associate with the rise in the graph and a numerical value to associate with the run in the graph. A quick study of Figure 2 shows that the coordinates of point $Q$ must be $\left(x_{2}, y_{1}\right)$, because $Q$ is directly below point $P_{2}$ and right across from point $P_{1}$. We can draw our diagram again in the manner shown in Figure 3. It is apparent from this graph that the rise can be expressed as $\left(y_{2}-y_{1}\right)$ and the run as $\left(x_{2}-x_{1}\right)$. We usually denote the slope of a line by the letter $m$. The complete definition of slope follows along with a diagram (Figure 3) that illustrates the definition.


FIGURE 3

DEFINITION If points $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ are any two different points, then the slope of the line on which they lie is

$$
\text { Slope }=m=\frac{\text { rise }}{\text { run }}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
$$

This definition of the slope of a line does just what we want it to do. If the line rises going from left to right, the slope will be positive. If the line falls from left to right, the slope will be negative. Also, the steeper the line, the larger numerical value the slope will have.

EXAMPLE 1 Find the slope of the line through $(1,2)$ and $(3,5)$.
SOLUTION We can let

$$
\left(x_{1}, y_{1}\right)=(1,2)
$$

and

$$
\left(x_{2}, y_{2}\right)=(3,5)
$$

then

$$
m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{5-2}{3-1}=\frac{3}{2}
$$

The slope is $\frac{3}{2}$. For every vertical change of 3 units, there will be a corresponding horizontal change of 2 units. (See Figure 4.)


FIGURE 4
EXAMPLE 2
Find the slope of the line through $(-2,1)$ and $(5,-4)$.


FIGURE 5

SOLUTION It makes no difference which ordered pair we call $\left(x_{1}, y_{1}\right)$ and which we call $\left(x_{2}, y_{2}\right)$.

$$
\text { Slope }=m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{-4-1}{5-(-2)}=\frac{-5}{7}
$$

The slope is $-\frac{5}{7}$. Every vertical change of -5 units (down 5 units) is accompanied by a horizontal change of 7 units (to the right 7 units). (See Figure 5.)

EXAMPLE $3 \quad$ Graph the line with slope $\frac{3}{2}$ and $y$-intercept 1.
SOLUTION Because the $y$-intercept is 1 , we know that one point on the line is $(0,1)$. So, we begin by plotting the point $(0,1)$, as shown in Figure 6.


FIGURE 6
There are many lines that pass through the point shown in Figure 6, but only one of those lines has a slope of $\frac{3}{2}$. The slope, $\frac{3}{2}$, can be thought of as the rise in
the graph divided by the run in the graph. Therefore, if we start at the point $(0,1)$ and move 3 units up (that's a rise of 3) and then 2 units to the right (a run of 2), we will be at another point on the graph. Figure 7 shows that the point we reach by doing so is the point $(2,4)$.

$$
\text { Slope }=m=\frac{\text { rise }}{\text { run }}=\frac{3}{2}
$$



FIGURE 7


FIGURE 8

To graph the line with slope $\frac{3}{2}$ and $y$-intercept 1 , we simply draw a line through the two points in Figure 7 to obtain the graph shown in Figure 8.

## The Slope-Intercept Form of an Equation of a Straight Line

To this point in the chapter, most of the problems we have worked have used the equation of a line to find different types of information about the line. For instance, given the equation of a line, we can find points on the line, the graph of the line, the intercepts, and the slope of the line. Now we reverse things somewhat and move in the other direction; we will use information about a line, such as its slope and $y$-intercept, to find the equation of a line.

EXAMPLE $4 \quad$ Find the equation of the line with slope $\frac{3}{2}$ and $y$-intercept 1.
SOLUTION We graphed the line with slope $\frac{3}{2}$ and $y$-intercept 1 in Example 3 .
Figure 9 shows that graph.


FIGURE 9

What we want to do now is find the equation of the line shown in Figure 9. To do so, we take any other point $(x, y)$ on the line and apply our slope formula to that point and the point $(0,1)$. We set that result equal to $\frac{3}{2}$, because $\frac{3}{2}$ is the slope of our line. The work is as follows, with a diagram of the situation following.

$$
\begin{aligned}
\frac{y-1}{x-0} & =\frac{3}{2} & & \text { Slope }=\frac{\text { vertical change }}{\text { horizontal change }} \\
\frac{y-1}{x} & =\frac{3}{2} & & x-0=x \\
y-1 & =\frac{3}{2} x & & \text { Multiply each side by } x \\
y & =\frac{3}{2} x+1 & & \text { Add } 1 \text { to each side }
\end{aligned}
$$



What is interesting and useful about the equation we found in the previous example is that the number in front of $x$ is the slope of the line and the constant term is the $y$-intercept. It is no coincidence that it turned out this way. Whenever an equation has the form $y=m x+b$, the graph is always a straight line with slope $m$ and $y$-intercept $b$. To see that this is true in general, suppose we want the equation of a line with slope $m$ and $y$-intercept $b$. Because the $y$-intercept is $b$, then the point $(0, b)$ is on the line. If $(x, y)$ is any other point on the line, then we apply our slope formula to get

$$
\begin{aligned}
\frac{y-b}{x-0} & =m & & \text { Slope }=\frac{\text { vertical change }}{\text { horizontal change }} \\
\frac{y-b}{x} & =m & & x-0=x \\
y-b & =m x & & \text { Multiply each side by } x \\
y & =m x+b & & \text { Add } b \text { to each side }
\end{aligned}
$$



Here is a summary of what we have just found.

Slope-Intercept Form of the Equation of a Line The equation of the line with slope $m$ and $y$-intercept $b$ is always given by

$$
y=m x+b
$$

EXAMPLE 5 Find the equation of the line with slope $-\frac{4}{3}$ and $y$-intercept 5. Then, graph the line.
SOLUTION Substituting $m=-\frac{4}{3}$ and $b=5$ into the equation $y=m x+b$, we have

$$
y=-\frac{4}{3} x+5
$$

Finding the equation from the slope and $y$-intercept is just that easy. If the slope is $m$ and the $y$-intercept is $b$, then the equation is always $y=m x+b$.
Because the $y$-intercept is 5 , the graph goes through the point $(0,5)$. To find a second point on the graph, we start at $(0,5)$ and move 4 units down (that's a rise of -4 ) and 3 units to the right (a run of 3 ). The point we reach is $(3,1)$. Drawing a line that passes through $(0,5)$ and $(3,1)$, we have the graph of our equation. (Note that we could also let the rise $=4$ and the run $=-3$ and obtain the same graph.) The graph is shown in Figure 10.


FIGURE 10

EXAMPLE $6 \quad$ Find the slope and $y$-intercept for $-2 x+y=-4$. Then,
use them to draw the graph.
SOLUTION To identify the slope and $y$-intercept from the equation, the equation must be in the form $y=m x+b$ (slope-intercept form). To write our equation in this form, we must solve the equation for $y$. To do so, we simply add $2 x$ to each side of the equation.

$$
\begin{aligned}
-2 x+y & =-4 & & \text { Original equation } \\
y & =2 x-4 & & \text { Add } 2 x \text { to each side }
\end{aligned}
$$

The equation is now in slope-intercept form, so the slope must be 2 and the $y$-intercept must be -4 . The graph, therefore, crosses the $y$-axis at $(0,-4)$. Because the slope is 2 , we can let the rise $=2$ and the run $=1$ and find a second point on the graph. The graph is shown in Figure 11.


FIGURE 11

EXAMPLE $7 \quad$ Find the slope and $y$-intercept for $3 x-2 y=6$.
SOLUTION To find the slope and $y$-intercept from the equation, we must write the equation in the form $y=m x+b$. This means we must solve the equation $3 x-2 y=6$ for $y$.

$$
\begin{aligned}
3 x-2 y & =6 & & \text { Original equation } \\
-2 y & =-3 x+6 & & \text { Add }-\mathbf{3 x} \text { to each side } \\
-\frac{\mathbf{1}}{\mathbf{2}}(-2 y) & =-\frac{\mathbf{1}}{\mathbf{2}}(-3 x+6) & & \text { Multiply each side by }-\frac{1}{2} \\
y & =\frac{3}{2} x-3 & & \text { Simplify each side }
\end{aligned}
$$

Now that the equation is written in slope-intercept form, we can identify the slope as $\frac{3}{2}$ and the $y$-intercept as -3 . The graph is shown in Figure 12 .

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,2
B 3
C ${ }_{4-7}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the slope of a line?
2. Would you rather climb a hill with a slope of 1 or a slope of 3 ? Explain why.
3. What do $m$ and $b$ represent in the equation $y=m x+b$ ?
4. Describe how you would find the slope and $y$-intercept of $3 y+2 x=6$.

## Problem Set 3.5

Find the slope of the line through the following pairs of points. Then plot each pair of points, draw a line through them, and indicate the rise and run in the graph in the same manner shown in Examples 1 and 2.

1. $(2,1),(4,4)$
2. $(3,1),(5,4)$
3. $(1,4),(5,2)$
4. $(1,3),(5,2)$
5. $(1,-3),(4,2)$
6. $(2,-3),(5,2)$
7. $(-3,-2),(1,3)$
8. $(-3,-1),(1,4)$
9. $(-3,2),(3,-2)$
10. $(-3,3),(3,-1)$
11. $(2,-5),(3,-2)$
12. $(2,-4),(3,-1)$

In each of the following problems, graph the line with the given slope and $y$-intercept $b$.
13. $m=\frac{2}{3}, b=1$
14. $m=\frac{3}{4}, b=-2$
15. $m=\frac{3}{2}, b=-3$
16. $m=\frac{4}{3}, b=2$
17. $m=-\frac{4}{3}, b=5$
18. $m=-\frac{3}{5}, b=4$
19. $m=2, b=1$
20. $m=-2, b=4$
21. $m=3, b=-1$
22. $m=3, b=-2$

Find the slope and $y$-intercept for each line.
23.

24.

25.

26.


Find the slope and $y$-intercept for each of the following equations by writing them in the form $y=m x+b$. Then, graph each equation.
27. $-2 x+y=4$
28. $-2 x+y=2$
29. $3 x+y=3$
30. $3 x+y=6$
31. $3 x+2 y=6$
32. $2 x+3 y=6$
33. $4 x-5 y=20$
34. $2 x-5 y=10$
35. $-2 x-5 y=10$
36. $-4 x+5 y=20$
37. Graph the line that has an $x$-intercept of 3 and a $y$-intercept of -2 . What is the slope of this line?
38. Graph the line that has an $x$-intercept of 2 and a $y$-intercept of -3 . What is the slope of this line?
39. Graph the line with $x$-intercept 4 and $y$-intercept 2 . What is the slope of this line?
40. Graph the line with $x$-intercept -4 and $y$-intercept -2 . What is the slope of this line?
41. Graph the line $y=2 x-3$, then name the slope and $y$-intercept by looking at the graph.
42. Graph the line $y=-2 x+3$, then name the slope and $y$-intercept by looking at the graph.
43. Graph the line $y=\frac{1}{2} x+1$, then name the slope and $y$-intercept by looking at the graph.
44. Graph the line $y=-\frac{1}{2} x-2$, then name the slope and $y$-intercept by looking at the graph.
45. Find $y$ if the line through $(4,2)$ and $(6, y)$ has a slope of 2 .
46. Find $y$ if the line through $(1, y)$ and $(7,3)$ has a slope of 6 .

For each equation in each table, give the slope of the graph.
47.

| Equation | Slope |
| :---: | :---: |
| $y=3$ |  |
| $y=3 x$ |  |

48. 

| Equation | Slope |
| :---: | :---: |
| $y=\frac{3}{2}$ |  |
| $y=\frac{3}{2} x$ |  |

49. 

| Equation | Slope |
| :---: | :---: |
| $y=-\frac{2}{3}$ |  |
| $y=-\frac{2}{3} x$ |  |

50. 

| Equation | Slope |
| :---: | :---: |
| $y=-2$ |  |
| $y=-2 x$ |  |

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working. Working these problems is a excellent way to get ready for a test or a quiz.
51. Work each problem according to the instructions given.
a. Solve: $-2 x+1=6$.
b. Write in slope-intercept form:
$-2 x+y=6$
c. Find the $y$-intercept: $-2 x+y=6$
d. Find the slope: $-2 x+y=6$
e. Graph: $-2 x+y=6$
52. Work each problem according to the instructions given.
a. Solve: $x+3=-6$.
b. Write in slope-intercept form:
$x+3 y=-6$
c. Find the $y$-intercept: $x+3 y=-6$
d. Find the slope: $x+3 y=-6$
e. Graph: $x+3 y=-6$

## Applying the Concepts

53. Value of a Copy Machine Cassandra buys a new color copier for her small business. It will cost $\$ 21,000$ and will decrease in value each year. The graph below shows the value of the copier after the first 5 years of ownership.

a. How much is the copier worth after 5 years?
b. After how many years is the copier worth \$12,000?
c. Find the slope of this line.
d. By how many dollars per year is the copier decreasing in value?
e. Find the equation of this line where $V$ is the value after $t$ years.
54. Value of a Forklift Elliot buys a new forklift for his business. It will cost $\$ 140,000$ and will decrease in value each year. The graph below shows the value of the forklift after the first 6 years of ownership.

a. How much is the forklift worth after 6 years?
b. After how many years is the forklift worth \$80,000?
c. Find the slope of this line.
d. By how many dollars per year is the forklift decreasing in value?
e. Find the equation of this line where $V$ is the value after $t$ years.
55. Salesperson's Income Kevin starts a new job in sales next month. He will earn $\$ 1,000$ per month plus a certain amount for each shirt he sells. The graph below shows the amount Kevin will earn per month based on how many shirts he sells.

a. How much will he earn for selling 1,000 shirts?
b. How many shirts must he sell to earn $\$ 2,000$ for a month?
c. Find the slope of this line.
d. How much money does Kevin earn for each shirt he sells?
e. Find the equation of this line where $y$ is the amount he earns for selling $x$ number of shirts.
56. Improving Your Quantitative Literacy Let $x$ represent the number of years past 2001 and $y$ represent health care costs as shown on the graph. (This means that when $x=0, y=5,039$.) Use the first and last points on the graph to find the equation of the line that connects them. Then use the equation to estimate health care costs in 2006.

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## Maintaining Your Skills

Use the equation $y=3 x-5$ to find $x$ if:
57. $y=13$
58. $y=-20$
59. $y=-11$
60. $y=16$

Use the equation $y=\frac{3}{7} x+4$ to find $y$ if:
61. $x=0$
62. $x=28$
63. $x=-35$
64. $x=-21$

Use the equation $y=3 x-5$ to find $y$ if:
65. $x=5$
66. $x=-5$
67. $x=-11$
68. $x=11$

Use the equation $y=\frac{x-6}{2}$ to find $x$ if:
69. $y=-9$
70. $y=9$
71. $y=-12$
72. $y=16$

## Getting Ready for the Next Section

Graph each of the following lines.
73. $x+y=4$
75. $y=2 x-3$
77. $y=2 x$
74. $x-y=-2$
76. $y=2 x+3$
78. $y=-2 x$

## 3.6

## OBJECTIVES

A Solve a system of linear equations in two variables by graphing.

## Solving Linear Systems by Graphing

Two linear equations considered at the same time make up what is called a system of linear equations. Both equations contain two variables and, of course, have graphs that are straight lines. The following are systems of linear equations:

$$
\begin{array}{rlr}
x+y=3 & y=2 x+1 & 2 x-y=1 \\
3 x+4 y=2 & y=3 x+2 & 3 x-2 y=6
\end{array}
$$

The solution set for a system of linear equations is all ordered pairs that are solutions to both equations. Because each linear equation has a graph that is a straight line, we can expect the intersection of the graphs to be a point whose coordinates are solutions to the system; that is, if we graph both equations on the same coordinate system, we can read the coordinates of the point of intersection and have the solution to our system. Here is an example.

$$
\begin{aligned}
& x+y=4 \\
& x-y=-2
\end{aligned}
$$

SOLUTION On the same set of coordinate axes we graph each equation separately. Figure 1 shows both graphs, without showing the work necessary to get them. We can see from the graphs that they intersect at the point $(1,3)$. The point $(1,3)$ therefore must be the solution to our system because it is the only ordered pair whose graph lies on both lines. Its coordinates satisfy both equations.


FIGURE 1
We can check our results by substituting the coordinates $x=1, y=3$ into both equations to see if they work.

| When | $X=1$ | When | $X=1$ |
| :---: | :---: | :---: | :---: |
| and | $y=3$ | and | $y=3$ |
| the equation | $x+y=4$ | the equation | $x-y=-2$ |
| becomes | $1+3 \stackrel{?}{\underline{?}} 4$ | becomes | $1-3 \stackrel{?}{\underline{?}}-2$ |
| or | $4=4$ | or | $-2=-2$ |

The point $(1,3)$ satisfies both equations.

Here are some steps to follow in solving linear systems by graphing.

## Strategy for Solving a Linear System by Graphing

Step 1: Graph the first equation by the methods described in Section 3.3 or 3.4.

Step 2: Graph the second equation on the same set of axes used for the first equation.

Step 3: Read the coordinates of the point of intersection of the two graphs.
Step 4: Check the solution in both equations.

## EXAMPLE 2 Solve the following system by graphing.

$$
\begin{array}{r}
x+2 y=8 \\
2 x-3 y=2
\end{array}
$$

SOLUTION Graphing each equation on the same coordinate system, we have the lines shown in Figure 2.


FIGURE 2

From Figure 2, we can see the solution for our system is $(4,2)$. We check this solution as follows.

| When | $x$ | $=4$ | When | $x$ |
| :--- | ---: | :--- | ---: | :--- |
| and | $=4$ | $=2$ | and | $=2$ |
| the equation | $x+2 y$ | $=8$ |  | the equation |
| becomes | $2 x-3 y$ | $=2$ |  |  |
| $4+2(2)$ | $\stackrel{?}{=} 8$ |  | becomes | $2(4)-3(2)$ |
| $4+4$ | $=8$ |  | $?$ |  |
| $8-6$ | $=2$ |  |  |  |
| 8 | $=8$ |  | 2 | $=2$ |

The point $(4,2)$ satisfies both equations and, therefore, must be the solution to our system.

EXAMPLE 3 Solve this system by graphing.


FIGURE 3

$$
\begin{aligned}
& y=2 x-3 \\
& x=3
\end{aligned}
$$

SOLUTION Graphing both equations on the same set of axes, we have Figure 3 .

## EXAMPLE 4 Solve by graphing.

$$
\begin{aligned}
& y=x-2 \\
& y=x+1
\end{aligned}
$$

## Note

We sometimes use special vocabulary to describe the special cases shown in Examples 4 and 5. When a system of equations has no solution because the lines are parallel (as in Example 4), we say the system is inconsistent. When the lines coincide (as in Example 5), we say the system is dependent.

SOLUTION Graphing both equations produces the lines shown in Figure 4. We can see in Figure 4 that the lines are parallel and therefore do not intersect. Our system has no ordered pair as a solution because there is no ordered pair that satisfies both equations. We say the solution set is the empty set and write $\varnothing$.


FIGURE 4
Example 4 is one example of two special cases associated with linear systems. The other special case happens when the two graphs coincide. Here is an example.

EXAMPLE 5 Graph the system.

$$
\begin{array}{r}
2 x+y=4 \\
4 x+2 y=8
\end{array}
$$

SOLUTION Both graphs are shown in Figure 5. The two graphs coincide. The reason becomes apparent when we multiply both sides of the first equation by 2 :

$$
\begin{aligned}
2 x+y & =4 \\
\mathbf{2}(2 x+y) & =\mathbf{2}(4) \quad \text { Multiply both sides by } \mathbf{2} \\
4 x+2 y & =8
\end{aligned}
$$

The equations have the same solution set. Any ordered pair that is a solution to one is a solution to the system. The system has an infinite number of solutions. (Any point on the line is a solution to the system.)

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $1-5$

The two special cases illustrated in the previous two examples do not happen often. Usually, a system has a single ordered pair as a solution. Solving a system of linear equations by graphing is useful only when the ordered pair in the solution set has integers for coordinates. Two other solution methods work well in all cases. We will develop the other two methods in the next two sections.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a system of two linear equations in two variables?
2. What is a solution to a system of linear equations?
3. How do we solve a system of linear equations by graphing?
4. Under what conditions will a system of linear equations not have a solution?

## Problem Set 3.6

Solve the following systems of linear equations by graphing.

1. $x+y=3$
$x-y=1$
2. $x+y=2$
$x-y=4$
3. $x+y=1$
$-x+y=3$
4. $x+y=1$
$x-y=-5$
5. $x+y=8$
$-x+y=2$
6. $x+y=6$
$-x+y=-2$
7. $3 x-2 y=6$
$x-y=1$
8. $5 x-2 y=10$
$x-y=-1$
9. $6 x-2 y=12$
$3 x+y=-6$
10. $4 x-2 y=8$
$2 x+y=-4$
11. $4 x+y=4$
$3 x-y=3$
12. $5 x-y=10$
$2 x+y=4$
13. $x+2 y=0$
$2 x-y=0$
14. $3 x+y=0$
$5 x-y=0$
15. $3 x-5 y=15$ $-2 x+y=4$
16. $2 x-4 y=8$
$2 x-y=-1$
17. $y=2 x+1$
$y=-2 x-3$
18. $y=3 x-4$
$y=-2 x+1$

$$
\text { 19. } \begin{aligned}
x+3 y & =3 \\
y & =x+5
\end{aligned}
$$

20. $2 x+y=-2$

$$
y=x+4
$$

21. $x+y=2$
$x=-3$
22. $\begin{aligned} x+y & =6 \\ y & =2\end{aligned}$
23. $x=-4$
$y=6$
24. $x=5$

$$
y=-1
$$

25. $x+y=4$
$2 x+2 y=-6$
26. $\begin{aligned} x-y & =3 \\ 2 x-2 y & =6\end{aligned}$
27. $4 x-2 y=8$
$2 x-y=4$
28. $3 x-6 y=6$
$x-2 y=4$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working.
29. Work each problem according to the instructions given.
a. Simplify: $(3 x-4 y)+(x-y)$
b. Find $y$ when $x$ is 4 in $3 x-4 y=8$
c. Find the $y$-intercept: $3 x-4 y=8$
d. Graph: $3 x-4 y=8$
e. Find the point where the graphs of $3 x-4 y=8$ and $x-y=2$ cross.
30. Work each problem according to the instructions given.
a. Simplify: $(x+4 y)+(-2 x+3 y)$
b. Find $y$ when $x$ is 3 in $-2 x+3 y=3$
c. Find the $y$-intercept: $-2 x+3 y=3$
d. Graph: $-2 x+3 y=3$
e. Find the point where the graphs of $-2 x+3 y=3$ and $x+4 y=4$ cross.
31. As you probably have guessed by now, it can be difficult to solve a system of equations by graphing if the solution to the system contains a fraction. The solution to the following system is $\left(\frac{1}{2}, 1\right)$. Solve the system by graphing.

$$
\begin{aligned}
& y=-2 x+2 \\
& y=4 x-1
\end{aligned}
$$

32. The solution to the following system is $\left(\frac{1}{3},-2\right)$.

Solve the system by graphing.

$$
\begin{aligned}
& y=3 x-3 \\
& y=-3 x-1
\end{aligned}
$$

33. A second difficulty can arise in solving a system of equations by graphing if one or both of the equations is difficult to graph. The solution to the following system is $(2,1)$. Solve the system by graphing.

$$
\begin{gathered}
3 x-8 y=-2 \\
x-y=1
\end{gathered}
$$

34. The solution to the following system is $(-3,2)$. Solve the system by graphing.

$$
\begin{aligned}
2 x+5 y & =4 \\
x-y & =-5
\end{aligned}
$$

## Applying the Concepts

35. Job Comparison Jane is deciding between two sales positions. She can work for Marcy's and receive $\$ 8.00$ per hour or for Gigi's, where she earns $\$ 6.00$
per hour but also receives a $\$ 50$ commission per week. The two lines in the following figure represent the money Jane will make for working at each of the jobs.

a. From the figure, how many hours would Jane have to work to earn the same amount at each of the positions?
b. If Jane expects to work less than 20 hours a week, which job should she choose?
c. If Jane expects to work more than 30 hours a week, which job should she choose?
36. Improving Your Quantitative Literacy The graph here shows the percentage of alcohol- and nonalcoholrelated fatal car crashes for the years 1982 through 2004. Although these graphs are not linear, they do intersect.
a. What is the point of intersection of these two lines?
b. Explain the significance of the point of intersection in terms of car crash fatalities.


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## Maintaining Your Skills

Simplify each expression.
37. $6 x+100(0.04 x+0.75)$
38. $5 x+100(0.03 x+0.65)$
39. $13 x-1,000(0.002 x+0.035)$
40. $9 x-1,000(0.023 x+0.015)$
41. $16 x-10(1.7 x-5.8)$
42. $43 x-10(3.1 x-2.7)$
43. $0.04 x+0.06(100-x)$
44. $0.07 x+0.03(100-x)$
45. $0.025 x-0.028(1,000+x)$
46. $0.065 x-0.037(1,000+x)$
47. $2.56 x-1.25(100+x)$
48. $8.42 x-6.68(100+x)$

## Getting Ready for the Next Section

Simplify each of the following.
49. $(x+y)+(x-y)$
50. $(x+2 y)+(-x+y)$
51. $(6 x-3 y)+(x+3 y)$
52. $(6 x+9 y)+(-6 x-10 y)$
53. $(-12 x-20 y)+(25 x+20 y)$
54. $(-3 x+2 y)+(3 x+8 y)$
55. $-4(3 x+5 y)$
56. $6\left(\frac{1}{2} x-\frac{1}{3} y\right)$
57. $12\left(\frac{1}{4} x+\frac{2}{3} y\right)$
58. $5(5 x+4 y)$
59. $-2(2 x-y)$
60. $-2(4 x-3 y)$
61. Let $x+y=4$. If $x=3$, find $y$.
62. Let $x+2 y=4$. If $x=3$, find $y$.
63. Let $x+3 y=3$. If $x=3$, find $y$.
64. Let $2 x+3 y=-1$. If $y=-1$, find $x$.
65. Let $3 x+5 y=-7$. If $x=6$, find $y$.
66. Let $3 x-2 y=12$. If $y=6$, find $x$.

## 3.7

## OBJECTIVES

A Use the elimination method to solve a system of linear equations in two variables.

## The Elimination Method

The addition property states that if equal quantities are added to both sides of an equation, the solution set is unchanged. In the past we have used this property to help solve equations in one variable. We will now use it to solve systems of linear equations. Here is another way to state the addition property of equality.

Let $A, B, C$, and $D$ represent algebraic expressions.
If

$$
\begin{gathered}
A=B \\
C=D \\
\hline A+C=B+D
\end{gathered}
$$

and
then
Because $C$ and $D$ are equal (that is, they represent the same number), what we have done is added the same amount to both sides of the equation $A=B$. Let's see how we can use this form of the addition property of equality to solve a system of linear equations.

## EXAMPLE 1 Solve the following system.

$$
\begin{aligned}
& x+y=4 \\
& x-y=2
\end{aligned}
$$

## Note

The graphs shown in the margin next to our first three examples are not part of the solution shown in each example. The graphs are there simply to show you that the results we obtain by the elimination method are consistent with the results we would obtain by graphing.


FIGURE 1

SOLUTION The system is written in the form of the addition property of equality as written in this section. It looks like this:

$$
\begin{aligned}
& A=B \\
& C=D
\end{aligned}
$$

where $A$ is $x+y, B$ is $4, C$ is $x-y$, and $D$ is 2 .
We use the addition property of equality to add the left sides together and the right sides together.

$$
\begin{array}{r}
x+y=4 \\
x-y=2 \\
\hline 2 x+0=6
\end{array}
$$

We now solve the resulting equation for $x$.

$$
\begin{aligned}
2 x+0 & =6 \\
2 x & =6 \\
x & =3
\end{aligned}
$$

The value we get for $x$ is the value of the $x$-coordinate of the point of intersection of the two lines $x+y=4$ and $x-y=2$. To find the $y$-coordinate, we simply substitute $x=3$ into either of the two original equations. Using the first equation, we get

$$
\begin{aligned}
3+y & =4 \\
y & =1
\end{aligned}
$$

The solution to our system is the ordered pair $(3,1)$. It satisfies both equations.

| When | $x$ | $=3$ | When | $x=3$ |
| :--- | ---: | :--- | ---: | :--- |
| and | $y$ | $=1$ |  | and |
| the equation | $x+y$ | $=4$ |  | the equation |
| becomes | $3+1$ | $\stackrel{?}{=} 4$ |  | becomes |
| or | 4 | $=4$ |  | or |
|  |  | $3-1 \stackrel{?}{=} 2$ |  |  |
|  |  | 2 | $=2$ |  |

Figure 1 is visual evidence that the solution to our system is $(3,1)$.
The most important part of this method of solving linear systems is eliminating one of the variables when we add the left and right sides together. In our first example, the equations were written so that the $y$ variable was eliminated when we added the left and right sides together. If the equations are not set up this way to begin with, we have to work on one or both of them separately before we can add them together to eliminate one variable.

EXAMPLE 2 Solve the following system.

$$
\begin{aligned}
x+2 y & =4 \\
x-y & =-5
\end{aligned}
$$

SOLUTION Notice that if we were to add the equations together as they are, the resulting equation would have terms in both $x$ and $y$. Let's eliminate the variable $x$ by multiplying both sides of the second equation by -1 before we add the equations together. (As you will see, we can choose to eliminate either the $x$ or the $y$ variable.) Multiplying both sides of the second equation by -1 will not


FIGURE 2
change its solution, so we do not need to be concerned that we have altered the system.

$$
\begin{aligned}
& \begin{aligned}
x+2 y=4 \\
x-y=-5 \xrightarrow[\text { Multiply by }-1]{\text { No change }} x+2 y=4 \quad \text { Add left and right sides to get } \\
0+3 y=9
\end{aligned} \\
& 3 y=9 \\
& y=3 \quad\left\{\begin{array}{l}
\boldsymbol{y} \text {-Coordinate of the } \\
\text { point of intersection }
\end{array}\right.
\end{aligned}
$$

Substituting $y=3$ into either of the two original equations, we get $x=-2$. The solution to the system is $(-2,3)$. It satisfies both equations. Figure 2 shows the solution to the system as the point where the two lines cross.

EXAMPLE 3 Solve the following system.

$$
\begin{aligned}
& 2 x-y=6 \\
& x+3 y=3
\end{aligned}
$$

SOLUTION Let's eliminate the $y$ variable from the two equations. We can do this by multiplying the first equation by 3 and leaving the second equation unchanged.

$$
\begin{aligned}
& 2 x-y=6 \xrightarrow{3 \text { times both sides }} 6 x-3 y=18 \\
& x+3 y=3 \xrightarrow[\text { No change }]{ } x+3 y=3
\end{aligned}
$$

The important thing about our system now is that the coefficients (the numbers in front) of the $y$ variables are opposites. When we add the terms on each side of the equal sign, then the terms in $y$ will add to zero and be eliminated.

$$
\begin{aligned}
6 x-3 y & =18 \\
x+3 y & =3 \\
\hline 7 x \quad & =21 \quad \text { Add corresponding terms }
\end{aligned}
$$

This gives us $x=3$. Using this value of $x$ in the second equation of our original system, we have

$$
\begin{array}{r}
3+3 y=3 \\
3 y=0 \\
y=0
\end{array}
$$

We could substitute $x=3$ into any of the equations with both $x$ and $y$ variables and also get $y=0$. The solution to our system is the ordered pair (3, 0). Figure 3 is a picture of the system of equations showing the solution $(3,0)$.
FIGURE 3

## EXAMPLE 4 Solve the system.

$$
\begin{aligned}
& 2 x+3 y=-1 \\
& 3 x+5 y=-2
\end{aligned}
$$

## Note

If you are having trouble understanding this method of solution, it is probably because you can't see why we chose to multiply by 3 and -2 in the first step of Example 4. Look at the result of doing so: the $6 x$ and $-6 x$ will add to 0 . We chose to multiply by 3 and -2 because they produce $6 x$ and $-6 x$, which will add to 0 .

SOLUTION Let's eliminate $x$ from the two equations. If we multiply the first equation by 3 and the second by -2 , the coefficients of $x$ will be 6 and -6 , respectively. The $x$ terms in the two equations will then add to zero.

$$
\begin{aligned}
& 2 x+3 y=-1 \xrightarrow{\text { Multiply by } 3} 6 x+9 y=-3 \\
& 3 x+5 y=-2 \xrightarrow[\text { Multiply by }-2]{ }-6 x-10 y=4
\end{aligned}
$$

We now add the left and right sides of our new system together.

$$
\begin{aligned}
6 x+9 y & =-3 \\
-6 x-10 y & =4 \\
\hline-y & =1 \\
y & =-1
\end{aligned}
$$

Substituting $y=-1$ into the first equation in our original system, we have

$$
\begin{aligned}
2 x+3(-1) & =-1 \\
2 x-3 & =-1 \\
2 x & =2 \\
x & =1
\end{aligned}
$$

The solution to our system is $(1,-1)$. It is the only ordered pair that satisfies both equations.

## EXAMPLE 5 Solve the system.

$$
\begin{aligned}
& 3 x+5 y=-7 \\
& 5 x+4 y=10
\end{aligned}
$$

SOLUTION Let's eliminate $y$ by multiplying the first equation by -4 and the second equation by 5 .

$$
\begin{aligned}
& 3 x+5 y=-7 \xrightarrow[\text { Multiply by }-4]{ }-12 x-20 y=28 \\
& 5 x+4 y=10 \xrightarrow[\text { Multiply by } 5]{ } \begin{aligned}
25 x+20 y & =50 \\
13 x & =78 \\
x & =6
\end{aligned}
\end{aligned}
$$

Substitute $x=6$ into either equation in our original system, and the result will be $y=-5$. The solution is therefore $(6,-5)$.

## EXAMPLE 6 Solve the system.

$$
\begin{aligned}
& \frac{1}{2} x-\frac{1}{3} y=2 \\
& \frac{1}{4} x+\frac{2}{3} y=6
\end{aligned}
$$

SOLUTION Although we could solve this system without clearing the equations of fractions, there is probably less chance for error if we have only integer coefficients to work with. So let's begin by multiplying both sides of the top equation by 6 and both sides of the bottom equation by 12 , to clear each equation of fractions.

$$
\begin{aligned}
& \frac{1}{2} x-\frac{1}{3} y=2 \xrightarrow{\text { Multiply by } 6} 3 x-2 y=12 \\
& \frac{1}{4} x+\frac{2}{3} y=6 \xrightarrow[\text { Multiply by } 12]{ } 3 x+8 y=72
\end{aligned}
$$

Now we can eliminate $x$ by multiplying the top equation by -1 and leaving the bottom equation unchanged.

$$
\begin{aligned}
& 3 x-2 y=12 \xrightarrow{\text { Multiply by }-1}-3 x+2 y=-12 \\
& 3 x+8 y=72 \xrightarrow[\text { No change }]{ } \begin{aligned}
3 x+8 y & =72 \\
10 y & =60 \\
y & =6
\end{aligned}
\end{aligned}
$$

We can substitute $y=6$ into any equation that contains both $x$ and $y$. Let's use $3 x-2 y=12$.

$$
\begin{aligned}
3 x-2(6) & =12 \\
3 x-12 & =12 \\
3 x & =24 \\
x & =8
\end{aligned}
$$

The solution to the system is $(8,6)$.
Our next two examples will show what happens when we apply the elimination method to a system of equations consisting of parallel lines and to a system in which the lines coincide.

## EXAMPLE 7 Solve the system.

$$
\begin{aligned}
& 2 x-y=2 \\
& 4 x-2 y=12
\end{aligned}
$$

SOLUTION Let us choose to eliminate $y$ from the system. We can do this by multiplying the first equation by -2 and leaving the second equation unchanged.

$$
\begin{aligned}
& 2 x-y=2 \xrightarrow{\text { Multiply by }-2}-4 x+2 y=-4 \\
& 4 x-2 y=12 \xrightarrow[\text { No change }]{ } \quad 4 x-2 y=12
\end{aligned}
$$



FIGURE 4

If we add both sides of the resulting system, we have

$$
\begin{aligned}
-4 x+2 y & =-4 \\
4 x-2 y & =12 \\
\hline 0+0 & =8
\end{aligned}
$$

or $\quad 0=8 \quad$ A false statement
Both variables have been eliminated and we end up with the false statement $0=8$. We have tried to solve a system that consists of two parallel lines. There is no solution, and that is the reason we end up with a false statement. Figure 4 is a visual representation of the situation and is conclusive evidence that there is no solution to our system.

EXAMPLE 8 Solve the system.

$$
\begin{aligned}
& 4 x-3 y=2 \\
& 8 x-6 y=4
\end{aligned}
$$

SOLUTION Multiplying the top equation by -2 and adding, we can eliminate the variable $x$.

$$
\begin{aligned}
& 4 x-3 y=2 \xrightarrow[\text { Multiply by }-2]{ }-8 x+6 y=-4 \\
& 8 x-6 y=4 \xrightarrow[\text { No change }]{ } \begin{array}{l}
8 x-6 y=4 \\
0=
\end{array}
\end{aligned}
$$

Both variables have been eliminated, and the resulting statement $0=0$ is true. In this case the lines coincide because the equations are equivalent. The solution set consists of all ordered pairs that satisfy either equation.

The preceding two examples illustrate the two special cases in which the graphs of the equations in the system either coincide or are parallel.
Here is a summary of our results from these two examples:
Both variables are eliminated and $\leftrightarrow$ The lines are parallel and there the resulting statement is false. is no solution to the system.

Both variables are eliminated and $\leftrightarrow \quad$ The lines coincide and there the resulting statement is true. is an infinite number of solutions to the system.

The main idea in solving a system of linear equations by the elimination method is to use the multiplication property of equality on one or both of the original equations, if necessary, to make the coefficients of either variable opposites. The following box shows some steps to follow when solving a system of linear equations by the elimination method.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-8

## Strategy for Solving a System of Linear Equations

 by the Elimination MethodStep 1: Decide which variable to eliminate. (In some cases one variable will be easier to eliminate than the other. With some practice you will notice which one it is.)

Step 2: Use the multiplication property of equality on each equation separately to make the coefficients of the variable that is to be eliminated opposites.
Step 3: Add the respective left and right sides of the system together.
Step 4: Solve for the variable remaining.
Step 5: Substitute the value of the variable from step 4 into an equation containing both variables and solve for the other variable.

Step 6: Check your solution in both equations, if necessary.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How is the addition property of equality used in the elimination method of solving a system of linear equations?
2. What happens when we use the elimination method to solve a system of linear equations consisting of two parallel lines?
3. What does it mean when we solve a system of linear equations by the elimination method and we end up with the statement $0=8$ ?
4. What is the first step in solving a system of linear equations that contains fractions?

## Problem Set 3.7

Solve the following systems of linear equations by elimination.

1. $x+y=3$
$x-y=1$
2. $x+y=-2$
$x-y=6$
3. $x+y=10$
$-x+y=4$
4. $x-y=1$
$-x-y=-7$
5. $x-y=7$
$-x-y=3$
6. $x-y=4$
$2 x+y=8$
7. $x+y=-1$
$3 x-y=-3$
8. $2 x-y=-2$
$-2 x-y=2$

$$
\text { 9. } \begin{aligned}
3 x+2 y & =1 \\
-3 x-2 y & =-1
\end{aligned}
$$

10. $-2 x-4 y=1$
$2 x+4 y=-1$

Solve each of the following systems by eliminating the $y$ variable.
11. $3 x-y=4$
$2 x+2 y=24$
12. $\begin{array}{r}2 x+y=3 \\ 3 x+2 y=1\end{array}$
13. $5 x-3 y=-2$
14. $4 x-y=-1$
$10 x-y=1$
$2 x+4 y=13$

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- Online student support materials available at www.thomsonedu.com/login

15. $11 x-4 y=11$

$$
5 x+y=5
$$

16. $3 x-y=7$
$10 x-5 y=25$

Solve each of the following systems by eliminating the $x$ variable.
17. $3 x-5 y=7$
18. $4 x+2 y=32$
$-x+y=-1$
$x+y=-2$
19. $-x-8 y=-1$
$-2 x+4 y=13$
20. $-x+10 y=1$
$-5 x+15 y=-9$
21. $-3 x-y=7$
$6 x+7 y=11$
22. $-5 x+2 y=-6$ $10 x+7 y=34$

Solve each of the following systems of linear equations by the elimination method.
23. $\begin{aligned} 6 x-y & =-8 \\ 2 x+y & =-16\end{aligned}$
25. $x+3 y=9$ $2 x-y=4$
27. $x-6 y=3$ $4 x+3 y=21$
29. $2 x+9 y=2$ $5 x+3 y=-8$
31. $\frac{1}{3} x+\frac{1}{4} y=\frac{7}{6}$ $\frac{3}{2} x-\frac{1}{3} y=\frac{7}{3}$
33. $3 x+2 y=-1$
$6 x+4 y=0$
35. $11 x+6 y=17$
$5 x-4 y=1$
37. $\frac{1}{2} x+\frac{1}{6} y=\frac{1}{3}$
$-x-\frac{1}{3} y=-\frac{1}{6}$

Solve each system.
39. $x+y=22$
$5 x+10 y=170$
41. $\begin{aligned} x+y & =14 \\ 5 x+25 y & =230\end{aligned}$
$5 x+25 y=230$
43. $x+y=15,000$ $6 x+7 y=98,000$
24. $5 x-3 y=-3$
$3 x+3 y=-21$
26. $x+2 y=0$
$2 x-y=0$
28. $8 x+y=-1$
$4 x-5 y=16$
30. $5 x+2 y=11$ $7 x+8 y=7$
32. $\frac{7}{12} x-\frac{1}{2} y=\frac{1}{6}$
$\frac{2}{5} x-\frac{1}{3} y=\frac{11}{15}$
34. $8 x-2 y=2$
$4 x-y=2$
36. $3 x-8 y=7$
$10 x-5 y=45$
38. $-\frac{1}{3} x-\frac{1}{2} y=-\frac{2}{3}$
$-\frac{2}{3} x-\quad y=-\frac{4}{3}$
40. $x+y=14$
$10 x+25 y=185$
42. $\begin{aligned} x+y & =11 \\ 5 x+10 y & =95\end{aligned}$
$5 x+10 y=95$
44. $x+y=10,000$
$6 x+7 y=63,000$
45. $x+y=11,000$
$4 x+7 y=68,000$
46. $x+y=20,000$
$8 x+6 y=138,000$
47. $x+y=23$
48. $x+y=45$
$5 x+10 y=175$
$25 x+5 y=465$
49. Multiply both sides of the second equation in the following system by 100, and then solve as usual.

$$
\begin{aligned}
x+y & =22 \\
0.05 x+0.10 y & =1.70
\end{aligned}
$$

50. Multiply both sides of the second equation in the following system by 100, and then solve as usual.

$$
x+y=15,000
$$

$$
0.06 x+0.07 y=980
$$

## Maintaining Your Skills

For each of the equations, determine the slope and $y$-intercept.
51. $3 x-y=3$
52. $2 x+y=-2$
53. $2 x-5 y=25$
54. $-3 x+4 y=-12$

Find the slope of the line through the given points.
55. $(-2,3)$ and $(6,-5)$
56. $(2,-4)$ and $(8,-2)$
57. $(5,3)$ and $(2,-3)$
58. $(-1,-4)$ and $(-4,-1)$

## Getting Ready for the Next Section

Solve.
59. $x+(2 x-1)=2$
60. $2 x-3(2 x-8)=12$
61. $2(3 y-1)-3 y=4$
62. $-2 x+4(3 x+6)=14$
63. $-2 x+3(5 x-1)=10$
64. $1.5 x+15=0.75 x+24.95$

Solve each equation for the indicated variable.
65. $x-3 y=-1$ for $x$
66. $-3 x+y=6$ for $y$
67. Let $y=2 x-1$. If $x=1$, find $y$.
68. Let $y=2 x-8$. If $x=5$, find $y$.
69. Let $x=3 y-1$. If $y=2$, find $x$.
70. Let $x=4 y-5$. If $y=2$, find $x$.

Let $y=1.5 x+15$.
71. If $x=13$, find $y$.
72. If $x=14$, find $y$.

Let $y=0.75 x+24.95$.
73. If $x=12$, find $y$.
74. If $x=16$, find $y$.


## OBJECTIVES

A Use the substitution method to solve a system of linear equations in two variables.

## The Substitution Method

There is a third method of solving systems of equations. It is the substitution method, and, like the elimination method, it can be used on any system of linear equations. Some systems, however, lend themselves more to the substitution method than others do.

## EXAMPLE 1 Solve the following system.

$$
\begin{aligned}
x+y & =2 \\
y & =2 x-1
\end{aligned}
$$

SOLUTION If we were to solve this system by the methods used in the previous section, we would have to rearrange the terms of the second equation so that similar terms would be in the same column. There is no need to do this, however, because the second equation tells us that $y$ is $2 x-1$. We can replace the $y$ variable in the first equation with the expression $2 x-1$ from the second equation; that is, we substitute $2 x-1$ from the second equation for $y$ in the first equation. Here is what it looks like:

$$
x+(2 x-1)=2
$$

The equation we end up with contains only the variable $x$. The $y$ variable has been eliminated by substitution.

Solving the resulting equation, we have

$$
\begin{array}{r}
x+(2 x-1)=2 \\
3 x-1=2 \\
3 x=3 \\
x=1
\end{array}
$$

This is the $x$-coordinate of the solution to our system. To find the $y$-coordinate, we substitute $x=1$ into the second equation of our system. (We could substitute $x=1$ into the first equation also and have the same result.)

$$
\begin{aligned}
& y=2(1)-1 \\
& y=2-1 \\
& y=1
\end{aligned}
$$

The solution to our system is the ordered pair $(1,1)$. It satisfies both of the original equations. Figure 1 provides visual evidence that the substitution method yields the correct solution.


EXAMPLE 2 Solve the following system by the substitution method.

$$
\begin{aligned}
2 x-3 y & =12 \\
y & =2 x-8
\end{aligned}
$$

SOLUTION Again, the second equation says $y$ is $2 x-8$. Because we are looking for the ordered pair that satisfies both equations, the $y$ in the first equation must also be $2 x-8$. Substituting $2 x-8$ from the second equation for $y$ in the first equation, we have

$$
2 x-3(2 x-8)=12
$$

This equation can still be read as $2 x-3 y=12$ because $2 x-8$ is the same as $y$. Solving the equation, we have

$$
\begin{aligned}
2 x-3(2 x-8) & =12 \\
2 x-6 x+24 & =12 \\
-4 x+24 & =12 \\
-4 x & =-12 \\
x & =3
\end{aligned}
$$

To find the $y$-coordinate of our solution, we substitute $x=3$ into the second equation in the original system.

$$
\begin{array}{ll}
\text { When } & x=3 \\
\text { the equation } & y=2 x-8 \\
\text { becomes } & y=2(3)-8 \\
& y=6-8=-2
\end{array}
$$

The solution to our system is $(3,-2)$.

EXAMPLE 3 Solve the following system by solving the first equation for $x$ and then using the substitution method:

$$
\begin{aligned}
x-3 y & =-1 \\
2 x-3 y & =4
\end{aligned}
$$

SOLUTION We solve the first equation for $x$ by adding $3 y$ to both sides to get

$$
x=3 y-1
$$

Using this value of $x$ in the second equation, we have

$$
\begin{aligned}
2(3 y-1)-3 y & =4 \\
6 y-2-3 y & =4 \\
3 y-2 & =4 \\
3 y & =6 \\
y & =2
\end{aligned}
$$

Next, we find $x$.

$$
\begin{array}{ll}
\text { When } & y=2 \\
\text { the equation } & x=3 y-1 \\
\text { becomes } & x=3(2)-1 \\
& x=6-1 \\
& x=5
\end{array}
$$

The solution to our system is $(5,2)$.
Here are the steps to use in solving a system of equations by the substitution method.

Strategy for Solving a System of Equations by the Substitution Method
Step 1: Solve either one of the equations for $x$ or $y$. (This step is not necessary if one of the equations is already in the correct form, as in Examples 1 and 2.)
Step 2: Substitute the expression for the variable obtained in step 1 into the other equation and solve it.

Step 3: Substitute the solution from step 2 into any equation in the system that contains both variables and solve it.

Step 4: Check your results, if necessary.

## EXAMPLE 4 Solve by substitution.

$$
\begin{gathered}
-2 x+4 y=14 \\
-3 x+y=6
\end{gathered}
$$

SOLUTION We can solve either equation for either variable. If we look at the system closely, it becomes apparent that solving the second equation for $y$ is the easiest way to go. If we add $3 x$ to both sides of the second equation, we have

$$
y=3 x+6
$$

Substituting the expression $3 x+6$ back into the first equation in place of $y$ yields the following result.

$$
\begin{aligned}
-2 x+4(3 x+6) & =14 \\
-2 x+12 x+24 & =14 \\
10 x+24 & =14 \\
10 x & =-10 \\
x & =-1
\end{aligned}
$$

Substituting $x=-1$ into the equation $y=3 x+6$ leaves us with

$$
\begin{aligned}
& y=3(-1)+6 \\
& y=-3+6 \\
& y=3
\end{aligned}
$$

The solution to our system is $(-1,3)$.

## EXAMPLE 5 Solve by substitution.

$$
\begin{aligned}
4 x+2 y & =8 \\
y & =-2 x+4
\end{aligned}
$$

SOLUTION Substituting the expression $-2 x+4$ for $y$ from the second equation into the first equation, we have

$$
\begin{aligned}
4 x+2(-2 x+4) & =8 \\
4 x-4 x+8 & =8 \\
8 & =8 \quad \text { A true statement }
\end{aligned}
$$

Both variables have been eliminated, and we are left with a true statement. Recall from the last section that a true statement in this situation tells us the lines coincide; that is, the equations $4 x+2 y=8$ and $y=-2 x+4$ have exactly the same graph. Any point on that graph has coordinates that satisfy both equations and is a solution to the system.

EXAMPLE 6 The following table shows two contract rates charged by GTE Wireless for cellular phone use. At how many minutes will the two rates cost the same amount?

|  | Flat Rate | Plus | Per Minute Charge |
| :--- | :---: | :---: | :---: |
| Plan 1 | $\$ 15$ |  | $\$ 1.50$ |
| Plan 2 | $\$ 24.95$ | $\$ 0.75$ |  |

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-6

SOLUTION If we let $y=$ the monthly charge for $x$ minutes of phone use, then the equations for each plan are

$$
\begin{array}{ll}
\text { Plan 1: } & y=1.5 x+15 \\
\text { Plan 2: } & y=0.75 x+24.95
\end{array}
$$

We can solve this system by substitution by replacing the variable $y$ in Plan 2 with the expression $1.5 x+15$ from Plan 1. If we do so, we have

$$
\begin{aligned}
1.5 x+15 & =0.75 x+24.95 \\
0.75 x+15 & =24.95 \\
0.75 x & =9.95 \\
x & =13.27 \quad \text { to the nearest hundredth }
\end{aligned}
$$

The monthly bill is based on the number of minutes you use the phone, with any fraction of a minute moving you up to the next minute. If you talk for a total of 13 minutes, you are billed for 13 minutes. If you talk for 13 minutes, 10 seconds, you are billed for 14 minutes. The number of minutes on your bill always will be a whole number. So, to calculate the cost for talking 13.27 minutes, we would replace $x$ with 14 and find $y$. Let's compare the two plans at $x=13$ minutes and at $x=14$ minutes.

$$
\begin{array}{ll}
\text { Plan 1: } & y=1.5 x+15 \\
\text { When } & x=13, y=\$ 34.50 \\
\text { When } & x=14, y=\$ 36.00 \\
\text { Plan 2: } & y=0.75 x+24.95 \\
\text { When } & x=13, y=\$ 34.70 \\
\text { When } & x=14, y=\$ 35.45
\end{array}
$$

The two plans will never give the same cost for talking $x$ minutes. If you talk 13 or less minutes, Plan 1 will cost less. If you talk for more than 13 minutes, you will be billed for 14 minutes, and Plan 2 will cost less than Plan 1.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the first step in solving a system of linear equations by substitution?
2. When would substitution be more efficient than the elimination method in solving two linear equations?
3. What does it mean when we solve a system of linear equations by the substitution method and we end up with the statement $8=8$ ?
4. How would you begin solving the following system using the substitution method?

$$
\begin{aligned}
x+y & =2 \\
y & =2 x-1
\end{aligned}
$$

## Problem Set 3.8

Solve the following systems by substitution. Substitute the expression in the second equation into the first equation and solve.

1. $x+y=11$

$$
y=2 x-1
$$

2. $x-y=-3$

$$
y=3 x+5
$$

3. $x+y=20$

$$
y=5 x+2
$$

4. $3 x-y=-1$
$x=2 y-7$
5. $-2 x+y=-1$

$$
y=-4 x+8
$$

6. $4 x-y=5$
$y=-4 x+1$
7. $3 x-2 y=-2$
$x=-y+6$
8. $2 x-3 y=17$
$x=-y+6$
9. $5 x-4 y=-16$
10. $6 x+2 y=18$
$y=4$
$x=3$
11. $5 x+4 y=7$
12. $10 x+2 y=-6$
$y=-3 x$
$y=-5 x$

Solve the following systems by solving one of the equations for $x$ or $y$ and then using the substitution method.
13. $x+3 y=4$
$x-2 y=-1$
14. $x-y=5$
$x+2 y=-1$
15. $\begin{aligned} 2 x+y & =1 \\ x-5 y & =17\end{aligned}$
16. $2 x-2 y=2$
$x-3 y=-7$
17. $3 x+5 y=-3$
18. $2 x-4 y=-4$
$x-5 y=-5$
$x+2 y=8$
19. $5 x+3 y=0$
$x-3 y=-18$
20. $x-3 y=-5$
$x-2 y=0$
21. $-3 x-9 y=7$
22. $2 x+6 y=-18$
$x+3 y=12$
$x+3 y=-9$

Solve the following systems using the substitution method.
23. $5 x-8 y=7$
$y=2 x-5$
24. $3 x+4 y=10$
$y=8 x-15$
25. $7 x-6 y=-1$
$x=2 y-1$
26. $4 x+2 y=3$
$x=4 y-3$
27. $-3 x+2 y=6$
$y=3 x$
28. $-2 x-y=-3$
$y=-3 x$
29. $5 x-6 y=-4$
30. $2 x-4 y=0$
$x=y$
$y=x$

$$
\text { 31. } \begin{aligned}
3 x+3 y & =9 \\
y & =2 x-12
\end{aligned}
$$

32. $7 x+6 y=-9$
$y=-2 x+1$
33. $7 x-11 y=16$
$y=10$
34. $9 x-7 y=-14$
$x=7$
35. $-4 x+4 y=-8$
$y=x-2$
36. $-4 x+2 y=-10$
$y=2 x-5$

Solve each system.
37. $2 x+5 y=36$
$y=12-x$
38. $3 x+6 y=120$
$y=25-x$
39. $5 x+2 y=54$
$y=18-x$
40. $10 x+5 y=320$
$y=40-x$
41. $2 x+2 y=96$
42. $x+y=22$
$y=2 x$
$y=x+9$

Solve each system by substitution. You can eliminate the decimals if you like, but you don't have to. The solution will be the same in either case.
43. $0.05 x+0.10 y=1.70$
44. $0.20 x+0.50 y=3.60$
$y=22-x$
$y=12-x$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working.
Working these problems is an excellent way to get ready for a test or quiz.
45. Work each problem according to the instructions given.
a. Solve: $4 y-5=20$
b. Solve for $y$ : $4 x-5 y=20$
c. Solve for $x: x-y=5$
d. Solve the system: $4 x-5 y=20$

$$
x-y=5
$$

46. Work each problem according to the instructions given.
a. Solve: $2 x-1=4$
b. Solve for $y$ : $2 x-y=4$
c. Solve for $x: x+3 y=9$
d. Solve the system: $2 x-y=4$

$$
x+3 y=9
$$

## Applying the Concepts

47. Gas Mileage Daniel is trying to decide whether to buy a car or a truck. The truck he is considering will cost him $\$ 150$ a month in loan payments, and it gets 20 miles per gallon in gas mileage. The car will cost $\$ 180$ a month in loan payments, but it gets 35 miles per gallon in gas mileage. Daniel estimates that he will pay $\$ 1.40$ per gallon for gas. This means that the monthly cost to drive the truck $x$ miles will be $y=\frac{1.40}{20} x+150$. The total monthly cost to drive the car $x$ miles will be $y=\frac{1.40}{35} x+180$. The following figure shows the graph of each equation.

a. At how many miles do the car and the truck cost the same to operate?
b. If Daniel drives more than 1,200 miles, which will be cheaper?
c. If Daniel drives fewer than 800 miles, which will be cheaper?
d. Why do the graphs appear in the first quadrant only?
48. Video Production Pat runs a small company that duplicates videotapes. The daily cost and daily revenue for a company duplicating videos are shown in the following figure. The daily cost for duplicating $x$ videos is $y=\frac{6}{5} x+20$; the daily revenue (the amount of money he brings in each day) for duplicating $x$ videos is $y=1.7 x$. The graphs of the two lines are shown in the following figure.

a. Pat will "break even" when his cost and his revenue are equal. How many videos does he need to duplicate to break even?
b. Pat will incur a loss when his revenue is less than his cost. If he duplicates 30 videos in one day, will he incur a loss?
c. Pat will make a profit when his revenue is larger than his costs. For what values of $x$ will Pat make a profit?
d. Why does the graph appear in the first quadrant only?

## Maintaining Your Skills

49. $6(3+4)+5$
50. $[(1+2)(2+3)]+(4 \div 2)$
51. $1^{2}+2^{2}+3^{2}$
52. $(1+2+3)^{2}$
53. $5(6+3 \cdot 2)+4+3 \cdot 2$
54. $(1+2)^{3}+[(2 \cdot 3)+(4 \cdot 5)]$
55. $\left(1^{3}+2^{3}\right)+[(2 \cdot 3)+(4 \cdot 5)]$
56. $[2(3+4+5)] \div 3$
57. $(2 \cdot 3+4+5) \div 3$
58. $10^{4}+10^{3}+10^{2}+10^{1}$
59. $6 \cdot 10^{3}+5 \cdot 10^{2}+4 \cdot 10^{1}$
60. $5 \cdot 10^{3}+2 \cdot 10^{2}+8 \cdot 10^{1}$
61. $1 \cdot 10^{3}+7 \cdot 10^{2}+6 \cdot 10^{1}+0$
62. $4(2-1)+5(3-2)$
63. $4 \cdot 2-1+5 \cdot 3-2$
64. $2^{3}+3^{2} \cdot 4-5$
65. $\left(2^{3}+3^{2}\right) \cdot 4-5$
66. $4^{2}-2^{4}+(2 \cdot 2)^{2}$
67. $2\left(2^{2}+3^{2}\right)+3\left(3^{2}\right)$
68. $2 \cdot 2^{2}+3^{2}+3 \cdot 3^{2}$

## Getting Ready for the Next Section

69. One number is eight more than five times another, their sum is 26 . Find the numbers.
70. One number is three less than four times another; their sum is 27 . Find the numbers.
71. The difference of two positive numbers is nine. The larger number is six less than twice the smaller number. Find the numbers.
72. The difference of two positive numbers is 17 . The larger number is one more than twice the smaller number. Find the numbers.
73. The length of a rectangle is 5 inches more than three times the width. The perimeter is 58 inches. Find the length and width.
74. The length of a rectangle is 3 inches less than twice the width. The perimeter is 36 inches. Find the length and width.
75. John has $\$ 1.70$ in nickels and dimes in his pocket. He has four more nickels than he does dimes. How many of each does he have?
76. Jamie has $\$ 2.65$ in dimes and quarters in her pocket. She has two more dimes than she does quarters. How many of each does she have?

Solve the systems by any method.
77. $y=5 x+2$
$x+y=20$
78. $y=2 x+3$
$x+y=9$
79. $x+y=15,000$
$0.06 x+0.07 y=980$
80. $x+y=22$
$0.05 x+0.1 y=1.70$

## 3.9

## Applications

## OBJECTIVES

A Apply the Blueprint for Problem Solving to a variety of application problems involving systems of equations.

I often have heard students remark about the word problems in elementary algebra: "What does this have to do with real life?" Most of the word problems we will encounter don't have much to do with "real life." We are actually just practicing. Ultimately, all problems requiring the use of algebra are word problems; that is, they are stated in words first, then translated to symbols. The problem then is solved by some system of mathematics, like algebra. Most real applications involve calculus or higher levels of mathematics. So, if the problems we solve are upsetting or frustrating to you, then you probably are taking them too seriously.
The word problems in this section have two unknown quantities. We will write two equations in two variables (each of which represents one of the unknown quantities), which of course is a system of equations. We then solve the system by one of the methods developed in the previous sections of this chapter. Here are the steps to follow in solving these word problems.

## Blueprint for Problem Solving Using a System of Equations

Step 1: Read the problem, and then mentally list the items that are known and the items that are unknown.
Step 2: Assign variables to each of the unknown items; that is, let $x=$ one of the unknown items and $y=$ the other unknown item. Then translate the other information in the problem to expressions involving the two variables.
Step 3: Reread the problem, and then write a system of equations, using the items and variables listed in steps 1 and 2 , that describes the situation.

Step 4: Solve the system found in step 3.
Step 5: Write your answers using complete sentences.
Step 6: Reread the problem, and check your solution with the original words in the problem.

## Note

We are using the substitution method here because the system we are solving is one in which the substitution method is the more convenient method.

Remember, the more problems you work, the more problems you will be able to work. If you have trouble getting started on the problem set, come back to the examples and work through them yourself. The examples are similar to the problems found in the problem set.

## Number Problem

EXAMPLE 1 One number is 2 more than 5 times another number. Their sum is 20 . Find the two numbers.

SOLUTION Applying the steps in our blueprint, we have
Step 1: We know that the two numbers have a sum of 20 and that one of them is 2 more than 5 times the other. We don't know what the numbers themselves are.
Step 2: Let $x$ represent one of the numbers and $y$ represent the other. "One number is 2 more than 5 times another" translates to

$$
y=5 x+2
$$

"Their sum is 20 " translates to

$$
x+y=20
$$

Step 3: The system that describes the situation must be

$$
\begin{aligned}
x+y & =20 \\
y & =5 x+2
\end{aligned}
$$

Step 4: We can solve this system by substituting the expression $5 x+2$ in the second equation for $y$ in the first equation:

$$
\begin{aligned}
x+(5 x+2) & =20 \\
6 x+2 & =20 \\
6 x & =18 \\
x & =3
\end{aligned}
$$

Using $x=3$ in either of the first two equations and then solving for $y$, we get $y=17$.

Step 5: So 17 and 3 are the numbers we are looking for.
Step 6: The number 17 is 2 more than 5 times 3 , and the sum of 17 and 3 is 20 .

## Interest Problem

EXAMPLE 2 Mr. Hicks had $\$ 15,000$ to invest. He invested part at $6 \%$ and the rest at $7 \%$. If he earns $\$ 980$ in interest, how much did he invest at each rate?

SOLUTION Remember, step 1 is done mentally.
Step 1: We do not know the specific amounts invested in the two accounts. We do know that their sum is $\$ 15,000$ and that the interest rates on the two accounts are 6\% and 7\%.

## Note

In this case we are using the elimination method. Notice also that multiplying the second equation by 100 clears it of decimals.

Step 2: Let $x=$ the amount invested at $6 \%$ and $y=$ the amount invested at $7 \%$. Because Mr. Hicks invested a total of \$15,000, we have

$$
x+y=15,000
$$

The interest he earns comes from 6\% of the amount invested at 6\% and $7 \%$ of the amount invested at $7 \%$. To find $6 \%$ of $x$, we multiply $x$ by 0.06, which gives us 0.06x. To find $7 \%$ of $y$, we multiply 0.07 times $y$ and get 0.07y.

$$
\begin{aligned}
& \text { Interest }+\begin{array}{c}
\text { interest } \\
\text { at } 6 \%
\end{array}=\begin{array}{c}
\text { total } \\
\text { at } 7 \%
\end{array} \\
& 0.06 x+0.07 y=980
\end{aligned}
$$

Step 3: The system is

$$
\begin{aligned}
x+y & =15,000 \\
0.06 x+0.07 y & =980
\end{aligned}
$$

Step 4: We multiply the first equation by -6 and the second by 100 to eliminate $x$ :

$$
\begin{aligned}
& x+\quad y=15,000 \xrightarrow{\text { Multiply by }-6}-6 x-6 y=-90,000 \\
& 0.06 x+0.07 y=980 \quad \begin{array}{l}
\text { Multiply by } 100
\end{array} \\
& \begin{array}{l}
6 x+7 y=98,000 \\
\hline
\end{array}
\end{aligned}
$$

Substituting $y=8,000$ into the first equation and solving for $x$, we get $x=7,000$.

Step 5: He invested $\$ 7,000$ at $6 \%$ and $\$ 8,000$ at $7 \%$.
Step 6: Checking our solutions in the original problem, we have: The sum of $\$ 7,000$ and $\$ 8,000$ is $\$ 15,000$, the total amount he invested. To complete our check, we find the total interest earned from the two accounts:

The interest on $\$ 7,000$ at $6 \%$ is $0.06(7,000)=420$
The interest on $\$ 8,000$ at $7 \%$ is $0.07(8,000)=560$
The total interest is $\$ 980$

## Coin Problem

EXAMPLE 3 John has $\$ 1.70$ all in dimes and nickels. He has a total of 22 coins. How many of each kind does he have?

## SOLUTION

Step 1: We know that John has 22 coins that are dimes and nickels. We know that a dime is worth 10 cents and a nickel is worth 5 cents. We do not know the specific number of dimes and nickels he has.

Step 2: Let $x=$ the number of nickels and $y=$ the number of dimes. The total number of coins is 22 , so

$$
x+y=22
$$

The total amount of money he has is $\$ 1.70$, which comes from nickels and dimes:

$$
\begin{gathered}
\begin{array}{c}
\text { Amount of money } \\
\text { in nickels }
\end{array} \\
\begin{array}{c}
\text { amount of money } \\
\text { in dimes }
\end{array}
\end{gathered} \begin{gathered}
\text { total amount } \\
\text { of money }
\end{gathered}
$$

Step 3: The system that represents the situation is

$$
\begin{aligned}
x+y & =22 \quad \text { The number of coins } \\
0.05 x+0.10 y & =1.70 \quad \text { The value of the coins }
\end{aligned}
$$

Step 4: We multiply the first equation by -5 and the second by 100 to eliminate the variable $x$ :

$$
\begin{aligned}
& x+y=22 \xrightarrow{\text { Multiply by }-5}-5 x-5 y=-110 \\
& 0.05 x+0.10 y=1.70 \xrightarrow[\text { Multiply by } 100]{ } \begin{array}{l}
5 x+10 y
\end{array}=170 \\
& 5 y=60 \\
& y=12
\end{aligned}
$$

Substituting $y=12$ into our first equation, we get $x=10$.
Step 5: John has 12 dimes and 10 nickels.
Step 6: Twelve dimes and 10 nickels total 22 coins.
12 dimes are worth $12(0.10)=1.20$
10 nickels are worth $10(0.05)=0.50$
The total value is $\$ 1.70$

## Mixture Problem

EXAMPLE 4 How much $20 \%$ alcohol solution and $50 \%$ alcohol solution must be mixed to get 12 gallons of $30 \%$ alcohol solution?
SOLUTION To solve this problem we must first understand that a $20 \%$ alcohol solution is $20 \%$ alcohol and $80 \%$ water.

Step 1: We know there are two solutions that together must total 12 gallons. $20 \%$ of one of the solutions is alcohol and the rest is water, whereas the other solution is $50 \%$ alcohol and $50 \%$ water. We do not know how many gallons of each individual solution we need.
Step 2: Let $x=$ the number of gallons of $20 \%$ alcohol solution needed and $y=$ the number of gallons of $50 \%$ alcohol solution needed. Because the total number of gallons we will end up with is 12 , and this 12 gallons must come from the two solutions we are mixing, our first equation is

$$
x+y=12
$$

To obtain our second equation, we look at the amount of alcohol in our two original solutions and our final solution. The amount of alcohol in the $x$ gallons of $20 \%$ solution is $0.20 x$, and the amount of alcohol
in $y$ gallons of $50 \%$ solution is $0.50 y$. The amount of alcohol in the 12 gallons of $30 \%$ solution is 0.30 (12). Because the amount of alcohol we start with must equal the amount of alcohol we end up with, our second equation is

$$
0.20 x+0.50 y=0.30(12)
$$

The information we have so far can also be summarized with a table. Sometimes by looking at a table like the one that follows it is easier to see where the equations come from.

|  | $\mathbf{2 0 \%}$ Solution | $\mathbf{5 0 \%}$ Solution | Final Solution |
| :--- | :---: | :---: | :---: |
| Number <br> of Gallons <br> Gallons of <br> Alcohol | $x$ | $y$ | 12 |
|  | $0.20 x$ | $0.50 y$ | $0.30(12)$ |

Step 3: Our system of equations is

$$
\begin{aligned}
x+y & =12 \\
0.20 x+0.50 y & =0.30(12)
\end{aligned}
$$

Step 4: We can solve this system by substitution. Solving the first equation for $y$ and substituting the result into the second equation, we have

$$
0.20 x+0.50(12-x)=0.30(12)
$$

Multiplying each side by 10 gives us an equivalent equation that is a little easier to work with.

$$
\begin{aligned}
2 x+5(12-x) & =3(12) \\
2 x+60-5 x & =36 \\
-3 x+60 & =36 \\
-3 x & =-24 \\
x & =8
\end{aligned}
$$

If $x$ is 8 , then $y$ must be 4 because $x+y=12$.
Step 5: It takes 8 gallons of $20 \%$ alcohol solution and 4 gallons of $50 \%$ alcohol solution to produce 12 gallons of $30 \%$ alcohol solution.

Step 6: Try it and see.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. If you were to apply the Blueprint for Problem Solving from Section 2.6 to the examples in this section, what would be the first step?
2. If you were to apply the Blueprint for Problem Solving from Section 2.6 to the examples in this section, what would be the last step?
3. Which method of solving these systems do you prefer? Why?
4. Write an application problem for which the solution depends on solving a system of equations.

## Problem Set 3.9

Solve the following word problems. Be sure to show the equations used.

## Number Problems

1. Two numbers have a sum of 25 . One number is 5 more than the other. Find the numbers.
2. The difference of two numbers is 6 . Their sum is 30. Find the two numbers.
3. The sum of two numbers is 15 . One number is 4 times the other. Find the numbers.
4. The difference of two positive numbers is 28 . One number is 3 times the other. Find the two numbers.
5. Two positive numbers have a difference of 5 . The larger number is one more than twice the smaller. Find the two numbers.
6. One number is 2 more than 3 times another. Their sum is 26 . Find the two numbers.
7. One number is 5 more than 4 times another. Their sum is 35 . Find the two numbers.
8. The difference of two positive numbers is 8 . The larger is twice the smaller decreased by 7 . Find the two numbers.

## Interest Problems

9. Mr. Wilson invested money in two accounts. His total investment was $\$ 20,000$. If one account pays $6 \%$ in interest and the other pays $8 \%$ in interest, how much does he have in each account if he earned a total of $\$ 1,380$ in interest in 1 year?
10. A total of $\$ 11,000$ was invested. Part of the $\$ 11,000$ was invested at $4 \%$, and the rest was invested at $7 \%$. If the investments earn $\$ 680$ per year, how much was invested at each rate?
11. A woman invested 4 times as much at $5 \%$ as she did at 6\%. The total amount of interest she earns in 1 year from both accounts is $\$ 520$. How much did she invest at each rate?
12. Ms. Hagan invested twice as much money in an account that pays $7 \%$ interest as she did in an account that pays $6 \%$ in interest. Her total investment pays her $\$ 1,000$ a year in interest. How much did she invest at each rate?

## Coin Problems

13. Ron has 14 coins with a total value of $\$ 2.30$. The coins are nickels and quarters. How many of each coin does he have?
14. Diane has $\$ 0.95$ in dimes and nickels. She has a total of 11 coins. How many of each kind does she have?
15. Suppose Tom has 21 coins totaling $\$ 3.45$. If he has only dimes and quarters, how many of each type does he have?
16. A coin collector has 31 dimes and nickels with a total face value of $\$ 2.40$. (They are actually worth a lot more.) How many of each coin does she have?

## Mixture Problems

17. How many liters of $50 \%$ alcohol solution and $20 \%$ alcohol solution must be mixed to obtain 18 liters of $30 \%$ alcohol solution?

|  | $50 \%$ <br> Solution | $\mathbf{2 0 \%}$ <br> Solution | Final <br> Solution |
| :---: | :---: | :---: | :---: |
| Number <br> of Liters <br> Liters <br> of Alcohol | $x$ | $y$ |  |

18. How many liters of $10 \%$ alcohol solution and $5 \%$ alcohol solution must be mixed to obtain 40 liters of $8 \%$ alcohol solution?

|  | $10 \%$ <br> Solution | $5 \%$ <br> Solution | Final <br> Solution |
| :---: | :---: | :---: | :---: |
| Number <br> of Liters <br> Liters <br> of Alcohol | $x$ | $y$ |  |

19. A mixture of $8 \%$ disinfectant solution is to be made from $10 \%$ and $7 \%$ disinfectant solutions. How much of each solution should be used if 30 gallons of $8 \%$ solution are needed?
20. How much 50\% antifreeze solution and $40 \%$ antifreeze solution should be combined to give 50 gallons of $46 \%$ antifreeze solution?

## Miscellaneous Problems

21. For a Saturday matinee, adult tickets cost $\$ 5.50$ and kids under 12 pay only $\$ 4.00$. If 70 tickets are sold for a total of $\$ 310$, how many of the tickets were adult tickets and how many were sold to kids under 12 ?
22. The Bishop's Peak $4-\mathrm{H}$ club is having its annual fundraising dinner. Adults pay $\$ 15$ apiece and children pay $\$ 10$ apiece. If the number of adult tickets sold is twice the number of children's tickets sold, and the total income for the dinner is $\$ 1,600$, how many of each kind of ticket did the 4-H club sell?
23. A farmer has 96 feet of fence with which to make a corral. If he arranges it into a rectangle that is twice as long as it is wide, what are the dimensions?

24. If a 22 -inch rope is to be cut into two pieces so that one piece is 3 inches longer than twice the other, how long is each piece?

25. A gambler finishes a session of blackjack with $\$ 5$ chips and $\$ 25$ chips. If he has 45 chips in all, with a total value of $\$ 465$, how many of each kind of chip does the gambler have?
26. Tyler has been saving his winning lottery tickets. He has 23 tickets that are worth a total of \$175. If each ticket is worth either $\$ 5$ or $\$ 10$, how many of each does he have?
27. Mary Jo spends $\$ 2,550$ to buy stock in two companies. She pays $\$ 11$ a share to one of the companies and $\$ 20$ a share to the other. If she ends up with a total of 150 shares, how many shares did she buy at $\$ 11$ a share and how many did she buy at $\$ 20$ a share?
28. Kelly sells 62 shares of stock she owns for a total of $\$ 433$. If the stock was in two different companies, one selling at $\$ 6.50$ a share and the other at $\$ 7.25$ a share, how many of each did she sell?

## Maintaining Your Skills

29. Fill in each ordered pair so that it is a solution to $y=\frac{1}{2} x+3 . \quad(-2, \quad),(0),,(2$,
30. Graph the line $y=\frac{1}{2} x+3$.
31. Graph the line $x=-2$.
32. Graph $3 x-2 y=6$.
33. Find the slope of the line through $(2,5)$ and $(0,1)$.
34. Find the slope and $y$-intercept for the line $2 x-5 y=10$.

## Chapter 3 sUMMARY

## EXAMPLES

1. The equation $3 x+2 y=6$ is an example of a linear equation in two variables.
2. The graph of $y=-\frac{2}{3} x-1$ is shown below.


## Intercepts [3.4]

The $x$-intercept of an equation is the $x$-coordinate of the point where the graph crosses the $x$-axis. The $y$-intercept is the $y$-coordinate of the point where the graph crosses the $y$-axis. We find the $y$-intercept by substituting $x=0$ into the equation and solving for $y$. The $x$-intercept is found by letting $y=0$ and solving for $x$.

## Slope of a Line [3.5]

The slope of the line containing the points $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ is given by

$$
\text { Slope }=m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{\text { rise }}{\text { run }}
$$

## Slope-Intercept Form of a Straight Line [3.5]

The equation of the line with a slope of $m$ and $a y$-intercept $b$ is

$$
y=m x+b
$$

## Definitions [3.6]

6. The solution to the system

$$
\begin{array}{r}
x+2 y=4 \\
x-y=1
\end{array}
$$

is the ordered pair $(2,1)$. It is the only ordered pair that satisfies both equations.
7. Solving the system in Example 6 by graphing looks like


## Strategy for Solving a System by the Elimination Method [3.7]

Step 1: Look the system over to decide which variable will be easier to eliminate.

Step 2: Use the multiplication property of equality on each equation separately to ensure that the coefficients of the variable to be eliminated are opposites.

Step 3: Add the left and right sides of the system produced in step 2, and solve the resulting equation.
Step 4: Substitute the solution from step 3 back into any equation with both $x$ and $y$ variables, and solve.
Step 5: Check your solution in both equations, if necessary.
9. We can apply the substitution method to the system in Example 6 by first solving the second equation for $x$ to get $x=y+1$.
Substituting this expression for $x$ into the first equation, we have

$$
\begin{array}{r}
(y+1)+2 y=4 \\
3 y+1=4 \\
3 y=3 \\
y=1
\end{array}
$$

Using $y=1$ in either of the original equations gives $x=2$.

## Special Cases [3.6, 3.7, 3.8]

In some cases, using the elimination or substitution method eliminates both variables. The situation is interpreted as follows.

1. If the resulting statement is false, then the lines are parallel and there is no solution to the system.
2. If the resulting statement is true, then the equations represent the same line (the lines coincide). In this case any ordered pair that satisfies either equation is a solution to the system.

## COMMON MISTAKES

The most common mistake encountered in solving linear systems is the failure to complete the problem. Here is an example.

$$
\begin{aligned}
x+y & =8 \\
x-y & =4 \\
2 x & =12 \\
x & =6
\end{aligned}
$$

This is only half the solution. To find the other half, we must substitute the 6 back into one of the original equations and then solve for $y$.

Remember, solutions to systems of linear equations always consist of ordered pairs. We need an $x$-coordinate and a $y$-coordinate; $x=6$ can never be a solution to a system of linear equations.

## Chapter 3 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

The numbers in brackets refer to the sections of the text in which similar problems can be found.

For each equation, complete the given ordered pairs. [3.2]

1. $3 x+y=6$
$(4),,(0),,(, 3),(, 0)$
2. $2 x-5 y=20$
$(5),,(0),,(, 2),(, 0)$
3. $y=2 x-6$
$(4),,(,-2),(, 3)$
4. $y=5 x+3$
$(2),,(, 0),(,-3)$
5. $y=-3$
$(2),,(-1),,(-3$,
6. $x=6$
$(, 5),(, 0),(,-1)$

For the following equations, tell which of the given ordered pairs are solutions. [3.2]

$$
\begin{array}{ll}
\text { 7. } 3 x-4 y=12 & \left(-2, \frac{9}{2}\right),(0,3),\left(2,-\frac{3}{2}\right) \\
\text { 8. } y=3 x+7 & \left(-\frac{8}{3},-1\right),\left(\frac{7}{3}, 0\right),(-3,-2)
\end{array}
$$

Graph the following ordered pairs. [3.1]
9. $(4,2)$
10. $(-3,1)$
11. $(0,5)$
12. $(-2,-3)$
13. $(-3,0)$
14. $\left(5,-\frac{3}{2}\right)$

For the following equations, complete the given ordered pairs, and use the results to graph the solution set for the equations. [3.3]
15. $x+y=-2$
$(, 0),(0),,(1$,
16. $y=3 x$
$(-1),,(1),,(, 0)$
17. $y=2 x-1$
$(1),,(0),,(,-3)$
18. $x=-3$
$(, 0),(, 5),(,-5)$
Graph the following equations. [3.3]
19. $3 x-y=3$
20. $x-2 y=2$
21. $y=-\frac{1}{3} x$
22. $y=\frac{3}{4} x$
23. $y=2 x+1$
24. $y=-\frac{1}{2} x+2$
25. $x=5$
26. $y=-3$
27. $2 x-3 y=3$
28. $5 x-2 y=5$

Find the $x$ - and $y$-intercepts for each equation. [3.4]
29. $3 x-y=6$
30. $2 x-6 y=24$
31. $y=x-3$
32. $y=3 x-6$
33. $y=-5$
34. $x=4$

Find the slope of the line through the given pair of points. [3.5]
35. $(2,3),(3,5)$
36. $(-2,3),(6,-5)$
37. $(-1,-4),(-3,-8)$
38. $\left(\frac{1}{2}, 4\right),\left(-\frac{1}{2}, 2\right)$

Find the equation of the line with the given slope and $y$-intercept. [3.5]
39. $m=3, b=2$
40. $m=-1, b=6$
41. $m=-\frac{1}{3}, b=\frac{3}{4}$
42. $m=0, b=0$

For each of the following equations, determine the slope and $y$-intercept. [3.5]
43. $y=4 x-1$
44. $2 x+y=-5$
45. $6 x+3 y=9$
46. $5 x+2 y=8$

Solve the following systems by graphing. [3.6]
47. $x+y=2$
$x-y=6$
48. $x+y=-1$
$-x+y=5$
49. $2 x-3 y=12$
50. $4 x-2 y=8$
$-2 x+y=-8$
$3 x+y=6$
51. $y=2 x-3$
$y=-2 x+5$
52. $y=-x-3$
$y=3 x+1$

Solve the following systems by the elimination method. [3.7]
53. $x-y=4$
$x+y=-2$
54. $-x-y=-3$
$2 x+y=1$
55. $5 x-3 y=2$
56. $2 x+3 y=-2$
$-10 x+6 y=-4$
$3 x-2 y=10$
57. $-3 x+4 y=1$ $-4 x+y=-3$
58. $-4 x-2 y=3$
$2 x+y=1$
59. $-2 x+5 y=-11$
$7 x-3 y=-5$
60. $-2 x+5 y=-15$
$3 x-4 y=19$

Solve the following systems by substitution. [3.8]
61. $x+y=5$
$y=-3 x+1$
62. $x-y=-2$
$y=-2 x-10$
63. $4 x-3 y=-16$

$$
y=3 x+7
$$

64. $5 x+2 y=-2$
$y=-8 x+10$
65. $x-4 y=2$
$-3 x+12 y=-8$
66. $4 x-2 y=8$
$3 x+y=-19$
67. $10 x-5 y=20$
$x+6 y=-11$
68. $3 x-y=2$
$-6 x+2 y=-4$

Solve the following word problems. Be sure to show the equations used. [3.9]
69. Number Problem The sum of two numbers is 18 . If twice the smaller number is 6 more than the larger, find the two numbers.
70. Number Problem The difference of two positive numbers is 16 . One number is 3 times the other. Find the two numbers.
71. Investing A total of $\$ 12,000$ was invested. Part of the $\$ 12,000$ was invested at $4 \%$, and the rest was invested at $5 \%$. If the interest for one year is $\$ 560$, how much was invested at each rate?
72. Investing A total of \$14,000 was invested. Part of the $\$ 14,000$ was invested at $6 \%$, and the rest was invested at $8 \%$. If the interest for one year is $\$ 1,060$, how much was invested at each rate?
73. Coin Problem Barbara has $\$ 1.35$ in dimes and nickels. She has a total of 17 coins. How many of each does she have?
74. Coin Problem Tom has $\$ 2.40$ in dimes and quarters. He has a total of 15 coins. How many of each does he have?
75. Mixture Problem How many liters of 20\% alcohol solution and $10 \%$ alcohol solution must be mixed to obtain 50 liters of a $12 \%$ alcohol solution?
76. Mixture Problem How many liters of $25 \%$ alcohol solution and $15 \%$ alcohol solution must be mixed to obtain 40 liters of a $20 \%$ alcohol solution?

## Chapter 3 Projects



Systems of Linear Equation

## GROUP PROJECT Tables and Graphs

Number of People 2-3
Time Needed 8-12 minutes
Equipment Pencil, ruler, and graph paper

Background In the Chapter 2 project, we built the table below. As you may recall, we built the table based on the different combinations of coins from a parking meter that accepts only dimes and quarters. The amount of money in the meter at the end of one particular day was $\$ 3.15$.

| Number of Dimes | Number of <br> Quarters | Total Coins | Value |
| :---: | :---: | :---: | :---: |
| 29 | 1 | 30 | $\$ 3.15$ |
| 24 | 3 | 27 | $\$ 3.15$ |
| 19 | 5 | 24 | $\$ 3.15$ |
| 14 | 7 | 21 | $\$ 3.15$ |
| 9 | 9 | 18 | $\$ 3.15$ |
| 4 | 11 | 15 | $\$ 3.15$ |
| $x$ | $y$ | $x+y$ |  |

## Procedure

1. From each row in the table, plot the points $(x, y)$ where $x$ is the number of dimes and $y$ is the number of quarters. For example, the first point to plot is $(29,1)$. Then connect the points by drawing a straight line through them. The line should pass through all the points.
2. What is the equation of the line that passes through all the points?
3. Draw six additional lines on the same grid using the following equations:
$x+y=30, x+y=27, x+y=24, x+y=21$, $x+y=18$, and $x+y=15$.
4. Write a parking meter application problem, the solution to which is the pair of lines $10 x+25 y=315$ and $x+y=24$.

## RESEARCH PROJECT

## Zeno's Paradoxes



zeno of Elea was born at about the same time that Pythagoras died. He is responsible for three paradoxes that have come to be known as Zeno's paradoxes. One of the three has to do with a race between Achilles and a tortoise. Achilles is much faster than the tortoise, but the tortoise has a head start. According to Zeno's method of reasoning, Achilles can never pass the tortoise because each time he reaches the place where the tortoise was, the tortoise is gone. Research Zeno's paradox concerning Achilles and the tortoise. Put your findings into essay form that begins with a definition for the word "paradox." Then use Zeno's method of reasoning to describe a race between Achilles and the tortoise-if Achilles runs at 10 miles per hour, the tortoise runs at 1 mile per hour, and the tortoise has a 1-mile head start. Next, use the methods shown in this chapter to find the distance at which Achilles reaches the tortoise and the time at which Achilles reaches the tortoise. Conclude your essay by summarizing what you have done and showing how the two results you have obtained form a paradox.

## Exponents and Polynomials

## 4


f you were given a penny on the first day of September, and then each day after that you were given twice the amount of money you received the day before, how much money would you receive on September 30? To begin, Table 1 and Figure 1 show the amount of money you would receive on each of the first 10 days of the month. As you can see, on the tenth day of the month you would receive \$5.12.

TABLE 1
Money That Doubles Each Day

| Day | Money <br> (in cents) |
| :--- | :---: |
| 1 | $1=2^{0}$ |
| 2 | $2=2^{1}$ |
| 3 | $4=2^{2}$ |
| 4 | $8=2^{3}$ |
| 5 | $16=2^{4}$ |
| 6 | $32=2^{5}$ |
| 7 | $64=2^{6}$ |
| 8 | $128=2^{7}$ |
| 9 | $256=2^{8}$ |
| 10 | $512=2^{9}$ |

To find the amount of money on day 30 , we could continue to double the amount on each of the next 20 days. Or, we could notice the pattern of exponents in the second column of the table and reason that the amount of money on day 30 would be $2^{29}$ cents, which is a very large number. In fact, $2^{29}$ cents is $\$ 5,368,709.12-a$ little less than $\$ 5.4$ million. When you are finished with this chapter, you will have a good working knowledge of exponents.

## CHAPTER OUTLINE

4.1 Multiplication with Exponents
4.2 Division with Exponents
4.3 Operations with Monomials
4.4 Addition and Subtraction of Polynomials
4.5 Multiplication with Polynomials
4.6 Binomial Squares and Other Special Products
4.7 Dividing a Polynomial by a Monomial
4.8 Dividing a Polynomial by a Polynomial

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## 1 Be Focused, Not Distracted

I have students who begin their assignments by asking themselves, "Why am I taking this class?" Or, "When am I ever going to use this stuff?" If you are asking yourself similar questions, you may be distracting yourself away from doing the things that will produce the results you want in this course. Don't dwell on questions and evaluations of the class that can be used as excuses for not doing well. If you want to succeed in this course, focus your energy and efforts toward success, rather than distracting yourself away from your goals.

## 2 Be Resilient

Don't let setbacks keep you from your goals. You want to put yourself on the road to becoming a person who can succeed in this class or any class in college. Failing a test or quiz, or having a difficult time on some topics, is normal. No one goes through college without some setbacks. Don't let a temporary disappointment keep you from succeeding in this course. A low grade on a test or quiz is simply a signal that some reevaluation of your study habits needs to take place.

## 3 Intend to Succeed

I always have a few students who simply go through the motions of studying without intending to master the material. It is more important to them to look like they are studying than to actually study. You need to study with the intention of being successful in the course. Intend to master the material, no matter what it takes.

## OBJECTIVES

A Use the definition of integer exponents to evaluate expressions containing exponents.

B Use Property 1 for exponents.

C Use Property 2 for exponents.

D Use Property 3 for exponents.

## E Write numbers in scientific notation and expanded form.

Recall that an exponent is a number written just above and to the right of another number, which is called the base. In the expression $5^{2}$, for example, the exponent is 2 and the base is 5 . The expression $5^{2}$ is read " 5 to the second power" or " 5 squared." The meaning of the expression is

$$
5^{2}=5 \cdot 5=25
$$

In the expression $5^{3}$, the exponent is 3 and the base is 5 . The expression $5^{3}$ is read " 5 to the third power" or " 5 cubed." The meaning of the expression is

$$
5^{3}=5 \cdot 5 \cdot 5=125
$$

Here are some further examples.

## EXAMPLES

1. $4^{3}=4 \cdot 4 \cdot 4=16 \cdot 4=64$
2. $-3^{4}=-3 \cdot 3 \cdot 3 \cdot 3=-81$
3. $(-2)^{5}=(-2)(-2)(-2)(-2)(-2)=-32$
4. $\left(-\frac{3}{4}\right)^{2}=\left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right)=\frac{9}{16}$

## Exponent 3, base 4

Exponent 4, base 3
Exponent 5, base - 2
Exponent 2, base $-\frac{3}{4}$

QUESTION: In what way are $(-5)^{2}$ and $-5^{2}$ different?
ANSWER: In the first case, the base is -5 . In the second case, the base is 5 . The answer to the first is 25 . The answer to the second is -25 . Can you tell why? Would there be a difference in the answers if the exponent in each case were changed to 3 ?

We can simplify our work with exponents by developing some properties of exponents. We want to list the things we know are true about exponents and then use these properties to simplify expressions that contain exponents.

The first property of exponents applies to products with the same base. We can use the definition of exponents, as indicating repeated multiplication, to simplify expressions like $7^{4} \cdot 7^{2}$.

$$
\begin{aligned}
7^{4} \cdot 7^{2} & =(7 \cdot 7 \cdot 7 \cdot 7)(7 \cdot 7) \\
& =(7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7) \\
& =7^{6}
\end{aligned}
$$

## Notice: $4+2=6$

As you can see, multiplication with the same base resulted in addition of exponents. We can summarize this result with the following property.

Property I for Exponents If $a$ is any real number and $r$ and $s$ are integers, then

$$
a^{r} \cdot a^{s}=a^{r+s}
$$

In words: To multiply two expressions with the same base, add exponents and use the common base.

## Note

In Examples 5, 6, and 7, notice that in each case the base in the original problem is the same base that appears in the answer and that it is written only once in the answer. A very common mistake that people make when they first begin to use Property 1 is to write a 2 in front of the base in the answer. For example, people making this mistake would get $2 x^{15}$ or $(2 x)^{15}$ as the result in Example 6. To avoid this mistake, you must be sure you understand the meaning of Property 1 exactly as it is written.

Here are some examples using Property 1.

## EXAMPLES Use Property 1 to simplify the following expressions.

 Leave your answers in terms of exponents:5. $5^{3} \cdot 5^{6}=5^{3+6}=5^{9}$
6. $x^{7} \cdot X^{8}=x^{7+8}=X^{15}$
7. $3^{4} \cdot 3^{8} \cdot 3^{5}=3^{4+8+5}=3^{17}$

Another common type of expression involving exponents is one in which an expression containing an exponent is raised to another power. The expression $\left(5^{3}\right)^{2}$ is an example:

$$
\begin{aligned}
\left(5^{3}\right)^{2} & =\left(5^{3}\right)\left(5^{3}\right) \\
& =5^{3+3} \\
& =5^{6} \quad \text { Notice: } \mathbf{3} \cdot \mathbf{2}=\mathbf{6}
\end{aligned}
$$

This result offers justification for the second property of exponents.

Property 2 for Exponents If $a$ is any real number and $r$ and $s$ are integers, then

$$
\left(a^{r}\right)^{s}=a^{r \cdot s}
$$

In words: A power raised to another power is the base raised to the product of the powers.

## EXAMPLES Simplify the following expressions:

8. $\left(4^{5}\right)^{6}=4^{5 \cdot 6}=4^{30}$
9. $\left(x^{3}\right)^{5}=x^{3.5}=x^{15}$

The third property of exponents applies to expressions in which the product of two or more numbers or variables is raised to a power. Let's look at how the expression $(2 x)^{3}$ can be simplified:

$$
\begin{aligned}
(2 x)^{3} & =(2 x)(2 x)(2 x) \\
& =(2 \cdot 2 \cdot 2)(x \cdot x \cdot x) \\
& =2^{3} \cdot x^{3} \quad \text { Notice: The exponent } 3 \text { distributes over the product } 2 x \\
& =8 x^{3}
\end{aligned}
$$

We can generalize this result into a third property of exponents.

Property 3 for Exponents If $a$ and $b$ are any two real numbers and $r$ is an integer, then

$$
(a b)^{r}=a^{r} b^{r}
$$

In words: The power of a product is the product of the powers.

Here are some examples using Properties 1-3 to simplify expressions.

## Note

If we include units with the dimensions of the diagrams, then the units for the area will be square units and the units for volume will be cubic units. More specifically:

If a square has a side 5 inches long, then its area will be

$$
A=\left(5 \text { inches }^{2}\right)^{2}=25 \text { inches }^{2}
$$

where the unit inches ${ }^{2}$ stands for square inches.
If a cube has a single side 5 inches long, then its volume will be

$$
V=\left(5 \text { inches }^{3}\right)^{3}=125 \text { inches }^{3}
$$

where the unit inches ${ }^{3}$ stands for cubic inches

If a rectangular solid has a length of 5 inches, a width of 4 inches, and a height of 3 inches, then its volume is

$$
\begin{aligned}
V & =(5 \text { inches })(4 \text { inches })(3 \text { inches }) \\
& =60 \text { inches }^{3}
\end{aligned}
$$

## EXAMPLES Simplify the following expressions:

10. $(5 x)^{2}=5^{2} \cdot x^{2}$

## Property 3

$$
=25 x^{2}
$$

11. $(2 x y)^{3}=2^{3} \cdot x^{3} \cdot y^{3}$

$$
=8 x^{3} y^{3}
$$

12. $\left(3 x^{2}\right)^{3}=3^{3}\left(x^{2}\right)^{3}$

$$
=27 x^{6}
$$

Property 3
Property 2
13. $\left(-\frac{1}{4} x^{2} y^{3}\right)^{2}=\left(-\frac{1}{4}\right)^{2}\left(x^{2}\right)^{2}\left(y^{3}\right)^{2} \quad$ Property 3

$$
=\frac{1}{16} x^{4} y^{6} \quad \text { Property } 2
$$

14. $\left(x^{4}\right)^{3}\left(x^{2}\right)^{5}=x^{12} \cdot x^{10}$

Property 2

$$
=x^{22}
$$

Property 1
15. $(2 y)^{3}\left(3 y^{2}\right)=2^{3} y^{3}\left(3 y^{2}\right)$

$$
\begin{aligned}
& =8 \cdot 3\left(y^{3} \cdot y^{2}\right) \\
& =24 y^{5}
\end{aligned}
$$

Property 3
Commutative and associative properties
Property 1
16. $\left(2 x^{2} y^{5}\right)^{3}\left(3 x^{4} y\right)^{2}=2^{3}\left(x^{2}\right)^{3}\left(y^{5}\right)^{3} \cdot 3^{2}\left(x^{4}\right)^{2} y^{2}$

Property 3
$=8 x^{6} y^{15} \cdot 9 x^{8} y^{2}$
Property 2
$=(8 \cdot 9)\left(x^{6} x^{8}\right)\left(y^{15} y^{2}\right)$
$=72 x^{14} y^{17}$
Commutative and associative properties
Property 1

## FACTS FROM GEOMETRY

## Volume of a Rectangular Solid

It is easy to see why the phrase "five squared" is associated with the expression $5^{2}$. Simply find the area of the square shown in Figure 1 with a side of 5 .


FIGURE 1


FIGURE 2

To see why the phrase "five cubed" is associated with the expression $5^{3}$, we have to find the volume of a cube for which all three dimensions are 5 units long. The volume of a cube is a measure of the space occupied by the cube. To calculate the volume of the cube shown in Figure 2, we multiply the three dimensions together to get $5 \cdot 5 \cdot 5=5^{3}$.

The cube shown in Figure 2 is a special case of a general category of threedimensional geometric figures called rectangular solids. Rectangular solids have rectangles for sides, and all connecting sides meet at right angles. The three dimensions are length, width, and height. To find the volume of a rectangular solid, we find the product of the three dimensions, as shown in Figure 3.


FIGURE 3

## Scientific Notation

Many branches of science require working with very large numbers. In astronomy, for example, distances commonly are given in light-years. A light-year is the distance light travels in a year. It is approximately

$$
5,880,000,000,000 \text { miles }
$$

This number is difficult to use in calculations because of the number of zeros it contains. Scientific notation provides a way of writing very large numbers in a more manageable form.

DEFINITION A number is in scientific notation when it is written as the product of a number between 1 and 10 and an integer power of 10. A number written in scientific notation has the form

$$
n \times 10^{r}
$$

where $1 \leq n<10$ and $r=$ an integer.

## EXAMPLE 17 Write 376,000 in scientific notation.

SOLUTION We must rewrite 376,000 as the product of a number between 1 and 10 and a power of 10 . To do so, we move the decimal point 5 places to the left so that it appears between the 3 and the 7 . Then we multiply this number by $10^{5}$. The number that results has the same value as our original number and is written in scientific notation:


EXAMPLE 18 Write $4.52 \times 10^{3}$ in expanded form.
SOLUTION Since $10^{3}$ is 1,000 , we can think of this as simply a multiplication problem; that is,

$$
4.52 \times 10^{3}=4.52 \times 1,000=4,520
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $1-4$
B 5-7, 14-16
C $8,9,12-14,16$
D $10-13,15,16$
E 17,18

On the other hand, we can think of the exponent 3 as indicating the number of places we need to move the decimal point to write our number in expanded form. Since our exponent is positive 3, we move the decimal point three places to the right:

$$
4.52 \times 10^{3}=4,520
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Explain the difference between $-5^{2}$ and $(-5)^{2}$.
2. How do you multiply two expressions containing exponents when they each have the same base?
3. What is Property 2 for exponents?
4. When is a number written in scientific notation?

## Problem Set 4.1

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Name the base and exponent in each of the following expressions. Then use the definition of exponents as repeated multiplication to simplify.

1. $4^{2}$
2. $6^{2}$
3. $(0.3)^{2}$
4. $(0.03)^{2}$
5. $4^{3}$
6. $10^{3}$
7. $(-5)^{2}$
8. $-5^{2}$
9. $-2^{3}$
10. $(-2)^{3}$
11. $3^{4}$
12. $(-3)^{4}$
13. $\left(\frac{2}{3}\right)^{2}$
14. $\left(\frac{2}{3}\right)^{3}$
15. $\left(\frac{1}{2}\right)^{4}$
16. $\left(\frac{4}{5}\right)^{2}$
17. a. Complete the following table.

| Number <br> $\boldsymbol{x}$ | Square <br> $\boldsymbol{x}^{\mathbf{2}}$ |
| :---: | :---: |
| 1 |  |
| 2 |  |
| 3 |  |
| 4 |  |
| 5 |  |
| 6 |  |
| 7 |  |

b. Using the results of part a, fill in the blank in the following statement: For numbers larger than 1, the square of the number is $\qquad$ than the number.
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## 236 Chapter 4 Exponents and Polynomials

18. a. Complete the following table.

| $\substack{\text { Number } \\ \boldsymbol{x}}$ | Square <br> $\boldsymbol{x}^{2}$ |
| :---: | :---: |
| $\frac{1}{2}$ |  |
| $\frac{1}{3}$ |  |
| $\frac{1}{4}$ |  |
| $\frac{1}{5}$ |  |
| $\frac{1}{6}$ |  |
| $\frac{1}{7}$ |  |
| $\frac{1}{8}$ |  |

b. Using the results of part $a$, fill in the blank in the following statement: For numbers between 0 and 1 , the square of the number is $\qquad$ than the number.

Use Property 1 for exponents to simplify each expression. Leave all answers in terms of exponents.
19. $x^{4} \cdot x^{5}$
20. $x^{7} \cdot x^{3}$
21. $y^{10} \cdot y^{20}$
22. $y^{30} \cdot y^{30}$
23. $2^{5} \cdot 2^{4} \cdot 2^{3}$
24. $4^{2} \cdot 4^{3} \cdot 4^{4}$
25. $x^{4} \cdot x^{6} \cdot x^{8} \cdot x^{10}$
26. $x^{20} \cdot x^{18} \cdot x^{16} \cdot x^{14}$

Use Property 2 for exponents to write each of the following problems with a single exponent. (Assume all variables are positive numbers.)
27. $\left(x^{2}\right)^{5}$
28. $\left(x^{5}\right)^{2}$
29. $\left(5^{4}\right)^{3}$
30. $\left(5^{3}\right)^{4}$
31. $\left(y^{3}\right)^{3}$
32. $\left(y^{2}\right)^{2}$
33. $\left(2^{5}\right)^{10}$
34. $\left(10^{5}\right)^{2}$
35. $\left(a^{3}\right)^{x}$
36. $\left(a^{5}\right)^{x}$
37. $\left(b^{x}\right)^{y}$
38. $\left(b^{r}\right)^{s}$

Use Property 3 for exponents to simplify each of the following expressions.
39. $(4 x)^{2}$
40. $(2 x)^{4}$
41. $(2 y)^{5}$
42. $(5 y)^{2}$
43. $(-3 x)^{4}$
44. $(-3 x)^{3}$
45. $(0.5 a b)^{2}$
46. $(0.4 a b)^{2}$
47. $(4 x y z)^{3}$
48. $(5 x y z)^{3}$

Simplify the following expressions by using the properties of exponents.
49. $\left(2 x^{4}\right)^{3}$
50. $\left(3 x^{5}\right)^{2}$
51. $\left(4 a^{3}\right)^{2}$
52. $\left(5 a^{2}\right)^{2}$
53. $\left(x^{2}\right)^{3}\left(x^{4}\right)^{2}$
54. $\left(x^{5}\right)^{2}\left(x^{3}\right)^{5}$
55. $\left(a^{3}\right)^{1}\left(a^{2}\right)^{4}$
56. $\left(a^{4}\right)^{1}\left(a^{1}\right)^{3}$
57. $(2 x)^{3}(2 x)^{4}$
58. $(3 x)^{2}(3 x)^{3}$
59. $\left(3 x^{2}\right)^{3}(2 x)^{4}$
60. $(3 x)^{3}\left(2 x^{3}\right)^{2}$
61. $\left(4 x^{2} y^{3}\right)^{2}$
62. $\left(9 x^{3} y^{5}\right)^{2}$
63. $\left(\frac{2}{3} a^{4} b^{5}\right)^{3}$
64. $\left(\frac{3}{4} a b^{7}\right)^{3}$

Write each expression as a perfect square.
65. $x^{4}=(\quad)^{2}$
66. $x^{6}=(\quad)^{2}$
67. $16 x^{2}=(\quad)^{2}$
68. $256 x^{4}=(\quad)^{2}$

Write each expression as a perfect cube.
69. $8=()^{3}$
70. $27=()^{3}$
71. $64 x^{3}=(\quad)^{3}$
72. $27 x^{6}=(\quad)^{3}$
73. Let $x=2$ in each of the following expressions and simplify.
a. $X^{3} X^{2}$
b. $\left(X^{3}\right)^{2}$
c. $X^{5}$
d. $X^{6}$
74. Let $x=-1$ in each of the following expressions and simplify.
a. $X^{3} X^{4}$
b. $\left(x^{3}\right)^{4}$
c. $X^{7}$
d. $X^{12}$
75. Complete the following table, and then construct a line graph of the information in the table.

| Number <br> $\boldsymbol{x}$ | Square <br> $\boldsymbol{x}^{2}$ |
| :---: | :---: |
| -3 |  |
| -2 |  |
| -1 |  |
| 0 |  |
| 1 |  |
| 2 |  |
| 3 |  |

76. Complete the table, and then construct a line graph of the information in the table.
\(\left.\begin{array}{c|c}Number <br>

\boldsymbol{x}\end{array}\right)\)| Cube |
| :---: |
| $\boldsymbol{x}^{3}$ |

77. Complete the table. When you are finished, notice how the points in this table could be used to refine the line graph you created in Problem 75.

| Number <br> $\boldsymbol{x}$ | Square <br> $\boldsymbol{x}^{2}$ |
| :---: | :---: |
| -2.5 |  |
| -1.5 |  |
| -0.5 |  |
| 0 |  |
| 0.5 |  |
| 1.5 |  |
| 2.5 |  |

78. Complete the following table. When you are finished, notice that this table contains exactly the same entries as the table from Problem 77. This table uses fractions, whereas the table from Problem 77 uses decimals.

| $\boldsymbol{N}$ Number |  |
| :---: | :---: |
| $\boldsymbol{x}$ | Square <br> $\boldsymbol{x}^{\mathbf{2}}$ |
| $-\frac{5}{2}$ |  |
| $-\frac{3}{2}$ |  |
| $-\frac{1}{2}$ |  |
| 0 |  |
| $\frac{1}{2}$ |  |
| $\frac{3}{2}$ |  |
| $\frac{5}{2}$ |  |

Write each number in scientific notation.
79. 43,200
80. 432,000
81. 570
82. 5,700
83. 238,000
84. 2,380,000

Write each number in expanded form.
85. $2.49 \times 10^{3}$
86. $2.49 \times 10^{4}$
87. $3.52 \times 10^{2}$
88. $3.52 \times 10^{5}$
89. $2.8 \times 10^{4}$
90. $2.8 \times 10^{3}$

## Applying the Concepts

91. Volume of a Cube Find the volume of a cube if each side is 3 inches long.
92. Volume of a Cube Find the volume of a cube if each side is 3 feet long.
93. Volume of a Cube A bottle of perfume is packaged in a box that is in the shape of a cube. Find the volume of the box if each side is 2.5 inches long. Round to the nearest tenth.

94. Volume of a Cube A television set is packaged in a box that is in the shape of a cube. Find the volume of the box if each side is 18 inches long.
95. Volume of a Box A rented videotape is in a plastic container that has the shape of a rectangular solid. Find the volume of the container if the length is 8 inches, the width is 4.5 inches, and the height is 1 inch.
96. Volume of a Box Your textbook is in the shape of a rectangular solid. Find the volume in cubic inches.
97. Volume of a Box If a box has a volume of 42 cubic feet, is it possible for you to fit inside the box? Explain your answer.
98. Volume of a Box A box has a volume of 45 cubic inches. Will a can of soup fit inside the box? Explain your answer.
99. Age in Seconds If you are 21 years old, you have been alive for more than 650,000,000 seconds. Write this last number in scientific notation.
100. Distance Around the Earth The distance around the Earth at the equator is more than $130,000,000$ feet. Write this number in scientific notation.
101. Lifetime Earnings If you earn at least $\$ 12$ an hour and work full-time for 30 years, you will make at least $7.4 \times 10^{5}$ dollars. Write this last number in expanded form.
102. Heart Beats per Year If your pulse is 72 , then in one year your heart will beat at least $3.78 \times 10^{7}$ times. Write this last number in expanded form.
103. Investing If you put $\$ 1,000$ into a savings account every year from the time you are 25 years old until you are 55 years old, you will have more than $1.8 \times 10^{5}$ dollars in the account when you reach 55 years of age (assuming $10 \%$ annual interest). Write $1.8 \times 10^{5}$ in expanded form.
104. Investing If you put $\$ 20$ into a savings account every month from the time you are 20 years old until you are 30 years old, you will have more than $3.27 \times 10^{3}$ dollars in the account when you reach 30 years of age (assuming 6\% annual interest compounded monthly). Write $3.27 \times 10^{3}$ in expanded form.
Displacement The displacement, in cubic inches, of a car engine is given by the formula

$$
d=\pi \cdot s \cdot c \cdot\left(\frac{1}{2} \cdot b\right)^{2}
$$

where $s$ is the stroke and $b$ is the bore, as shown in the figure, and $c$ is the number of cylinders.


Calculate the engine displacement for each of the following cars. Use 3.14 to approximate $\pi$. Round your answers to the nearest cubic inch.
105. Ferrari Modena 8 cylinders, 3.35 inches of bore, 3.11 inches of stroke
106. Audi A8 8 cylinders, 3.32 inches of bore, 3.66 inches of stroke
107. Mitsubishi Eclipse 6 cylinders, 3.59 inches of bore, 2.99 inches of stroke
108. Porsche 911 GT3 6 cylinders, 3.94 inches of bore, 3.01 inches of stroke

## Maintaining Your Skills

Factor each of the following into its product of prime factors.
109. 128
110. 200
111. 250
112. 512
113. 720
114. 555
115. 820
116. 1,024

Factor the following by first factoring the base and then raising each of its factors to the third power.
117. $6^{3}$
118. $10^{3}$
119. $30^{3}$
120. $42^{3}$
121. $25^{3}$
122. $8^{3}$
123. $12^{3}$
124. $36^{3}$

## Getting Ready for the Next Section

Subtract.
125. 4-7
126. $-4-7$
127. $4-(-7)$
128. $-4-(-7)$
129. $15-20$
130. $15-(-20)$
131. $-15-(-20)$
132. $-15-20$

Simplify.
133. 2(3) - 4
134. 5(3) - 10
135. $4(3)-3(2)$
136. $-8-2(3)$
137. $2(5-3)$
138. $2(3)-4-3(-4)$
139. $5+4(-2)-2(-3)$
140. $2(3)+4(5)-5(2)$

## Division with Exponents

## OBJECTIVES

A Apply the definition for negative exponents.

B Use Property 4 for exponents.

C Use Property 5 for exponents.

D Simplify expressions involving exponents of 0 and 1.

E Simplify expressions using combinations of the properties of exponents.

F Write numbers in scientific notation and expanded form.

In Section 4.1 we found that multiplication with the same base results in addition of exponents; that is, $a^{r} \cdot a^{s}=a^{r+s}$. Since division is the inverse operation of multiplication, we can expect division with the same base to result in subtraction of exponents.
To develop the properties for exponents under division, we again apply the definition of exponents:

$$
\begin{aligned}
\frac{X^{5}}{x^{3}} & =\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} & \frac{2^{4}}{2^{7}} & =\frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} \\
& =\frac{x \cdot x \cdot x}{X \cdot x \cdot x}(x \cdot x) & & =\frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2} \cdot \frac{1}{2 \cdot 2 \cdot 2} \\
& =1(x \cdot x) & & =\frac{1}{2 \cdot 2 \cdot 2} \\
& =X^{2} \quad \text { Notice: } 5-\mathbf{3}=\mathbf{2} & & =\frac{1}{2^{3}} \quad \text { Notice: } \mathbf{7}-\mathbf{4}=\mathbf{3}
\end{aligned}
$$

In both cases division with the same base resulted in subtraction of the smaller exponent from the larger. The problem is deciding whether the answer is a fraction. The problem is resolved easily by the following definition.

$$
\text { DEFINITION If } r \text { is a positive integer, then } a^{-r}=\frac{1}{a^{r}}=\left(\frac{1}{a}\right)^{r} \quad(a \neq 0)
$$

The following examples illustrate how we use this definition to simplify expressions that contain negative exponents.

## EXAMPLES

simplify:

1. $2^{-3}=\frac{1}{2^{3}}=\frac{1}{8}$

Notice: Negative exponents do not indicate negative numbers. They indicate reciprocals
2. $5^{-2}=\frac{1}{5^{2}}=\frac{1}{25}$
3. $3 x^{-6}=3 \cdot \frac{1}{x^{6}}=\frac{3}{X^{6}}$

Now let us look back to our original problem and try to work it again with the help of a negative exponent. We know that $\frac{2^{4}}{2^{7}}=\frac{1}{2^{3}}$. Let us decide now that with division of the same base, we will always subtract the exponent in the denominator from the exponent in the numerator and see if this conflicts with what we know is true.

$$
\begin{aligned}
\frac{2^{4}}{2^{7}} & =2^{4-7} & & \text { Subtracting the bottom exponent from the top exponent } \\
& =2^{-3} & & \text { Subtraction } \\
& =\frac{1}{2^{3}} & & \text { Definition of negative exponents }
\end{aligned}
$$

Subtracting the exponent in the denominator from the exponent in the numerator and then using the definition of negative exponents gives us the same result we obtained previously. We can now continue the list of properties of exponents we started in Section 4.1.

Property 4 for Exponents If $a$ is any real number and $r$ and $s$ are integers, then

$$
\frac{a^{r}}{a^{s}}=a^{r-s} \quad(a \neq 0)
$$

In words: To divide with the same base, subtract the exponent in the denominator from the exponent in the numerator and raise the base to the exponent that results.

The following examples show how we use Property 4 and the definition for negative exponents to simplify expressions involving division.

## EXAMPLES Simplify the following expressions:

4. $\frac{x^{9}}{x^{6}}=x^{9-6}=x^{3}$
5. $\frac{X^{4}}{X^{10}}=x^{4-10}=x^{-6}=\frac{1}{X^{6}}$
6. $\frac{2^{15}}{2^{20}}=2^{15-20}=2^{-5}=\frac{1}{2^{5}}=\frac{1}{32}$

Our final property of exponents is similar to Property 3 from Section 4.1, but it involves division instead of multiplication. After we have stated the property, we will give a proof of it. The proof shows why this property is true.

Property 5 for Exponents If $a$ and $b$ are any two real numbers $(b \neq 0)$ and $r$ is an integer, then

$$
\left(\frac{a}{b}\right)^{r}=\frac{a^{r}}{b^{r}}
$$

In words: A quotient raised to a power is the quotient of the powers.

## Proof

$$
\begin{aligned}
\left(\frac{a}{b}\right)^{r} & =\left(a \cdot \frac{1}{b}\right)^{r} & & \text { By the definition of division } \\
& =a^{r} \cdot\left(\frac{1}{b}\right)^{r} & & \text { By Property } 3 \\
& =a^{r} \cdot b^{-r} & & \text { By the definition of negative exponents } \\
& =a^{r} \cdot \frac{1}{b^{r}} & & \text { By the definition of negative exponents } \\
& =\frac{a^{r}}{b^{r}} & & \text { By the definition of division }
\end{aligned}
$$

## EXAMPLES

7. $\left(\frac{x}{2}\right)^{3}=\frac{x^{3}}{2^{3}}=\frac{x^{3}}{8}$
8. $\left(\frac{5}{y}\right)^{2}=\frac{5^{2}}{y^{2}}=\frac{25}{y^{2}}$
9. $\left(\frac{2}{3}\right)^{4}=\frac{2^{4}}{3^{4}}=\frac{16}{81}$

## Zero and One as Exponents

We have two special exponents left to deal with before our rules for exponents are complete: 0 and 1. To obtain an expression for $x^{1}$, we will solve a problem two different ways:

$$
\left.\begin{array}{l}
\frac{x^{3}}{x^{2}}=\frac{x \cdot x \cdot x}{x \cdot x}=x \\
\frac{x^{3}}{x^{2}}=X^{3-2}=x^{1}
\end{array}\right\} \quad \text { Hence } x^{1}=x
$$

Stated generally, this rule says that $a^{1}=a$. This seems reasonable and we will use it since it is consistent with our property of division using the same base.

We use the same procedure to obtain an expression for $x^{0}$ :

$$
\left.\begin{array}{l}
\frac{5^{2}}{5^{2}}=\frac{25}{25}=1 \\
\frac{5^{2}}{5^{2}}=5^{2-2}=5^{0}
\end{array}\right\} \quad \text { Hence } 5^{0}=\mathbf{1}
$$

It seems, therefore, that the best definition of $x^{0}$ is 1 for all $x$ except $x=0$. In the case of $x=0$, we have $0^{\circ}$, which we will not define. This definition will probably seem awkward at first. Most people would like to define $x^{0}$ as 0 when they first encounter it. Remember, the zero in this expression is an exponent, so $x^{0}$ does not mean to multiply by zero. Thus, we can make the general statement that $a^{0}=1$ for all real numbers except $a=0$.

Here are some examples involving the exponents 0 and 1 .

## EXAMPLES <br> Simplify the following expressions:

10. $8^{0}=1$
11. $8^{1}=8$
12. $4^{0}+4^{1}=1+4=5$
13. $\left(2 x^{2} y\right)^{0}=1$

Here is a summary of the definitions and properties of exponents we have developed so far. For each definition or property in the list, $a$ and $b$ are real numbers, and $r$ and $s$ are integers.

## Definitions

$$
\begin{aligned}
& a^{-r}=\frac{1}{a^{r}}=\left(\frac{1}{a}\right)^{r} \quad a \neq 0 \\
& a^{1}=a \\
& a^{0}=1 \quad a \neq 0
\end{aligned}
$$

## Properties

1. $a^{r} \cdot a^{s}=a^{r+s}$
2. $\left(a^{r}\right)^{s}=a^{r s}$
3. $(a b)^{r}=a^{r} b^{r}$
4. $\frac{a^{r}}{a^{s}}=a^{r-s} \quad a \neq 0$
5. $\left(\frac{a}{b}\right)^{r}=\frac{a^{r}}{b^{r}} \quad b \neq 0$

Here are some additional examples. These examples use a combination of the preceding properties and definitions.

## EXAMPLES

exponents only:
14. $\frac{\left(5 x^{3}\right)^{2}}{x^{4}}=\frac{25 x^{6}}{x^{4}} \quad$ Properties 2 and 3
$=25 x^{2} \quad$ Property 4
15. $\frac{x^{-8}}{\left(x^{2}\right)^{3}}=\frac{x^{-8}}{x^{6}} \quad$ Property 2
$\begin{array}{ll}=x^{-8-6} & \\ =x^{-14} & \\ =\frac{\text { Property } 4}{X^{14}} & \\ \text { Subtraction }\end{array}$
16. $\left(\frac{y^{5}}{y^{3}}\right)^{2}=\frac{\left(y^{5}\right)^{2}}{\left(y^{3}\right)^{2}} \quad$ Property 5

$$
\begin{array}{ll}
=\frac{y^{10}}{y^{6}} & \text { Property } 2 \\
=y^{4} & \text { Property } 4
\end{array}
$$

Notice in Example 16 that we could have simplified inside the parentheses first and then raised the result to the second power:

$$
\left(\frac{y^{5}}{y^{3}}\right)^{2}=\left(y^{2}\right)^{2}=y^{4}
$$

17. $\left(3 x^{5}\right)^{-2}=\frac{1}{\left(3 x^{5}\right)^{2}} \quad$ Definition of negative exponents

$$
=\frac{1}{9 X^{10}} \quad \text { Properties } 2 \text { and } 3
$$

18. $x^{-8} \cdot X^{5}=X^{-8+5} \quad$ Property 1
$=x^{-3} \quad$ Addition
$=\frac{1}{X^{3}} \quad$ Definition of negative exponents
19. $\frac{\left(a^{3}\right)^{2} a^{-4}}{\left(a^{-4}\right)^{3}}=\frac{a^{6} a^{-4}}{a^{-12}} \quad$ Property 2

$$
\begin{array}{ll}
=\frac{a^{2}}{a^{-12}} & \text { Property } 1 \\
=a^{14} & \text { Property } 4
\end{array}
$$

In the next two examples we use division to compare the area and volume of geometric figures.

EXAMPLE 20 suppose you have two squares, one of which is larger than the other. If the length of a side of the larger square is 3 times as long as the length of a side of the smaller square, how many of the smaller squares will it take to cover up the larger square?

SOLUTION If we let $x$ represent the length of a side of the smaller square, then the length of a side of the larger square is $3 x$. The area of each square, along with a diagram of the situation, is given in Figure 1.


Square 1: $A=x^{2}$


Square 2: $A=(3 x)^{2}=9 x^{2}$

To find out how many smaller squares it will take to cover up the larger square, we divide the area of the larger square by the area of the smaller square.

$$
\frac{\text { Area of square } 2}{\text { Area of square } 1}=\frac{9 x^{2}}{x^{2}}=9
$$

It will take 9 of the smaller squares to cover the larger square.

EXAMPLE 21 Suppose you have two boxes, each of which is a cube. If the length of a side in the second box is 3 times as long as the length of a side of the first box, how many of the smaller boxes will fit inside the larger box?

SOLUTION If we let $x$ represent the length of a side of the smaller box, then the length of a side of the larger box is $3 x$. The volume of each box, along with a diagram of the situation, is given in Figure 2.


To find out how many smaller boxes will fit inside the larger box, we divide the volume of the larger box by the volume of the smaller box.

$$
\frac{\text { Volume of box } 2}{\text { Volume of box } 1}=\frac{27 x^{3}}{x^{3}}=27
$$

We can fit 27 of the smaller boxes inside the larger box.

## More on Scientific Notation

Now that we have completed our list of definitions and properties of exponents, we can expand the work we did previously with scientific notation.

Recall that a number is in scientific notation when it is written in the form

$$
n \times 10^{r}
$$

where $1 \leq n<10$ and $r$ is an integer.
Since negative exponents give us reciprocals, we can use negative exponents to write very small numbers in scientific notation. For example, the number 0.00057 , when written in scientific notation, is equivalent to $5.7 \times 10^{-4}$. Here's why:

$$
5.7 \times 10^{-4}=5.7 \times \frac{1}{10^{4}}=5.7 \times \frac{1}{10,000}=\frac{5.7}{10,000}=0.00057
$$

The table below lists some other numbers in both scientific notation and expanded form.

## EXAMPLE 22

| Number Written <br> the Long Way |  | Number Written Again <br> in Scientific Notation |
| :---: | :---: | :---: |
| 376,000 | $=$ | $3.76 \times 10^{5}$ |
| 49,500 | $=$ | $4.95 \times 10^{4}$ |
| 3,200 | $=$ | $3.2 \times 10^{3}$ |
| 591 | $=$ | $5.91 \times 10^{2}$ |
| 46 | $=$ | $4.6 \times 10^{1}$ |
| 8 | $=$ | $8 \times 10^{0}$ |
| 0.47 | $=$ | $9.7 \times 10^{-1}$ |
| 0.093 | $=$ | $6.88 \times 10^{-2}$ |
| 0.00688 | $2.30^{-3}$ |  |
| 0.0002 | $=$ | $9.8 \times 10^{-5}$ |
| 0.000098 |  |  |

Notice that in each case, when the number is written in scientific notation, the decimal point in the first number is placed so that the number is between 1 and 10. The exponent on 10 in the second number keeps track of the number of places we moved the decimal point in the original number to get a number between 1 and 10:


LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-3
B 4-6
C 7-9
D 10-13
E $14-19$
F 22

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How do you divide two expressions containing exponents when they each have the same base?
2. Explain the difference between $3^{2}$ and $3^{-2}$.
3. If a positive base is raised to a negative exponent, can the result be a negative number?
4. Explain what happens when we use $O$ as an exponent.

## Problem Set 4.2

Online support materials can be found at www.thomsonedu.com/login

Write each of the following with positive exponents, and then simplify, when possible.

1. $3^{-2}$
2. $6^{-2}$
3. $8^{-2}$
4. $5^{-3}$
5. $2 x^{-3}$
6. $(2 x)^{-3}$
7. $(5 y)^{-2}$
8. $10^{-2}$
9. Complete the following table.

| Number <br> $\boldsymbol{x}$ | Cube <br> $\boldsymbol{x}^{3}$ | Power of 3 <br> $3^{\boldsymbol{x}}$ |
| :---: | :---: | :---: |
| -3 |  |  |
| -2 |  |  |
| -1 |  |  |
| 0 |  |  |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |

17. Complete the following table.

| Number <br> $\boldsymbol{x}$ | Square <br> $\boldsymbol{x}^{\mathbf{2}}$ | Power of 2 <br> $\mathbf{2}^{\boldsymbol{x}}$ |
| :---: | :---: | :---: |
| -3 |  |  |
| -2 |  |  |
| -1 |  |  |
| 0 |  |  |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |

Use Property 4 to simplify each of the following expressions. Write all answers that contain exponents with positive exponents only.
19. $\frac{5^{1}}{5^{3}}$
20. $\frac{7^{6}}{7^{8}}$
21. $\frac{X^{10}}{X^{4}}$
22. $\frac{X^{4}}{X^{10}}$

[^2]23. $\frac{4^{3}}{4^{0}}$
24. $\frac{4^{0}}{4^{3}}$
25. $\frac{(2 x)^{7}}{(2 x)^{4}}$
26. $\frac{(2 x)^{4}}{(2 x)^{7}}$
27. $\frac{6^{11}}{6}$
28. $\frac{8^{7}}{8}$
29. $\frac{6}{6^{11}}$
30. $\frac{8}{8^{7}}$
31. $\frac{2^{-5}}{2^{3}}$
32. $\frac{2^{-5}}{2^{-3}}$
33. $\frac{2^{5}}{2^{-3}}$
34. $\frac{2^{-3}}{2^{-5}}$
35. $\frac{(3 x)^{-5}}{(3 x)^{-8}}$
36. $\frac{(2 x)^{-10}}{(2 x)^{-15}}$

Simplify the following expressions. Any answers that contain exponents should contain positive exponents only.
37. $(3 x y)^{4}$
38. $(4 x y)^{3}$
39. $10^{\circ}$
40. $10^{1}$
41. $\left(2 a^{2} b\right)^{1}$
42. $\left(2 a^{2} b\right)^{0}$
43. $\left(7 y^{3}\right)^{-2}$
44. $\left(5 y^{4}\right)^{-2}$
45. $X^{-3} X^{-5}$
46. $x^{-6} \cdot x^{8}$
47. $y^{7} \cdot y^{-10}$
49. $\frac{\left(X^{2}\right)^{3}}{X^{4}}$
48. $y^{-4} \cdot y^{-6}$
51. $\frac{\left(a^{4}\right)^{3}}{\left(a^{3}\right)^{2}}$
50. $\frac{\left(x^{5}\right)^{3}}{X^{10}}$
53. $\frac{y^{7}}{\left(y^{2}\right)^{8}}$
55. $\left(\frac{y^{7}}{y^{2}}\right)^{8}$
52. $\frac{\left(a^{5}\right)^{3}}{\left(a^{5}\right)^{2}}$
57. $\frac{\left(x^{-2}\right)^{3}}{X^{-5}}$
54. $\frac{y^{2}}{\left(y^{3}\right)^{4}}$
59. $\left(\frac{x^{-2}}{x^{-5}}\right)^{3}$
60. $\left(\frac{X^{2}}{X^{-5}}\right)^{-3}$
61. $\frac{\left(a^{3}\right)^{2}\left(a^{4}\right)^{5}}{\left(a^{5}\right)^{2}}$
62. $\frac{\left(a^{4}\right)^{8}\left(a^{2}\right)^{5}}{\left(a^{3}\right)^{4}}$
63. $\frac{\left(a^{-2}\right)^{3}\left(a^{4}\right)^{2}}{\left(a^{-3}\right)^{-2}}$
64. $\frac{\left(a^{-5}\right)^{-3}\left(a^{7}\right)^{-1}}{\left(a^{-3}\right)^{5}}$
65. Let $x=2$ in each of the following expressions and simplify.
a. $\frac{X^{7}}{X^{2}}$
b. $X^{5}$
c. $\frac{x^{2}}{x^{7}}$
d. $X^{-5}$
66. Let $x=-1$ in each of the following expressions and simplify.
a. $\frac{X^{12}}{X^{9}}$
b. $X^{3}$
c. $\frac{X^{11}}{X^{9}}$
d. $X^{2}$
67. Write each expression as a perfect square.
a. $\frac{1}{25}=(-)^{2}$
b. $\frac{1}{64}=(-)^{2}$
c. $\frac{1}{X^{2}}=(-)^{2}$
d. $\frac{1}{X^{4}}=(-)^{2}$
68. Write each expression as a perfect cube.
a. $\frac{1}{125}=(-)^{3}$
b. $\frac{1}{27}=(-)^{3}$
c. $\frac{X^{6}}{125}=(-)^{3}$
d. $\frac{x^{3}}{27}=(-)^{3}$
69. Complete the following table, and then construct a line graph of the information in the table.

| Number <br> $\boldsymbol{x}$ | Power of 2 <br> $\mathbf{2}^{\boldsymbol{x}}$ |
| :---: | :---: |
| -3 |  |
| -2 |  |
| -1 |  |
| 0 |  |
| 1 |  |
| 2 |  |
| 3 |  |

70. Complete the following table, and then construct a line graph of the information in the table.

| Number <br> $\boldsymbol{x}$ | Power of 3 <br> $3^{\boldsymbol{x}}$ |
| :---: | :---: |
| -3 |  |
| -2 |  |
| -1 |  |
| 0 |  |
| 1 |  |
| 2 |  |
| 3 |  |

Write each of the following numbers in scientific notation.
71. 0.0048
72. 0.000048
73. 25
74. 35
75. 0.000009
76. 0.0009
77. Complete the following table.

| Expanded Form | Scientific Notation <br> $\boldsymbol{n} \times \mathbf{1 0}^{\boldsymbol{r}}$ |
| :---: | :---: |
| 0.000357 | $3.57 \times 10^{-4}$ |
| 0.00357 |  |
| 0.0357 |  |
| 0.357 |  |
| 3.57 |  |
| 35.7 |  |
| 357 |  |
| 3,570 |  |
| 35,700 |  |

78. Complete the following table.

| Expanded Form | Scientific Notation <br> $\boldsymbol{n} \times \mathbf{1 0}^{\boldsymbol{r}}$ |
| :---: | :---: |
| 0.000123 | $1.23 \times 10^{-4}$ |
|  | $1.23 \times 10^{-3}$ |
|  | $1.23 \times 10^{-2}$ |
|  | $1.23 \times 10^{-1}$ |
|  | $1.23 \times 10^{0}$ |
|  | $1.23 \times 10^{1}$ |
|  | $1.23 \times 10^{2}$ |
|  | $1.23 \times 10^{3}$ |
|  | $1.23 \times 10^{4}$ |

Write each of the following numbers in expanded form.
79. $4.23 \times 10^{-3}$
80. $4.23 \times 10^{3}$
81. $8 \times 10^{-5}$
82. $8 \times 10^{5}$
83. $4.2 \times 10^{0}$
84. $4.2 \times 10^{1}$

## Applying the Concepts

85. Some home computers can do a calculation in $2 \times 10^{-3}$ seconds. Write this number in expanded form.
86. Some of the cells in the human body have a radius of $3 \times 10^{-5}$ inches. Write this number in expanded form.
87. Margin of Victory In this graph are the five closest margins of victory of the IRL IndyCar Series. Write each number in scientific notation.

88. Some cameras used in scientific research can take one picture every 0.000000167 second. Write this number in scientific notation.
89. The number $25 \times 10^{3}$ is not in scientific notation because 25 is larger than 10 . Write $25 \times 10^{3}$ in scientific notation.
90. The number $0.25 \times 10^{3}$ is not in scientific notation because 0.25 is less than 1 . Write $0.25 \times 10^{3}$ in scientific notation.
91. The number $23.5 \times 10^{4}$ is not in scientific notation because 23.5 is not between 1 and 10. Rewrite $23.5 \times 10^{4}$ in scientific notation.
92. The number $375 \times 10^{3}$ is not in scientific notation because 375 is not between 1 and 10. Rewrite $375 \times 10^{3}$ in scientific notation.
93. The number $0.82 \times 10^{-3}$ is not in scientific notation because 0.82 is not between 1 and 10 . Rewrite $0.82 \times 10^{-3}$ in scientific notation.
94. The number $0.93 \times 10^{-2}$ is not in scientific notation because 0.93 is not between 1 and 10. Rewrite $0.93 \times 10^{-2}$ in scientific notation.

Comparing Areas Suppose you have two squares, one of which is larger than the other. Suppose further that the side of the larger square is twice as long as the side of the smaller square.
95. If the length of the side of the smaller square is 10 inches, give the area of each square. Then find the number of smaller squares it will take to cover the larger square.
96. How many smaller squares will it take to cover the larger square if the length of the side of the smaller square is 1 foot?
97. If the length of the side of the smaller square is $x$, find the area of each square. Then find the number of smaller squares it will take to cover the larger square.
98. Suppose the length of the side of the larger square is 1 foot. How many smaller squares will it take to cover the larger square?

Comparing Volumes Suppose you have two boxes, each of which is a cube. Suppose further that the length of a side of the second box is twice as long as the length of a side of the first box.
99. If the length of a side of the first box is 6 inches, give the volume of each box. Then find the number of smaller boxes that will fit inside the larger box.
100. How many smaller boxes can be placed inside the larger box if the length of a side of the second box is 1 foot?
101. If the length of a side of the first box is $x$, find the volume of each box. Then find the number of smaller boxes that will fit inside the larger box.
102. Suppose the length of a side of the larger box is 12 inches. How many smaller boxes will fit inside the larger box?

## Maintaining Your Skills

Simplify the following expressions.
103. $4 x+3 x$
104. $9 x+7 x$
105. $5 a-3 a$
106. $10 a-2 a$
107. $4 y+5 y+y$
108. $6 y-y+2 y$

## Getting Ready for the Next Section

Simplify.
109. $3(4.5)$
110. $\frac{1}{2} \cdot \frac{5}{7}$
111. $\frac{4}{5}(10)$
112. $\frac{9.6}{3}$
113. 6.8(3.9)
114. $9-20$
115. $-3+15$
116. $2 x \cdot x \cdot \frac{1}{2} x$
117. $x^{5} \cdot x^{3}$
118. $y^{2} \cdot y$
119. $\frac{x^{3}}{X^{2}}$
120. $\frac{x^{2}}{x}$
121. $\frac{y^{3}}{y^{5}}$
122. $\frac{x^{2}}{x^{5}}$

Write in expanded form.
123. $3.4 \times 10^{2}$
124. $6.0 \times 10^{-4}$

## 4.3 <br> Operations with Monomials

## OBJECTIVES

A Multiply monomials.
B Divide monomials.
C Multiply and divide numbers written in scientific notation.

D Add and subtract monomials.

We have developed all the tools necessary to perform the four basic operations on the simplest of polynomials: monomials.

DEFINITION A monomial is a one-term expression that is either a constant (number) or the product of a constant and one or more variables raised to whole number exponents.

The following are examples of monomials:

$$
\begin{array}{lllll}
-3 & 15 x & -23 x^{2} y & 49 x^{4} y^{2} z^{4} & \frac{3}{4} a^{2} b^{3}
\end{array}
$$

The numerical part of each monomial is called the numerical coefficient, or just coefficient. Monomials are also called terms.

## Multiplication and Division of Monomials

There are two basic steps involved in the multiplication of monomials. First, we rewrite the products using the commutative and associative properties. Then, we simplify by multiplying coefficients and adding exponents of like bases.

## EXAMPLES Multiply:

1. $\left(-3 x^{2}\right)\left(4 X^{3}\right)=(-3 \cdot 4)\left(x^{2} \cdot X^{3}\right)$

$$
=-12 x^{5}
$$

2. $\left(\frac{4}{5} x^{5} y^{2}\right)\left(10 x^{3} y\right)=\left(\frac{4}{5} 10\right)\left(x^{5} x^{3}\right)\left(y^{2} y\right)$

$$
=8 x^{8} y^{3}
$$

Commutative and associative properties
Multiply coefficients, add exponents
Commutative and associative properties
Multiply coefficients, add exponents

You can see that in each case the work was the same-multiply coefficients and add exponents of the same base. We can expect division of monomials to proceed in a similar way. Since our properties are consistent, division of monomials will result in division of coefficients and subtraction of exponents of like bases.

## EXAMPLES

Divide:
3. $\frac{15 x^{3}}{3 x^{2}}=\frac{15}{3} \cdot \frac{x^{3}}{x^{2}}$

$$
=5 x
$$

Write as separate fractions
4. $\frac{39 x^{2} y^{3}}{3 x y^{5}}=\frac{39}{3} \cdot \frac{x^{2}}{x} \cdot \frac{y^{3}}{y^{5}}$

Write as separate fractions

$$
=13 x \cdot \frac{1}{y^{2}}
$$

$$
=\frac{13 x}{y^{2}} \quad \text { Write answer as a single fraction }
$$

In Example 4, the expression $\frac{y^{3}}{y^{5}}$ simplifies to $\frac{1}{y^{2}}$ because of Property 4 for exponents and the definition of negative exponents. If we were to show all the work in this simplification process, it would look like this:

$$
\begin{aligned}
\frac{y^{3}}{y^{5}} & =y^{3-5} & & \text { Property } 4 \text { for exponents } \\
& =y^{-2} & & \text { Subtraction } \\
& =\frac{1}{y^{2}} & & \text { Definition of negative exponents }
\end{aligned}
$$

The point of this explanation is this: Even though we may not show all the steps when simplifying an expression involving exponents, the result we obtain still can be justified using the properties of exponents. We have not introduced any new properties in Example 4; we have just not shown the details of each simplification.

## EXAMPLE 5 Divide:

$$
\begin{aligned}
\frac{25 a^{5} b^{3}}{50 a^{2} b^{7}} & =\frac{25}{50} \cdot \frac{a^{5}}{a^{2}} \cdot \frac{b^{3}}{b^{7}} & & \text { Write as separate fractions } \\
& =\frac{1}{2} \cdot a^{3} \cdot \frac{1}{b^{4}} & & \text { Divide coefficients, subtract exponents } \\
& =\frac{a^{3}}{2 b^{4}} & & \text { Write answer as a single fraction }
\end{aligned}
$$

Notice in Example 5 that dividing 25 by 50 results in $\frac{1}{2}$. This is the same result we would obtain if we reduced the fraction $\frac{25}{50}$ to lowest terms, and there is no harm in thinking of it that way. Also, notice that the expression $\frac{b^{3}}{b^{7}}$ simplifies to $\frac{1}{b^{4}}$ by Property 4 for exponents and the definition of negative exponents, even though we have not shown the steps involved in doing so.

## Multiplication and Division of Numbers Written in Scientific Notation

We multiply and divide numbers written in scientific notation using the same steps we used to multiply and divide monomials.

## EXAMPLE $6 \quad$ Multiply $\left(4 \times 10^{7}\right)\left(2 \times 10^{-4}\right)$.

SOLUTION Since multiplication is commutative and associative, we can rearrange the order of these numbers and group them as follows:

$$
\begin{aligned}
\left(4 \times 10^{7}\right)\left(2 \times 10^{-4}\right) & =(4 \times 2)\left(10^{7} \times 10^{-4}\right) \\
& =8 \times 10^{3}
\end{aligned}
$$

Notice that we add exponents, $7+(-4)=3$, when we multiply with the same base.

EXAMPLE 7 Divide $\frac{9.6 \times 10^{12}}{3 \times 10^{4}}$.
SOLUTION We group the numbers between 1 and 10 separately from the powers of 10 and proceed as we did in Example 6:

$$
\begin{aligned}
\frac{9.6 \times 10^{12}}{3 \times 10^{4}} & =\frac{9.6}{3} \times \frac{10^{12}}{10^{4}} \\
& =3.2 \times 10^{8}
\end{aligned}
$$

Notice that the procedure we used in both of these examples is very similar to multiplication and division of monomials, for which we multiplied or divided coefficients and added or subtracted exponents.

## Addition and Subtraction of Monomials

Addition and subtraction of monomials will be almost identical since subtraction is defined as addition of the opposite. With multiplication and division of monomials, the key was rearranging the numbers and variables using the commutative and associative properties. With addition, the key is application of the distributive property. We sometimes use the phrase combine monomials to describe addition and subtraction of monomials.

DEFINITION Two terms (monomials) with the same variable part (same variables raised to the same powers) are called similar (or like) terms.

You can add only similar terms. This is because the distributive property (which is the key to addition of monomials) cannot be applied to terms that are not similar.

## EXAMPLES Combine the following monomials.

8. $-3 x^{2}+15 x^{2}=(-3+15) X^{2} \quad$ Distributive property

$$
=12 x^{2} \quad \text { Add coefficients }
$$

9. $9 x^{2} y-20 x^{2} y=(9-20) x^{2} y$

Distributive property $=-11 x^{2} y$

Add coefficients
10. $5 x^{2}+8 y^{2} \quad$ In this case we cannot apply the distributive property, so we cannot add the monomials

The next examples show how we simplify expressions containing monomials when more than one operation is involved.

## EXAMPLE 11 Simplify $\frac{\left(6 x^{4} y\right)\left(3 x^{7} y^{5}\right)}{9 x^{5} y^{2}}$.

SOLUTION We begin by multiplying the two monomials in the numerator:

$$
\begin{aligned}
\frac{\left(6 x^{4} y\right)\left(3 x^{7} y^{5}\right)}{9 x^{5} y^{2}} & =\frac{18 x^{11} y^{6}}{9 x^{5} y^{2}} & & \text { Simplify numerator } \\
& =2 x^{6} y^{4} & & \text { Divide }
\end{aligned}
$$

## EXAMPLE 12 Simplify $\frac{\left(6.8 \times 10^{5}\right)\left(3.9 \times 10^{-7}\right)}{7.8 \times 10^{-4}}$.

SOLUTION We group the numbers between 1 and 10 separately from the powers of 10 :

$$
\begin{aligned}
\frac{(6.8)(3.9)}{7.8} \times \frac{\left(10^{5}\right)\left(10^{-7}\right)}{10^{-4}} & =3.4 \times 10^{5+(-7)-(-4)} \\
& =3.4 \times 10^{2}
\end{aligned}
$$

EXAMPLE 13 simplify $\frac{14 x^{5}}{2 x^{2}}+\frac{15 x^{8}}{3 x^{5}}$.
SOLUTION Simplifying each expression separately and then combining similar terms gives

$$
\begin{aligned}
\frac{14 x^{5}}{2 x^{2}}+\frac{15 x^{8}}{3 x^{5}} & =7 x^{3}+5 x^{3} & & \text { Divide } \\
& =12 x^{3} & & \text { Add }
\end{aligned}
$$

Our work with exponents and division allows us to multiply fractions and other expressions involving exponents. For example,

$$
x^{5} \cdot \frac{6}{x^{2}}=\frac{x^{5}}{1} \cdot \frac{6}{x^{2}}=\frac{6 x^{5}}{x^{2}}=6 x^{3}
$$

It is not necessary to show the intermediate steps in a problem like this. We are showing them here just so you can see why we subtract exponents to get the $x^{3}$ in the answer.

## EXAMPLES Apply the distributive property, then simplify, if possible.

14. $x^{2}\left(1-\frac{6}{x}\right)=x^{2} \cdot 1-x^{2} \cdot \frac{6}{x}=x^{2}-\frac{6 x^{2}}{x}=x^{2}-6 x$
15. $a b\left(\frac{1}{b}-\frac{1}{a}\right)=a b \cdot \frac{1}{b}-a b \cdot \frac{1}{a}=\frac{a b}{b}-\frac{a b}{a}=a-b$

EXAMPLE 16 A rectangular solid is twice as long as it is wide and onehalf as high as it is wide. Write an expression for the volume.
SOLUTION We begin by making a diagram of the object (Figure 1) with the dimensions labeled as given in the problem.


FIGURE 1
The volume is the product of the three dimensions:

$$
V=2 x \cdot x \cdot \frac{1}{2} x=x^{3}
$$

The box has the same volume as a cube with side $x$, as shown in Figure 2 .


Equal Volumes
FIGURE 2

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $1,2,11$
B $3-5,11$
C $6,7,12$
D 8-10

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a monomial?
2. Describe how you would multiply $3 x^{2}$ and $5 x^{2}$.
3. Describe how you would add $3 x^{2}$ and $5 x^{2}$.
4. Describe how you would multiply two numbers written in scientific notation.

## Problem Set 4.3

Multiply.

1. $\left(3 x^{4}\right)\left(4 x^{3}\right)$
2. $\left(6 x^{5}\right)\left(-2 x^{2}\right)$
3. $\left(-2 y^{4}\right)\left(8 y^{7}\right)$
4. $\left(5 y^{10}\right)\left(2 y^{5}\right)$
5. $(8 x)(4 x)$
6. $(7 x)(5 x)$
7. $\left(10 a^{3}\right)(10 a)\left(2 a^{2}\right)$
8. $\left(5 a^{4}\right)(10 a)\left(10 a^{4}\right)$
9. $\left(6 a b^{2}\right)\left(-4 a^{2} b\right)$
10. $\left(-5 a^{3} b\right)\left(4 a b^{4}\right)$
11. $\left(4 x^{2} y\right)\left(3 x^{3} y^{3}\right)\left(2 x y^{4}\right)$
12. $\left(5 x^{6}\right)\left(-10 x y^{4}\right)\left(-2 x^{2} y^{6}\right)$

Divide. Write all answers with positive exponents only.
13. $\frac{15 x^{3}}{5 x^{2}}$
14. $\frac{25 x^{5}}{5 x^{4}}$
15. $\frac{18 y^{9}}{3 y^{12}}$
16. $\frac{24 y^{4}}{-8 y^{7}}$
17. $\frac{32 a^{3}}{64 a^{4}}$
18. $\frac{25 a^{5}}{75 a^{6}}$
19. $\frac{21 a^{2} b^{3}}{-7 a b^{5}}$
20. $\frac{32 a^{5} b^{6}}{8 a b^{5}}$
21. $\frac{3 x^{3} y^{2} z}{27 x y^{2} z^{3}}$
22. $\frac{5 x^{5} y^{4} z}{30 x^{3} y z^{2}}$
23. Fill in the table.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $\boldsymbol{a b}$ | $\frac{a}{b}$ | $\frac{b}{a}$ |
| :---: | :---: | :---: | :---: | :---: |
| 10 | $5 x$ |  |  |  |
| $20 x^{3}$ | $6 x^{2}$ |  |  |  |
| $25 x^{5}$ | $5 x^{4}$ |  |  |  |
| $3 x^{-2}$ | $3 x^{2}$ |  |  |  |
| $-2 y^{4}$ | $8 y^{7}$ |  |  |  |

24. Fill in the table.

| $a$ | $b$ | $a b$ | $\frac{a}{b}$ | $\frac{b}{a}$ |
| :--- | :---: | :---: | :---: | :---: |
| $10 y$ | $2 y^{2}$ |  |  |  |
| $10 y^{2}$ | $2 y$ |  |  |  |
| $5 y^{3}$ | 15 |  |  |  |
| 5 | $15 y^{3}$ |  |  |  |
| $4 y^{-3}$ | $4 y^{3}$ |  |  |  |

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Find each product. Write all answers in scientific notation.
25. $\left(3 \times 10^{3}\right)\left(2 \times 10^{5}\right)$
26. $\left(4 \times 10^{8}\right)\left(1 \times 10^{6}\right)$
27. $\left(3.5 \times 10^{4}\right)\left(5 \times 10^{-6}\right)$
28. $\left(7.1 \times 10^{5}\right)\left(2 \times 10^{-8}\right)$
29. $\left(5.5 \times 10^{-3}\right)\left(2.2 \times 10^{-4}\right)$
30. $\left(3.4 \times 10^{-2}\right)\left(4.5 \times 10^{-6}\right)$

Find each quotient. Write all answers in scientific notation.
31. $\frac{8.4 \times 10^{5}}{2 \times 10^{2}}$
32. $\frac{9.6 \times 10^{20}}{3 \times 10^{6}}$
33. $\frac{6 \times 10^{8}}{2 \times 10^{-2}}$
34. $\frac{8 \times 10^{12}}{4 \times 10^{-3}}$
35. $\frac{2.5 \times 10^{-6}}{5 \times 10^{-4}}$
36. $\frac{4.5 \times 10^{-8}}{9 \times 10^{-4}}$

Combine by adding or subtracting as indicated.
37. $3 x^{2}+5 x^{2}$
38. $4 x^{3}+8 x^{3}$
39. $8 x^{5}-19 x^{5}$
40. $75 x^{6}-50 x^{6}$
41. $2 a+a-3 a$
42. $5 a+a-6 a$
43. $10 x^{3}-8 x^{3}+2 x^{3}$
44. $7 x^{5}+8 x^{5}-12 x^{5}$
45. $20 a b^{2}-19 a b^{2}+30 a b^{2}$
46. $18 a^{3} b^{2}-20 a^{3} b^{2}+10 a^{3} b^{2}$
47. Fill in the table.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $\boldsymbol{a} \boldsymbol{b}$ | $\boldsymbol{a}+\boldsymbol{b}$ |
| :--- | :--- | :--- | :--- |
| $5 x$ | $3 x$ |  |  |
| $4 x^{2}$ | $2 x^{2}$ |  |  |
| $3 x^{3}$ | $6 x^{3}$ |  |  |
| $2 x^{4}$ | $-3 x^{4}$ |  |  |
| $x^{5}$ | $7 x^{5}$ |  |  |

48. Fill in the table.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $\boldsymbol{a} \boldsymbol{b}$ | $\boldsymbol{a}-\boldsymbol{b}$ |
| :--- | :---: | :---: | :---: |
| $2 y$ | $3 y$ |  |  |
| $-2 y$ | $3 y$ |  |  |
| $4 y^{2}$ | $5 y^{2}$ |  |  |
| $y^{3}$ | $-3 y^{3}$ |  |  |
| $5 y^{4}$ | $7 y^{4}$ |  |  |

Simplify. Write all answers with positive exponents only.
49. $\frac{\left(3 x^{2}\right)\left(8 x^{5}\right)}{6 x^{4}}$
50. $\frac{\left(7 x^{3}\right)\left(6 x^{8}\right)}{14 x^{5}}$
51. $\frac{\left(9 a^{2} b\right)\left(2 a^{3} b^{4}\right)}{18 a^{5} b^{7}}$
52. $\frac{\left(21 a^{5} b\right)\left(2 a^{8} b^{4}\right)}{14 a b}$
53. $\frac{\left(4 x^{3} y^{2}\right)\left(9 x^{4} y^{10}\right)}{\left(3 x^{5} y\right)\left(2 x^{6} y\right)}$
54. $\frac{\left(5 x^{4} y^{4}\right)\left(10 x^{3} y^{3}\right)}{\left(25 x y^{5}\right)\left(2 x y^{7}\right)}$

Simplify each expression, and write all answers in scientific notation.
55. $\frac{\left(6 \times 10^{8}\right)\left(3 \times 10^{5}\right)}{9 \times 10^{7}}$
56. $\frac{\left(8 \times 10^{4}\right)\left(5 \times 10^{10}\right)}{2 \times 10^{7}}$
57. $\frac{\left(5 \times 10^{3}\right)\left(4 \times 10^{-5}\right)}{2 \times 10^{-2}}$
58. $\frac{\left(7 \times 10^{6}\right)\left(4 \times 10^{-4}\right)}{1.4 \times 10^{-3}}$
59. $\frac{\left(2.8 \times 10^{-7}\right)\left(3.6 \times 10^{4}\right)}{2.4 \times 10^{3}}$
60. $\frac{\left(5.4 \times 10^{2}\right)\left(3.5 \times 10^{-9}\right)}{4.5 \times 10^{6}}$

Simplify.
61. $\frac{18 x^{4}}{3 x}+\frac{21 x^{7}}{7 x^{4}}$
62. $\frac{24 x^{10}}{6 x^{4}}+\frac{32 x^{7}}{8 x}$
63. $\frac{45 a^{6}}{9 a^{4}}-\frac{50 a^{8}}{2 a^{6}}$
64. $\frac{16 a^{9}}{4 a}-\frac{28 a^{12}}{4 a^{4}}$
65. $\frac{6 x^{7} y^{4}}{3 x^{2} y^{2}}+\frac{8 x^{5} y^{8}}{2 y^{6}}$
66. $\frac{40 x^{10} y^{10}}{8 x^{2} y^{5}}+\frac{10 x^{8} y^{8}}{5 y^{3}}$

Apply the distributive property.
67. $x y\left(x+\frac{1}{y}\right)$
68. $x y\left(y+\frac{1}{x}\right)$
69. $x y\left(\frac{1}{y}+\frac{1}{x}\right)$
70. $x y\left(\frac{1}{x}-\frac{1}{y}\right)$
71. $x^{2}\left(1-\frac{4}{x^{2}}\right)$
72. $x^{2}\left(1-\frac{9}{x^{2}}\right)$
73. $x^{2}\left(1-\frac{1}{x}-\frac{6}{x^{2}}\right)$
74. $x^{2}\left(1-\frac{5}{x}+\frac{6}{x^{2}}\right)$
75. $x^{2}\left(1-\frac{5}{x}\right)$
76. $x^{2}\left(1-\frac{3}{x}\right)$
77. $x^{2}\left(1-\frac{8}{x}\right)$
78. $x^{2}\left(1-\frac{6}{x}\right)$
79. Divide each monomial by $5 a^{2}$.
a. $10 a^{2}$
b. $-15 a^{2} b$
c. $25 a^{2} b^{2}$
80. Divide each monomial by $36 x^{2}$.
a. $6 x^{2} a$
b. $12 x^{2} a$
c. $-6 x^{2} a$
81. Divide each monomial by $8 x^{2} y$.
a. $24 x^{3} y^{2}$
b. $16 x^{2} y^{2}$
c. $-4 x^{2} y^{3}$
82. Divide each monomial by $7 x^{2} y$.
a. $21 x^{3} y^{2}$
b. $14 x^{2} y^{2}$
c. $-7 x^{2} y^{3}$

## Maintaining Your Skills

Find the value of each expression when $x=-2$.
83. $4 x$
84. $-3 x$
85. $-2 x+5$
86. $-4 x-1$
87. $x^{2}+5 x+6$
88. $x^{2}-5 x+6$

For each of the following equations complete the given ordered pairs so each is a solution to the equation, and then use the ordered pairs to graph the equation.
89. $y=2 x+2 \quad(-2, \quad),(0),,(2$,
90. $y=2 x-3 \quad(-1, \quad),(0, \quad),(2$,
91. $y=\frac{1}{3} x+1 \quad(-3),,(0),,(3$,
92. $y=\frac{1}{2} x-2 \quad(-2, \quad),(0, \quad),(2, \quad)$

## Getting Ready for the Next Section

Simplify.
93. $3-8$
95. $-1+7$
97. $3(5)^{2}+1$
99. $2 x^{2}+4 x^{2}$
101. $-5 x+7 x$
103. $-(2 x+9)$
104. $-\left(4 x^{2}-2 x-6\right)$
105. Find the value of $2 x+3$ when $x=4$.
106. Find the value of $(3 x)^{2}$ when $x=3$.

## 4.4

## OBJECTIVES

A Add and subtract polynomials.

B Find the value of a polynomial for a given value of the variable.

## Addition and Subtraction of Polynomials

In this section we will extend what we learned in Section 4.3 to expressions called polynomials. We begin this section with the definition of a polynomial.

DEFINITION A polynomial is a finite sum of monomials (terms).

Examples The following are polynomials:

$$
3 x^{2}+2 x+1 \quad 15 x^{2} y+21 x y^{2}-y^{2} \quad 3 a-2 b+4 c-5 d
$$

Polynomials can be further classified by the number of terms they contain. A polynomial with two terms is called a binomial. If it has three terms, it is a trinomial. As stated before, a monomial has only one term.

DEFINITION The degree of a polynomial in one variable is the highest power to which the variable is raised.

## Examples

| $3 x^{5}+2 x^{3}+1$ | A trinomial of degree $\mathbf{5}$ |
| :--- | :--- |
| $2 x+1$ | A binomial of degree $\mathbf{1}$ |
| $3 x^{2}+2 x+1$ | A trinomial of degree $\mathbf{2}$ |
| $3 x^{5}$ | A monomial of degree 5 |
| -9 | A monomial of degree $\mathbf{0}$ |

There are no new rules for adding one or more polynomials. We rely only on our previous knowledge. Here are some examples.

## EXAMPLE 1 Add $\left(2 x^{2}-5 x+3\right)+\left(4 x^{2}+7 x-8\right)$.

SOLUTION We use the commutative and associative properties to group similar terms together and then apply the distributive property to add:

$$
\begin{aligned}
\left(2 x^{2}-\right. & 5 x+3)+\left(4 x^{2}+7 x-8\right) & & \\
& =\left(2 x^{2}+4 x^{2}\right)+(-5 x+7 x)+(3-8) & & \begin{array}{l}
\text { Commutative and } \\
\text { associative properties }
\end{array} \\
& =(2+4) x^{2}+(-5+7) x+(3-8) & & \text { Distributive property } \\
& =6 x^{2}+2 x-5 & & \text { Addition }
\end{aligned}
$$

The results here indicate that to add two polynomials, we add coefficients of similar terms.

## EXAMPLE $2 \quad$ Add $x^{2}+3 x+2 x+6$.

SOLUTION The only similar terms here are the two middle terms. We combine them as usual to get

$$
\begin{equation*}
x^{2}+3 x+2 x+6=x^{2}+5 x+6 \tag{111415}
\end{equation*}
$$

You will recall from Chapter 1 the definition of subtraction: $a-b=a+(-b)$. To subtract one expression from another, we simply add its opposite. The letters $a$ and $b$ in the definition can each represent polynomials. The opposite of a polynomial is the polynomial with opposite terms. When you subtract one polynomial from another you subtract each of its terms.

## EXAMPLE 3 Subtract $\left(3 x^{2}+x+4\right)-\left(x^{2}+2 x+3\right)$.

SOLUTION To subtract $x^{2}+2 x+3$, we change the sign of each of its terms and add. If you are having trouble remembering why we do this, remember that we can think of $-\left(x^{2}+2 x+3\right)$ as $-1\left(x^{2}+2 x+3\right)$. If we distribute the -1 across $x^{2}+2 x+3$, we get $-x^{2}-2 x-3$ :

$$
\begin{aligned}
\left(3 x^{2}\right. & +x+4)-\left(x^{2}+2 x+3\right) \\
& =3 x^{2}+x+4-x^{2}-2 x-3 \\
& =\left(3 x^{2}-x^{2}\right)+(x-2 x)+(4-3) \\
& =2 x^{2}-x+1
\end{aligned}
$$

Take the opposite of each term in the second polynomial

## EXAMPLE 4 Subtract $-4 x^{2}+5 x-7$ from $x^{2}-x-1$.

SOLUTION The polynomial $x^{2}-x-1$ comes first, then the subtraction sign, and finally the polynomial $-4 x^{2}+5 x-7$ in parentheses.

$$
\begin{aligned}
\left(x^{2}\right. & -x-1)-\left(-4 x^{2}+5 x-7\right) \\
& =x^{2}-x-1+4 x^{2}-5 x+7 \quad \begin{array}{l}
\text { Take the opposite of each term in } \\
\text { the second polynomial }
\end{array} \\
& =\left(x^{2}+4 x^{2}\right)+(-x-5 x)+(-1+7) \\
& =5 x^{2}-6 x+6
\end{aligned}
$$

The last topic we want to consider in this section is finding the value of a polynomial for a given value of the variable.
To find the value of the polynomial $3 x^{2}+1$ when $x$ is 5 , we replace $x$ with 5 and simplify the result:

|  | When | $x$ | $=5$ |
| ---: | :--- | ---: | :--- |
| the polynomial | $3 x^{2}$ | +1 |  |
|  | becomes | $3(5)^{2}+1$ | $=3(25)+1$ |
|  |  | $=75+1=76$ |  |

There are two important points to remember when adding or subtracting polynomials. First, to add or subtract two polynomials, you always add or subtract coefficients of similar terms. Second, the exponents never increase in value when you are adding or subtracting similar terms.

## EXAMPLE $5 \quad$ Find the value of $3 x^{2}-5 x+4$ when $x=-2$.

## SOLUTION

$$
\begin{aligned}
& \text { When } \begin{aligned}
x & =-2 \\
\text { the polynomial } 3 x^{2}-5 x & +4 \\
\text { becomes } 3(-2)^{2}-5(-2)+4 & =3(4)+10+4 \\
& =12+10+4=26
\end{aligned}
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-4

B 5

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What are similar terms?
2. What is the degree of a polynomial?
3. Describe how you would subtract one polynomial from another.
4. How would you find the value of $3 x^{2}-5 x+4$ when $x$ is -2 ?

Identify each of the following polynomials as a trinomial, binomial, or monomial, and give the degree in each case.

1. $2 x^{3}-3 x^{2}+1$
2. $4 x^{2}-4 x+1$
3. $5+8 a-9 a^{3}$
4. $6+12 x^{3}+x^{4}$
5. $2 x-1$
6. $4+7 x$
7. $45 x^{2}-1$
8. $3 a^{3}+8$
9. $7 a^{2}$
10. $90 x$
11. -4
12. 56

Perform the following additions and subtractions.
13. $\left(2 x^{2}+3 x+4\right)+\left(3 x^{2}+2 x+5\right)$
14. $\left(x^{2}+5 x+6\right)+\left(x^{2}+3 x+4\right)$
15. $\left(3 a^{2}-4 a+1\right)+\left(2 a^{2}-5 a+6\right)$
16. $\left(5 a^{2}-2 a+7\right)+\left(4 a^{2}-3 a+2\right)$
17. $x^{2}+4 x+2 x+8$
18. $x^{2}+5 x-3 x-15$
19. $6 x^{2}-3 x-10 x+5$
20. $10 x^{2}+30 x-2 x-6$
21. $x^{2}-3 x+3 x-9$
22. $x^{2}-5 x+5 x-25$
23. $3 y^{2}-5 y-6 y+10$
24. $y^{2}-18 y+2 y-12$
25. $\left(6 x^{3}-4 x^{2}+2 x\right)+\left(9 x^{2}-6 x+3\right)$
26. $\left(5 x^{3}+2 x^{2}+3 x\right)+\left(2 x^{2}+5 x+1\right)$
27. $\left(\frac{2}{3} x^{2}-\frac{1}{5} x-\frac{3}{4}\right)+\left(\frac{4}{3} x^{2}-\frac{4}{5} x+\frac{7}{4}\right)$
28. $\left(\frac{3}{8} x^{3}-\frac{5}{7} x^{2}-\frac{2}{5}\right)+\left(\frac{5}{8} x^{3}-\frac{2}{7} x^{2}+\frac{7}{5}\right)$
29. $\left(a^{2}-a-1\right)-\left(-a^{2}+a+1\right)$
30. $\left(5 a^{2}-a-6\right)-\left(-3 a^{2}-2 a+4\right)$
31. $\left(\frac{5}{9} x^{3}+\frac{1}{3} x^{2}-2 x+1\right)-\left(\frac{2}{3} x^{3}+x^{2}+\frac{1}{2} x-\frac{3}{4}\right)$
32. $\left(4 x^{3}-\frac{2}{5} x^{2}+\frac{3}{8} x-1\right)-\left(\frac{9}{2} x^{3}+\frac{1}{4} x^{2}-x+\frac{5}{6}\right)$
33. $\left(4 y^{2}-3 y+2\right)+\left(5 y^{2}+12 y-4\right)-\left(13 y^{2}-6 y+20\right)$
34. $\left(2 y^{2}-7 y-8\right)-\left(6 y^{2}+6 y-8\right)+\left(4 y^{2}-2 y+3\right)$

Simplify.
35. $\left(x^{2}-5 x\right)-\left(x^{2}-3 x\right)$
36. $(-2 x+8)-(-2 x+6)$
37. $\left(6 x^{2}-11 x\right)-\left(6 x^{2}-15 x\right)$
38. $\left(10 x^{2}-3 x\right)-\left(10 x^{2}-50 x\right)$
39. $\left(x^{3}+3 x^{2}+9 x\right)-\left(3 x^{2}+9 x+27\right)$
40. $\left(x^{3}+2 x^{2}+4 x\right)-\left(2 x^{2}+4 x-8\right)$
41. $\left(x^{3}+4 x^{2}+4 x\right)+\left(2 x^{2}+8 x+8\right)$
42. $\left(x^{3}+2 x^{2}+x\right)+\left(x^{2}+2 x+1\right)$
43. $\left(x^{2}-4\right)-\left(x^{2}-4 x+4\right)$
44. $\left(4 x^{2}-9\right)-\left(4 x^{2}-12 x+9\right)$
45. Subtract $10 x^{2}+23 x-50$ from $11 x^{2}-10 x+13$.
46. Subtract $2 x^{2}-3 x+5$ from $4 x^{2}-5 x+10$.
47. Subtract $3 y^{2}+7 y-15$ from $11 y^{2}+11 y+11$.
48. Subtract $15 y^{2}-8 y-2$ from $3 y^{2}-3 y+2$.
49. Add $50 x^{2}-100 x-150$ to $25 x^{2}-50 x+75$.
50. Add $7 x^{2}-8 x+10$ to $-8 x^{2}+2 x-12$.
51. Subtract $2 x+1$ from the sum of $3 x-2$ and $11 x+5$.
52. Subtract $3 x-5$ from the sum of $5 x+2$ and $9 x-1$.
53. Find the value of the polynomial $x^{2}-2 x+1$ when $x$ is 3 .
54. Find the value of the polynomial $(x-1)^{2}$ when $x$ is 3 .
55. Find the value of $100 p^{2}-1,300 p+4,000$ when
a. $p=5$
b. $p=8$
56. Find the value of $100 p^{2}-800 p+1,200$ when
a. $p=2$
b. $p=6$
57. Find the value of $600+1,000 x-100 x^{2}$ when
a. $x=8$
b. $x=-2$
58. Find the value of $500+800 x-100 x^{2}$ when
a. $x=6$
b. $x=-1$

## Applying the Concepts

59. Packaging A crystal ball with a diameter of 6 inches is being packaged for shipment. If the crystal ball is placed inside a circular cylinder with radius 3
 inches and height 6 inches, how much volume will need to be filled with padding? (The volume of a sphere with radius $r$ is $\frac{4}{3} \pi r^{3}$, and the volume of a right circular cylinder with radius $r$ and height $h$ is $\pi r^{2} h$.)
60. Packaging Suppose the circular cylinder of Problem 45 has a radius of 4 inches and a height of 7 inches. How much volume will need to be filled with padding?

## Maintaining Your Skills

61. $3 x(-5 x)$
62. $-3 x(-7 x)$
63. $2 x\left(3 x^{2}\right)$
64. $x^{2}(3 x)$
65. $3 x^{2}\left(2 x^{2}\right)$
66. $4 x^{2}\left(2 x^{2}\right)$

## Getting Ready for the Next Section

Simplify.
67. $(-5)(-1)$
68. $3(-4)$
69. $(-1)(6)$
70. $(-7) 8$
71. $(5 x)(-4 x)$
72. $(3 x)(2 x)$
73. $3 x(-7)$
74. $3 x(-1)$
75. $5 x+(-3 x)$
76. $-3 x-10 x$

Multiply.
77. $3(2 x-6)$
78. $-4 x(x+5)$

## 4.5

## Multiplication with Polynomials

## OBJECTIVES

A Multiply a monomial with a polynomial.
B Multiply two binomials.
C Multiply two polynomials.

We begin our discussion of multiplication of polynomials by finding the product of a monomial and a trinomial.

## EXAMPLE 1 Multiply $3 x^{2}\left(2 x^{2}+4 x+5\right)$.

SOLUTION Applying the distributive property gives us

$$
\begin{array}{rlrl}
3 x^{2}\left(2 x^{2}+4 x+5\right) & =3 x^{2}\left(2 x^{2}\right)+3 x^{2}(4 x)+3 x^{2}(5) & & \text { Distributive property } \\
& =6 x^{4}+12 x^{3}+15 x^{2} & \text { Multiplication }
\end{array}
$$

The distributive property is the key to multiplication of polynomials. We can use it to find the product of any two polynomials. There are some shortcuts we can use in certain situations, however. Let's look at an example that involves the product of two binomials.

## EXAMPLE $2 \quad$ Multiply $(3 x-5)(2 x-1)$.

## SOLUTION

$$
\begin{aligned}
(3 x-5)(2 x-1) & =3 x(2 x-1)-5(2 x-1) \\
& =3 x(2 x)+3 x(-1)+(-5)(2 x)+(-5)(-1) \\
& =6 x^{2}-3 x-10 x+5 \\
& =6 x^{2}-13 x+5
\end{aligned}
$$

If we look closely at the second and third lines of work in this example, we can see that the terms in the answer come from all possible products of terms in the first binomial with terms in the second binomial. This result is generalized as follows.

Rule To multiply any two polynomials, multiply each term in the first with each term in the second.

There are two ways we can put this rule to work.

## FOIL Method

If we look at the original problem in Example 2 and then at the answer, we see that the first term in the answer came from multiplying the first terms in each binomial:

$$
3 x \cdot 2 x=6 x^{2} \quad \text { FIRST }
$$

The middle term in the answer came from adding the products of the two outside terms and the two inside terms in each binomial:

$$
\begin{aligned}
3 x(-1) & =-3 x & & \text { OUTSIDE } \\
-5(2 x) & =\frac{-10 x}{-13 x} & & \text { Inside }
\end{aligned}
$$

The last term in the answer came from multiplying the two last terms:

$$
-5(-1)=5 \quad \text { LAST }
$$

To summarize the foil method, we will multiply another two binomials.

## EXAMPLE 3 Multiply $(2 x+3)(5 x-4)$.

## SOLUTION



$$
\begin{aligned}
& =10 x^{2}-8 x+15 x-12 \\
& =10 x^{2}+7 x-12
\end{aligned}
$$

With practice $-8 x+15 x=7 x$ can be done mentally.

## COLUMN Method

The forl method can be applied only when multiplying two binomials. To find products of polynomials with more than two terms, we use what is called the column method.

The column method of multiplying two polynomials is very similar to long multiplication with whole numbers. It is just another way of finding all possible products of terms in one polynomial with terms in another polynomial.

## EXAMPLE $4 \quad$ Multiply $(2 x+3)\left(3 x^{2}-2 x+1\right)$.

## SOLUTION

$$
\begin{aligned}
& \begin{aligned}
3 x^{2}-2 x+1 \\
2 x+3
\end{aligned} \\
& \frac{6 x^{3}-4 x^{2}+2 x}{} \leftarrow 2 x\left(3 x^{2}-2 x+1\right) \\
& \frac{9 x^{2}-6 x+3}{6 x^{3}+5 x^{2}-4 x+3} \leftarrow 3\left(3 x^{2}-2 x+1\right) \\
& \leftarrow \text { Add similar terms }
\end{aligned}
$$

It will be to your advantage to become very fast and accurate at multiplying polynomials. You should be comfortable using either method. The following examples illustrate the three types of multiplication.

## EXAMPLES <br> Multiply:

5. $4 a^{2}\left(2 a^{2}-3 a+5\right)=4 a^{2}\left(2 a^{2}\right)+4 a^{2}(-3 a)+4 a^{2}(5)$

$$
=8 a^{4}-12 a^{3}+20 a^{2}
$$

6. $(x-2)(y+3)=x(y)+x(3)+(-2)(y)+(-2)(3)$

$$
=x y+3 x-2 y-6
$$

7. $(x+y)(a-b)=x(a)+x(-b)+y(a)+y(-b)$

$$
\begin{array}{cccc}
\mathrm{F} & \mathrm{O} & \mathrm{I} & \mathrm{~L} \\
=x a-x b+y a-y b & &
\end{array}
$$

8. $(5 x-1)(2 x+6)=5 x(2 x)+5 x(6)+(-1)(2 x)+(-1)(6)$

$$
\begin{aligned}
& \text { F } \quad \text { O } \\
=10 x^{2}+30 x+(-2 x)+(-6) & \text { L } \\
= & 10 x^{2}+28 x-6
\end{aligned}
$$

EXAMPLE 9 The length of a rectangle is 3 more than twice the width.
Write an expression for the area of the rectangle.
SOLUTION We begin by drawing a rectangle and labeling the width with $x$. Since the length is 3 more than twice the width, we label the length with $2 x+3$.


Since the area $A$ of a rectangle is the product of the length and width, we write our formula for the area of this rectangle as

$$
\begin{aligned}
& A=x(2 x+3) \\
& A=2 x^{2}+3 x \quad \text { Multiply }
\end{aligned}
$$

## Revenue

Suppose that a store sells $x$ items at $p$ dollars per item. The total amount of money obtained by selling the items is called the revenue. It can be found by multiplying the number of items sold, $x$, by the price per item, $p$. For example, if 100 items are sold for $\$ 6$ each, the revenue is $100(6)=\$ 600$. Similarly, if 500 items are sold for $\$ 8$ each, the total revenue is $500(8)=\$ 4,000$. If we denote the revenue with the letter $R$, then the formula that relates $R, x$, and $p$ is

> Revenue = (number of items sold) (price of each item)

In symbols: $R=x p$.

EXAMPLE 10 A store selling diskettes for home computers knows from past experience that it can sell $x$ diskettes each day at a price of $p$ dollars per diskette, according to the equation $x=800-100 p$. Write a formula for the daily revenue that involves only the variables $R$ and $p$.
SOLUTION From our previous discussion we know that the revenue $R$ is given by the formula

$$
R=x p
$$

But, since $x=800-100 p$, we can substitute $800-100 p$ for $x$ in the revenue equation to obtain

$$
\begin{aligned}
& R=(800-100 p) p \\
& R=800 p-100 p^{2}
\end{aligned}
$$

This last formula gives the revenue, $R$, in terms of the price, $p$.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

$$
\begin{array}{ll}
\text { A } & 1,5 \\
\text { B } & 2,3,6-8 \\
\text { C } & 1-10
\end{array}
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How do we multiply two polynomials?
2. Describe how the distributive property is used to multiply a monomial and a polynomial.
3. Describe how you would use the Foll method to multiply two binomials.
4. Show how the product of two binomials can be a trinomial.

Multiply the following by applying the distributive property.

1. $2 x(3 x+1)$
2. $4 x(2 x-3)$
3. $2 x^{2}\left(3 x^{2}-2 x+1\right)$
4. $5 x\left(4 x^{3}-5 x^{2}+x\right)$
5. $2 a b\left(a^{2}-a b+1\right)$
6. $3 a^{2} b\left(a^{3}+a^{2} b^{2}+b^{3}\right)$
7. $y^{2}\left(3 y^{2}+9 y+12\right)$
8. $5 y\left(2 y^{2}-3 y+5\right)$
9. $4 x^{2} y\left(2 x^{3} y+3 x^{2} y^{2}+8 y^{3}\right)$
10. $6 x y^{3}\left(2 x^{2}+5 x y+12 y^{2}\right)$

Multiply the following binomials. You should do about half the problems using the forl method and the other half using the column method. Remember, you want to be comfortable using both methods.
11. $(x+3)(x+4)$
12. $(x+2)(x+5)$
13. $(x+6)(x+1)$
14. $(x+1)(x+4)$
15. $\left(x+\frac{1}{2}\right)\left(x+\frac{3}{2}\right)$
16. $\left(x+\frac{3}{5}\right)\left(x+\frac{2}{5}\right)$
17. $(a+5)(a-3)$
18. $(a-8)(a+2)$
19. $(x-a)(y+b)$
20. $(x+a)(y-b)$
21. $(x+6)(x-6)$
23. $\left(y+\frac{5}{6}\right)\left(y-\frac{5}{6}\right)$
22. $(x+3)(x-3)$
25. $(2 x-3)(x-4)$
26. $(3 x-5)(x-2)$
27. $(a+2)(2 a-1)$
28. $(a-6)(3 a+2)$
29. $(2 x-5)(3 x-2)$
30. $(3 x+6)(2 x-1)$
31. $(2 x+3)(a+4)$
$=$ Videos available by instructor request
$>$ = Online student support materials available at www.thomsonedu.com/login
32. $(2 x-3)(a-4)$
33. $(5 x-4)(5 x+4)$
34. $(6 x+5)(6 x-5)$
35. $\left(2 x-\frac{1}{2}\right)\left(x+\frac{3}{2}\right)$
36. $\left(4 x-\frac{3}{2}\right)\left(x+\frac{1}{2}\right)$
37. $(1-2 a)(3-4 a)$
38. $(1-3 a)(3+2 a)$

For each of the following problems, fill in the area of each small rectangle and square, and then add the results together to find the indicated product.
39. $(x+2)(x+3)$

40. $(x+4)(x+5)$

41. $(x+1)(2 x+2)$

42. $(2 x+1)(2 x+2)$

4.5 Multiplication with Polynomials

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Multiply the following.
43. $(a-3)\left(a^{2}-3 a+2\right)$
44. $(a+5)\left(a^{2}+2 a+3\right)$
45. $(x+2)\left(x^{2}-2 x+4\right)$
46. $(x+3)\left(x^{2}-3 x+9\right)$
47. $(2 x+1)\left(x^{2}+8 x+9\right)$
48. $(3 x-2)\left(x^{2}-7 x+8\right)$
49. $\left(5 x^{2}+2 x+1\right)\left(x^{2}-3 x+5\right)$
50. $\left(2 x^{2}+x+1\right)\left(x^{2}-4 x+3\right)$

Multiply.
51. $\left(x^{2}+3\right)\left(2 x^{2}-5\right)$
52. $\left(4 x^{3}-8\right)\left(5 x^{3}+4\right)$
53. $\left(3 a^{4}+2\right)\left(2 a^{2}+5\right)$
54. $\left(7 a^{4}-8\right)\left(4 a^{3}-6\right)$
55. $(x+3)(x+4)(x+5)$
56. $(x-3)(x-4)(x-5)$

Simplify.
57. $(x-3)(x-2)+2$
58. $(2 x-5)(3 x+2)-4$
59. $(2 x-3)(4 x+3)+4$
60. $(3 x+8)(5 x-7)+52$
61. $(x+4)(x-5)+(-5)(2)$
62. $(x+3)(x-4)+(-4)(2)$
63. $2(x-3)+x(x+2)$
64. $5(x+3)+1(x+4)$
65. $3 x(x+1)-2 x(x-5)$
66. $4 x(x-2)-3 x(x-4)$
67. $x(x+2)-3$
68. $2 x(x-4)+6$
69. $a(a-3)+6$
70. $a(a-4)+8$
71. Find each product.
a. $(x+1)(x-1)$
b. $(x+1)(x+1)$
c. $(x+1)\left(x^{2}+2 x+1\right)$
d. $(x+1)\left(x^{3}+3 x^{2}+3 x+1\right)$
72. Find each product.
a. $(x+1)\left(x^{2}-x+1\right)$
b. $(x+2)\left(x^{2}-2 x+4\right)$
c. $(x+3)\left(x^{2}-3 x+9\right)$
d. $(x+4)\left(x^{2}-4 x+16\right)$
73. Find each product.
a. $(x+1)(x-1)$
b. $(x+1)(x-2)$
c. $(x+1)(x-3)$
d. $(x+1)(x-4)$
74. Find each product.
a. $(x+2)(x-2)$
b. $(x-2)\left(x^{2}+2 x+4\right)$
c. $\left(x^{2}+4\right)\left(x^{2}-4\right)$
d. $\left(x^{3}+8\right)\left(x^{3}-8\right)$

If the product of two expressions is 0 , then one or both of the expressions must be zero. That is, the only way to multiply and get 0 , is to multiply by 0 . For each expression below, find all values of $x$ that make the expression 0. (If the expression cannot be 0 , say so.)
75. $5 x$
77. $x+5$
79. $(x-3)(x+2)$
81. $x^{2}+16$
83. $(x-a)(x-b)$
76. $3 x^{2}$
78. $x^{2}+5$
80. $x(x-5)$
82. $x^{2}-100$

## Applying the Concepts

85. Area The length of a rectangle is 5 units more than twice the width. Write an expression for the area of the rectangle.

86. Area The length of a rectangle is 2 more than three times the width. Write an expression for the area of the rectangle.
87. Area The width and length of a rectangle are given by two consecutive integers. Write an expression for the area of the rectangle.
88. Area The width and length of a rectangle are given by two consecutive even integers. Write an expression for the area of the rectangle.
89. Revenue A store selling typewriter ribbons knows that the number of ribbons it can sell each week, $x$, is related to the price per ribbon, $p$, by the equation $x=1,200-100 p$. Write an expression for the weekly revenue that involves only the variables $R$ and $p$. (Remember: The equation for revenue is $R=x p$.
90. Revenue A store selling small portable radios knows from past experience that the number of radios it can sell each week, $x$, is related to the price per radio, $p$, by the equation $x=1,300-100 p$. Write an expression for the weekly revenue that involves only the variables $R$ and $p$.

## Maintaining Your Skills

Solve the following systems by the graphing method.
91. $x+y=4$
$x-y=2$
92. $x-y=4$
$x+y=2$

Solve each system by the elimination method.
93. $3 x+2 y=1$
$2 x+y=3$
94. $2 x+3 y=-1$
$3 x+5 y=-2$

Solve each system by the substitution method.
95. $x+y=20$
96. $2 x-6 y=2$
$y=5 x+2$
$y=3 x+1$
97. Investing A total of $\$ 1,200$ is invested in two accounts. One of the accounts pays $8 \%$ interest annually and the other pays $10 \%$ interest annually. If the total amount of interest earned from both accounts for the year is $\$ 104$, how much is invested in each account?
98. Coin Problem Amy has $\$ 1.85$ in dimes and quarters. If she has a total of 11 coins, how many of each coin does she have?

## Getting Ready for the Next Section

Simplify.
99. $13 \cdot 13$
101. $2(x)(-5)$
103. $6 x+(-6 x)$
105. $(2 x)(-3)+(2 x)(3)$ Multiply.
107. $-4(3 x-4)$
109. $(x-1)(x+2)$
111. $(x+3)(x+3)$
112. $(3 x-2)(3 x-2)$
12. $(3 x-2)(3 x-2)$
100. $3 x \cdot 3 x$
102. $2(2 x)(-3)$
104. $3 x+(-3 x)$
106. $(2 x)(-5 y)+(2 x)(5 y)$
108. $-2 x(2 x+7)$
110. $(x+5)(x-6)$

## Binomial Squares and Other Special Products

In this section we will combine the results of the last section with our definition of exponents to find some special products.

A $\begin{aligned} & \text { Find the square of a } \\ & \text { binomial. }\end{aligned}$
B Multiply expressions of the form $(a+b)(a-b)$.

SOLUTION To square $(3 x-2)$, we multiply it by itself:

$$
\begin{aligned}
(3 x-2)^{2} & =(3 x-2)(3 x-2) & & \text { Definition of exponents } \\
& =9 x^{2}-6 x-6 x+4 & & \text { Foll method } \\
& =9 x^{2}-12 x+4 & & \text { Combine similar terms }
\end{aligned}
$$

## Note

A very common mistake when squaring binomials is to write $(a+b)^{2}=a^{2}+b^{2}$
which just isn't true. The mistake becomes obvious when we substitute 2 for $a$ and 3 for $b$ :

$$
\begin{aligned}
(2+3)^{2} & \neq 2^{2}+3^{2} \\
25 & \neq 13
\end{aligned}
$$

Exponents do not distribute over addition or subtraction.

Notice that the first and last terms in the answer are the squares of the first and last terms in the original binomial and that the middle term is twice the product of the two terms in the original binomial.

## EXAMPLES

$$
\text { 2. } \begin{aligned}
(a+b)^{2} & =(a+b)(a+b) \\
& =a^{2}+2 a b+b^{2} \\
\text { 3. } a-b)^{2} & =(a-b)(a-b) \\
& =a^{2}-2 a b+b^{2}
\end{aligned}
$$

Binomial squares having the form of Examples 2 and 3 occur very frequently in algebra. It will be to your advantage to memorize the following rule for squaring a binomial.

Rule The square of a binomial is the sum of the square of the first term, the square of the last term, and twice the product of the two original terms. In symbols this rule is written as follows:

$$
(x+y)^{2}=\underset{\text { Square of first term }}{x^{2}}+\underset{\text { Twice product of the two terms }}{\text { Square of last term }}
$$

## EXAMPLES Multiply using the preceding rule:

|  | First term squared |  | Twice their product |  | Last term squared | Answer |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4. $(x-5)^{2}$ | $X^{2}$ | + | $2(x)(-5)$ | + | 25 | $=x^{2}-10 x+25$ |
| 5. $(x+2)^{2}$ | $\chi^{2}$ | + | $2(x)(2)$ | + | 4 | $=x^{2}+4 x$ |
| 6. $(2 x-3)^{2}$ | $4 x^{2}$ | + | $2(2 x)(-3)$ | + | 9 | $=4 x^{2}-12 x+9$ |
| 7. $(5 x-4)^{2}$ | $25 x^{2}$ |  | $2(5 x)(-4)$ | $+$ | 16 | $=25 x^{2}-40 x+16$ |

Another special product that occurs frequently is $(a+b)(a-b)$. The only difference in the two binomials is the sign between the two terms. The interesting thing about this type of product is that the middle term is always zero. Here are some examples.

## EXAMPLES Multiply using the Foil method:

8. $(2 x-3)(2 x+3)=4 x^{2}+6 x-6 x-9 \quad$ Foll method

$$
=4 x^{2}-9
$$

9. $(x-5)(x+5)=x^{2}+5 x-5 x-25 \quad$ Foll method

$$
=x^{2}-25
$$

10. $(3 x-1)(3 x+1)=9 x^{2}+3 x-3 x-1 \quad$ Foil method

$$
=9 x^{2}-1
$$

Notice that in each case the middle term is zero and therefore doesn't appear in the answer. The answers all turn out to be the difference of two squares. Here is a rule to help you memorize the result.

Rule When multiplying two binomials that differ only in the sign between their terms, subtract the square of the last term from the square of the first term.

$$
(a-b)(a+b)=a^{2}-b^{2}
$$

Here are some problems that result in the difference of two squares.

## EXAMPLES Multiply using the preceding rule:

11. $(x+3)(x-3)=x^{2}-9$
12. $(a+2)(a-2)=a^{2}-4$
13. $(9 a+1)(9 a-1)=81 a^{2}-1$
14. $(2 x-5 y)(2 x+5 y)=4 x^{2}-25 y^{2}$
15. $(3 a-7 b)(3 a+7 b)=9 a^{2}-49 b^{2}$

Although all the problems in this section can be worked correctly using the methods in the previous section, they can be done much faster if the two rules are memorized. Here is a summary of the two rules:

$$
\begin{aligned}
(a+b)^{2}= & (a+b)(a+b)=a^{2}+2 a b+b^{2} \\
(a-b)^{2}= & (a-b)(a-b)=a^{2}-2 a b+b^{2} \\
& (a-b)(a+b)=a^{2}-b^{2}
\end{aligned}
$$

EXAMPLE 16 Write an expression in symbols for the sum of the squares of three consecutive even integers. Then, simplify that expression.
SOLUTION If we let $x=$ the first of the even integers, then $x+2$ is the next consecutive even integer, and $x+4$ is the one after that. An expression for the sum of their squares is

$$
\begin{aligned}
x^{2}+ & (x+2)^{2}+(x+4)^{2} & & \text { Sum of squares } \\
& =x^{2}+\left(x^{2}+4 x+4\right)+\left(x^{2}+8 x+16\right) & & \text { Expand squares } \\
& =3 x^{2}+12 x+20 & & \text { Add similar terms }
\end{aligned}
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Describe how you would square the binomial $a+b$.
2. Explain why $(x+3)^{2}$ is not equal to $x^{2}+9$.
3. What kind of products result in the difference of two squares?
4. When multiplied out, how will $(x+3)^{2}$ and $(x-3)^{2}$ differ?

## Problem Set 4.6

Perform the indicated operations.

1. $(x-2)^{2}$
2. $(x+2)^{2}$
3. $(a+3)^{2}$
4. $(a-3)^{2}$
5. $(x-5)^{2}$
6. $(x-4)^{2}$
7. $\left(a-\frac{1}{2}\right)^{2}$
8. $\left(a+\frac{1}{2}\right)^{2}$
9. $(x+10)^{2}$
10. $(x-10)^{2}$
11. $(a+0.8)^{2}$
12. $(a-0.4)^{2}$
13. $(2 x-1)^{2}$
14. $(3 x+2)^{2}$
15. $(4 a+5)^{2}$
16. $(4 a-5)^{2}$
17. $(3 x-2)^{2}$
18. $(2 x-3)^{2}$
19. $(3 a+5 b)^{2}$
20. $(5 a-3 b)^{2}$
21. $(4 x-5 y)^{2}$
22. $(5 x+4 y)^{2}$
23. $(7 m+2 n)^{2}$
24. $(2 m-7 n)^{2}$
25. $(6 x-10 y)^{2}$
26. $(10 x+6 y)^{2}$
27. $\left(x^{2}+5\right)^{2}$
28. $\left(x^{2}+3\right)^{2}$
29. $\left(a^{2}+1\right)^{2}$
30. $\left(a^{2}-2\right)^{2}$
31. $\left(y+\frac{3}{2}\right)^{2}$
32. $\left(y-\frac{3}{2}\right)^{2}$
33. $\left(a+\frac{1}{2}\right)^{2}$
34. $\left(a-\frac{5}{2}\right)^{2}$
35. $\left(x+\frac{3}{4}\right)^{2}$
36. $\left(t+\frac{1}{5}\right)^{2}$
37. $\left(x-\frac{3}{8}\right)^{2}$

Comparing Expressions Fill in each table.
39.

| $x$ | $(x+3)^{2}$ | $x^{2}+9$ | $x^{2}+6 x+9$ |
| :--- | :--- | :--- | :--- |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

40. 

| $\boldsymbol{x}$ | $(x-\mathbf{5})^{2}$ | $\boldsymbol{x}^{2}+\mathbf{2 5}$ | $\mathbf{x}^{2}-\mathbf{1 0 x}+\mathbf{2 5}$ |
| :--- | :--- | :--- | :--- |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

41. 

| $a$ | $b$ | $(a+b)^{2}$ | $a^{2}+b^{2}$ | $a^{2}+a b+b^{2}$ | $a^{2}+2 a b+b^{2}$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 |  |  |  |  |
| 3 | 5 |  |  |  |  |
| 3 | 4 |  |  |  |  |
| 4 | 5 |  |  |  |  |

42. 

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $(\boldsymbol{a}-\boldsymbol{b})^{2}$ | $\boldsymbol{a}^{2}-\boldsymbol{b}^{2}$ | $\boldsymbol{a}^{2}-\mathbf{2 a b}+\boldsymbol{b}^{2}$ |
| :--- | :--- | :--- | :--- | :--- |
| 2 | 1 |  |  |  |
| 5 | 2 |  |  |  |
| 2 | 5 |  |  |  |
| 4 | 3 |  |  |  |

Multiply.
43. $(a+5)(a-5)$
44. $(a-6)(a+6)$
45. $(y-1)(y+1)$
46. $(y-2)(y+2)$
47. $(9+x)(9-x)$
48. $(10-x)(10+x)$
49. $(2 x+5)(2 x-5)$
50. $(3 x+5)(3 x-5)$
51. $\left(4 x+\frac{1}{3}\right)\left(4 x-\frac{1}{3}\right)$
52. $\left(6 x+\frac{1}{4}\right)\left(6 x-\frac{1}{4}\right)$
53. $(2 a+7)(2 a-7)$
54. $(3 a+10)(3 a-10)$
55. $(6-7 x)(6+7 x)$
56. $(7-6 x)(7+6 x)$
57. $\left(x^{2}+3\right)\left(x^{2}-3\right)$
58. $\left(x^{2}+2\right)\left(x^{2}-2\right)$
59. $\left(a^{2}+4\right)\left(a^{2}-4\right)$
60. $\left(a^{2}+9\right)\left(a^{2}-9\right)$
61. $\left(5 y^{4}-8\right)\left(5 y^{4}+8\right)$
62. $\left(7 y^{5}+6\right)\left(7 y^{5}-6\right)$

Multiply and simplify.
63. $(x+3)(x-3)+(x-5)(x+5)$
64. $(x-7)(x+7)+(x-4)(x+4)$
65. $(2 x+3)^{2}-(4 x-1)^{2}$
66. $(3 x-5)^{2}-(2 x+3)^{2}$
67. $(a+1)^{2}-(a+2)^{2}+(a+3)^{2}$
68. $(a-1)^{2}+(a-2)^{2}-(a-3)^{2}$
69. $(2 x+3)^{3}$
70. $(3 x-2)^{3}$
71. Find the value of each expression when $x$ is 6 .
a. $x^{2}-25$
b. $(x-5)^{2}$
c. $(x+5)(x-5)$
72. Find the value of each expression when $x$ is 5 .
a. $x^{2}-9$
b. $(x-3)^{2}$
c. $(x+3)(x-3)$
73. Evaluate each expression when $x$ is -2 .
a. $(x+3)^{2}$
b. $x^{2}+9$
c. $x^{2}+6 x+9$
74. Evaluate each expression when $x$ is -3 .
a. $(x+2)^{2}$
b. $x^{2}+4$
c. $x^{2}+4 x+4$

## Applying the Concepts

75. Number Problem Write an expression for the sum of the squares of two consecutive integers. Then, simplify that expression.
76. Number Problem Write an expression for the sum of the squares of two consecutive odd integers. Then, simplify that expression.
77. Number Problem Write an expression for the sum of the squares of three consecutive integers. Then, simplify that expression.
78. Number Problem Write an expression for the sum of the squares of three consecutive odd integers. Then, simplify that expression.
79. Area We can use the concept of area to further justify our rule for squaring a binomial. The length of each side of the square shown in the figure is $a+b$. (The longer line segment has length $a$ and the shorter line segment has length $b$.) The area of the whole square is $(a+b)^{2}$. However, the whole area is the sum of the areas of the two smaller squares and the two smaller rectangles that make it up. Write the area of the two smaller squares and the two smaller rectangles and then add them together to verify the formula $(a+b)^{2}=a^{2}+2 a b+b^{2}$.

80. Area The length of each side of the large square shown in the figure is $x+5$. Therefore, its area is $(x+5)^{2}$. Find the area of the two smaller squares and the two smaller rectangles that make up the large square, then add them together to verify the formula $(x+5)^{2}=x^{2}+10 x+25$.


## Maintaining Your Skills

Solve each system by graphing.
81. $x+y=2$
$x-y=4$
82. $x+y=1$
$x-y=-3$
83. $y=2 x+3$
$y=-2 x-1$
84. $y=2 x-1$
$y=-2 x+3$
87. $\frac{3 x^{2}}{3}$
88. $\frac{4 x^{2}}{2}$
89. $\frac{9 x^{2}}{3 x}$
90. $\frac{3 x^{4}}{9 x^{2}}$
91. $\frac{24 x^{3} y^{2}}{8 x^{2} y}$
92. $-\frac{4 x^{2} y^{3}}{8 x^{2} y}$
93. $\frac{15 x^{2} y}{3 x y}$
94. $\frac{21 x y^{2}}{3 x y}$
95. $\frac{35 a^{6} b^{8}}{70 a^{2} b^{10}}$
96. $\frac{75 a^{2} b^{6}}{25 a^{4} b^{3}}$

## Getting Ready for the Next Section

Simplify each expression (divide).
85. $\frac{10 x^{3}}{5 x}$
86. $\frac{15 x^{2}}{5 x}$

## 4.7

## OBJECTIVES

A Divide a polynomial by a monomial.

## Dividing a Polynomial by a Monomial

To divide a polynomial by a monomial, we will use the definition of division and apply the distributive property. Follow the steps in this example closely.

## EXAMPLE 1 Divide $10 x^{3}-15 x^{2}$ by $5 x$.

SOLUTION

$$
\begin{array}{rlr}
\frac{10 x^{3}-15 x^{2}}{5 x} & =\left(10 x^{3}-15 x^{2}\right) \frac{1}{5 x} & \\
\begin{array}{l}
\text { Division by } 5 x \text { is the same } \\
\text { as multiplication by } \frac{1}{5 x}
\end{array} \\
& =10 x^{3}\left(\frac{1}{5 x}\right)-15 x^{2}\left(\frac{1}{5 x}\right) & \\
& \begin{array}{l}
\text { Distribute } \frac{1}{5 x} \text { to both term } \\
5 x
\end{array} \frac{10 x^{3}}{5 x} & \begin{array}{l}
\text { Multiplication by } \frac{1}{5 x} \text { is the } \\
\text { same as division by } 5 x
\end{array} \\
& =2 x^{2}-3 x &
\end{array} \begin{aligned}
& \text { Division of monomials as } \\
& \text { done in Section } 4.3
\end{aligned}
$$

If we were to leave out the first steps, the problem would look like this:

$$
\begin{aligned}
\frac{10 x^{3}-15 x^{2}}{5 x} & =\frac{10 x^{3}}{5 x}-\frac{15 x^{2}}{5 x} \\
& =2 x^{2}-3 x
\end{aligned}
$$

The problem is much shorter and clearer this way. You may leave out the first two steps from Example 1 when working problems in this section. They are part of Example 1 only to help show you why the following rule is true.

Rule To divide a polynomial by a monomial, simply divide each term in the polynomial by the monomial.

Here are some further examples using our rule for division of a polynomial by a monomial.

EXAMPLE 2 Divide $\frac{3 x^{2}-6}{3}$.
SOLUTION We begin by writing the 3 in the denominator under each term in the numerator. Then we simplify the result:

$$
\begin{aligned}
\frac{3 x^{2}-6}{3} & =\frac{3 x^{2}}{3}-\frac{6}{3} & & \text { Divide each term in the numerator by } 3 \\
& =x^{2}-2 & & \text { Simplify }
\end{aligned}
$$

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## EXAMPLE 3 Divide $\frac{4 x^{2}-2}{2}$.

SOLUTION Dividing each term in the numerator by 2, we have

$$
\begin{aligned}
\frac{4 x^{2}-2}{2} & =\frac{4 x^{2}}{2}-\frac{2}{2} & & \text { Divide each term in the numerator by } 2 \\
& =2 x^{2}-1 & & \text { Simplify }
\end{aligned}
$$

## EXAMPLE 4 Find the quotient of $27 x^{3}-9 x^{2}$ and $3 x$.

SOLUTION We again are asked to divide the first polynomial by the second one:

$$
\begin{aligned}
\frac{27 x^{3}-9 x^{2}}{3 x} & =\frac{27 x^{3}}{3 x}-\frac{9 x^{2}}{3 x} & & \text { Divide each term by } 3 x \\
& =9 x^{2}-3 x & & \text { Simplify }
\end{aligned}
$$

## EXAMPLE 5 Divide $\left(15 x^{2} y-21 x y^{2}\right) \div(-3 x y)$.

SOLUTION This is the same type of problem we have shown in the first four examples; it is just worded a little differently. Note that when we divide each term in the first polynomial by $-3 x y$, the negative sign must be taken into account:

$$
\begin{aligned}
\frac{15 x^{2} y-21 x y^{2}}{-3 x y} & =\frac{15 x^{2} y}{-3 x y}-\frac{21 x y^{2}}{-3 x y} & & \text { Divide each term by }-3 x y \\
& =-5 x-(-7 y) & & \text { Simplify } \\
& =-5 x+7 y & & \text { Simplify }
\end{aligned}
$$

EXAMPLE 6 Divide $\frac{24 x^{3} y^{2}+16 x^{2} y^{2}-4 x^{2} y^{3}}{8 x^{2} y}$.
SOLUTION Writing $8 x^{2} y$ under each term in the numerator and then simplifying, we have

$$
\begin{aligned}
\frac{24 x^{3} y^{2}+16 x^{2} y^{2}-4 x^{2} y^{3}}{8 x^{2} y} & =\frac{24 x^{3} y^{2}}{8 x^{2} y}+\frac{16 x^{2} y^{2}}{8 x^{2} y}-\frac{4 x^{2} y^{3}}{8 x^{2} y} \\
& =3 x y+2 y-\frac{y^{2}}{2}
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-6

From the examples in this section, it is clear that to divide a polynomial by a monomial, we must divide each term in the polynomial by the monomial. Often, students taking algebra for the first time will make the following mistake:

$$
\frac{x+22}{2}=x+1 \quad \text { Mistake }
$$

The mistake here is in not dividing both terms in the numerator by 2 . The correct way to divide $x+2$ by 2 looks like this:

$$
\frac{x+2}{2}=\frac{x}{2}+\frac{2}{2}=\frac{x}{2}+1 \quad \text { Correct }
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What property of real numbers is the key to dividing a polynomial by a monomial?
2. Describe how you would divide a polynomial by $5 x$.
3. Is our answer in Example 6 a polynomial?
4. Explain the mistake in the problem $\frac{x+2}{2}=x+1$.

## Problem Set 4.7

Divide the following polynomials by $5 x$.

1. $5 x^{2}-10 x$
2. $10 x^{3}-15 x$
3. $15 x-10 x^{3}$
4. $50 x^{3}-20 x^{2}$
5. $25 x^{2} y-10 x y$
6. $15 x y^{2}+20 x^{2} y$
7. $35 x^{5}-30 x^{4}+25 x^{3}$
8. $40 x^{4}-30 x^{3}+20 x^{2}$
9. $50 x^{5}-25 x^{3}+5 x$
10. $75 x^{6}+50 x^{3}-25 x$

Divide the following by $-2 a$.
11. $8 a^{2}-4 a$
13. $16 a^{5}+24 a^{4}$
15. $8 a b+10 a^{2}$
12. $a^{3}-6 a^{2}$
14. $30 a^{6}+20 a^{3}$
16. $6 a^{2} b-10 a b^{2}$
19. $a^{2}+2 a b+b^{2}$
20. $a^{2} b-2 a b^{2}+b^{3}$

Perform the following divisions (find the following quotients).
21. $\frac{6 x+8 y}{2}$
22. $\frac{9 x-3 y}{3}$
23. $\frac{7 y-21}{-7}$
24. $\frac{14 y-12}{2}$
25. $\frac{2 x^{2}+16 x-18}{2}$
26. $\frac{3 x^{2}-3 x-18}{3}$
27. $\frac{3 y^{2}-9 y+3}{3}$
28. $\frac{2 y^{2}-8 y+2}{2}$
29. $\frac{x^{2} y-x^{3} y^{2}}{-x^{2} y}$
30. $\frac{a b+a^{2} b^{2}}{a b}$
31. $\frac{a^{2} b^{2}-a b^{2}}{-a b^{2}} \quad$ 32. $\frac{a^{2} b^{2} c+a b^{2} c^{2}}{a b c}$
33. $\frac{x^{3}-3 x^{2} y+x y^{2}}{x}$
34. $\frac{x^{2}-3 x y^{2}+x y^{3}}{x}$
35. $\frac{10 a^{2}-15 a^{2} b+25 a^{2} b^{2}}{5 a^{2}}$
36. $\frac{11 a^{2} b^{2}-33 a b}{-11 a b}$
37. $\frac{26 x^{2} y^{2}-13 x y}{-13 x y}$
38. $\frac{6 x^{2} y^{2}-3 x y}{6 x y}$
39. $\frac{4 x^{2} y^{2}-2 x y}{4 x y}$
40. $\frac{6 x^{2} a+12 x^{2} b-6 x^{2} c}{36 x^{2}}$
41. $\frac{5 a^{2} x-10 a x^{2}+15 a^{2} x^{2}}{20 a^{2} x^{2}}$
42. $\frac{12 a x-9 b x+18 c x}{6 x^{2}}$
43. $\frac{16 x^{5}+8 x^{2}+12 x}{12 x^{3}}$
44. $\frac{27 x^{2}-9 x^{3}-18 x^{4}}{-18 x^{3}}$

Divide. Assume all variables represent positive numbers.
45. $\frac{9 a^{5 m}-27 a^{3 m}}{3 a^{2 m}}$
46. $\frac{26 a^{3 m}-39 a^{5 m}}{13 a^{3 m}}$
47. $\frac{10 x^{5 m}-25 x^{3 m}+35 x^{m}}{5 x^{m}}$
48. $\frac{18 x^{2 m}+24 x^{4 m}-30 x^{6 m}}{6 x^{2 m}}$

Simplify each numerator, and then divide.
49. $\frac{2 x^{3}(3 x+2)-3 x^{2}(2 x-4)}{2 x^{2}}$
50. $\frac{5 x^{2}(6 x-3)+6 x^{3}(3 x-1)}{3 x}$
51. $\frac{(x+2)^{2}-(x-2)^{2}}{2 x}$
52. $\frac{(x-3)^{2}-(x+3)^{2}}{3 x}$
53. $\frac{(x+5)^{2}+(x+5)(x-5)}{2 x}$
54. $\frac{(x-4)^{2}+(x+4)(x-4)}{2 x}$
55. Find the value of each expression when $x$ is 2 .
a. $2 x+3$
b. $\frac{10 x+15}{5}$
c. $10 x+3$
56. Find the value of each expression when $x$ is 5 .
a. $3 x+2$
b. $\frac{6 x^{2}+4 x}{2 x}$
c. $6 x^{2}+2$
57. Evaluate each expression for $x=10$.
a. $\frac{3 x+8}{2}$
b. $3 x+4$
c. $\frac{3}{2} x+4$
58. Evaluate each expression for $x=10$.
a. $\frac{5 x-6}{2}$
b. $5 x-3$
c. $\frac{5}{2} x-3$
59. Find the value of each expression when $x$ is 2 .
a. $2 x^{2}-3 x$
b. $\frac{10 x^{3}-15 x^{2}}{5 x}$
60. Find the value of each expression when $a$ is 3 .
a. $-4 a+2$
b. $\frac{8 a^{2}-4 a}{-2 a}$

## Maintaining Your Skills

Solve each system of equations by the elimination method.
61. $x+y=6$
62. $2 x+y=5$
$x-y=8$
$-x+y=-4$
63. $2 x-3 y=-5$
$x+y=5$
64. $2 x-4 y=10$
$3 x-2 y=-1$

Solve each system by the substitution method.
65. $x+y=2$
$y=2 x-1$
66. $2 x-3 y=4$
$x=3 y-1$
67. $4 x+2 y=8$
$y=-2 x+4$
68. $4 x+2 y=8$
$y=-2 x+5$

## Getting Ready for the Next Section

Divide.
69. $2 7 \longdiv { 3 , 9 6 2 }$
70. $1 3 \longdiv { 1 8 , 7 8 0 }$
71. $\frac{2 x^{2}+5 x}{x}$
72. $\frac{7 x^{2}+9 x^{3}+3 x^{7}}{x^{3}}$

Multiply.
73. $(x-3) x$
74. $(x-3)(-2)$
75. $2 x^{2}(x-5)$
76. $10 x(x-5)$

Subtract.
77. $\left(x^{2}-5 x\right)-\left(x^{2}-3 x\right)$
78. $\left(2 x^{3}+0 x^{2}\right)-\left(2 x^{3}-10 x^{2}\right)$
79. $(-2 x+8)-(-2 x+6)$
80. $(4 x-14)-(4 x-10)$

## 4.8

## OBJECTIVES

A Divide a polynomial by a polynomial.

## Dividing a Polynomial by a Polynomial

Since long division for polynomials is very similar to long division with whole numbers, we will begin by reviewing a division problem with whole numbers. You may realize when looking at Example 1 that you don't have a very good idea why you proceed as you do with long division. What you do know is that the process always works. We are going to approach the explanations in this section in much the same manner; that is, we won't always be sure why the steps we will use are important, only that they always produce the correct result.

## EXAMPLE 1 Divide $2 7 \longdiv { 3 , 9 6 2 . }$

## SOLUTION

$$
\begin{aligned}
\frac{1}{2 7 \longdiv { 3 , 9 6 2 }} & \leftarrow \text { Estimate } 27 \text { into } 39 \\
\frac{27}{12} & \leftarrow \text { Multiply } 1 \times 27=27 \\
& \leftarrow \text { Subtract } 39-27=12 \\
\frac{1}{2 7 \longdiv { 3 , 9 6 2 }} & \\
\frac{27 \downarrow}{126} & \leftarrow \text { Bring down the 6 }
\end{aligned}
$$

These are the four basic steps in long division. Estimate, multiply, subtract, and bring down the next term. To finish the problem, we simply perform the same four steps again:

$$
\begin{aligned}
& 14 \\
& 2 7 \longdiv { 3 , 9 6 2 } \leftarrow 4 \text { is the estimate } \\
& \frac{27 \downarrow}{126} \\
& \frac{108}{182} \leftarrow \text { Multiply to get } \mathbf{1 0 8} \\
& \leftarrow \text { Subtract to get } \mathbf{1 8} \text {, then bring down the } \mathbf{2}
\end{aligned}
$$

One more time.

$$
\begin{aligned}
146 & \leftarrow 6 \text { is the estimate } \\
2 7 \longdiv { 3 , 9 6 2 } & \begin{aligned}
& \frac{27}{126} \\
& \frac{108}{182} \\
& \frac{162}{20} \leftarrow \text { Multiply to get } 162 \\
& \leftarrow \text { Subtract to get } 20
\end{aligned}
\end{aligned}
$$

Since there is nothing left to bring down, we have our answer.

$$
\frac{3,962}{27}=146+\frac{20}{27} \quad \text { or } \quad 146 \frac{20}{27}
$$

Here is how it works with polynomials.
EXAMPLE 2 Divide $\frac{x^{2}-5 x+8}{x-3}$.

## SOLUTION

$$
\begin{aligned}
& x - 3 \longdiv { x ^ { 2 } - 5 x + 8 } \leftarrow \text { Estimate } x^{2} \div x=x \\
& \begin{aligned}
\frac{-}{\not+} x^{2} \not+3 x & \leftarrow \text { Multiply } x(x-3)=x^{2}-3 x \\
-2 x & \leftarrow \text { Subtract }\left(x^{2}-5 x\right)-\left(x^{2}-3 x\right)=-2 x
\end{aligned} \\
& x - 3 \longdiv { x } \begin{array} { c } 
{ x } \\
{ x ^ { 2 } - 5 x + 8 }
\end{array} \\
& \frac{-x^{2}+3 x}{-2 x+8} \leftarrow \text { Bring down the } 8
\end{aligned}
$$

Notice that to subtract one polynomial from another, we add its opposite. That is why we change the signs on $x^{2}-3 x$ and add what we get to $x^{2}-5 x$. (To subtract the second polynomial, simply change the signs and add.)

We perform the same four steps again:

$$
\begin{aligned}
& x - 3 \longdiv { x ^ { 2 } - 5 x + 8 } \leftarrow - 2 \text { is the estimate }(-2 x \div x=-2) \\
& \frac{-x^{2} \neq 3 x \downarrow}{-2 x+8} \\
& \begin{aligned}
\frac{+2 x+6}{\not-2} & \leftarrow \text { Multiply }-2(x-3)=-2 x+6 \\
2 & \leftarrow \text { Subtract }(-2 x+8)-(-2 x+6)=2
\end{aligned}
\end{aligned}
$$

Since there is nothing left to bring down, we have our answer:

$$
\frac{x^{2}-5 x+8}{x-3}=x-2+\frac{2}{x-3}
$$

To check our answer, we multiply $(x-3)(x-2)$ to get $x^{2}-5 x+6$. Then, adding on the remainder, 2 , we have $x^{2}-5 x+8$.

## EXAMPLE 3 Divide $\frac{6 x^{2}-11 x-14}{2 x-5}$.

## SOLUTION

$$
\begin{gathered}
3 x+2 \\
2 x - 5 \longdiv { 6 x ^ { 2 } - 1 1 x - 1 4 } \\
\frac{+6 x^{2}+15 x \downarrow}{+} \downarrow \\
\frac{+4 x-14}{-4} \\
\frac{+4 x+10}{+} \\
\frac{6 x^{2}-11 x-14}{2 x-5}=3 x+2+\frac{-4}{2 x-5}
\end{gathered}
$$

One last step is sometimes necessary. The two polynomials in a division problem must both be in descending powers of the variable and cannot skip any powers from the highest power down to the constant term.

EXAMPLE $4 \quad$ Divide $\frac{2 x^{3}-3 x+2}{x-5}$.
SOLUTION The problem will be much less confusing if we write $2 x^{3}-3 x+2$ as $2 x^{3}+0 x^{2}-3 x+2$. Adding $0 x^{2}$ does not change our original problem.

Adding the term $0 x^{2}$ gives us a column in which to write $10 x^{2}$. (Remember, you can add and subtract only similar terms.)
Here is the completed problem:

$$
\begin{array}{r}
2 x^{2}+10 x+47 \\
x - 5 \longdiv { 2 x ^ { 3 } + 0 x ^ { 2 } - 3 x + 2 } \\
\frac{\bar{H} 2 x^{3}+10 x^{2}}{+10 x^{2}}-3 x \\
\frac{-}{+10 x^{2}+50 x} \\
\frac{+47 x}{+}+2 \\
\frac{-47++235}{237}
\end{array}
$$

Our answer is $\frac{2 x^{3}-3 x+2}{x-5}=2 x^{2}+10 x+47+\frac{237}{x-5}$.

## LINKING OBJECTIVES

 AND EXAMPLESNext to each objective we have listed the examples that are best described by that objective.

A 2-4

As you can see, long division with polynomials is a mechanical process. Once you have done it correctly a couple of times, it becomes very easy to produce the correct answer.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What are the four steps used in long division with whole numbers?
2. How is division of two polynomials similar to long division with whole numbers?
3. What are the four steps used in long division with polynomials?
4. How do we use $O$ when dividing the polynomial $2 x^{3}-3 x+2$ by $x-5$ ?

## Problem Set 4.8

Divide.

1. $\frac{x^{2}-5 x+6}{x-3}$
2. $\frac{x^{2}-5 x+6}{x-2}$
3. $\frac{a^{2}+9 a+20}{a+5}$
4. $\frac{a^{2}+9 a+20}{a+4}$
5. $\frac{x^{2}-6 x+9}{x-3}$
6. $\frac{x^{2}+10 x+25}{x+5}$
7. $\frac{2 x^{2}+5 x-3}{2 x-1}$
8. $\frac{4 x^{2}+4 x-3}{2 x-1}$
9. $\frac{2 a^{2}-9 a-5}{2 a+1}$
10. $\frac{4 a^{2}-8 a-5}{2 a+1}$
11. $\frac{x^{2}+5 x+8}{x+3}$
12. $\frac{x^{2}+5 x+4}{x+3}$
13. $\frac{a^{2}+3 a+2}{a+5}$
14. $\frac{a^{2}+4 a+3}{a+5}$
15. $\frac{x^{2}+2 x+1}{x-2}$
16. $\frac{x^{2}+6 x+9}{x-3}$
17. $\frac{x^{2}+5 x-6}{x+1}$
18. $\frac{x^{2}-x-6}{x+1}$
19. $\frac{a^{2}+3 a+1}{a+2}$
20. $\frac{a^{2}-a+3}{a+1}$
21. $\frac{2 x^{2}-2 x+5}{2 x+4}$
22. $\frac{15 x^{2}+19 x-4}{3 x+8}$
23. $\frac{6 a^{2}+5 a+1}{2 a+3}$
24. $\frac{4 a^{2}+4 a+3}{2 a+1}$
25. $\frac{6 a^{3}-13 a^{2}-4 a+15}{3 a-5}$
26. $\frac{2 a^{3}-a^{2}+3 a+2}{2 a+1}$

Fill in the missing terms in the numerator, and then use long division to find the quotients (see Example 4).
27. $\frac{x^{3}+4 x+5}{x+1}$
28. $\frac{x^{3}+4 x^{2}-8}{x+2}$
29. $\frac{x^{3}-1}{x-1}$
30. $\frac{x^{3}+1}{x+1}$
31. $\frac{x^{3}-8}{x-2}$
32. $\frac{x^{3}+27}{x+3}$

$=$ Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
33. Find the value of each expression when $x$ is 3 .
a. $x^{2}+2 x+4$
b. $\frac{x^{3}-8}{x-2}$
c. $x^{2}-4$
34. Find the value of each expression when $x$ is 2 .
a. $x^{2}-3 x+9$
b. $\frac{x^{3}+27}{x+3}$
c. $x^{2}+9$
35. Find the value of each expression when $x$ is 4 .
a. $x+3$
b. $\frac{x^{2}+9}{x+3}$
c. $x-3+\frac{18}{x+3}$
36. Find the value of each expression when $x$ is 2 .
a. $X+1$
b. $\frac{x^{2}+1}{x+1}$
c. $x-1+\frac{2}{x+1}$

Long Division Use the information in the table to find the monthly payment for auto insurance for the cities below. Round to the nearest cent.

37. Detroit
38. Philadelphia
39. Newark, N.J.
40. Los Angeles

## Maintaining Your Skills

Use systems of equations to solve the following word problems.
41. Number Problem The sum of two numbers is 25 . One of the numbers is 4 times the other. Find the numbers.
42. Number Problem The sum of two numbers is 24 . One of the numbers is 3 more than twice the other. Find the numbers.
43. Investing Suppose you have a total of $\$ 1,200$ invested in two accounts. One of the accounts pays $8 \%$ annual interest and the other pays $9 \%$ annual interest. If your total interest for the year is $\$ 100$, how much money did you invest in each of the accounts?
44. Investing If you invest twice as much money in an account that pays $12 \%$ annual interest as you do in an account that pays $11 \%$ annual interest, how much do you have in each account if your total interest for a year is $\$ 210$ ?
45. Money Problem If you have a total of $\$ 160$ in $\$ 5$ bills and $\$ 10$ bills, how many of each type of bill do you have if you have 4 more $\$ 10$ bills than $\$ 5$ bills?
46. Coin Problem Suppose you have 20 coins worth a total of $\$ 2.80$. If the coins are all nickels and quarters, how many of each type do you have?
47. Mixture Problem How many gallons of $20 \%$ antifreeze solution and $60 \%$ antifreeze solution must be mixed to get 16 gallons of $35 \%$ antifreeze solution?
48. Mixture Problem A chemist wants to obtain 80 liters of a solution that is $12 \%$ hydrochloric acid. How many liters of $10 \%$ hydrochloric acid solution and $20 \%$ hydrochloric acid solution should he mix to do so?

## Chapter 4 SUMMARY

## Exponents: Definition and Properties [4.1, 4.2]

Integer exponents indicate repeated multiplications.

| $a^{r} \cdot a^{s}$ | $=a^{r+s}$ |  | To multiply with the same base, you add exponents |
| ---: | :--- | ---: | :--- |
| $\frac{a^{r}}{a^{s}}$ | $=a^{r-s}$ |  | To divide with the same base, you subtract exponents |
| $(a b)^{r}$ | $=a^{r} \cdot b^{r}$ |  | Exponents distribute over multiplication |
| $\left(\frac{a}{b}\right)^{r}$ | $=\frac{a^{r}}{b^{r}}$ |  | Exponents distribute over division |
| $\left(a^{r}\right)^{s}$ | $=a^{r \cdot s}$ |  | A power of a power is the product of the powers |
| $a^{-r}$ | $=\frac{1}{a^{r}}$ |  | Negative exponents imply reciprocals |

## Multiplication of Monomials [4.3]

To multiply two monomials, multiply coefficients and add exponents.

## Division of Monomials [4.3]

To divide two monomials, divide coefficients and subtract exponents.

## Scientific Notation [4.1, 4.2, 4.3]

A number is in scientific notation when it is written as the product of a number between 1 and 10 and an integer power of 10 .

## Addition of Polynomials [4.4]

5. $\left(3 x^{2}-2 x+1\right)+\left(2 x^{2}+7 x-3\right)$

$$
=5 x^{2}+5 x-2
$$

6. $(3 x+5)-(4 x-3)$

$$
\begin{aligned}
& =3 x+5-4 x+3 \\
& =-x+8
\end{aligned}
$$

## Subtraction of Polynomials [4.4]

To subtract one polynomial from another, add the opposite of the second to the first.
7. a. $2 a^{2}\left(5 a^{2}+3 a-2\right)$

$$
=10 a^{4}+6 a^{3}-4 a^{2}
$$

b. $(x+2)(3 x-1)$

$$
=3 x^{2}-x+6 x-2
$$

$$
=3 x^{2}+5 x-2
$$

c. $2 x^{2}-3 x+4$
$\frac{3 x-2}{6 x^{3}-9 x^{2}+12 x}$
$\frac{-4 x^{2}+6 x-8}{6 x^{3}-13 x^{2}+18 x-8}$

## Multiplication of Polynomials [4.5]

To multiply a polynomial by a monomial, we apply the distributive property. To multiply two binomials we use the foil method. In other situations we use the column method. Each method achieves the same result: To multiply any two polynomials, we multiply each term in the first polynomial by each term in the second polynomial.

## Special Products [4.6]

8. $(x+3)^{2}=x^{2}+6 x+9$
$(x-3)^{2}=x^{2}-6 x+9$ $(x+3)(x-3)=x^{2}-9$

$$
\left.\begin{array}{l}
(a+b)^{2}=a^{2}+2 a b+b^{2} \\
(a-b)^{2}=a^{2}-2 a b+b^{2} \\
(a+b)(a-b)=a^{2}-b^{2}
\end{array}\right\}
$$

Binomial squares

Difference of two squares

## Dividing a Polynomial by a Monomial [4.7]

9. $\frac{12 x^{3}-18 x^{2}}{6 x}=2 x^{2}-3 x$

To divide a polynomial by a monomial, divide each term in the polynomial by the monomial.

## Long Division with Polynomials [4.8]

10. $\frac{x-2}{x - 3 \longdiv { x ^ { 2 } - 5 x + 8 }}$

$$
\begin{array}{r}
\frac{\mp x^{2}+3 x \downarrow}{-2 x+8} \\
\frac{+2 x \neq 6}{2}
\end{array}
$$

## COMMON MISTAKES

1. If a term contains a variable that is raised to a power, then the exponent on the variable is associated only with that variable, unless there are parentheses; that is, the expression $3 x^{2}$ means $3 \cdot x \cdot x$, not $3 x \cdot 3 x$. It is a mistake to write $3 x^{2}$ as $9 x^{2}$. The only way to end up with $9 x^{2}$ is to start with $(3 x)^{2}$.
2. It is a mistake to add nonsimilar terms. For example, $2 x$ and $3 x^{2}$ are nonsimilar terms and therefore cannot be combined; that is, $2 x+3 x^{2} \neq 5 x^{3}$. If you were to substitute 10 for $x$ in the preceding expression, you would see that the two sides are not equal.
3. It is a mistake to distribute exponents over sums and differences; that is, $(a+b)^{2}$ is not $a^{2}+b^{2}$. Convince yourself of this by letting $a=2$ and $b=3$ and then simplifying both sides.
4. Another common mistake can occur when dividing a polynomial by a monomial. Here is an example:

$$
\frac{x+2}{2}=x+1 \quad \text { Mistake }
$$

The mistake here is in not dividing both terms in the numerator by 2 . The correct way to divide $x+2$ by 2 looks like this:

$$
\begin{aligned}
\frac{x+2}{2} & =\frac{x}{2}+\frac{2}{2} \quad \text { Correct } \\
& =\frac{x}{2}+1
\end{aligned}
$$

## Chapter 4 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

The numbers in brackets refer to the sections of the text in which similar problems can be found.

Simplify. [4.1]

1. $(-1)^{3}$
2. $-8^{2}$
3. $\left(\frac{3}{7}\right)^{2}$
4. $y^{3} \cdot y^{9}$
5. $x^{15} \cdot x^{7} \cdot x^{5} \cdot x^{3}$
6. $\left(x^{7}\right)^{5}$
7. $\left(2^{6}\right)^{4}$
8. $(3 y)^{3}$
9. $(-2 x y z)^{3}$

Simplify each expression. Any answers that contain exponents should contain positive exponents only. [4.2]
10. $7^{-2}$
11. $4 x^{-5}$
12. $(3 y)^{-3}$
13. $\frac{a^{9}}{a^{3}}$
14. $\left(\frac{X^{3}}{x^{5}}\right)^{2}$
15. $\frac{x^{9}}{x^{-6}}$
16. $\frac{x^{-7}}{x^{-2}}$
17. $(-3 x y)^{0}$
18. $3^{0}-5^{1}+5^{0}$

Simplify. Any answers that contain exponents should contain positive exponents only. [4.1, 4.2]
19. $\left(3 x^{3} y^{2}\right)^{2}$
20. $\left(2 a^{3} b^{2}\right)^{4}\left(2 a^{5} b^{6}\right)^{2}$
21. $\left(-3 x y^{2}\right)^{-3}$
22. $\frac{\left(b^{3}\right)^{4}\left(b^{2}\right)^{5}}{\left(b^{7}\right)^{3}}$
23. $\frac{\left(x^{-3}\right)^{3}\left(x^{6}\right)^{-1}}{\left(x^{-5}\right)^{-4}}$

Simplify. Write all answers with positive exponents only. [4.3]
24. $\frac{\left(2 x^{4}\right)\left(15 x^{9}\right)}{6 x^{6}}$
25. $\frac{\left(10 x^{3} y^{5}\right)\left(21 x^{2} y^{6}\right)}{\left(7 x y^{3}\right)\left(5 x^{9} y\right)}$
26. $\frac{21 a^{10}}{3 a^{4}}-\frac{18 a^{17}}{6 a^{11}}$
27. $\frac{8 x^{8} y^{3}}{2 x^{3} y}-\frac{10 x^{6} y^{9}}{5 x y^{7}}$

Simplify, and write all answers in scientific notation. [4.3]
28. $\left(3.2 \times 10^{3}\right)\left(2 \times 10^{4}\right)$
29. $\frac{4.6 \times 10^{5}}{2 \times 10^{-3}}$
30. $\frac{\left(4 \times 10^{6}\right)\left(6 \times 10^{5}\right)}{3 \times 10^{8}}$

Perform the following additions and subtractions. [4.4]
31. $\left(3 a^{2}-5 a+5\right)+\left(5 a^{2}-7 a-8\right)$
32. $\left(-7 x^{2}+3 x-6\right)-\left(8 x^{2}-4 x+7\right)+\left(3 x^{2}-2 x-1\right)$
33. Subtract $8 x^{2}+3 x-2$ from $4 x^{2}-3 x-2$.
34. Find the value of $2 x^{2}-3 x+5$ when $x=3$.

Multiply. [4.5]
35. $3 x(4 x-7)$
36. $8 x^{3} y\left(3 x^{2} y-5 x y^{2}+4 y^{3}\right)$
37. $(a+1)\left(a^{2}+5 a-4\right)$
38. $(x+5)\left(x^{2}-5 x+25\right)$
39. $(3 x-7)(2 x-5)$
40. $\left(5 y+\frac{1}{5}\right)\left(5 y-\frac{1}{5}\right)$
41. $\left(a^{2}-3\right)\left(a^{2}+3\right)$

Perform the indicated operations. [4.6]
42. $(a-5)^{2}$
43. $(3 x+4)^{2}$
44. $\left(y^{2}+3\right)^{2}$
45. Divide $10 a b+20 a^{2}$ by $-5 a$. [4.7]
46. Divide $40 x^{5} y^{4}-32 x^{3} y^{3}-16 x^{2} y$ by $-8 x y$. [4.7]

Divide using long division. [4.8]
47. $\frac{x^{2}+15 x+54}{x+6} \quad$ 48. $\frac{6 x^{2}+13 x-5}{3 x-1}$
49. $\frac{x^{3}+64}{x+4}$
50. $\frac{3 x^{2}-7 x+10}{3 x+2}$
51. $\frac{2 x^{3}-7 x^{2}+6 x+10}{2 x+1}$

## Chapter 4 Projects

Exponents and Polynomials

## GROUP PROJECT Discovering Pascal's Triangle

## Number of People 3

Time Needed 20 minutes

## Equipment Paper and pencils

Background The triangular array of numbers shown here is known as Pascal's triangle, after the French philosopher Blaise Pascal (1623-1662).

Procedure Look at Pascal's triangle and discover how the numbers in each row of the triangle are obtained from the numbers in the row above it.

1. Once you have discovered how to extend the triangle, write the next two rows.
2. Pascal's triangle can be linked to the Fibonacci sequence by rewriting Pascal's triangle so that the 1s on the left side of the triangle line up under one another, and the other columns are equally spaced to the right of the first column. Rewrite Pascal's triangle as indicated and then look along the diagonals of the new array until you discover how the Fibonacci sequence can be obtained from it.
3. The diagram below shows Pascal's triangle as written in Japanese in 1781 . Use your knowledge of Pascal's triangle to translate the numbers written in Japanese into our number system. Then write down the Japanese numbers from 1 to 20.


Pascal's triangle in Japanese (1781)

## Binomial Expansions

T.
he title on the following diagram is Binomial Expansions because each line gives the expansion of the binomial $x+y$ raised to a whole-number power.

Binomial Expansions

$$
\begin{array}{lc}
(x+y)^{0}= & 1 \\
(x+y)^{1}= & x+y \\
(x+y)^{2}= & x^{2}+2 x y+y^{2} \\
(x+y)^{3}= & x^{3}+3 x^{2} y+3 x y^{2}+y^{3} \\
(x+y)^{4}= & \\
(x+y)^{5}= &
\end{array}
$$

The fourth row in the diagram was completed by expanding $(x+y)^{3}$ using the methods developed in this chapter. Next, complete the diagram by expanding the binomials $(x+y)^{4}$ and $(x+y)^{5}$ using the multiplication procedures you have learned in this chapter. Finally, study the completed diagram until you see patterns that will allow you to continue the diagram one more row without using multiplication. (One pattern that you will see is Pascal's triangle, which we mentioned in the preceding group project.) When you are finished, write an essay in which you describe what you have done and the results you have obtained.

## Factoring

## 5


f you watch professional football on television, you will hear the announcers refer to "hang time" when the punter punts the ball. Hang time is the amount of time the ball is in the air, and it depends on only one thing-the initial vertical velocity imparted to the ball by the kicker's foot. We can find the hang time of a football by solving equations. Table 1 shows the equations to solve for hang time, given various initial vertical velocities. Figure 1 is a visual representation of some equations associated with the ones in Table 1. In Figure 1, you can find hang time on the horizontal axis.

## TABLE 1

Hang Time for a Football

| Initial Vertical <br> Velocity <br> (feet/second) | Equation in <br> Factored Form | Hang Time <br> (seconds) |
| :---: | :---: | :---: |
| 16 | $16 t(1-t)=0$ | 1 |
| 32 | $16 t(2-t)=0$ | 2 |
| 48 | $16 t(3-t)=0$ | 3 |
| 64 | $16 t(4-t)=0$ | 4 |
| 80 | $16 t(5-t)=0$ | 5 |



The equations in the second column of the table are in what is called "factored form." Once the equation is in factored form, hang time can be read from the second factor. In this chapter we develop techniques that allow us to factor a variety of polynomials. Factoring is the key to solving equations like the ones in Table 1.

## CHAPTER OUTLINE

5.1 The Greatest Common Factor and Factoring by Grouping
5.2 Factoring Trinomials
5.3 More Trinomials to Factor
5.4 The Difference of Two Squares
5.5 The Sum and Difference of Two Cubes
5.6 Factoring: A General Review
5.7 Solving Equations by Factoring

### 5.8 Applications

- Improve your grade and save time!

Go online to www.thomsonedu.com/login where you can

- Watch videos of instructors working through the in-text examples
- Follow step-by-step online tutorials of in-text examples and review questions
- Work practice problems
- Check your readiness for an exam by taking a pre-test and exploring the modules recommended in your Personalized Study plan
- Receive help from a live tutor online through vMentor ${ }^{\text {TM }}$
Try it out! Log in with an access code or purchase access at www.ichapters.com.


## 1 Continue to Set and Keep a Schedule

Sometimes I find students do well at first and then become overconfident. They begin to put in less time with their homework. Don't do that. Keep to the same schedule that you started with.

## 2 Increase Effectiveness

You want to become more and more effective with the time you spend on your homework. You want to increase the amount of learning you obtain in the time you have set aside. Increase those activities that you feel are the most beneficial and decrease those that have not given you the results you want.

## 3 Continue to List Difficult Problems

This study skill was started in Chapter 2. You should continue to list and rework the problems that give you the most difficulty. It is this list that you will use to study for the next exam. Your goal is to go into the next exam knowing that you successfully can work any problem from your list of hard problems.

## OBJECTIVES

A Factor the greatest common factor from a polynomial.

B Factor by grouping.

In Chapter 1 we used the following diagram to illustrate the relationship between multiplication and factoring.


A similar relationship holds for multiplication of polynomials. Reading the following diagram from left to right, we say the product of the binomials $x+2$ and $x+3$ is the trinomial $x^{2}+5 x+6$. However, if we read in the other direction, we can say that $x^{2}+5 x+6$ factors into the product of $x+2$ and $x+3$.

$$
\text { Factors } \rightarrow(x+2)(\underbrace{\text { Multiplication }}_{\text {Factoring }}
$$

In this chapter we develop a systematic method of factoring polynomials. In this section we will apply the distributive property to polynomials to factor from them what is called the greatest common factor.

DEFINITION The greatest common factor for a polynomial is the largest monomial that divides (is a factor of) each term of the polynomial.

We use the term largest monomial to mean the monomial with the greatest coefficient and highest power of the variable.

## EXAMPLE 1 Find the greatest common factor for the polynomial:

$$
3 x^{5}+12 x^{2}
$$

SOLUTION The terms of the polynomial are $3 x^{5}$ and $12 x^{2}$. The largest number that divides the coefficients is 3 , and the highest power of $x$ that is a factor of $x^{5}$ and $x^{2}$ is $x^{2}$. Therefore, the greatest common factor for $3 x^{5}+12 x^{2}$ is $3 x^{2}$; that is, $3 x^{2}$ is the largest monomial that divides each term of $3 x^{5}+12 x^{2}$.

## EXAMPLE 2 Find the greatest common factor for:

$$
8 a^{3} b^{2}+16 a^{2} b^{3}+20 a^{3} b^{3}
$$

SOLUTION The largest number that divides each of the coefficients is 4 . The highest power of the variable that is a factor of $a^{3} b^{2}, a^{2} b^{3}$, and $a^{3} b^{3}$ is $a^{2} b^{2}$. The greatest common factor for $8 a^{3} b^{2}+16 a^{2} b^{3}+20 a^{3} b^{3}$ is $4 a^{2} b^{2}$. It is the largest monomial that is a factor of each term.

Once we have recognized the greatest common factor of a polynomial, we can apply the distributive property and factor it out of each term. We rewrite the polynomial as the product of its greatest common factor with the polynomial that remains after the greatest common factor has been factored from each term in the original polynomial.

## EXAMPLE 3 Factor the greatest common factor from $3 x-15$.

SOLUTION The greatest common factor for the terms $3 x$ and 15 is 3 . We can rewrite both $3 x$ and 15 so that the greatest common factor 3 is showing in each term. It is important to realize that $3 x$ means $3 \cdot x$. The 3 and the $x$ are not "stuck" together:

$$
3 x-15=3 \cdot x-3 \cdot 5
$$

Now, applying the distributive property, we have:

$$
3 \cdot x-3 \cdot 5=3(x-5)
$$

To check a factoring problem like this, we can multiply 3 and $x-5$ to get $3 x-15$, which is what we started with. Factoring is simply a procedure by which we change sums and differences into products. In this case we changed the difference $3 x-15$ into the product $3(x-5)$. Note, however, that we have not changed the meaning or value of the expression. The expression we end up with is equivalent to the expression we started with.

## EXAMPLE 4 Factor the greatest common factor from:

$$
5 x^{3}-15 x^{2}
$$

SOLUTION The greatest common factor is $5 x^{2}$. We rewrite the polynomial as:

$$
5 x^{3}-15 x^{2}=5 x^{2} \cdot x-5 x^{2} \cdot 3
$$

Then we apply the distributive property to get:

$$
5 x^{2} \cdot x-5 x^{2} \cdot 3=5 x^{2}(x-3)
$$

To check our work, we simply multiply $5 x^{2}$ and $(x-3)$ to get $5 x^{3}-15 x^{2}$, which is our original polynomial.

## EXAMPLE 5 Factor the greatest common factor from:

$$
16 x^{5}-20 x^{4}-8 x^{3}
$$

SOLUTION The greatest common factor is $4 x^{3}$. We rewrite the polynomial so we can see the greatest common factor $4 x^{3}$ in each term; then we apply the distributive property to factor it out.

$$
\begin{aligned}
16 x^{5}-20 x^{4}-8 x^{3} & =4 x^{3} \cdot 4 x^{2}-4 x^{3} \cdot 5 x-4 x^{3} \cdot 2 \\
& =4 x^{3}\left(4 x^{2}-5 x-2\right)
\end{aligned}
$$

EXAMPLE 6 Factor the greatest common factor from:

$$
6 x^{3} y-18 x^{2} y^{2}-12 x y^{3}
$$

SOLUTION The greatest common factor is $6 x y$. We rewrite the polynomial in terms of $6 x y$ and then apply the distributive property as follows:

$$
\begin{aligned}
6 x^{3} y-18 x^{2} y^{2}-12 x y^{3} & =6 x y \cdot x^{2}-6 x y \cdot 3 x y-6 x y \cdot 2 y^{2} \\
& =6 x y\left(x^{2}-3 x y-2 y^{2}\right)
\end{aligned}
$$

EXAMPLE 7 Factor the greatest common factor from:

$$
3 a^{2} b-6 a^{3} b^{2}+9 a^{3} b^{3}
$$

SOLUTION The greatest common factor is $3 a^{2} b$ :

$$
\begin{aligned}
3 a^{2} b-6 a^{3} b^{2}+9 a^{3} b^{3} & =3 a^{2} b(1)-3 a^{2} b(2 a b)+3 a^{2} b\left(3 a b^{2}\right) \\
& =3 a^{2} b\left(1-2 a b+3 a b^{2}\right)
\end{aligned}
$$

## Factoring by Grouping

To develop our next method of factoring, called factoring by grouping, we start by examining the polynomial $x c+y c$. The greatest common factor for the two terms is $c$. Factoring $c$ from each term we have:

$$
x c+y c=c(x+y)
$$

But suppose that $c$ itself was a more complicated expression, such as $a+b$, so that the expression we were trying to factor was $x(a+b)+y(a+b)$, instead of $x c+y c$. The expression $x(a+b)+y(a+b)$ has $(a+b)$ common to each term. Factoring this common factor from each term looks like this:

$$
x(a+b)+y(a+b)=(a+b)(x+y)
$$

To see how all of this applies to factoring polynomials, consider the polynomial

$$
x y+3 x+2 y+6
$$

There is no greatest common factor other than the number 1 . However, if we group the terms together two at a time, we can factor an $x$ from the first two terms and a 2 from the last two terms:

$$
x y+3 x+2 y+6=x(y+3)+2(y+3)
$$

The expression on the right can be thought of as having two terms: $x(y+3)$ and $2(y+3)$. Each of these expressions contains the common factor $y+3$, which can be factored out using the distributive property:

$$
x(y+3)+2(y+3)=(y+3)(x+2)
$$

This last expression is in factored form. The process we used to obtain it is called factoring by grouping. Here are some additional examples.

## EXAMPLE 8 Factor $a x+b x+a y+b y$.

SOLUTION We begin by factoring $x$ from the first two terms and $y$ from the last two terms:

$$
\begin{aligned}
a x+b x+a y+b y & =x(a+b)+y(a+b) \\
& =(a+b)(x+y)
\end{aligned}
$$

To convince yourself that this is factored correctly, multiply the two factors $(a+b)$ and $(x+y)$.

## EXAMPLE $9 \quad$ Factor by grouping: $3 a x-2 a+15 x-10$.

SOLUTION First, we factor $a$ from the first two terms and 5 from the last two terms. Then, we factor $3 x-2$ from the remaining two expressions:

$$
\begin{aligned}
3 a x-2 a+15 x-10 & =a(3 x-2)+5(3 x-2) \\
& =(3 x-2)(a+5)
\end{aligned}
$$

Again, multiplying $(3 x-2)$ and $(a+5)$ will convince you that these are the correct factors.

## EXAMPLE 10 Factor $2 x^{2}+5 a x-2 x y-5 a y$.

SOLUTION From the first two terms we factor $x$. From the second two terms we must factor $-y$ so that the binomial that remains after we do so matches the binomial produced by the first two terms:

$$
\begin{aligned}
2 x^{2}+5 a x-2 x y-5 a y & =x(2 x+5 a)-y(2 x+5 a) \\
& =(2 x+5 a)(x-y)
\end{aligned}
$$

Another way to accomplish the same result is to use the commutative property to interchange the middle two terms, and then factor by grouping:

$$
\begin{aligned}
2 x^{2}+5 a x-2 x y-5 a y & =2 x^{2}-2 x y+5 a x-5 a y \quad \begin{array}{l}
\text { Commutative } \\
\text { property }
\end{array} \\
& =2 x(x-y)+5 a(x-y) \\
& =(x-y)(2 x+5 a)
\end{aligned}
$$

This is the same result we obtained previously.

## EXAMPLE 11 Factor $6 x^{2}-3 x-4 x+2$ by grouping.

SOLUTION The first two terms have $3 x$ in common, and the last two terms have either a 2 or a -2 in common. Suppose we factor $3 x$ from the first two terms and 2 from the last two terms. We get:

$$
6 x^{2}-3 x-4 x+2=3 x(2 x-1)+2(-2 x+1)
$$

We can't go any further because there is no common factor that will allow us to factor further. However, if we factor -2 , instead of 2 , from the last two terms, our problem is solved:

$$
\begin{aligned}
6 x^{2}-3 x-4 x+2 & =3 x(2 x-1)-2(2 x-1) \\
& =(2 x-1)(3 x-2)
\end{aligned}
$$

In this case, factoring -2 from the last two terms gives us an expression that can be factored further.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A ${ }^{1-7}$
B ${ }^{8-11}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the greatest common factor for a polynomial?
2. After factoring a polynomial, how can you check your result?
3. When would you try to factor by grouping?
4. What is the relationship between multiplication and factoring?

## Problem Set 5.1

Factor the following by taking out the greatest common factor.

1. $15 x+25$
2. $6 a+9$
3. $4 x-8 y$
4. $3 x^{2}-6 x-9$
5. $2 x^{2}+6 x+4$
6. $3 a^{2}-3 a-60$
7. $2 a^{2}-18 a+28$
8. $24 y^{2}-52 y+24$
9. $18 y^{2}+48 y+32$
10. $9 x^{2}-8 x^{3}$
11. $7 x^{3}-4 x^{2}$
12. $13 a^{2}-26 a^{3}$
13. $21 x^{2} y-28 x y^{2}$
14. $30 x y^{2}-25 x^{2} y$
15. $22 a^{2} b^{2}-11 a b^{2}$
16. $15 x^{3}-25 x^{2}+30 x$
17. $7 x^{3}+21 x^{2}-28 x$
18. $16 x^{4}-20 x^{2}-16 x$
19. $121 y^{4}-11 x^{4}$
20. $25 a^{4}-5 b^{4}$
21. $100 x^{4}-50 x^{3}+25 x^{2}$
22. $36 x^{5}+72 x^{3}-81 x^{2}$
23. $8 a^{2}+16 b^{2}+32 c^{2}$
24. $9 a^{2}-18 b^{2}-27 c^{2}$
25. $4 a^{2} b-16 a b^{2}+32 a^{2} b^{2}$
26. $5 a b^{2}+10 a^{2} b^{2}+15 a^{2} b$

Factor the following by taking out the greatest common factor.
31. $121 a^{3} b^{2}-22 a^{2} b^{3}+33 a^{3} b^{3}$
32. $20 a^{4} b^{3}-18 a^{3} b^{4}+22 a^{4} b^{4}$
33. $12 x^{2} y^{3}-72 x^{5} y^{3}-36 x^{4} y^{4}$
34. $49 x y-21 x^{2} y^{2}+35 x^{3} y^{3}$

Factor by grouping.
35. $x y+5 x+3 y+15$
36. $x y+2 x+4 y+8$
37. $x y+6 x+2 y+12$
38. $x y+2 y+6 x+12$
39. $a b+7 a-3 b-21$
40. $a b+3 b-7 a-21$
41. $a x-b x+a y-b y$
42. $a x-a y+b x-b y$
43. $2 a x+6 x-5 a-15$
44. $3 a x+21 x-a-7$
45. $3 x b-4 b-6 x+8$
46. $3 x b-4 b-15 x+20$
47. $x^{2}+a x+2 x+2 a$
48. $x^{2}+a x+3 x+3 a$
49. $x^{2}-a x-b x+a b$
50. $x^{2}+a x-b x-a b$

Factor by grouping. You can group the terms together two at a time or three at a time. Either way will produce the same result.
51. $a x+a y+b x+b y+c x+c y$
52. $a x+b x+c x+a y+b y+c y$

Factor the following polynomials by grouping the terms together two at a time.
53. $6 x^{2}+9 x+4 x+6$
54. $6 x^{2}-9 x-4 x+6$
55. $20 x^{2}-2 x+50 x-5$
56. $20 x^{2}+25 x+4 x+5$
57. $20 x^{2}+4 x+25 x+5$
58. $20 x^{2}+4 x-25 x-5$
59. $x^{3}+2 x^{2}+3 x+6$
60. $x^{3}-5 x^{2}-4 x+20$
61. $6 x^{3}-4 x^{2}+15 x-10$
62. $8 x^{3}-12 x^{2}+14 x-21$
63. The greatest common factor of the binomial $3 x+6$ is 3 . The greatest common factor of the binomial $2 x+4$ is 2 . What is the greatest common factor of their product $(3 x+6)(2 x+4)$ when it has been multiplied out?
64. The greatest common factors of the binomials $4 x+2$ and $5 x+10$ are 2 and 5 , respectively. What is the greatest common factor of their product $(4 x+2)(5 x+10)$ when it has been multiplied out?
65. The following factorization is incorrect. Find the mistake, and correct the right-hand side:

$$
12 x^{2}+6 x+3=3\left(4 x^{2}+2 x\right)
$$

66. Find the mistake in the followzing factorization, and then rewrite the right-hand side correctly:

$$
10 x^{2}+2 x+6=2\left(5 x^{2}+3\right)
$$

## Maintaining Your Skills

Divide.
67. $\frac{y^{3}-16 y^{2}+64 y}{y}$
68. $\frac{5 x^{3}+35 x^{2}+60 x}{5 x}$
69. $\frac{-12 x^{4}+48 x^{3}+144 x^{2}}{-12 x^{2}}$
70. $\frac{16 x^{5}+20 x^{4}+60 x^{3}}{4 x^{3}}$
71. $\frac{-18 y^{5}+63 y^{4}-108 y^{3}}{-9 y^{3}}$
72. $\frac{36 y^{6}-66 y^{5}+54 y^{4}}{6 y^{4}}$

Subtract.
73. $\left(5 x^{2}+5 x-4\right)-\left(3 x^{2}-2 x+7\right)$
74. $\left(7 x^{4}-4 x^{2}-5\right)-\left(2 x^{4}-4 x^{2}+5\right)$
75. Subtract $4 x-5$ from $7 x+3$.
76. Subtract $3 x+2$ from $-6 x+1$.
77. Subtract $2 x^{2}-4 x$ from $5 x^{2}-5$.
78. Subtract $6 x^{2}+3$ from $2 x^{2}-4 x$.

## Getting Ready for the Next Section

Multiply each of the following.
79. $(x-7)(x+2)$
80. $(x-7)(x-2)$
81. $(x-3)(x+2)$
82. $(x+3)(x-2)$
83. $(x+3)\left(x^{2}-3 x+9\right)$
84. $(x-2)\left(x^{2}+2 x+4\right)$
85. $(2 x+1)\left(x^{2}+4 x-3\right)$
86. $(3 x+2)\left(x^{2}-2 x-4\right)$
87. $3 x^{4}\left(6 x^{3}-4 x^{2}+2 x\right)$
88. $2 x^{4}\left(5 x^{3}+4 x^{2}-3 x\right)$
89. $\left(x+\frac{1}{3}\right)\left(x+\frac{2}{3}\right)$
90. $\left(x+\frac{1}{4}\right)\left(x+\frac{3}{4}\right)$
91. $(6 x+4 y)(2 x-3 y)$
92. $(8 a-3 b)(4 a-5 b)$
93. $(9 a+1)(9 a-1)$
94. $(7 b+1)(7 b+1)$
95. $(x-9)(x-9)$
96. $(x-8)(x-8)$
97. $(x+2)\left(x^{2}-2 x+4\right)$
98. $(x-3)\left(x^{2}+3 x+9\right)$

## 5.2

## OBJECTIVES

A Factor a trinomial whose leading coefficient is the number 1.

B Factor a polynomial by first factoring out the greatest common factor and then factoring the polynomial that remains.

## Note

As you will see as we progress through the book, factoring is a tool that is used in solving a number of problems. Before seeing how it is used, however, we first must learn how to do it. So, in this section and the two sections that follow, we will be developing our factoring skills.

## Factoring Trinomials

In this section we will factor trinomials in which the coefficient of the squared term is 1 . The more familiar we are with multiplication of binomials the easier factoring trinomials will be.

Recall multiplication of binomials from Chapter 4:

$$
\begin{aligned}
& (x+3)(x+4)=x^{2}+7 x+12 \\
& (x-5)(x+2)=x^{2}-3 x-10
\end{aligned}
$$

The first term in the answer is the product of the first terms in each binomial. The last term in the answer is the product of the last terms in each binomial. The middle term in the answer comes from adding the product of the outside terms to the product of the inside terms.

Let's have $a$ and $b$ represent real numbers and look at the product of $(x+a)$ and $(x+b)$ :

$$
\begin{aligned}
(x+a)(x+b) & =x^{2}+a x+b x+a b \\
& =x^{2}+(a+b) x+a b
\end{aligned}
$$

The coefficient of the middle term is the sum of $a$ and $b$. The last term is the product of $a$ and $b$. Writing this as a factoring problem, we have:

$$
\begin{gathered}
x^{2}+(a+b) x+\underset{\text { Sum }}{(a b \text { Product }} \quad a b=(x+a)(x+b)
\end{gathered}
$$

To factor a trinomial in which the coefficient of $x^{2}$ is 1 , we need only find the numbers $a$ and $b$ whose sum is the coefficient of the middle term and whose product is the constant term (last term).

## EXAMPLE 1 Factor $x^{2}+8 x+12$.

SOLUTION The coefficient of $x^{2}$ is 1 . We need two numbers whose sum is 8 and whose product is 12 . The numbers are 6 and 2 :

$$
x^{2}+8 x+12=(x+6)(x+2)
$$

We can easily check our work by multiplying $(x+6)$ and $(x+2)$
Check: $(x+6)(x+2)=x^{2}+6 x+2 x+12$

$$
=x^{2}+8 x+12
$$

## Note

Again, we can check our results by multiplying our factors to see if their product is the original polynomial.

## EXAMPLE 2 Factor $x^{2}-2 x-15$.

SOLUTION The coefficient of $x^{2}$ is again 1 . We need to find a pair of numbers whose sum is -2 and whose product is -15 . Here are all the possibilities for products that are -15 .

| Products | Sums |
| :---: | :---: |
| $-1(15)=-15$ | $-1+15=14$ |
| $1(-15)=-15$ | $1+(-15)=-14$ |
| $-5(3)=-15$ | $-5+3=-2$ |
| $5(-3)=-15$ | $5+(-3)=2$ |

The third line gives us what we want. The factors of $x^{2}-2 x-15$ are $(x-5)$ and $(x+3)$ :

$$
x^{2}-2 x-15=(x-5)(x+3)
$$

## EXAMPLE 3 Factor $2 x^{2}+10 x-28$.

SOLUTION The coefficient of $x^{2}$ is 2 . We begin by factoring out the greatest common factor, which is 2 :

$$
2 x^{2}+10 x-28=2\left(x^{2}+5 x-14\right)
$$

Now, we factor the remaining trinomial by finding a pair of numbers whose sum is 5 and whose product is -14 . Here are the possibilities:

| Products | Sums |
| :---: | :---: |
| $-1(14)=-14$ | $-1+14=13$ |
| $1(-14)=-14$ | $1+(-14)=-13$ |
| $-7(2)=-14$ | $-7+2=-5$ |
| $7(-2)=-14$ | $7+(-2)=5$ |

From the last line we see that the factors of $x^{2}+5 x-14$ are $(x+7)$ and $(x-2)$. Here is the complete problem:

$$
\begin{aligned}
2 x^{2}+10 x-28 & =2\left(x^{2}+5 x-14\right) \\
& =2(x+7)(x-2)
\end{aligned}
$$

## EXAMPLE 4

Factor $3 x^{3}-3 x^{2}-18 x$.

## Note

Trinomials in which the coefficient of the seconddegree term is 1 are the easiest to factor. Success in factoring any type of polynomial is directly related to the amount of time spent working the problems. The more we practice, the more accomplished we become at factoring.

SOLUTION We begin by factoring out the greatest common factor, which is $3 x$. Then we factor the remaining trinomial. Without showing the table of products and sums as we did in Examples 2 and 3, here is the complete solution:

$$
\begin{aligned}
3 x^{3}-3 x^{2}-18 x & =3 x\left(x^{2}-x-6\right) \\
& =3 x(x-3)(x+2)
\end{aligned}
$$

## EXAMPLE $5 \quad$ Factor $x^{2}+8 x y+12 y^{2}$.

SOLUTION This time we need two expressions whose product is $12 y^{2}$ and whose sum is $8 y$. The two expressions are $6 y$ and $2 y$ (see Example 1 in this section):

$$
x^{2}+8 x y+12 y^{2}=(x+6 y)(x+2 y)
$$

You should convince yourself that these factors are correct by finding their product.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

$$
\text { A } \quad 1,2,5
$$

B 3,4

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. When the leading coefficient of a trinomial is 1 , what is the relationship between the other two coefficients and the factors of the trinomial?
2. When factoring polynomials, what should you look for first?
3. How can you check to see that you have factored a trinomial correctly?
4. Describe how you would find the factors of $x^{2}+8 x+12$.

Factor the following trinomials.

1. $x^{2}+7 x+12$
2. $x^{2}+7 x+10$
3. $x^{2}+3 x+2$
4. $x^{2}+7 x+6$
5. $a^{2}+10 a+21$
6. $a^{2}-7 a+12$
7. $x^{2}-7 x+10$
8. $x^{2}-3 x+2$
9. $y^{2}-10 y+21$
10. $y^{2}-7 y+6$
11. $x^{2}-x-12$
12. $x^{2}-4 x-5$
13. $y^{2}+y-12$
14. $y^{2}+3 y-18$
15. $x^{2}+5 x-14$
16. $x^{2}-5 x-24$
17. $r^{2}-8 r-9$
18. $r^{2}-r-2$
19. $x^{2}-x-30$
20. $x^{2}+8 x+12$
21. $a^{2}+15 a+56$
22. $a^{2}-9 a+20$
23. $y^{2}-y-42$
24. $y^{2}+y-42$
25. $x^{2}+13 x+42$
26. $x^{2}-13 x+42$

Factor the following problems completely. First, factor out the greatest common factor, and then factor the remaining trinomial.
27. $2 x^{2}+6 x+4$
28. $3 x^{2}-6 x-9$
29. $3 a^{2}-3 a-60$
30. $2 a^{2}-18 a+28$
31. $100 x^{2}-500 x+600$
32. $100 x^{2}-900 x+2,000$
33. $100 p^{2}-1,300 p+4,000$
34. $100 p^{2}-1,200 p+3,200$
35. $x^{4}-x^{3}-12 x^{2}$
36. $x^{4}-11 x^{3}+24 x^{2}$
37. $2 r^{3}+4 r^{2}-30 r$
38. $5 r^{3}+45 r^{2}+100 r$
39. $2 y^{4}-6 y^{3}-8 y^{2}$
40. $3 r^{3}-3 r^{2}-6 r$
41. $x^{5}+4 x^{4}+4 x^{3}$
42. $x^{5}+13 x^{4}+42 x^{3}$
43. $3 y^{4}-12 y^{3}-15 y^{2}$
44. $5 y^{4}-10 y^{3}+5 y^{2}$
45. $4 x^{4}-52 x^{3}+144 x^{2}$
46. $3 x^{3}-3 x^{2}-18 x$

Factor the following trinomials.
47. $x^{2}+5 x y+6 y^{2}$
48. $x^{2}-5 x y+6 y^{2}$
49. $x^{2}-9 x y+20 y^{2}$
50. $x^{2}+9 x y+20 y^{2}$
51. $a^{2}+2 a b-8 b^{2}$
52. $a^{2}-2 a b-8 b^{2}$
53. $a^{2}-10 a b+25 b^{2}$
54. $a^{2}+6 a b+9 b^{2}$
55. $a^{2}+10 a b+25 b^{2}$
56. $a^{2}-6 a b+9 b^{2}$
57. $x^{2}+2 x a-48 a^{2}$
58. $x^{2}-3 x a-10 a^{2}$
59. $x^{2}-5 x b-36 b^{2}$
60. $x^{2}-13 x b+36 b^{2}$

Factor completely.
61. $x^{4}-5 x^{2}+6$
62. $x^{6}-2 x^{3}-15$
63. $x^{2}-80 x-2,000$
64. $x^{2}-190 x-2,000$
65. $x^{2}-x+\frac{1}{4}$
66. $x^{2}-\frac{2}{3} x+\frac{1}{9}$
67. $x^{2}+0.6 x+0.08$
68. $x^{2}+0.8 x+0.15$
69. If one of the factors of $x^{2}+24 x+128$ is $x+8$, what is the other factor?
70. If one factor of $x^{2}+260 x+2,500$ is $x+10$, what is the other factor?
71. What polynomial, when factored, gives $(4 x+3)(x-1) ?$
72. What polynomial factors to $(4 x-3)(x+1)$ ?

## Maintaining Your Skills

Simplify each expression. Write using only positive exponents.
73. $\left(-\frac{2}{5}\right)^{2}$
74. $\left(-\frac{3}{8}\right)^{2}$
75. $\left(3 a^{3}\right)^{2}\left(2 a^{2}\right)^{3}$
76. $\left(-4 x^{4}\right)^{2}\left(2 x^{5}\right)^{4}$
77. $\frac{(4 x)^{-7}}{(4 x)^{-5}}$
78. $\frac{(2 x)^{-3}}{(2 x)^{-5}}$
79. $\frac{12 a^{5} b^{3}}{72 a^{2} b^{5}}$
80. $\frac{25 x^{5} y^{3}}{50 x^{2} y^{7}}$
81. $\frac{15 x^{-5} y^{3}}{45 x^{2} y^{5}}$
82. $\frac{25 a^{2} b^{7}}{75 a^{5} b^{3}}$
83. $\left(-7 x^{3} y\right)\left(3 x y^{4}\right)$
84. $\left(9 a^{6} b^{4}\right)\left(6 a^{4} b^{3}\right)$
85. $\left(-5 a^{3} b^{-1}\right)\left(4 a^{-2} b^{4}\right)$
86. $\left(-3 a^{2} b^{-4}\right)\left(6 a^{5} b^{-2}\right)$
87. $\left(9 a^{2} b^{3}\right)\left(-3 a^{3} b^{5}\right)$
88. $\left(-7 a^{5} b^{8}\right)\left(6 a^{7} b^{4}\right)$

## Getting Ready for the Next Section

Multiply using the foil method.
89. $(6 a+1)(a+2)$
90. $(6 a-1)(a-2)$
91. $(3 a+2)(2 a+1)$
92. $(3 a-2)(2 a-1)$
93. $(6 a+2)(a+1)$
94. $(3 a+1)(2 a+2)$

## More Trinomials to Factor

## OBJECTIVES

A Factor a trinomial whose leading coefficient is other than 1.

B Factor a polynomial by first factoring out the greatest common factor and then factoring the polynomial that remains.

## Note

Remember, we can always check our results by multiplying the factors we have and comparing that product with our original polynomial.

We now will consider trinomials whose greatest common factor is 1 and whose leading coefficient (the coefficient of the squared term) is a number other than 1. We present two methods for factoring trinomials of this type. The first method involves listing possible factors until the correct pair of factors is found. This requires a certain amount of trial and error. The second method is based on the factoring by grouping process that we covered previously. Either method can be used to factor trinomials whose leading coefficient is a number other than 1.

## Method 1: Factoring $a x^{2}+b x+c$ by trial and error

Suppose we want to factor the trinomial $2 x^{2}-5 x-3$. We know the factors (if they exist) will be a pair of binomials. The product of their first terms is $2 x^{2}$ and the product of their last term is -3 . Let us list all the possible factors along with the trinomial that would result if we were to multiply them together. Remember, the middle term comes from the product of the inside terms plus the product of the outside terms.

| Binomial <br> Factors | First <br> Term | Middle <br> Term | Last <br> Term |
| :---: | :---: | :---: | :---: |
| $(2 x-3)(x+1)$ | $2 x^{2}$ | $-x$ | -3 |
| $(2 x+3)(x-1)$ | $2 x^{2}$ | $+x$ | -3 |
| $(2 x-1)(x+3)$ | $2 x^{2}$ | $+5 x$ | -3 |
| $(2 x+1)(x-3)$ | $2 x^{2}$ | $-5 x$ | -3 |

We can see from the last line that the factors of $2 x^{2}-5 x-3$ are $(2 x+1)$ and $(x-3)$. There is no straightforward way, as there was in the previous section, to find the factors, other than by trial and error or by simply listing all the possibilities. We look for possible factors that, when multiplied, will give the correct first and last terms, and then we see if we can adjust them to give the correct middle term.

## EXAMPLE 1 Factor $6 a^{2}+7 a+2$

SOLUTION We list all the possible pairs of factors that, when multiplied together, give a trinomial whose first term is $6 a^{2}$ and whose last term is +2 .

| Binomial <br> Factors | First <br> Term | Middle <br> Term | Last <br> Term |
| :---: | :---: | :---: | :---: |
| $(6 a+1)(a+2)$ | $6 a^{2}$ | $+13 a$ | +2 |
| $(6 a-1)(a-2)$ | $6 a^{2}$ | $-13 a$ | +2 |
| $(3 a+2)(2 a+1)$ | $6 a^{2}$ | $+7 a$ | +2 |
| $(3 a-2)(2 a-1)$ | $6 a^{2}$ | $-7 a$ | +2 |

The factors of $6 a^{2}+7 a+2$ are $(3 a+2)$ and $(2 a+1)$.

$$
\text { Check: } \quad(3 a+2)(2 a+1)=6 a^{2}+7 a+2
$$

Notice that in the preceding list we did not include the factors $(6 a+2)$ and $(a+1)$. We do not need to try these since the first factor has a 2 common to
each term and so could be factored again, giving $2(3 a+1)(a+1)$. Since our original trinomial, $6 a^{2}+7 a+2$, did not have a greatest common factor of 2 , neither of its factors will.

## EXAMPLE 2 Factor $4 x^{2}-x-3$.

SOLUTION We list all the possible factors that, when multiplied, give a trinomial whose first term is $4 x^{2}$ and whose last term is -3 .

| Binomial <br> Factors | First <br> Term | Middle <br> Term | Last <br> Term |
| :---: | :---: | :---: | :---: |
| $(4 x+1)(x-3)$ | $4 x^{2}$ | $-11 x$ | -3 |
| $(4 x-1)(x+3)$ | $4 x^{2}$ | $+11 x$ | -3 |
| $(4 x+3)(x-1)$ | $4 x^{2}$ | $-x$ | -3 |
| $(4 x-3)(x+1)$ | $4 x^{2}$ | $+x$ | -3 |
| $(2 x+1)(2 x-3)$ | $4 x^{2}$ | $-4 x$ | -3 |
| $(2 x-1)(2 x+3)$ | $4 x^{2}$ | $+4 x$ | -3 |

The third line shows that the factors are $(4 x+3)$ and $(x-1)$.
Check: $(4 x+3)(x-1)=4 x^{2}-x-3$
You will find that the more practice you have at factoring this type of trinomial, the faster you will get the correct factors. You will pick up some shortcuts along the way, or you may come across a system of eliminating some factors as possibilities. Whatever works best for you is the method you should use. Factoring is a very important tool, and you must be good at it.

## Note

Once again, the first step in any factoring problem is to factor out the greatest common factor if it is other than 1 .

## EXAMPLE 3 Factor $12 y^{3}+10 y^{2}-12 y$.

SOLUTION We begin by factoring out the greatest common factor, $2 y$ :

$$
12 y^{3}+10 y^{2}-12 y=2 y\left(6 y^{2}+5 y-6\right)
$$

We now list all possible factors of a trinomial with the first term $6 y^{2}$ and last term -6 , along with the associated middle terms.

| Possible <br> Factors | Middle Term <br> When Multiplied |
| :---: | :---: |
| $(3 y+2)(2 y-3)$ | $-5 y$ |
| $(3 y-2)(2 y+3)$ | $+5 y$ |
| $(6 y+1)(y-6)$ | $-35 y$ |
| $(6 y-1)(y+6)$ | $+35 y$ |

The second line gives the correct factors. The complete problem is:

$$
\begin{aligned}
12 y^{3}+10 y^{2}-12 y & =2 y\left(6 y^{2}+5 y-6\right) \\
& =2 y(3 y-2)(2 y+3)
\end{aligned}
$$

EXAMPLE 4 Factor $30 x^{2} y-5 x y^{2}-10 y^{3}$.
SOLUTION The greatest common factor is $5 y$ :

$$
\begin{aligned}
30 x^{2} y-5 x y^{2}-10 y^{3} & =5 y\left(6 x^{2}-x y-2 y^{2}\right) \\
& =5 y(2 x+y)(3 x-2 y)
\end{aligned}
$$

## Method 2: Factoring $a x^{2}+b x+c$ by grouping

Recall previously that we can use factoring by grouping to factor the polynomial $6 x^{2}-3 x-4 x+2$. We begin by factoring $3 x$ from the first two terms and -2 from the last two terms. For review, here is the complete problem:

$$
\begin{aligned}
6 x^{2}-3 x-4 x+2 & =3 x(2 x-1)-2(2 x-1) \\
& =(2 x-1)(3 x-2)
\end{aligned}
$$

Now, let's back up a little and notice that our original polynomial $6 x^{2}-3 x-4 x+2$ can be simplified to $6 x^{2}-7 x+2$ by adding $-3 x$ and $-4 x$. This means that $6 x^{2}-7 x+2$ can be factored to $(2 x-1)(3 x-2)$ by the grouping method shown previously. The key to using this process is to rewrite the middle term $-7 x$ as $-3 x-4 x$.
To generalize this discussion, here are the steps we use to factor trinomials by grouping.

## Strategy for Factoring $a x^{2}+b x+c$ by Grouping

Step 1: Form the product $a c$.
Step 2: Find a pair of numbers whose product is $a c$ and whose sum is $b$.
Step 3: Rewrite the polynomial to be factored so that the middle term $b x$ is written as the sum of two terms whose coefficients are the two numbers found in step 2.

Step 4: Factor by grouping.

## EXAMPLE 5 Factor $3 x^{2}-10 x-8$ using these steps.

SOLUTION The trinomial $3 x^{2}-10 x-8$ has the form $a x^{2}+b x+c$, where $a=3$, $b=-10$, and $c=-8$.

Step 1: The product $a c$ is $3(-8)=-24$.
Step 2: We need to find two numbers whose product is -24 and whose sum is -10 . Let's systematically begin to list all the pairs of numbers whose product is -24 to find the pair whose sum is -10 .

$$
\begin{array}{ll}
\text { Product } & \text { Sum } \\
-24(1)=-24 & -24+1=-23 \\
-12(2)=-24 & -12+2=-10
\end{array}
$$

We stop here because we have found the pair of numbers whose product is -24 and whose sum is -10 . The numbers are -12 and 2 .
Step 3: We now rewrite our original trinomial so the middle term $-10 x$ is written as the sum of $-12 x$ and $2 x$ :

$$
3 x^{2}-10 x-8=3 x^{2}-12 x+2 x-8
$$

Step 4: Factoring by grouping, we have:

$$
\begin{aligned}
3 x^{2}-12 x+2 x-8 & =3 x(x-4)+2(x-4) \\
& =(x-4)(3 x+2)
\end{aligned}
$$

We can check our work by multiplying $x-4$ and $3 x+2$ to get $3 x^{2}-10 x-8$.

## EXAMPLE 6 Factor $4 x^{2}-x-3$.

SOLUTION In this case, $a=4, b=-1$, and $c=-3$. The product $a c$ is $4(-3)=-12$. We need a pair of numbers whose product is -12 and whose sum is -1 . We begin listing pairs of numbers whose product is -12 and whose sum is -1 .

$$
\begin{array}{ll}
\text { Product } & \text { Sum } \\
-12(1)=-12 & -12+1=-11 \\
-6(2)=-12 & -6+2=-4 \\
-4(3)=-12 & -4+3=-1
\end{array}
$$

We stop here because we have found the pair of numbers for which we are looking. They are -4 and 3. Next, we rewrite the middle term $-x$ as the sum $-4 x$ $+3 x$ and proceed to factor by grouping.

$$
\begin{aligned}
4 x^{2}-x-3 & =4 x^{2}-4 x+3 x-3 \\
& =4 x(x-1)+3(x-1) \\
& =(x-1)(4 x+3)
\end{aligned}
$$

Compare this procedure and the result with those shown in Example 2 of this section.

## EXAMPLE 7 Factor $8 x^{2}-2 x-15$.

SOLUTION The product $a c$ is $8(-15)=-120$. There are many pairs of numbers whose product is -120 . We are looking for the pair whose sum is also -2 . The numbers are -12 and 10 . Writing $-2 x$ as $-12 x+10 x$ and then factoring by grouping, we have:

$$
\begin{aligned}
8 x^{2}-2 x-15 & =8 x^{2}-12 x+10 x-15 \\
& =4 x(2 x-3)+5(2 x-3) \\
& =(2 x-3)(4 x+5)
\end{aligned}
$$

EXAMPLE 8 A ball is tossed into the air with an upward velocity of 16 feet per second from the top of a building 32 feet high. The equation that gives the height of the ball above the ground at any time $t$ is

$$
h=32+16 t-16 t^{2}
$$

Factor the right side of this equation and then find $h$ when $t$ is 2 .
SOLUTION We begin by factoring out the greatest common factor, 16. Then, we factor the trinomial that remains:

$$
\begin{aligned}
& h=32+16 t-16 t^{2} \\
& h=16\left(2+t-t^{2}\right) \\
& h=16(2-t)(1+t)
\end{aligned}
$$

Letting $t=2$ in the equation, we have

$$
h=16(0)(3)=0
$$

When $t$ is $2, h$ is 0 .

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-8
B $3,4,8$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the first step in factoring a trinomial?
2. Describe the criteria you would use to set up a table of possible factors of a trinomial.
3. What does it mean if you factor a trinomial and one of your factors has a greatest common factor of 3 ?
4. Describe how you would look for possible factors of $6 a^{2}+7 a+2$.

## Problem Set 5.3

## Online support materials can be found at www.thomsonedu.com/login

Factor the following trinomials.

1. $2 x^{2}+7 x+3$
2. $2 x^{2}+5 x+3$
3. $2 a^{2}-a-3$
4. $2 a^{2}+a-3$
5. $3 x^{2}+2 x-5$
6. $3 x^{2}-2 x-5$
7. $3 y^{2}-14 y-5$
8. $3 y^{2}+14 y-5$
9. $6 x^{2}+13 x+6$
10. $6 x^{2}-13 x+6$
11. $4 x^{2}-12 x y+9 y^{2}$
12. $4 x^{2}+12 x y+9 y^{2}$
13. $4 y^{2}-11 y-3$
14. $4 y^{2}+y-3$
15. $20 x^{2}-41 x+20$
16. $20 x^{2}+9 x-20$
17. $20 a^{2}+48 a b-5 b^{2}$
18. $20 a^{2}+29 a b+5 b^{2}$
19. $20 x^{2}-21 x-5$
20. $20 x^{2}-48 x-5$
21. $12 m^{2}+16 m-3$
22. $12 m^{2}+20 m+3$
23. $20 x^{2}+37 x+15$
24. $20 x^{2}+13 x-15$
25. $12 a^{2}-25 a b+12 b^{2}$
26. $12 a^{2}+7 a b-12 b^{2}$
27. $3 x^{2}-x y-14 y^{2}$
28. $3 x^{2}+19 x y-14 y^{2}$
29. $14 x^{2}+29 x-15$
30. $14 x^{2}+11 x-15$
31. $6 x^{2}-43 x+55$
32. $6 x^{2}-7 x-55$
33. $15 t^{2}-67 t+38$
34. $15 t^{2}-79 t-34$

Factor each of the following completely. Look first for the greatest common factor.
35. $4 x^{2}+2 x-6$
36. $6 x^{2}-51 x+63$
37. $24 a^{2}-50 a+24$
38. $18 a^{2}+48 a+32$
39. $10 x^{3}-23 x^{2}+12 x$
40. $10 x^{4}+7 x^{3}-12 x^{2}$
41. $6 x^{4}-11 x^{3}-10 x^{2}$
42. $6 x^{3}+19 x^{2}+10 x$
43. $10 a^{3}-6 a^{2}-4 a$
44. $6 a^{3}+15 a^{2}+9 a$
45. $15 x^{3}-102 x^{2}-21 x$
46. $2 x^{4}-24 x^{3}+64 x^{2}$
47. $35 y^{3}-60 y^{2}-20 y$
48. $14 y^{4}-32 y^{3}+8 y^{2}$
49. $15 a^{4}-2 a^{3}-a^{2}$
50. $10 a^{5}-17 a^{4}+3 a^{3}$
51. $24 x^{2} y-6 x y-45 y$
52. $8 x^{2} y^{2}+26 x y^{2}+15 y^{2}$
53. $12 x^{2} y-34 x y^{2}+14 y^{3}$
54. $12 x^{2} y-46 x y^{2}+14 y^{3}$
55. Evaluate the expression $2 x^{2}+7 x+3$ and the expression $(2 x+1)(x+3)$ for $x=2$.
56. Evaluate the expression $2 a^{2}-a-3$ and the expression $(2 a-3)(a+1)$ for $a=5$.
57. What polynomial factors to $(2 x+3)(2 x-3)$ ?
58. What polynomial factors to $(5 x+4)(5 x-4)$ ?
59. What polynomial factors to $(x+3)(x-3)\left(x^{2}+9\right)$ ?
60. What polynomial factors to $(x+2)(x-2)\left(x^{2}+4\right)$ ?
61. One factor of $12 x^{2}-71 x+105$ is $x-3$. Find the other factor.
62. One factor of $18 x^{2}+121 x-35$ is $x+7$. Find the other factor.
63. One factor of $54 x^{2}+111 x+56$ is $6 x+7$. Find the other factor.
64. One factor of $63 x^{2}+110 x+48$ is $7 x+6$. Find the other factor.
65. One factor of $16 t^{2}-64 t+48$ is $t-1$. Find the other factors, then write the polynomial in factored form.
66. One factor of $16 t^{2}-32 t+12$ is $2 t-1$. Find the other factors, then write the polynomial in factored form.

## Applying the Concepts

67. Archery Margaret shoots an arrow into the air. The equation for the height (in feet) of the tip of the arrow is:

$$
h=8+62 t-16 t^{2}
$$

Factor the right side of this equation. Then fill in the table for various heights of the arrow, using the factored form of the equation.

68. Coin Toss At the beginning of every football game, the referee flips a coin to see who will kick off. The equation that gives the height (in feet) of the coin tossed in the air is:

$$
h=6+29 t-16 t^{2}
$$


a. Factor this equation.
b. Use the factored form of the equation to find the height of the quarter after 0 seconds, 1 second, and 2 seconds.

## Maintaining Your Skills

Perform the following additions and subtractions.
69. $\left(6 x^{3}-4 x^{2}+2 x\right)+\left(9 x^{2}-6 x+3\right)$
70. $\left(6 x^{3}-4 x^{2}+2 x\right)-\left(9 x^{2}-6 x+3\right)$
71. $\left(-7 x^{4}+4 x^{3}-6 x\right)+\left(8 x^{4}+7 x^{3}-9\right)$
72. $\left(-7 x^{4}+4 x^{3}-6 x\right)-\left(8 x^{4}+7 x^{3}-9\right)$
73. $\left(2 x^{5}+3 x^{3}+4 x\right)+\left(5 x^{3}-6 x-7\right)$
74. $\left(2 x^{5}+3 x^{3}+4 x\right)-\left(5 x^{3}-6 x-7\right)$
75. $\left(-8 x^{5}-5 x^{4}+7\right)+\left(7 x^{4}+2 x^{2}+5\right)$
76. $\left(-8 x^{5}-5 x^{4}+7\right)-\left(7 x^{4}+2 x^{2}+5\right)$
77. $\frac{24 x^{3} y^{7}}{6 x^{-2} y^{4}}+\frac{27 x^{-2} y^{10}}{9 x^{-7} y^{7}}$
78. $\frac{15 x^{8} y^{4}}{5 x^{2} y^{2}}-\frac{4 x^{7} y^{5}}{2 x y^{3}}$
79. $\frac{18 a^{5} b^{9}}{3 a^{3} b^{6}}-\frac{48 a^{-3} b^{-1}}{16 a^{-5} b^{-4}}$
80. $\frac{54 a^{-3} b^{5}}{6 a^{-7} b^{-2}}-\frac{32 a^{6} b^{5}}{8 a^{2} b^{-2}}$

## Getting Ready for the Next Section

Multiply each of the following.
81. $(x+3)(x-3)$
83. $(x+5)(x-5)$
85. $(x+7)(x-7)$
87. $(x+9)(x-9)$
89. $(2 x-3 y)(2 x+3 y)$
90. $(5 x-6 y)(5 x+6 y)$
91. $\left(x^{2}+4\right)(x+2)(x-2)$
92. $\left(x^{2}+9\right)(x+3)(x-3)$
93. $(x+3)^{2}$
95. $(x+5)^{2}$
97. $(x+7)^{2}$
99. $(x+9)^{2}$
101. $(2 x+3)^{2}$
103. $(4 x-2 y)^{2}$
104. $(5 x-6 y)^{2}$

## 5.4

## OBJECTIVES

A Factor the difference of two squares.

B Factor a perfect square trinomial.

C Factor a polynomial by first factoring out the greatest common factor and then factoring the polynomial that remains.

## The Difference of Two Squares

In Chapter 5 we listed the following three special products:

$$
\begin{aligned}
(a+b)^{2}= & (a+b)(a+b)=a^{2}+2 a b+b^{2} \\
(a-b)^{2}= & (a-b)(a-b)=a^{2}-2 a b+b^{2} \\
& (a+b)(a-b)=a^{2}-b^{2}
\end{aligned}
$$

Since factoring is the reverse of multiplication, we can also consider the three special products as three special factorings:

$$
\begin{aligned}
a^{2}+2 a b+b^{2} & =(a+b)^{2} \\
a^{2}-2 a b+b^{2} & =(a-b)^{2} \\
a^{2}-b^{2} & =(a+b)(a-b)
\end{aligned}
$$

Any trinomial of the form $a^{2}+2 a b+b^{2}$ or $a^{2}-2 a b+b^{2}$ can be factored by the methods of Section 6.3. The last line is the factoring to obtain the difference of two squares. The difference of two squares always factors in this way. Again, these are patterns you must be able to recognize on sight.

## EXAMPLE 1 Factor $16 x^{2}-25$.

SOLUTION We can see that the first term is a perfect square, and the last term is also. This fact becomes even more obvious if we rewrite the problem as:

$$
16 x^{2}-25=(4 x)^{2}-(5)^{2}
$$

The first term is the square of the quantity $4 x$, and the last term is the square of 5. The completed problem looks like this:

$$
\begin{aligned}
16 x^{2}-25 & =(4 x)^{2}-(5)^{2} \\
& =(4 x+5)(4 x-5)
\end{aligned}
$$

To check our results, we multiply:

$$
\begin{aligned}
(4 x+5)(4 x-5) & =16 x^{2}+20 x-20 x-25 \\
& =16 x^{2}-25
\end{aligned}
$$

## EXAMPLE 2 Factor $36 a^{2}-1$.

SOLUTION We rewrite the two terms to show they are perfect squares and then factor. Remember, 1 is its own square, $1^{2}=1$.

$$
\begin{aligned}
36 a^{2}-1 & =(6 a)^{2}-(1)^{2} \\
& =(6 a+1)(6 a-1)
\end{aligned}
$$

To check our results, we multiply:

$$
\begin{aligned}
(6 a+1)(6 a-1) & =36 a^{2}+6 a-6 a-1 \\
& =36 a^{2}-1
\end{aligned}
$$

## EXAMPLE 3 Factor $x^{4}-y^{4}$.

SOLUTION $x^{4}$ is the perfect square $\left(x^{2}\right)^{2}$, and $y^{4}$ is $\left(y^{2}\right)^{2}$ :

$$
\begin{aligned}
x^{4}-y^{4} & =\left(x^{2}\right)^{2}-\left(y^{2}\right)^{2} \\
& =\left(x^{2}-y^{2}\right)\left(x^{2}+y^{2}\right)
\end{aligned}
$$

The factor $\left(x^{2}-y^{2}\right)$ is itself the difference of two squares and therefore can be factored again. The factor $\left(x^{2}+y^{2}\right)$ is the sum of two squares and cannot be factored again. The complete solution is this:

$$
\begin{aligned}
x^{4}-y^{4} & =\left(x^{2}\right)^{2}-\left(y^{2}\right)^{2} \\
& =\left(x^{2}-y^{2}\right)\left(x^{2}+y^{2}\right) \\
& =(x+y)(x-y)\left(x^{2}+y^{2}\right)
\end{aligned}
$$

## EXAMPLE $4 \quad$ Factor $25 x^{2}-60 x+36$.

SOLUTION Although this trinomial can be factored by the method we used in Section 6.3, we notice that the first and last terms are the perfect squares $(5 x)^{2}$ and $(6)^{2}$. Before going through the method for factoring trinomials by listing all

## Note

As we have indicated before, perfect square trinomials like the ones in Examples 4 and 5 can be factored by the methods developed in previous sections. Recognizing that they factor to binomial squares simply saves time in factoring.
possible factors, we can check to see if $25 x^{2}-60 x+36$ factors to $(5 x-6)^{2}$. We need only multiply to check:

$$
\begin{aligned}
(5 x-6)^{2} & =(5 x-6)(5 x-6) \\
& =25 x^{2}-30 x-30 x+36 \\
& =25 x^{2}-60 x+36
\end{aligned}
$$

The trinomial $25 x^{2}-60 x+36$ factors to $(5 x-6)(5 x-6)=(5 x-6)^{2}$.

## EXAMPLE 5 Factor $5 x^{2}+30 x+45$.

SOLUTION We begin by factoring out the greatest common factor, which is 5 . Then we notice that the trinomial that remains is a perfect square trinomial:

$$
\begin{aligned}
5 x^{2}+30 x+45 & =5\left(x^{2}+6 x+9\right) \\
& =5(x+3)^{2}
\end{aligned}
$$

## EXAMPLE 6 Factor $(x-3)^{2}-25$.

SOLUTION This example has the form $a^{2}-b^{2}$, where $a$ is $x-3$ and $b$ is 5 . We factor it according to the formula for the difference of two squares:

$$
\begin{aligned}
(x-3)^{2}-25 & =(x-3)^{2}-5^{2} & & \text { Write } \mathbf{2 5} \text { as } 5^{2} \\
& =[(x-3)-5][(x-3)+5] & & \text { Factor } \\
& =(x-8)(x+2) & & \text { Simplify }
\end{aligned}
$$

Notice in this example we could have expanded $(x-3)^{2}$, subtracted 25 , and then factored to obtain the same result:

$$
\begin{aligned}
(x-3)^{2}-25 & =x^{2}-6 x+9-25 & & \text { Expand }(x-3)^{2} \\
& =x^{2}-6 x-16 & & \text { Simplify } \\
& =(x-8)(x+2) & & \text { Factor }
\end{aligned}
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Describe how you factor the difference of two squares.
2. What is a perfect square trinomial?
3. How do you know when you've factored completely?
4. Describe how you would factor $25 x^{2}-60 x+36$.

Factor the following.

1. $x^{2}-9$
2. $a^{2}-36$
3. $x^{2}-49$
4. $4 a^{2}-16$
5. $9 x^{2}+25$
6. $16 x^{2}-36$
7. $25 x^{2}-169$
8. $x^{2}-y^{2}$
9. $9 a^{2}-16 b^{2}$
10. $49 a^{2}-25 b^{2}$
11. $9-m^{2}$
12. $16-m^{2}$
13. $25-4 x^{2}$
14. $36-49 y^{2}$
15. $2 x^{2}-18$
16. $3 x^{2}-27$
17. $32 a^{2}-128$
18. $3 a^{3}-48 a$
19. $8 x^{2} y-18 y$
20. $50 a^{2} b-72 b$
21. $a^{4}-b^{4}$
22. $a^{4}-16$
23. $16 m^{4}-81$
24. $81-m^{4}$
25. $3 x^{3} y-75 x y^{3}$
26. $2 x y^{3}-8 x^{3} y$

Factor the following.
31. $x^{2}-2 x+1$
33. $x^{2}+2 x+1$
35. $a^{2}-10 a+25$
37. $y^{2}+4 y+4$
39. $x^{2}-4 x+4$
41. $m^{2}-12 m+36$
43. $4 a^{2}+12 a+9$
45. $49 x^{2}-14 x+1$
47. $9 y^{2}-30 y+25$
32. $x^{2}-6 x+9$
34. $x^{2}+6 x+9$
36. $a^{2}+10 a+25$
38. $y^{2}-8 y+16$
40. $x^{2}+8 x+16$
42. $m^{2}+12 m+36$
44. $9 a^{2}-12 a+4$
46. $64 x^{2}-16 x+1$
48. $25 y^{2}+30 y+9$
49. $x^{2}+10 x y+25 y^{2}$
50. $25 x^{2}+10 x y+y^{2}$
51. $9 a^{2}+6 a b+b^{2}$
52. $9 a^{2}-6 a b+b^{2}$

Factor.
53. $y^{2}-3 y+\frac{9}{4}$
54. $y^{2}+3 y+\frac{9}{4}$
55. $a^{2}+a+\frac{1}{4}$
56. $a^{2}-5 a+\frac{25}{4}$
57. $x^{2}-7 x+\frac{49}{4}$
58. $x^{2}+9 x+\frac{81}{4}$
59. $x^{2}-\frac{3}{4} x+\frac{9}{64}$
60. $x^{2}-\frac{3}{2} x+\frac{9}{16}$
61. $t^{2}-\frac{2}{5} t+\frac{1}{25}$
62. $t^{2}+\frac{6}{5} t+\frac{9}{25}$

Factor the following by first factoring out the greatest common factor.
63. $3 a^{2}+18 a+27$
64. $4 a^{2}-16 a+16$
65. $2 x^{2}+20 x y+50 y^{2}$
66. $3 x^{2}+30 x y+75 y^{2}$
67. $5 x^{3}+30 x^{2} y+45 x y^{2}$
68. $12 x^{2} y-36 x y^{2}+27 y^{3}$

Factor by grouping the first three terms together.
69. $x^{2}+6 x+9-y^{2}$
70. $x^{2}+10 x+25-y^{2}$
71. $x^{2}+2 x y+y^{2}-9$
72. $a^{2}+2 a b+b^{2}-25$
73. Find a value for $b$ so that the polynomial $x^{2}+b x+49$ factors to $(x+7)^{2}$.
74. Find a value of $b$ so that the polynomial $x^{2}+b x+81$ factors to $(x+9)^{2}$.
75. Find the value of $c$ for which the polynomial $x^{2}+10 x+c$ factors to $(x+5)^{2}$.
76. Find the value of $a$ for which the polynomial $a x^{2}+12 x+9$ factors to $(2 x+3)^{2}$.

## Maintaining Your Skills

Divide.
77. $\frac{24 y^{3}-36 y^{2}-18 y}{6 y}$
78. $\frac{77 y^{3}+35 y^{2}+14 y}{-7 y}$
79. $\frac{48 x^{7}-36 x^{5}+12 x^{2}}{4 x^{2}}$
80. $\frac{-50 x^{5}+15 x^{4}+10 x^{2}}{5 x^{2}}$
81. $\frac{18 x^{7}+12 x^{6}-6 x^{5}}{-3 x^{4}}$
82. $\frac{-64 x^{5}-18 x^{4}-56 x^{3}}{2 x^{3}}$
83. $\frac{-42 x^{5}+24 x^{4}-66 x^{2}}{6 x^{2}}$
84. $\frac{63 x^{7}-27 x^{6}-99 x^{5}}{-9 x^{4}}$

Use long division to divide.
85. $\frac{x^{2}-5 x+8}{x-3}$
86. $\frac{x^{2}+7 x+12}{x+4}$
87. $\frac{6 x^{2}+5 x+3}{2 x+3}$
88. $\frac{x^{3}+27}{x+3}$

## Getting Ready for the Next Section

Multiply each of the following
89. a. $1^{3}$
b. $2^{3}$
c. $3^{3}$
d. $4^{3}$
e. $5^{3}$
90. a. $(-1)^{3}$
b. $(-2)^{3}$
c. $(-3)^{3}$
d. $(-4)^{3}$
e. $(-5)^{3}$
91. a. $x\left(x^{2}-x+1\right)$
b. $1\left(x^{2}-x+1\right)$
c. $(x+1)\left(x^{2}-x+1\right)$
92. a. $x\left(x^{2}+x+1\right)$
b. $-1\left(x^{2}+x+1\right)$
c. $(x-1)\left(x^{2}+x+1\right)$
93. a. $x\left(x^{2}-2 x+4\right)$
b. $2\left(x^{2}-2 x+4\right)$
c. $(x+2)\left(x^{2}-2 x+4\right)$
94. a. $x\left(x^{2}+2 x+4\right)$
b. $-2\left(x^{2}+2 x+4\right)$
c. $(x-2)\left(x^{2}+2 x+4\right)$
95. a. $x\left(x^{2}-3 x+9\right)$
b. $3\left(x^{2}-3 x+9\right)$
c. $(x+3)\left(x^{2}-3 x+9\right)$
96. a. $x\left(x^{2}+3 x+9\right)$
b. $-3\left(x^{2}+3 x+9\right)$
c. $(x-3)\left(x^{2}+3 x+9\right)$
97. a. $x\left(x^{2}-4 x+16\right)$
b. $4\left(x^{2}-4 x+16\right)$
c. $(x+4)\left(x^{2}-4 x+16\right)$
98. a. $x\left(x^{2}+4 x+16\right)$
b. $-4\left(x^{2}+4 x+16\right)$
c. $(x-4)\left(x^{2}+4 x+16\right)$
99. a. $x\left(x^{2}-5 x+25\right)$
b. $5\left(x^{2}-5 x+25\right)$
c. $(x+5)\left(x^{2}-5 x+25\right)$
100. a. $x\left(x^{2}+5 x+25\right)$
b. $-5\left(x^{2}+5 x+25\right)$
c. $(x-5)\left(x^{2}+5 x+25\right)$

## The Sum and Difference of Two Cubes

## OBJECTIVES

A Factor the sum or difference of two cubes.

Previously, we factored a variety of polynomials. Among the polynomials we factored were polynomials that were the difference of two squares. The formula we used to factor the difference of two squares looks like this:

$$
a^{2}-b^{2}=(a+b)(a-b)
$$

If we ran across a binomial that had the form of the difference of two squares, we factored it by applying this formula. For example, to factor $x^{2}-25$, we simply notice that it can be written in the form $x^{2}-5^{2}$, which looks like the difference of two squares. According to the formula above, this binomial factors into $(x+5)(x-5)$.

In this section we want to use two new formulas that will allow us to factor the sum and difference of two cubes. For example, we want to factor the binomial $x^{3}-8$, which is the difference of two cubes. (To see that it is the differrence of two cubes, notice that it can be written $x^{3}-2^{3}$.) We also want to factor $y^{3}+27$, which is the sum of two cubes. (To see this, notice that $y^{3}+27$ can be written as $y^{3}+3^{3}$.)

The formulas that allow us to factor the sum of two cubes and the difference of two cubes are not as simple as the formula for factoring the difference of two squares. Here is what they look like:

$$
\begin{aligned}
& a^{3}+b^{3}=(a+b)\left(a^{2}-a b+b^{2}\right) \\
& a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right)
\end{aligned}
$$

Let's begin our work with these two formulas by showing that they are true. To do so, we multiply out the right side of each formula.

## EXAMPLE 1 Verify the two formulas.

SOLUTION We verify the formulas by multiplying the right sides and comparing the results with the left sides:

$$
\begin{aligned}
& a^{2}-a b+b^{2} \\
& \frac{a+b}{a^{3}-a^{2} b+a b^{2}} \\
& \frac{a^{2} b-a b^{2}+b^{3}}{a^{3}}+\quad+b^{3}
\end{aligned}
$$

The first formula is correct.

$$
\begin{aligned}
& \begin{array}{l}
a^{2}+a b+b^{2} \\
\frac{a-b}{a^{3}+a^{2} b+a b^{2}} \\
\frac{-a^{2} b-a b^{2}-b^{3}}{a^{3}}
\end{array} \frac{-b^{3}}{}
\end{aligned}
$$

The second formula is correct.
Here are some examples that use the formulas for factoring the sum and difference of two cubes.

## EXAMPLE 2 Factor $x^{3}-8$.

SOLUTION Since the two terms are perfect cubes, we write them as such and apply the formula:

$$
\begin{aligned}
x^{3}-8 & =x^{3}-2^{3} \\
& =(x-2)\left(x^{2}+2 x+2^{2}\right) \\
& =(x-2)\left(x^{2}+2 x+4\right)
\end{aligned}
$$

## EXAMPLE $3 \quad$ Factor $y^{3}+27$.

SOLUTION Proceeding as we did in Example 2, we first write 27 as $3^{3}$. Then, we apply the formula for factoring the sum of two cubes, which is $a^{3}+b^{3}=$ $(a+b)\left(a^{2}-a b+b^{2}\right):$

$$
\begin{aligned}
y^{3}+27 & =y^{3}+3^{3} \\
& =(y+3)\left(y^{2}-3 y+3^{2}\right) \\
& =(y+3)\left(y^{2}-3 y+9\right)
\end{aligned}
$$

## EXAMPLE 4

Factor $64+t^{3}$.
SOLUTION The first term is the cube of 4 and the second term is the cube of $t$. Therefore,

$$
\begin{aligned}
64+t^{3} & =4^{3}+t^{3} \\
& =(4+t)\left(16-4 t+t^{2}\right)
\end{aligned}
$$

## EXAMPLE 5 Factor $27 x^{3}+125 y^{3}$.

SOLUTION Writing both terms as perfect cubes, we have

$$
\begin{aligned}
27 x^{3}+125 y^{3} & =(3 x)^{3}+(5 y)^{3} \\
& =(3 x+5 y)\left(9 x^{2}-15 x y+25 y^{2}\right)
\end{aligned}
$$

EXAMPLE $6 \quad$ Factor $a^{3}-\frac{1}{8}$.
SOLUTION The first term is the cube of $a$, whereas the second term is the cube of $\frac{1}{2}$ :

$$
\begin{aligned}
a^{3}-\frac{1}{8} & =a^{3}-\left(\frac{1}{2}\right)^{3} \\
& =\left(a-\frac{1}{2}\right)\left(a^{2}+\frac{1}{2} a+\frac{1}{4}\right)
\end{aligned}
$$

## EXAMPLE 7 Factor $x^{6}-y^{6}$.

SOLUTION We have a choice of how we want to write the two terms to begin. We can write the expression as the difference of two squares, $\left(x^{3}\right)^{2}-\left(y^{3}\right)^{2}$, or as the difference of two cubes, $\left(x^{2}\right)^{3}-\left(y^{2}\right)^{3}$. It is better to use the difference of two squares if we have a choice:

$$
\begin{aligned}
x^{6}-y^{6} & =\left(x^{3}\right)^{2}-\left(y^{3}\right)^{2} \\
& =\left(x^{3}-y^{3}\right)\left(x^{3}+y^{3}\right) \\
& =(x-y)\left(x^{2}+x y+y^{2}\right)(x+y)\left(x^{2}-x y+y^{2}\right)
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A ${ }^{2-7}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How can you check your work when factoring?
2. Why are the numbers $8,27,64$, and 125 used so frequently in this section?
3. List the cubes of the numbers 1 through 10 .
4. How are you going to remember that the sum of two cubes factors, while the sum of two squares is prime?

## Problem Set 5.5

## Online support materials can be found at www.thomsonedu.com/login

Factor each of the following as the sum or difference of two cubes.

1 1. $x^{3}-y^{3}$
2. $x^{3}+y^{3}$
3. $a^{3}+8$
4. $a^{3}-8$
5. $27+x^{3}$
6. $27-x^{3}$
7. $y^{3}-1$
8. $y^{3}+1$
9. $y^{3}-64$
10. $y^{3}+64$
11. $125 h^{3}-t^{3}$
12. $t^{3}+125 h^{3}$
13. $x^{3}-216$
14. $216+X^{3}$
15. $2 y^{3}-54$
16. $81+3 y^{3}$
17. $2 a^{3}-128 b^{3}$
18. $128 a^{3}+2 b^{3}$
19. $2 x^{3}+432 y^{3}$
20. $432 x^{3}-2 y^{3}$
21. $10 a^{3}-640 b^{3}$
22. $640 a^{3}+10 b^{3}$
23. $10 r^{3}-1,250$
24. $10 r^{3}+1,250$
25. $64+27 a^{3}$
26. $27-64 a^{3}$
27. $8 x^{3}-27 y^{3}$
28. $27 x^{3}-8 y^{3}$
29. $t^{3}+\frac{1}{27}$
30. $t^{3}-\frac{1}{27}$
31. $27 x^{3}-\frac{1}{27}$
32. $8 x^{3}+\frac{1}{8}$
33. $64 a^{3}+125 b^{3}$
34. $125 a^{3}-27 b^{3}$
35. $\frac{1}{8} X^{3}-\frac{1}{27} y^{3}$
36. $\frac{1}{27} x^{3}+\frac{1}{8} y^{3}$
37. $a^{6}-b^{6}$
38. $x^{6}-64 y^{6}$
39. $64 x^{6}-y^{6}$
40. $x^{6}-(3 y)^{6}$
41. $x^{6}-(5 y)^{6}$
42. $(4 x)^{6}-(7 y)^{6}$

## Maintaining Your Skills

Solve each equation for $x$.
43. $2 x-6 y=8$
44. $-3 x+9 y=12$
45. $4 x-6 y=8$
46. $-20 x+15 y=-10$

Solve each equation for $y$.
47. $3 x-6 y=-18$
48. $-3 x+9 y=-18$
49. $4 x-6 y=24$
50. $-20 x+5 y=-10$

## Getting Ready for the Next Section

Multiply each of the following.
51. $2 x^{3}(x+2)(x-2)$
52. $3 x^{2}(x+3)(x-3)$
53. $3 x^{2}(x-3)^{2}$
54. $2 x^{3}(x+5)^{2}$
55. $y\left(y^{2}+25\right)$
56. $y^{3}\left(y^{2}+36\right)$
57. $(5 a-2)(3 a+1)$
58. $(3 a-4)(2 a-1)$
59. $4 x^{2}(x-5)(x+2)$
60. $6 x(x-4)(x+2)$
61. $2 a b^{3}\left(b^{2}-4 b+1\right)$
62. $2 a^{3} b\left(a^{2}+3 a+1\right)$

## 

## OBJECTIVES

A Factor a variety of polynomials.

## Factoring: A General Review

In this section we will review the different methods of factoring that we presented in the previous sections of the chapter. This section is important because it will give you an opportunity to factor a variety of polynomials. Prior to this section, the polynomials you worked with were grouped together according to the method used to factor them; that is, in Section 6.4 all the polynomials you factored were either the difference of two squares or perfect square trinomials. What usually happens in a situation like this is that you become proficient at factoring the kind of polynomial you are working with at the time but have trouble when given a variety of polynomials to factor.
We begin this section with a checklist that can be used in factoring polynomials of any type. When you have finished this section and the problem set that follows, you want to be proficient enough at factoring that the checklist is second nature to you.

## Strategy for Factoring a Polynomial

Step 1: If the polynomial has a greatest common factor other than 1, then factor out the greatest common factor.

Step 2: If the polynomial has two terms (it is a binomial), then see if it is the difference of two squares or the sum or difference of two cubes, and then factor accordingly. Remember, if it is the sum of two squares, it will not factor.

Step 3: If the polynomial has three terms (a trinomial), then either it is a perfect square trinomial, which will factor into the square of a binomial, or it is not a perfect square trinomial, in which case you use the trial and error method developed in Section 5.3.

Step 4: If the polynomial has more than three terms, try to factor it by grouping.

Step 5: As a final check, see if any of the factors you have written can be factored further. If you have overlooked a common factor, you can catch it here.

Here are some examples illustrating how we use the checklist.

## EXAMPLE 1 Factor $2 x^{5}-8 x^{3}$.

SOLUTION First, we check to see if the greatest common factor is other than 1. Since the greatest common factor is $2 x^{3}$, we begin by factoring it out. Once we have done so, we notice that the binomial that remains is the difference of two squares:

$$
\begin{aligned}
2 x^{5}-8 x^{3} & =2 x^{3}\left(x^{2}-4\right) & & \begin{array}{l}
\text { Factor out the greatest common } \\
\text { factor, } 2 x^{3}
\end{array} \\
& =2 x^{3}(x+2)(x-2) & & \begin{array}{l}
\text { Factor the difference } \\
\text { of two squares }
\end{array}
\end{aligned}
$$

Note that the greatest common factor $2 x^{3}$ that we factored from each term in the first step of Example 1 remains as part of the answer to the problem; that is because it is one of the factors of the original binomial. Remember, the expression we end up with when factoring must be equal to the expression we start with. We can't just drop a factor and expect the resulting expression to equal the original expression.

## EXAMPLE 2 Factor $3 x^{4}-18 x^{3}+27 x^{2}$.

SOLUTION Step 1 is to factor out the greatest common factor, $3 x^{2}$. After we have done so, we notice that the trinomial that remains is a perfect square trinomial, which will factor as the square of a binomial:

$$
\begin{array}{rlrl}
3 x^{4}-18 x^{3}+27 x^{2} & =3 x^{2}\left(x^{2}-6 x+9\right) \\
& =3 x^{2}(x-3)^{2} & &
\end{array}
$$

## EXAMPLE 3 Factor $y^{3}+25 y$.

SOLUTION We begin by factoring out the $y$ that is common to both terms. The binomial that remains after we have done so is the sum of two squares, which does not factor, so after the first step we are finished:

$$
y^{3}+25 y=y\left(y^{2}+25\right)
$$

Factor out the greatest common factor, $y$; then notice that $y^{2}+25$ cannot be factored further

## EXAMPLE $4 \quad$ Factor $6 a^{2}-11 a+4$.

SOLUTION Here we have a trinomial that does not have a greatest common factor other than 1 . Since it is not a perfect square trinomial, we factor it by trial and error; that is, we look for binomial factors the product of whose first terms is $6 a^{2}$ and the product of whose last terms is 4 . Then we look for the combination of these types of binomials whose product gives us a middle term of $-11 a$. Without showing all the different possibilities, here is the answer:

$$
6 a^{2}-11 a+4=(3 a-4)(2 a-1)
$$

## EXAMPLE 5 Factor $6 x^{3}-12 x^{2}-48 x$.

SOLUTION This trinomial has a greatest common factor of $6 x$. The trinomial that remains after the $6 x$ has been factored from each term must be factored by trial and error:

$$
\begin{aligned}
6 x^{3}-12 x^{2}-48 x & =6 x\left(x^{2}-2 x-8\right) \\
& =6 x(x-4)(x+2)
\end{aligned}
$$

EXAMPLE $6 \quad$ Factor $2 a b^{5}+8 a b^{4}+2 a b^{3}$.
SOLUTION The greatest common factor is $2 a b^{3}$. We begin by factoring it from each term. After that we find the trinomial that remains cannot be factored further:

$$
2 a b^{5}+8 a b^{4}+2 a b^{3}=2 a b^{3}\left(b^{2}+4 b+1\right)
$$

## EXAMPLE $7 \quad$ Factor $x y+8 x+3 y+24$.

SOLUTION Since our polynomial has four terms, we try factoring by grouping:

$$
\begin{aligned}
x y+8 x+3 y+24 & =x(y+8)+3(y+8) \\
& =(y+8)(x+3)
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A ${ }_{1-7}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the first step in factoring any polynomial?
2. If a polynomial has four terms, what method of factoring should you try?
3. If a polynomial has two terms, what method of factoring should you try?
4. What is the last step in factoring any polynomial?

Factor each of the following polynomials completely; that is, once you are finished factoring, none of the factors you obtain should be factorable. Also, note that the even-numbered problems are not necessarily similar to the odd-numbered problems that precede them in this problem set.

1. $x^{2}-81$
2. $x^{2}-18 x+81$
3. $x^{2}+2 x-15$
4. $15 x^{2}+11 x-6$
5. $x^{2}+6 x+9$
6. $12 x^{2}-11 x+2$
7. $y^{2}-10 y+25$
8. $21 y^{2}-25 y-4$
9. $2 a^{3} b+6 a^{2} b+2 a b$
10. $6 a^{2}-a b-15 b^{2}$
11. $x^{2}+x+1$
12. $2 x^{2}-4 x+2$
13. $12 a^{2}-75$
14. $18 a^{2}-50$
15. $9 x^{2}-12 x y+4 y^{2}$
16. $x^{3}-x^{2}$
17. $4 x^{3}+16 x y^{2}$
18. $16 x^{2}+49 y^{2}$
19. $2 y^{3}+20 y^{2}+50 y$
20. $3 y^{2}-9 y-30$
21. $a^{6}+4 a^{4} b^{2}$
22. $5 a^{2}-45 b^{2}$
23. $x y+3 x+4 y+12$
24. $x y+7 x+6 y+42$
25. $x^{3}-27$
26. $x^{4}-81$
27. $x y-5 x+2 y-10$
28. $x y-7 x+3 y-21$
29. $5 a^{2}+10 a b+5 b^{2}$
30. $3 a^{3} b^{2}+15 a^{2} b^{2}+3 a b^{2}$
31. $x^{2}+49$
32. $16-x^{4}$
33. $3 x^{2}+15 x y+18 y^{2}$
34. $3 x^{2}+27 x y+54 y^{2}$
35. $2 x^{2}+15 x-38$
36. $2 x^{2}+7 x-85$
37. $100 x^{2}-300 x+200$
38. $100 x^{2}-400 x+300$
39. $x^{2}-64$
40. $9 x^{2}-4$
41. $x^{2}+3 x+a x+3 a$
42. $x^{2}+4 x+b x+4 b$
43. $49 a^{7}-9 a^{5}$
44. $8 a^{3}+1$
45. $49 x^{2}+9 y^{2}$
46. $12 x^{4}-62 x^{3}+70 x^{2}$
47. $25 a^{3}+20 a^{2}+3 a$
48. $36 a^{4}-100 a^{2}$
49. $x a-x b+a y-b y$
50. $x y-b x+a y-a b$
51. $48 a^{4} b-3 a^{2} b$
52. $18 a^{4} b^{2}-12 a^{3} b^{3}+8 a^{2} b^{4}$
53. $20 x^{4}-45 x^{2}$
54. $16 x^{3}+16 x^{2}+3 x$
55. $3 x^{2}+35 x y-82 y^{2}$
56. $3 x^{2}+37 x y-86 y^{2}$
57. $16 x^{5}-44 x^{4}+30 x^{3}$
58. $16 x^{2}+16 x-1$
59. $2 x^{2}+2 a x+3 x+3 a$
60. $2 x^{2}+2 a x+5 x+5 a$
61. $y^{4}-1$
62. $a^{7}+8 a^{4} b^{3}$
63. $12 x^{4} y^{2}+36 x^{3} y^{3}+27 x^{2} y^{4}$
64. $16 x^{3} y^{2}-4 x y^{2}$
65. $16 t^{2}-64 t+48$
66. $16 t^{2}-32 t+12$
67. $54 x^{2}+111 x+56$
68. $63 x^{2}+110 x+48$

## Maintaining Your Skills

Solve each equation.
69. $-2(x+4)=-10$
70. $\frac{3}{4}(-4 x-8)=21$
71. $\frac{3}{5} x+4=22$
72. $-10=4-\frac{7}{4} x$
73. $6 x-4(9-x)=-96$
74. $-2(x-5)+5 x=4-9$
75. $2 x-3(4 x-7)=-3 x$
76. $\frac{3}{4}(8+x)=\frac{1}{5}(5 x-15)$
77. $\frac{1}{2} x-\frac{5}{12}=\frac{1}{12} x+\frac{5}{12}$
78. $\frac{3}{10} x+\frac{5}{2}=\frac{3}{5} x-\frac{1}{2}$

## Getting Ready for the Next Section

Solve each equation.
79. $3 x-6=9$
80. $5 x-1=14$
81. $2 x+3=0$
82. $4 x-5=0$
83. $4 x+3=0$
84. $3 x-1=0$

## 5.7

## OBJECTIVES

A Solve an equation by writing it in standard form and then factoring.

## Solving Equations by Factoring

In this section we will use the methods of factoring developed in previous sections, along with a special property of 0 , to solve quadratic equations.

DEFINITION Any equation that can be put in the form $a x^{2}+b x+c=0$, where $a, b$, and $c$ are real numbers $(a \neq 0)$, is called a quadratic equation. The equation $a x^{2}+b x+c=0$ is called standard form for a quadratic equation:

$$
\begin{gathered}
\text { an } x^{2} \text { term an } x \text { term } \quad \text { and a constant term } \\
a(\text { variable })^{2}+b(\text { variable })+(\text { absence of the variable })=0
\end{gathered}
$$

The number 0 has a special property. If we multiply two numbers and the product is 0 , then one or both of the original two numbers must be 0 . In symbols, this property looks like this.

Zero-Factor Property Let $a$ and $b$ represent real numbers. If $a \cdot b=0$, then $a=0$ or $b=0$.

Suppose we want to solve the quadratic equation $x^{2}+5 x+6=0$. We can factor the left side into $(x+2)(x+3)$. Then we have:

$$
\begin{array}{r}
x^{2}+5 x+6=0 \\
(x+2)(x+3)=0
\end{array}
$$

## Note

Notice that to solve a quadratic equation by this method, it must be possible to factor it. If we can't factor it, we can't solve it by this method. We will learn how to solve quadratic equations that do not factor when we get to Chapter 10 .

Now, $(x+2)$ and $(x+3)$ both represent real numbers. Their product is 0 ; therefore, either $(x+3)$ is 0 or $(x+2)$ is 0 . Either way we have a solution to our equation. We use the property of 0 stated to finish the problem:

$$
\begin{array}{rlrlrl} 
& x^{2}+5 x+6 & =0 \\
& & (x+2)(x+3) & =0 \\
x+2 & =0 & & \text { or } & x+3 & =0 \\
x & =-2 & & \text { or } & x & =-3
\end{array}
$$

Our solution set is $\{-2,-3\}$. Our equation has two solutions. To check our solutions we have to check each one separately to see that they both produce a true statement when used in place of the variable:
$\left.\begin{array}{lrl}\text { When } & x & =-3 \\ \text { the equation } & x^{2}+5 x+6 & =0 \\ \text { becomes } & (-3)^{2}+5(-3)+6 & \stackrel{?}{=} 0 \\ 9+(-15)+6 & =0 \\ 0 & =0 \\ & =-2 \\ & \text { When } & \\ \text { the equation } & & x^{2}+5 x+6\end{array}\right) 0$

We have solved a quadratic equation by replacing it with two linear equations in one variable.

## Strategy for Solving a Quadratic Equation by Factoring

Step 1: Put the equation in standard form; that is, 0 on one side and decreasing powers of the variable on the other.

Step 2: Factor completely.
Step 3: Use the zero-factor property to set each variable factor from step 2 to 0 .

Step 4: Solve each equation produced in step 3.
Step 5: Check each solution, if necessary.

## EXAMPLE 1 Solve the equation $2 x^{2}-5 x=12$.

## SOLUTION

Step 1: Begin by adding -12 to both sides, so the equation is in standard form:

$$
\begin{aligned}
2 x^{2}-5 x & =12 \\
2 x^{2}-5 x-12 & =0
\end{aligned}
$$

Step 2: Factor the left side completely:

$$
(2 x+3)(x-4)=0
$$

Step 3: Set each factor to 0:

$$
2 x+3=0 \quad \text { or } \quad x-4=0
$$

Step 4: Solve each of the equations from step 3:

$$
\begin{array}{rlrl}
2 x+3 & =0 & x-4 & =0 \\
2 x & =-3 & x=4 \\
x & =-\frac{3}{2} &
\end{array}
$$

Step 5: Substitute each solution into $2 x^{2}-5 x=12$ to check:

$$
\begin{array}{rlrl}
\text { Check: }-\frac{3}{2} & & \text { Check: } 4 \\
2\left(-\frac{3}{2}\right)^{2}-5\left(-\frac{3}{2}\right) & \stackrel{?}{=} 12 & 2(4)^{2}-5(4) \stackrel{?}{=} 12 \\
2\left(\frac{9}{4}\right)+5\left(\frac{3}{2}\right) & =12 & 2(16)-20 & =12 \\
\frac{9}{2}+\frac{15}{2} & =12 & 32-20 & =12 \\
\frac{24}{2} & =12 & 12 & =12 \\
12 & =12 &
\end{array}
$$

EXAMPLE $2 \quad$ Solve for $a: 16 a^{2}-25=0$.
SOLUTION The equation is already in standard form:

$$
\begin{aligned}
16 a^{2}-25 & =0 & & \\
4 a-5 & =0 & \text { or } \quad 4 a-5)(4 a+5) & =0 \\
4 a+5 & =0 & & \text { Factor left side } \\
4 a & =5 & & \text { Set each factor to 0 } \\
a & =\frac{5}{4} & &
\end{aligned}
$$

## EXAMPLE 3 Solve $4 x^{2}=8 x$.

SOLUTION We begin by adding $-8 x$ to each side of the equation to put it in standard form. Then we factor the left side of the equation by factoring out the greatest common factor.

\[

\]

The solutions are 0 and 2 .

## EXAMPLE $4 \quad$ Solve $x(2 x+3)=44$.

SOLUTION We must multiply out the left side first and then put the equation in standard form:

$$
\begin{aligned}
x(2 x+3) & =44 & & \\
2 x^{2}+3 x & =44 & & \text { Multiply out the left side } \\
2 x^{2}+3 x-44 & =0 & & \text { Add }-44 \text { to each side } \\
(2 x+11)(x-4) & =0 & & \text { Factor the left side } \\
2 x+11 & =0 & \text { or } \quad x-4 & =0 \\
2 x & =-11 & & \text { or }
\end{aligned} \begin{array}{ll}
\text { Set each factor to } 0 \\
x & =4
\end{array} \begin{array}{ll}
\text { Solve the resulting equations } \\
x & =-\frac{11}{2}
\end{array}
$$

The two solutions are $-\frac{11}{2}$ and 4 .

## EXAMPLE 5 Solve for $x: 5^{2}=x^{2}+(x+1)^{2}$.

SOLUTION Before we can put this equation in standard form we must square the binomial. Remember, to square a binomial, we use the formula $(a+b)^{2}=$ $a^{2}+2 a b+b^{2}:$

$$
\begin{array}{rlrl}
5^{2} & =x^{2}+(x+1)^{2} & & \\
25 & =x^{2}+x^{2}+2 x+1 & & \text { Expand } 5^{2} \text { and }(x+\mathbf{1})^{2} \\
25 & =2 x^{2}+2 x+1 & & \text { Simplify the right side } \\
0 & =2 x^{2}+2 x-24 & & \text { Add }-\mathbf{2 5} \text { to each side } \\
0 & =2\left(x^{2}+x-12\right) & & \text { Begin factoring } \\
0 & =2(x+4)(x-3) & & \text { Factor completely } \\
x+4 & =0 \quad \text { or } x-3 & =0 & \\
\text { Set each variable factor to } \mathbf{0} \\
x & =-4 \quad \text { or } & x & =3
\end{array}
$$

Note, in the second to the last line, that we do not set 2 equal to 0 . That is because 2 can never be 0 . It is always 2 . We only use the zero-factor property to set variable factors to 0 because they are the only factors that can possibly be 0 .

Also notice that it makes no difference which side of the equation is 0 when we write the equation in standard form.

Although the equation in the next example is not a quadratic equation, it can be solved by the method shown in the first five examples.

## EXAMPLE 6 Solve $24 x^{3}=-10 x^{2}+6 x$ for $x$.

SOLUTION First, we write the equation in standard form:

|  |  | $24 x^{3}+10 x^{2}-6 x=0$ |  |  | Standard form |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $2 x\left(12 x^{2}+5 x-3\right)=0$ |  |  | Factor out $2 x$ |  |
|  |  | $2 x(3 x-1)(4 x+3)=0$ |  |  | Factor remaining trinomial |  |
| $2 x=0$ | or | $3 x-1=0$ | or | $4 x+3=0$ | Set factors to 0 |  |
| $x=0$ | or | $x=\frac{1}{3}$ | or | $x=$ | Solutions | $111 / 2$ |

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A ${ }_{1-6}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. When is an equation in standard form?
2. What is the first step in solving an equation by factoring?
3. Describe the zero-factor property in your own words.
4. Describe how you would solve the equation $2 x^{2}-5 x=12$.

## Problem Set 5.7

Online support materials can be found at www.thomsonedu.com/login

The following equations are already in factored form. Use the special zero factor property to set the factors to 0 and solve.

1. $(x+2)(x-1)=0$
2. $(x+3)(x+2)=0$
3. $(a-4)(a-5)=0$
4. $(a+6)(a-1)=0$
5. $x(x+1)(x-3)=0$
6. $x(2 x+1)(x-5)=0$
7. $(3 x+2)(2 x+3)=0$
8. $(4 x-5)(x-6)=0$
9. $m(3 m+4)(3 m-4)=0$
10. $m(2 m-5)(3 m-1)=0$
11. $2 y(3 y+1)(5 y+3)=0$
12. $3 y(2 y-3)(3 y-4)=0$

Solve the following equations
13. $x^{2}+3 x+2=0$
15. $x^{2}-9 x+20=0$
17. $a^{2}-2 a-24=0$
19. $100 x^{2}-500 x+600=0$
20. $100 x^{2}-300 x+200=0$
21. $x^{2}=-6 x-9$
23. $a^{2}-16=0$
25. $2 x^{2}+5 x-12=0$
26. $3 x^{2}+14 x-5=0$
27. $9 x^{2}+12 x+4=0$
29. $a^{2}+25=10 a$
31. $2 x^{2}=3 x+20$
33. $3 m^{2}=20-7 m$
35. $4 x^{2}-49=0$
37. $x^{2}+6 x=0$
39. $x^{2}-3 x=0$
41. $2 x^{2}=8 x$
43. $3 x^{2}=15 x$
28. $12 x^{2}-24 x+9=0$
30. $a^{2}+16=8 a$
32. $6 x^{2}=x+2$
34. $2 m^{2}=-18+15 m$
36. $16 x^{2}-25=0$
38. $x^{2}-8 x=0$
40. $x^{2}+5 x=0$
42. $2 x^{2}=10 x$
44. $5 x^{2}=15 x$
45. $1,400=400+700 x-100 x^{2}$
46. $2,700=700+900 x-100 x^{2}$
47. $6 x^{2}=-5 x+4$
48. $9 x^{2}=12 x-4$
49. $x(2 x-3)=20$
50. $x(3 x-5)=12$
51. $t(t+2)=80$
52. $t(t+2)=99$
53. $4,000=(1,300-100 p) p$
54. $3,200=(1,200-100 p) p$
55. $x(14-x)=48$
56. $x(12-x)=32$
57. $(x+5)^{2}=2 x+9$
58. $(x+7)^{2}=2 x+13$
59. $(y-6)^{2}=y-4$
60. $(y+4)^{2}=y+6$
61. $10^{2}=(x+2)^{2}+x^{2}$
62. $15^{2}=(x+3)^{2}+x^{2}$
63. $2 x^{3}+11 x^{2}+12 x=0$
64. $3 x^{3}+17 x^{2}+10 x=0$
65. $4 y^{3}-2 y^{2}-30 y=0$
= Online student support materials available at www.thomsonedu.com/login
66. $9 y^{3}+6 y^{2}-24 y=0$
67. $8 x^{3}+16 x^{2}=10 x$
68. $24 x^{3}-22 x^{2}=-4 x$
69. $20 a^{3}=-18 a^{2}+18 a$
70. $12 a^{3}=-2 a^{2}+10 a$
71. $16 t^{2}-32 t+12=0$
72. $16 t^{2}-64 t+48=0$

Simplify each side as much as possible, then solve the equation.
73. $(a-5)(a+4)=-2 a$
74. $(a+2)(a-3)=-2 a$
75. $3 x(x+1)-2 x(x-5)=-42$
76. $4 x(x-2)-3 x(x-4)=-3$
77. $2 x(x+3)=x(x+2)-3$
78. $3 x(x-3)=2 x(x-4)+6$
79. $a(a-3)+6=2 a$
80. $a(a-4)+8=2 a$
81. $15(x+20)+15 x=2 x(x+20)$
82. $15(x+8)+15 x=2 x(x+8)$
83. $15=a(a+2)$
84. $6=a(a-5)$

Use factoring by grouping to solve the following equations.
85. $x^{3}+3 x^{2}-4 x-12=0$
86. $x^{3}+5 x^{2}-9 x-45=0$
87. $x^{3}+x^{2}-16 x-16=0$
88. $4 x^{3}+12 x^{2}-9 x-27=0$
89. Find a quadratic equation that has two solutions: $x=3$ and $x=5$. Write your answer in standard form.
90. Find a quadratic equation that has two solutions: $x=9$ and $x=1$. Write your answer in standard form.
91. Find a quadratic equation that has the two given solutions.
a. $x=3$ and $x=2$.
b. $x=1$ and $x=6$.
c. $x=3$ and $x=-2$.
92. Find a quadratic equation that has the two given solutions.
a. $x=4$ and $x=5$.
b. $x=2$ and $x=10$.
c. $x=-4$ and $x=5$.

## Maintaining Your Skills

Use the properties of exponents to simplify each expression.
93. $2^{-3}$
95. $\frac{X^{5}}{X^{-3}}$
97. $\frac{\left(x^{2}\right)^{3}}{\left(x^{-3}\right)^{4}}$
99. Write the number 0.0056 in scientific notation.
100. Write the number $2.34 \times 10^{-4}$ in expanded form.
101. Write the number $5,670,000,000$ in scientific notation.
102. Write the number 0.00000567 in scientific notation.

## Getting Ready for the Next Section

Write each sentence as an algebraic equation.
103. The product of two consecutive integers is 72 .
104. The product of two consecutive even integers is 80.
105. The product of two consecutive odd integers is 99.
106. The product of two consecutive odd integers is 63 .
107. The product of two consecutive even integers is 10 less than 5 times their sum.
108. The product of two consecutive odd integers is 1 less than 4 times their sum.

The following word problems are taken from the book Academic Algebra, written by William J. Milne and published by the American Book Company in 1901. Solve each problem.
109. Cost of a Bicycle and a Suit A bicycle and a suit cost $\$ 90$. How much did each cost, if the bicycle cost 5 times as much as the suit?
110. Cost of a Cow and a Calf A man bought a cow and a calf for $\$ 36$, paying 8 times as much for the cow as for the calf. What was the cost of each?
111. Cost of a House and a Lot A house and a lot cost $\$ 3,000$. If the house cost 4 times as much as the lot, what was the cost of each?
112. Daily Wages A plumber and two helpers together earned $\$ 7.50$ per day. How much did each earn per day, if the plumber earned 4 times as much as each helper?

## 5.8

## Applications

## OBJECTIVES

A Solve a variety of application problems

In this section we will look at some application problems, the solutions to which require solving a quadratic equation. We will also introduce the Pythagorean theorem, one of the oldest theorems in the history of mathematics. The person whose name we associate with the theorem, Pythagoras (of Samos), was a Greek philosopher and mathematician who lived from about 560 в.c. to 480 в.с. According to the British philosopher Bertrand Russell, Pythagoras was "intellectually one of the most important men that ever lived."

Also in this section, the solutions to the examples show only the essential steps from our Blueprint for Problem Solving. Recall that step 1 is done mentally; we read the problem and mentally list the items that are known and the items that are unknown. This is an essential part of problem solving. However, now that you have had experience with application problems, you are doing step 1 automatically.

## Number Problems

## EXAMPLE 1 The product of two consecutive odd integers is 63. Find

the integers.
SOLUTION Let $x=$ the first odd integer; then $x+2=$ the second odd integer. An equation that describes the situation is:

$$
x(x+2)=63 \quad \text { Their product is } 63
$$

We solve the equation:

$$
\begin{array}{rlrl}
x(x+2) & =63 \\
x^{2}+2 x & =63 \\
x^{2}+2 x-63 & =0 \\
(x-7)(x+9) & =0 \\
x-7=0 & & \text { or } \quad x+9 & =0 \\
x=7 & & \text { or } \quad x & =-9
\end{array}
$$

If the first odd integer is 7 , the next odd integer is $7+2=9$. If the first odd integer is -9 , the next consecutive odd integer is $-9+2=-7$. We have two pairs of consecutive odd integers that are solutions. They are 7,9 and $-9,-7$.

We check to see that their products are 63:

$$
\begin{aligned}
7(9) & =63 \\
-7(-9) & =63
\end{aligned}
$$

Suppose we know that the sum of two numbers is 50 . We want to find a way to represent each number using only one variable. If we let $x$ represent one of the two numbers, how can we represent the other? Let's suppose for a moment that $x$ turns out to be 30 . Then the other number will be 20 , because their sum is 50 ; that is, if two numbers add up to 50 and one of them is 30 , then the other must be $50-30=20$. Generalizing this to any number $x$, we see that if two numbers have a sum of 50 and one of the numbers is $x$, then the other must be $50-x$. The table that follows shows some additional examples.

| If Two Numbers <br> Have a Sum of | And One of <br> Them Is | Then the <br> Other Must Be |
| :---: | :---: | :---: |
| 50 | $x$ | $50-x$ |
| 100 | $x$ | $100-x$ |
| 10 | $y$ | $10-y$ |
| 12 | $n$ | $12-n$ |

Now, let's look at an example that uses this idea.

## EXAMPLE 2

The sum of two numbers is 13 . Their product is 40 . Find the numbers.

SOLUTION If we let $x$ represent one of the numbers, then $13-x$ must be the other number because their sum is 13 . Since their product is 40 , we can write:

$$
\begin{array}{rlrl}
x(13-x) & =40 & \text { The product of the two numbers is } 40 \\
13 x-x^{2} & =40 & & \text { Multiply the left side } \\
x^{2}-13 x & =-40 & \begin{array}{l}
\text { Multiply both sides by }-1 \text { and } \\
\text { reverse the order of the terms on } \\
\text { the left side }
\end{array} \\
x^{2}-13 x+40 & =0 & \text { Add 40 to each side } \\
(x-8)(x-5) & =0 & \text { Factor the left side } \\
x-8=0 & \text { or } \begin{aligned}
x-5 & =0
\end{aligned} & \\
x=8 & x & =5 &
\end{array}
$$

The two solutions are 8 and 5. If $x$ is 8 , then the other number is $13-x=$ $13-8=5$. Likewise, if $x$ is 5 , the other number is $13-x=13-5=8$. Therefore, the two numbers we are looking for are 8 and 5 . Their sum is 13 and their product is 40 .

## Geometry Problems

Many word problems dealing with area can best be described algebraically by quadratic equations.

EXAMPLE 3 The length of a rectangle is 3 more than twice the width. The area is 44 square inches. Find the dimensions (find the length and width).
SOLUTION As shown in Figure 1, let $x=$ the width of the rectangle. Then $2 x+3=$ the length of the rectangle because the length is three more than twice the width.


FIGURE 1
Since the area is 44 square inches, an equation that describes the situation is

$$
x(2 x+3)=44 \quad \text { Length } \cdot \text { width }=\text { area }
$$

We now solve the equation:

$$
\begin{aligned}
x(2 x+3) & =44 \\
2 x^{2}+3 x & =44 \\
2 x^{2}+3 x-44 & =0 \\
(2 x+11)(x-4) & =0 \\
2 x+11=0 \quad & \text { or } \quad x-4
\end{aligned}=0
$$

The solution $x=-\frac{11}{2}$ cannot be used since length and width are always given in positive units. The width is 4 . The length is 3 more than twice the width or $2(4)+3=11$.

$$
\begin{aligned}
\text { Width } & =4 \text { inches } \\
\text { Length } & =11 \text { inches }
\end{aligned}
$$

The solutions check in the original problem since $4(11)=44$.

EXAMPLE 4 The numerical value of the area of a square is twice its perimeter. What is the length of its side?
SOLUTION As shown in Figure 2, let $x=$ the length of its side. Then $x^{2}=$ the area of the square and $4 x=$ the perimeter of the square:


FIGURE 2

An equation that describes the situation is

$$
\left.\begin{array}{rl}
x^{2} & =2(4 x) \quad \text { The area is } 2 \text { times the perimeter } \\
x^{2} & =8 x \\
x^{2}-8 x & =0 \\
x(x-8) & =0 \\
x(x-8) & =0 \\
x=0 \quad \text { or } & x-8
\end{array}\right)
$$

Since $x=0$ does not make sense in our original problem, we use $x=8$. If the side has length 8 , then the perimeter is $4(8)=32$ and the area is $8^{2}=64$. Since 64 is twice 32 , our solution is correct.

## FACTS FROM GEOMETRY

## The Pythagorean Theorem

Next, we will work some problems involving the Pythagorean theorem, which we mentioned in the introduction to this section. It may interest you to know that Pythagoras formed a secret society around the year 540 в.c. Known as the Pythagoreans, members kept no written record of their work; everything was handed down by spoken word. They influenced not only mathematics, but religion, science, medicine, and music as well. Among other things, they discovered the correlation between musical notes and the reciprocals of counting numbers, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and so on. In their daily lives, they followed strict dietary and moral rules to achieve a higher rank in future lives.

Pythagorean Theorem In any right triangle (Figure 3), the square of the longer side (called the hypotenuse) is equal to the sum of the squares of the other two sides (called legs).


FIGURE 3

EXAMPLE 5
The three sides of a right triangle are three consecutive integers. Find the lengths of the three sides.

SOLUTION Let $x=$ the first integer (shortest side) then $x+1=$ the next consecutive integer and $x+2=$ the third consecutive integer (longest side)

A diagram of the triangle is shown in Figure 4.


FIGURE 4

The Pythagorean theorem tells us that the square of the longest side $(x+2)^{2}$ is equal to the sum of the squares of the two shorter sides, $(x+1)^{2}+x^{2}$. Here is the equation:

$$
\begin{array}{rlrl}
(x+2)^{2} & =(x+1)^{2}+x^{2} & & \\
x^{2}+4 x+4 & =x^{2}+2 x+1+x^{2} & & \text { Expand squares } \\
x^{2}-2 x-3 & =0 & & \text { Standard form } \\
(x-3)(x+1) & =0 & & \text { Factor } \\
x-3=0 \quad \text { or } x+1=0 & & \text { Set factors to 0 } \\
x=3 \quad \text { or } & x=-1 &
\end{array}
$$

Since a triangle cannot have a side with a negative number for its length, we must not use -1 for a solution to our original problem; therefore, the shortest side is 3 . The other two sides are the next two consecutive integers, 4 and 5.

## EXAMPLE 6 The hypotenuse of a right triangle is 5 inches, and the

 lengths of the two legs (the other two sides) are given by two consecutive integers. Find the lengths of the two legs.SOLUTION If we let $x=$ the length of the shortest side, then the other side must be $x+1$. A diagram of the triangle is shown in Figure 5 .


FIGURE 5
The Pythagorean theorem tells us that the square of the longest side, $5^{2}$, is equal to the sum of the squares of the two shorter sides, $x^{2}+(x+1)^{2}$. Here is the equation:

$$
\begin{aligned}
5^{2} & =x^{2}+(x+1)^{2} & & \text { Pythagorean theorem } \\
25 & =x^{2}+x^{2}+2 x+1 & & \text { Expand } 5^{2} \text { and }(x+\mathbf{1})^{2} \\
25 & =2 x^{2}+2 x+1 & & \text { Simplify the right side } \\
0 & =2 x^{2}+2 x-24 & & \text { Add } \mathbf{- 2 5} \text { to each side } \\
0 & =2\left(x^{2}+x-12\right) & & \text { Begin factoring } \\
0 & =2(x+4)(x-3) & & \text { Factor completely } \\
x+4 & =0 \quad \text { or } x-3=0 & & \text { Set variable factors to 0 } \\
x & =-4 \quad \text { or } & x=3 &
\end{aligned}
$$

Since a triangle cannot have a side with a negative number for its length, we cannot use -4 ; therefore, the shortest side must be 3 inches. The next side is

## Note

If you are planning on taking finite mathematics, statistics, or business calculus in the future, Examples 7 and 8 will give you a head start on some of the problems you will see in those classes.
$x+1=3+1=4$ inches. Since the hypotenuse is 5 , we can check our solutions with the Pythagorean theorem as shown in Figure 6.


FIGURE 6

EXAMPLE 7
A company can manufacture $x$ hundred items for a total
cost of $C=300+500 x-100 x^{2}$. How many items were manufactured if the total cost is $\$ 900$ ?

SOLUTION We are looking for $x$ when $C$ is 900 . We begin by substituting 900 for $C$ in the cost equation. Then we solve for $x$ :

| When | $C$ | $=900$ |
| :--- | ---: | :--- |
| the equation | $C$ | $=300+500 x-100 x^{2}$ |
|  | becomes | 900 |

We can write this equation in standard form by adding $-300,-500 x$, and $100 x^{2}$ to each side. The result looks like this:

$$
\begin{array}{rlrl}
100 x^{2}-500 x+600 & =0 & & \\
100\left(x^{2}-5 x+6\right) & =0 & & \text { Begin factoring } \\
100(x-2)(x-3) & =0 & & \text { Factor completely } \\
x-2=0 & \text { or } \quad x-3 & =0 & \\
\text { Set variable factors to } 0 \\
x=2 & \text { or } & x & =3
\end{array} r l a l
$$

Our solutions are 2 and 3, which means that the company can manufacture 2 hundred items or 3 hundred items for a total cost of $\$ 900$.

## EXAMPLE 8

A manufacturer of small portable radios knows that the number of radios she can sell each week is related to the price of the radios by the equation $x=1,300-100 p$ ( $x$ is the number of radios and $p$ is the price per radio). What price should she charge for the radios to have a weekly revenue of $\$ 4,000$ ?
SOLUTION First, we must find the revenue equation. The equation for total revenue is $R=x p$, where $x$ is the number of units sold and $p$ is the price per unit. Since we want $R$ in terms of $p$, we substitute $1,300-100 p$ for $x$ in the equation $R=x p$ :

| If | $R=x p$ |
| :--- | :--- |
| and | $x=1,300-100 p$ |
| then | $R=(1,300-100 p) p$ |

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-8

We want to find $p$ when $R$ is 4,000. Substituting 4,000 for $R$ in the equation gives us:

$$
4,000=(1,300-100 p) p
$$

If we multiply out the right side, we have:

$$
4,000=1,300 p-100 p^{2}
$$

To write this equation in standard form, we add $100 p^{2}$ and $-1,300 p$ to each side:

$$
\begin{array}{rlrl}
100 p^{2}-1,300 p+4,000 & =0 & & \text { Add } 100 p^{2} \text { and }-\mathbf{1 , 3 0 0} p \text { to each side } \\
100\left(p^{2}-13 p+40\right) & =0 & \text { Begin factoring } \\
100(p-5)(p-8) & =0 & \text { Factor completely } \\
p-5=0 & \text { or } \quad p-8 & =0 & \text { Set variable factors to } 0 \\
p=5 & \text { or } & p & =8
\end{array}
$$

If she sells the radios for $\$ 5$ each or for $\$ 8$ each, she will have a weekly revenue of \$4,000.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What are consecutive integers?
2. Explain the Pythagorean theorem in words.
3. Write an application problem for which the solution depends on solving the equation $x(x+1)=12$.
4. Write an application problem for which the solution depends on solving the equation $x(2 x-3)=40$.

## Problem Set 5.8

Online support materials can be found at www.thomsonedu.com/login

Solve the following word problems. Be sure to show the equation used.

## Number Problems

1. The product of two consecutive even integers is 80 . Find the two integers.
2. The product of two consecutive integers is 72 . Find the two integers.
3. The product of two consecutive odd integers is 99 . Find the two integers.
4. The product of two consecutive integers is 132 . Find the two integers.
5. The product of two consecutive even integers is 10 less than 5 times their sum. Find the two integers.
6. The product of two consecutive odd integers is 1 less than 4 times their sum. Find the two integers.
7. The sum of two numbers is 14 . Their product is 48 . Find the numbers.
8. The sum of two numbers is 12 . Their product is 32 . Find the numbers.

[^3]9. One number is 2 more than 5 times another. Their product is 24 . Find the numbers.
10. One number is 1 more than twice another. Their product is 55 . Find the numbers.
11. One number is 4 times another. Their product is 4 times their sum. Find the numbers.
12. One number is 2 more than twice another. Their product is 2 more than twice their sum. Find the numbers.

## Geometry Problems

13. The length of a rectangle is 1 more than the width. The area is 12 square inches. Find the dimensions.
14. The length of a rectangle is 3 more than twice the width. The area is 44 square inches. Find the dimensions.
15. The height of a triangle is twice the base. The area is 9 square inches. Find the base.
16. The height of a triangle is 2 more than twice the base. The area is 20 square feet. Find the base.
17. The hypotenuse of a right triangle is 10 inches. The lengths of the two legs are given by two consecutive even integers. Find the lengths of the two legs.
18. The hypotenuse of a right triangle is 15 inches. One of the legs is 3 inches more than the other. Find the lengths of the two legs.
19. The shorter leg of a right triangle is 5 meters. The hypotenuse is 1 meter longer than the longer leg. Find the length of the longer leg.
20. The shorter leg of a right triangle is 12 yards. If the hypotenuse is 20 yards, how long is the other leg?

## Business Problems

21. A company can manufacture $x$ hundred items for a total cost of $C=400+700 x-100 x^{2}$. Find $x$ if the total cost is $\$ 1,400$.
22. If the total cost $C$ of manufacturing $x$ hundred items is given by the equation $C=700+900 x-100 x^{2}$, find $x$ when $C$ is $\$ 2,700$.
23. The total cost $C$ of manufacturing $x$ hundred videotapes is given by the equation

$$
C=600+1,000 x-100 x^{2}
$$

Find $x$ if the total cost is $\$ 2,200$.
24. The total cost $C$ of manufacturing $x$ hundred pen and pencil sets is given by the equation

$$
C=500+800 x-100 x^{2} .
$$

Find $x$ when $C$ is $\$ 1,700$.
25. A company that manufactures typewriter ribbons knows that the number of ribbons it can sell each week, $x$, is related to the price $p$ per ribbon by the equation $x=1,200-100 p$. At what price should the company sell the ribbons if it wants the weekly revenue to be $\$ 3,200$ ? (Remember: The equation for revenue is $R=x p$.)
26. A company manufactures diskettes for home computers. It knows from experience that the number of diskettes it can sell each day, $x$, is related to the price $p$ per diskette by the equation $x=800-100 p$. At what price should the company sell the diskettes if it wants the daily revenue to be $\$ 1,200$ ?
27. The relationship between the number of calculators a company sells per week, $x$, and the price $p$ of each calculator is given by the equation $x=1,700-100 p$. At what price should the calculators be sold if the weekly revenue is to be $\$ 7,000$ ?
28. The relationship between the number of pencil sharpeners a company can sell each week, $x$, and the price $p$ of each sharpener is given by the equation $x=1,800-100 p$. At what price should the sharpeners be sold if the weekly revenue is to be $\$ 7,200$ ?
29. Pythagorean Theorem A 13-foot ladder is placed so that it reaches to a point on the wall that is 2 feet higher than twice the distance from the base of the wall to the base of the ladder.
a. How far from the wall is the base of the ladder?
b. How high does the ladder reach?

30. Height of a Projectile If a rocket is fired vertically into the air with a speed of 240 feet per second, its height at time $t$ seconds is given by $h(t)=-16 t^{2}+240 t$. Here is a graph of its height at various times, with the details left out:


At what time(s) will the rocket be the following number of feet above the ground?
a. 704 feet
b. 896 feet
c. Why do parts (a) and (b) each have two answers?
d. How long will the rocket be in the air? (Hint: How high is it when it hits the ground?)
e. When the equation for part (d) is solved, one of the answers is $t=0$ second. What does this represent?
31. Projectile Motion A gun fires a bullet almost straight up from the edge of a 100 -foot cliff. If the bullet leaves the gun with a speed of 396 feet per second, its height at time $t$ is given by $h(t)=-16 t^{2}+396 t+100$, measured from the ground below the cliff.
a. When will the bullet land on the ground below the cliff? (Hint: What is its height when it lands? Remember that we are measuring from the ground below, not from the cliff.)
b. Make a table showing the bullet's height every five seconds, from the time it is fired $(t=0)$ to the time it lands. (Note: It is faster to substitute into the factored form.)

| $\boldsymbol{t}$ (sec) $\boldsymbol{h}$ (feet) |
| :---: |
| 0 |
| 5 |
| 10 |
| 15 |
| 20 |
| 25 |

32. Constructing a Box I have a piece of cardboard that is twice as long as it is wide. If I cut a 2 -inch by 2-inch square from each corner and fold up the resulting flaps, I get a box with a volume of 32 cubic inches. What are the dimensions of the cardboard?


## Maintaining Your Skills

Simplify each expression. (Write all answers with positive exponents only.)
33. $\left(5 x^{3}\right)^{2}\left(2 x^{6}\right)^{3}$
35. $\frac{X^{4}}{x^{-3}}$
34. $2^{-3}$
36. $\frac{\left(20 x^{2} y^{3}\right)\left(5 x^{4} y\right)}{\left(2 x y^{5}\right)\left(10 x^{2} y^{3}\right)}$
37. $\left(2 \times 10^{-4}\right)\left(4 \times 10^{5}\right)$
38. $\frac{9 \times 10^{-3}}{3 \times 10^{-2}}$
39. $20 a b^{2}-16 a b^{2}+6 a b^{2}$
40. Subtract $6 x^{2}-5 x-7$ from $9 x^{2}+3 x-2$.

Multiply.
41. $2 x^{2}\left(3 x^{2}+3 x-1\right)$
42. $(2 x+3)(5 x-2)$
43. $(3 y-5)^{2}$
44. $(a-4)\left(a^{2}+4 a+16\right)$
45. $\left(2 a^{2}+7\right)\left(2 a^{2}-7\right)$
46. Divide $15 x^{10}-10 x^{8}+25 x^{6}$ by $5 x^{6}$.

## Chapter 5 SUMMARY

## Greatest Common Factor [5.1]

The largest monomial that divides each term of a polynomial is called the greatest common factor for that polynomial. We begin all factoring by factoring out the greatest common factor.

## Factoring Trinomials [5.2, 5.3]

One method of factoring a trinomial is to list all pairs of binomials the product of whose first terms gives the first term of the trinomial and the product of whose last terms gives the last term of the trinomial. We then choose the pair that gives the correct middle term for the original trinomial.

## Special Factoring [5.3]

3. $x^{2}+10 x+25=(x+5)^{2}$
$x^{2}-10 x+25=(x-5)^{2}$ $x^{2}-25=(x+5)(x-5)$
4. $x^{3}-27=(x-3)\left(x^{2}+3 x+9\right)$
$x^{3}+27=(x+3)\left(x^{2}-3 x+9\right)$

## Sum and Difference of Two Cubes [5.5]

$$
\begin{array}{ll}
a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right) & \text { Difference of two cubes } \\
a^{3}+b^{3}=(a+b)\left(a^{2}-a b+b^{2}\right) & \text { Sum of two cubes }
\end{array}
$$

## Strategy for Factoring a Polynomial [5.6]

5. a. $2 x^{5}-8 x^{3}=2 x^{3}\left(x^{2}-4\right)$

$$
=2 x^{3}(x+2)(x-2)
$$

b. $3 x^{4}-18 x^{3}+27 x^{2}$

$$
\begin{aligned}
& =3 x^{2}\left(x^{2}-6 x+9\right) \\
& =3 x^{2}(x-3)^{2}
\end{aligned}
$$

c. $6 x^{3}-12 x^{2}-48 x$

$$
\begin{aligned}
& =6 x\left(x^{2}-2 x-8\right) \\
& =6 x(x-4)(x+2)
\end{aligned}
$$

d. $x^{2}+a x+b x+a b$

$$
\begin{aligned}
& =x(x+a)+b(x+a) \\
& =(x+a)(x+b)
\end{aligned}
$$

Step 1: If the polynomial has a greatest common factor other than 1, then factor out the greatest common factor.

Step 2: If the polynomial has two terms (it is a binomial), then see if it is the difference of two squares or the sum or difference of two cubes, and then factor accordingly. Remember, if it is the sum of two squares, it will not factor.

Step 3: If the polynomial has three terms (a trinomial), then it is either a perfect square trinomial that will factor into the square of a binomial, or it is not a perfect square trinomial, in which case you use the trial and error method developed in Section 6.3.

Step 4: If the polynomial has more than three terms, then try to factor it by grouping.

Step 5: As a final check, see if any of the factors you have written can be factored further. If you have overlooked a common factor, you can catch it here.

## Strategy for Solving a Quadratic Equation [5.7]

6. Solve $x^{2}-6 x=-8$.

$$
\begin{array}{rlrl} 
& x^{2}-6 x+8 & =0 \\
(x-4)(x-2) & =0 \\
x-4 & =0 & \text { or } & x-2
\end{array}=0
$$

Both solutions check.
7. The hypotenuse of a right triangle is 5 inches, and the lengths of the two legs (the other two sides) are given by two consecutive integers. Find the lengths of the two legs.
If we let $x=$ the length of the shortest side, then the other side must be $x+1$. The Pythagorean theorem tells us that

$$
\begin{aligned}
5^{2} & =x^{2}+(x+1)^{2} \\
25 & =x^{2}+x^{2}+2 x+1 \\
25 & =2 x^{2}+2 x+1 \\
0 & =2 x^{2}+2 x-24 \\
0 & =2\left(x^{2}+x-12\right) \\
0 & =2(x+4)(x-3) \\
x+4 & =0 \quad \text { or } \quad x-3 \\
x & =-4 \quad \text { or } \quad x
\end{aligned}
$$

Since a triangle cannot have a side with a negative number for its length, we cannot use -4 . One leg is $x=3$ and the other leg is $x+1=3+1=4$.

Step 1: Write the equation in standard form:

$$
a x^{2}+b x+c=0
$$

Step 2: Factor completely.
Step 3: Set each variable factor equal to 0 .
Step 4: Solve the equations found in step 3.
Step 5: Check solutions, if necessary.

## The Pythagorean Theorem [5.8]

In any right triangle, the square of the longest side (called the hypotenuse) is equal to the sum of the squares of the other two sides (called legs).


## COMMON MISTAKES

It is a mistake to apply the zero-factor property to numbers other than zero. For example, consider the equation $(x-3)(x+4)=18$. A fairly common mistake is to attempt to solve it with the following steps:

$$
\begin{array}{rlrlrl} 
& & (x-3)(x+4) & =18 \\
x-3 & =18 & & \text { or } & x+4 & =18 \leftarrow \text { Mistake } \\
x & =21 & \text { or } & & x & =14
\end{array}
$$

These are obviously not solutions, as a quick check will verify:

$$
\begin{array}{rlrl}
\text { Check: } x & =21 & \text { Check: } x & =14 \\
(21-3)(21+4) & \stackrel{?}{=} 18 & (14-3)(14+4) & \stackrel{?}{=} 18 \\
18 \cdot 25 & =18 & 11 \cdot 18 & =18 \\
450 & =18 \xrightarrow{\text { false statements }} 198 & =18
\end{array}
$$

The mistake is in setting each factor equal to 18 . It is not necessarily true that when the product of two numbers is 18 , either one of them is itself 18 . The correct solution looks like this:

$$
\left.\begin{array}{rlrl}
(x-3)(x+4) & =18 \\
x^{2}+x-12 & =18 \\
x^{2}+x-30 & =0 \\
(x+6)(x-5) & =0 \\
x+6=0 & & \text { or } x-5 & =0 \\
x=-6 & & \text { or } & x
\end{array}\right)=5
$$

To avoid this mistake, remember that before you factor a quadratic equation, you must write it in standard form. It is in standard form only when 0 is on one side and decreasing powers of the variable are on the other.

## Chapter 5 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

The numbers in brackets refer to the sections of the text in which similar problems can be found.
Factor the following by factoring out the greatest common factor. [5.1]

1. $10 x-20$
2. $4 x^{3}-9 x^{2}$
3. $5 x-5 y$
4. $7 x^{3}+2 x$
5. $8 x+4$
6. $2 x^{2}+14 x+6$
7. $24 y^{2}-40 y+48$
8. $30 x y^{3}-45 x^{3} y^{2}$
9. $49 a^{3}-14 b^{3}$
10. $6 a b^{2}+18 a^{3} b^{3}-24 a^{2} b$

Factor by grouping. [5.1]
11. $x y+b x+a y+a b$
12. $x y+4 x-5 y-20$
13. $2 x y+10 x-3 y-15$
14. $5 x^{2}-4 a x-10 b x+8 a b$

Factor the following trinomials. [5.2]
15. $y^{2}+9 y+14$
16. $w^{2}+15 w+50$
17. $a^{2}-14 a+48$
18. $r^{2}-18 r+72$
19. $y^{2}+20 y+99$
20. $y^{2}+8 y+12$

Factor the following trinomials. [5.3]
21. $2 x^{2}+13 x+15$
22. $4 y^{2}-12 y+5$
23. $5 y^{2}+11 y+6$
24. $20 a^{2}-27 a+9$
25. $6 r^{2}+5 r t-6 t^{2}$
26. $10 x^{2}-29 x-21$

Factor the following if possible. [5.4]
27. $n^{2}-81$
28. $4 y^{2}-9$
29. $x^{2}+49$
31. $64 a^{2}-121 b^{2}$
30. $36 y^{2}-121 x^{2}$
32. $64-9 m^{2}$

Factor the following. [5.4]
33. $y^{2}+20 y+100$
34. $m^{2}-16 m+64$
35. $64 t^{2}+16 t+1$
36. $16 n^{2}-24 n+9$
37. $4 r^{2}-12 r t+9 t^{2}$
38. $9 m^{2}+30 m n+25 n^{2}$

Factor the following. [5.2]
39. $2 x^{2}+20 x+48$
40. $a^{3}-10 a^{2}+21 a$
41. $3 m^{3}-18 m^{2}-21 m$

Factor the following trinomials. [5.3]
43. $8 x^{2}+16 x+6$
44. $3 a^{3}-14 a^{2}-5 a$
45. $20 m^{3}-34 m^{2}+6 m$
46. $30 x^{2} y-55 x y^{2}+15 y^{3}$

Factor the following. [5.4]
47. $4 x^{2}+40 x+100$
48. $4 x^{3}+12 x^{2}+9 x$
49. $5 x^{2}-45$
50. $12 x^{3}-27 x y^{2}$

Factor the following. [5.5]
51. $8 a^{3}-b^{3}$
52. $27 x^{3}+8 y^{3}$
53. $125 x^{3}-64 y^{3}$

Factor the following polynomials completely. [5.6]
54. $6 a^{3} b+33 a^{2} b^{2}+15 a b^{3}$
55. $x^{5}-x^{3}$
56. $4 y^{6}+9 y^{4}$
57. $12 x^{5}+20 x^{4} y-8 x^{3} y^{2}$
58. $30 a^{4} b+35 a^{3} b^{2}-15 a^{2} b^{3}$
59. $18 a^{3} b^{2}+3 a^{2} b^{3}-6 a b^{4}$

Solve. [5.7]
60. $(x-5)(x+2)=0$
61. $3(2 y+5)(2 y-5)=0$
62. $m^{2}+3 m=10$
63. $a^{2}-49=0$
64. $m^{2}-9 m=0$
65. $6 y^{2}=-13 y-6$
66. $9 x^{4}+9 x^{3}=10 x^{2}$

Solve the following word problems. [5.8]
67. Number Problem The product of two consecutive even integers is 120 . Find the two integers.
68. Number Problem The product of two consecutive integers is 110 . Find the two integers.
69. Number Problem The product of two consecutive odd integers is 1 less than 3 times their sum. Find the integers.
70. Number Problem The sum of two numbers is 20 . Their product is 75 . Find the numbers.
71. Number Problem One number is 1 less than twice another. Their product is 66 . Find the numbers.
72. Geometry The height of a triangle is 8 times the base. The area is 16 square inches. Find the base.

## Chapter 5 Projects

Factoring

## GROUP PROJECT Visual Factoring

## Number of People 2 or 3

## Time Needed 10-15 minutes

Equipment Pencil, graph paper, and scissors
Background When a geometric figure is divided into smaller figures, the area of the original figure and the area of any rearrangement of the smaller figures must be the same. We can use this fact to help visualize some factoring problems.

Procedure Use the diagram below to work the following problems.


1. Write a polynomial involving $x$ that gives the area of the diagram.
2. Factor the polynomial found in Part 1.
3. Copy the figure onto graph paper, then cut along the lines so that you end up with 2 squares and 6 rectangles.
4. Rearrange the pieces from Part 3 to show that the factorization you did in Part 2 is correct.

# Factoring and Internet Security 

T.
he security of the information on computers is directly related to factoring of whole numbers. The key lies in the fact that multiplying whole numbers is a straightforward, simple task, whereas factoring can be very time-consuming. For example, multiplying the numbers 1,234 and 3,433 to obtain $4,236,322$ takes very little time, even if done by hand. But given the number $4,236,322$, finding its factors, even with a calculator or computer, is more than you want to try. The discipline that studies how to make and break codes is cryptography. The current Web browsers, such as Internet Explorer and Netscape, use a system called RSA public-key cryptosystem invented by Adi Shamir of Israel's Weizmann Institute of Science. In 1999 Shamir announced that he had found a method of factoring large numbers quickly that will put the current Internet security system at risk.
Research the connection between computer security and factoring, the RSA cryptosystem, and the current state of security on the Internet, and then write an essay summarizing your results.

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## Rational Expressions

## 6


irst Bank of San Luis Obispo charges $\$ 2.00$ per month and $\$ 0.15$ per check for a regular checking account. If we write $x$ checks in one month, the total monthly cost of the checking account will be $C=2.00+0.15 x$. From this formula we see that the more checks we write in a month, the more we pay for the account. But, it is also true that the more checks we write in a month, the lower the cost per check. To find the average cost per check, we divide the total cost by the number of checks written:

$$
\text { Average cost }=A=\frac{C}{X}=\frac{2.00+0.15 x}{X}
$$

We can use this formula to create Table 1 and Figure 1, giving us a visual interpretation of the relationship between the number of checks written and the average cost per check.

TABLE 1
Average Cost

| Number of <br> Checks | Average Cost <br> Per Check |
| :---: | :---: |
| 1 | 2.15 |
| 2 | 1.15 |
| 5 | 0.55 |
| 10 | 0.35 |
| 15 | 0.28 |
| 20 | 0.25 |



As you can see, if we write one check per month, the cost per check is relatively high, $\$ 2.15$. However, if we write 20 checks per month, each check costs us only $\$ 0.25$. Using average cost per check is a good way to compare different checking accounts. The expression $\frac{2.00+0.15 x}{x}$ in the average cost formula is a rational expression. When you have finished this chapter you will have a good working knowledge of rational expressions.

## CHAPTER OUTLINE

6.1 Reducing Rational Expressions to Lowest Terms
6.2 Multiplication and Division of Rational Expressions
6.3 Addition and Subtraction of Rational Expressions
6.4 Equations Involving Rational Expressions
6.5 Applications
6.6 Complex Fractions
6.7 Proportions

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This is the last chapter in which we will mention study skills. You know by now what works best for you and what you have to do to achieve your goals for this course. From now on, it is simply a matter of sticking with the things that work for you, and avoiding the things that do not work. It seems simple, but as with anything that takes effort, it is up to you to see that you maintain the skills that get you where you want to be in the course.

If you intend to take more classes in mathematics, and you want to ensure your success in those classes, then you can work toward this goal: Become a student who can learn mathematics on your own. Most people who have degrees in mathematics were students who could learn mathematics on their own. This doesn't mean that you have to learn it all on your own; it simply means that if you have to, you can learn it on your own. Attaining this goal gives you independence and puts you in control of your success in any math class you take.

## Reducing Rational Expressions to Lowest Terms

## OBJECTIVES

A Find the restrictions on the variable in a rational expression.

B Reduce a rational expression to lowest terms. ratios.

In Chapter 1 we defined the set of rational numbers to be the set of all numbers that could be put in the form $\frac{a}{b}$, where $a$ and $b$ are integers $(b \neq 0)$ :

$$
\text { Rational numbers }=\left\{\left.\frac{a}{b} \right\rvert\, a \text { and } b \text { are integers, } b \neq 0\right\}
$$

A rational expression is any expression that can be put in the form $\frac{P}{Q}$, where $P$ and $Q$ are polynomials and $Q \neq 0$ :

$$
\text { Rational expressions }=\left\{\left.\frac{P}{Q} \right\rvert\, P \text { and } Q \text { are polynomials, } Q \neq 0\right\}
$$

Each of the following is an example of a rational expression:

$$
\frac{2 x+3}{x} \quad \frac{x^{2}-6 x+9}{x^{2}-4} \quad \frac{5}{x^{2}+6} \quad \frac{2 x^{2}+3 x+4}{2}
$$

For the rational expression

$$
\frac{x^{2}-6 x+9}{x^{2}-4}
$$

the polynomial on top, $x^{2}-6 x+9$, is called the numerator, and the polynomial on the bottom, $x^{2}-4$, is called the denominator. The same is true of the other rational expressions.
We must be careful that we do not use a value of the variable that will give us a denominator of zero. Remember, division by zero is not defined.

## EXAMPLES

State the restrictions on the variable in the following rational expressions:

1. $\frac{x+2}{x-3}$

SOLUTION The variable $x$ can be any real number except $x=3$ since, when $x=3$, the denominator is $3-3=0$. We state this restriction by writing $x \neq 3$.
2. $\frac{5}{x^{2}-x-6}$

SOLUTION If we factor the denominator, we have $x^{2}-x-6=(x-3)(x+2)$. If either of the factors is zero, the whole denominator is zero. Our restrictions are $x \neq 3$ and $x \neq-2$ since either one makes $x^{2}-x-6=0$.

We will not always list each restriction on a rational expression, but we should be aware of them and keep in mind that no rational expression can have a denominator of zero.

The two fundamental properties of rational expressions are listed next. We will use these two properties many times in this chapter.

## Properties of Rational Expressions

## Property 1

Multiplying the numerator and denominator of a rational expression by the same nonzero quantity will not change the value of the rational expression.

## Property 2

Dividing the numerator and denominator of a rational expression by the same nonzero quantity will not change the value of the rational expression.

We can use Property 2 to reduce rational expressions to lowest terms. Since this process is almost identical to the process of reducing fractions to lowest terms, let's recall how the fraction $\frac{6}{15}$ is reduced to lowest terms:

$$
\begin{array}{rlrl}
\frac{6}{15} & =\frac{2 \cdot 3}{5 \cdot 3} & \text { Factor numerator and denominator } \\
& =\frac{2 \cdot \not 3}{5 \cdot 3} & \text { Divide out the common factor, 3 } \\
& =\frac{2}{5} & & \text { Reduce to lowest terms }
\end{array}
$$

The same procedure applies to reducing rational expressions to lowest terms. The process is summarized in the following rule.

Rule To reduce a rational expression to lowest terms, first factor the numerator and denominator completely and then divide both the numerator and denominator by any factors they have in common.

EXAMPLE 3 Reduce $\frac{x^{2}-9}{x^{2}+5 x+6}$ to lowest terms.
SOLUTION We begin by factoring:

$$
\frac{x^{2}-9}{x^{2}+5 x+6}=\frac{(x-3)(x+3)}{(x+2)(x+3)}
$$

Notice that both polynomials contain the factor $(x+3)$. If we divide the numerator by $(x+3)$, we are left with $(x-3)$. If we divide the denominator by $(x+3)$, we are left with $(x+2)$. The complete solution looks like this:

$$
\begin{aligned}
\frac{x^{2}-9}{x^{2}+5 x+6} & =\frac{(x-3)(x-3)}{(x+2)(x-3)} & \begin{array}{l}
\text { Factor the numerator and } \\
\text { denominator completely }
\end{array} \\
& =\frac{x-3}{x+2} & \begin{array}{l}
\text { Divide out the common } \\
\text { factor, } x+3
\end{array}
\end{aligned}
$$

It is convenient to draw a line through the factors as we divide them out. It is especially helpful when the problems become longer.

## EXAMPLE 4 Reduce to lowest terms $\frac{10 a+20}{5 a^{2}-20}$.

SOLUTION We begin by factoring out the greatest common factor from the numerator and denominator:

$$
\begin{array}{rlr}
\frac{10 a+20}{5 a^{2}-20} & =\frac{10(a+2)}{5\left(a^{2}-4\right)} & \begin{array}{l}
\text { Factor the greatest common factor } \\
\text { from the numerator and denominator }
\end{array} \\
& =\frac{10(a+2)}{5(a+2)(a-2)} & \begin{array}{l}
\text { Factor the denominator as the } \\
\text { difference of two squares }
\end{array} \\
& =\frac{2}{a-2} & \begin{array}{l}
\text { Divide out the common } \\
\text { factors 5 and } a+2
\end{array}
\end{array}
$$

EXAMPLE $5 \quad$ Reduce $\frac{2 x^{3}+2 x^{2}-24 x}{x^{3}+2 x^{2}-8 x}$ to lowest terms.
SOLUTION We begin by factoring the numerator and denominator completely. Then we divide out all factors common to the numerator and denominator. Here is what it looks like:

$$
\begin{array}{rlrl}
\frac{2 x^{3}+2 x^{2}-24 x}{x^{3}+2 x^{2}-8 x} & =\frac{2 x\left(x^{2}+x-12\right)}{x\left(x^{2}+2 x-8\right)} & \begin{array}{l}
\text { Factor out the greatest } \\
\text { common factor first }
\end{array} \\
& =\frac{2 x(x-3)(x+4)}{x(x-2)(x+4)} & \text { Factor the remaining trinomials } \\
& =\frac{2(x-3)}{x-2} & & \begin{array}{l}
\text { Divide out the factors common to } \\
\text { the numerator and denominator }
\end{array}
\end{array}
$$

EXAMPLE $6 \quad$ Reduce $\frac{x-5}{x^{2}-25}$ to lowest terms.

## SOLUTION

$$
\begin{aligned}
\frac{x-5}{x^{2}-25} & =\frac{x-5}{(x-5)(x+5)} & & \begin{array}{l}
\text { Factor numerator and denominator } \\
\text { completely }
\end{array} \\
& =\frac{1}{x+5} & & \begin{array}{l}
\text { Divide out the common factor, } \\
x-5
\end{array}
\end{aligned}
$$

## EXAMPLE 7 Reduce $\frac{x^{3}+y^{3}}{x^{2}-y^{2}}$ to lowest terms.

SOLUTION We begin by factoring the numerator and denominator completely. (Remember, we can only reduce to lowest terms when the numerator and denominator are in factored form. Trying to reduce before factoring will only lead to mistakes.)

$$
\begin{aligned}
\frac{x^{3}+y^{3}}{x^{2}-y^{2}} & =\frac{(x+y)\left(x^{2}-x y+y^{2}\right)}{(x+y)(x-y)} & & \text { Factor } \\
& =\frac{x^{2}-x y+y^{2}}{x-y} & & \text { Divide out the common factor }
\end{aligned}
$$

## Note

With ratios it is common to leave the 1 in the denominator.

## Ratios

For the rest of this section we will concern ourselves with ratios, a topic closely related to reducing fractions and rational expressions to lowest terms. Let's start with a definition.

## DEFINITION If $a$ and $b$ are any two numbers, $b \neq 0$, then the ratio of $a$

 and $b$ is$$
\frac{a}{b}
$$

As you can see, ratios are another name for fractions or rational numbers. They are a way of comparing quantities. Since we also can think of $\frac{a}{b}$ as the quotient of $a$ and $b$, ratios are also quotients. The following table gives some ratios in words and as fractions.

| Ratio | As a Fraction | In Lowest Terms |
| :---: | :---: | :---: |
| 25 to 75 | $\frac{25}{75}$ | $\frac{1}{3}$ |
| 8 to 2 | $\frac{8}{2}$ | $\frac{4}{1}$ |
| 20 to 16 | $\frac{20}{16}$ | $\frac{5}{4}$ |

EXAMPLE 8 A solution of hydrochloric acid $(\mathrm{HCl})$ and water contains 49 milliliters of water and 21 milliliters of HCl . Find the ratio of HCl to water and of HCl to the total volume of the solution.

SOLUTION The ratio of HCl to water is 21 to 49 , or

$$
\frac{21}{49}=\frac{3}{7}
$$

The amount of total solution volume is $49+21=70$ milliliters. Therefore, the ratio of HCl to total solution is 21 to 70 , or

$$
\frac{21}{70}=\frac{3}{10}
$$

## Rate Equation

Many of the problems in this chapter will use what is called the rate equation. You use this equation on an intuitive level when you are estimating how long it will take you to drive long distances. For example, if you drive at 50 miles per hour for 2 hours, you will travel 100 miles. Here is the rate equation:

$$
\begin{aligned}
\text { Distance } & =\text { rate } \cdot \text { time } \\
d & =r \cdot t
\end{aligned}
$$

The rate equation has two equivalent forms, the most common of which is obtained by solving for $r$. Here it is:

$$
r=\frac{d}{t}
$$

The rate $r$ in the rate equation is the ratio of distance to time and also is referred to as average speed. The units for rate are miles per hour, feet per second, kilometers per hour, and so on.


## EXAMPLE 9

The Forest chair lift at the Northstar ski resort in Lake Tahoe is 5,603 feet long. If a ride on this chair lift takes 11 minutes, what is the average speed of the lift in feet per minute?

SOLUTION To find the speed of the lift, we find the ratio of distance covered to time. (Our answer is rounded to the nearest whole number.)

$$
\text { Rate }=\frac{\text { distance }}{\text { time }}=\frac{5,603 \text { feet }}{11 \text { minutes }}=\frac{5,603}{11} \text { feet } / \text { minute }=509 \text { feet } / \text { minute }
$$

Note how we separate the numerical part of the problem from the units. In the next section, we will convert this rate to miles per hour.

EXAMPLE 10 A Ferris wheel was built in St. Louis in 1986. It is named Colossus. The circumference of the wheel is 518 feet. It has 40 cars, each of which holds 6 passengers. A trip around the wheel takes 40 seconds. Find the average speed of a rider on Colossus.

Solution To find the average speed, we divide the distance traveled, which in this case is the circumference, by the time it takes to
 travel once around the wheel.

$$
r=\frac{d}{t}=\frac{518 \text { feet }}{40 \text { seconds }}=13.0(\text { rounded })
$$

The average speed of a rider on the Colossus is 13.0 feet per second.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,2
B 3-7
C ${ }^{8-10}$
In the next section, you will convert the ratio into an equivalent ratio that gives the speed of the rider in miles per hour.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How do you reduce a rational expression to lowest terms?
2. What are the properties we use to manipulate rational expressions?
3. For what values of the variable is a rational expression undefined?
4. What is a ratio?
5. Simplify each expression.
a. $\frac{5+1}{25-1}$
b. $\frac{x+1}{x^{2}-1}$
c. $\frac{x^{2}-x}{x^{2}-1}$
d. $\frac{x^{3}-1}{x^{2}-1}$
e. $\frac{x^{3}-1}{x^{3}-x^{2}}$
6. Simplify each expression.
a. $\frac{25-30+9}{25-9}$
b. $\frac{x^{2}-6 x+9}{x^{2}-9}$
c. $\frac{x^{2}-10 x+9}{x^{2}-9 x}$
d. $\frac{x^{2}+3 x+a x+3 a}{x^{2}-9}$
e. $\frac{x^{3}+27}{x^{3}-9 x}$

Reduce the following rational expressions to lowest terms, if possible. Also, specify any restrictions on the variable in Problems 1 through 10.
3. $\frac{a-3}{a^{2}-9}$
4. $\frac{a+4}{a^{2}-16}$
5. $\frac{x+5}{x^{2}-25}$
6. $\frac{x-2}{x^{2}-4}$
7. $\frac{2 x^{2}-8}{4}$
8. $\frac{5 x-10}{x-2}$
9. $\frac{2 x-10}{3 x-6}$
10. $\frac{4 x-8}{x-2}$
11. $\frac{10 a+20}{5 a+10}$
12. $\frac{11 a+33}{6 a+18}$
13. $\frac{5 x^{2}-5}{4 x+4}$
14. $\frac{7 x^{2}-28}{2 x+4}$
15. $\frac{x-3}{x^{2}-6 x+9}$
16. $\frac{x^{2}-10 x+25}{x-5}$
17. $\frac{3 x+15}{3 x^{2}+24 x+45}$
18. $\frac{5 x+15}{5 x^{2}+40 x+75}$
19. $\frac{a^{2}-3 a}{a^{3}-8 a^{2}+15 a}$
20. $\frac{a^{2}+3 a}{a^{3}-2 a^{2}-15 a}$
21. $\frac{3 x-2}{9 x^{2}-4}$
22. $\frac{2 x-3}{4 x^{2}-9}$
23. $\frac{x^{2}+8 x+15}{x^{2}+5 x+6}$
24. $\frac{x^{2}-8 x+15}{x^{2}-x-6}$
25. $\frac{2 m^{3}-2 m^{2}-12 m}{m^{2}-5 m+6}$
26. $\frac{2 m^{3}+4 m^{2}-6 m}{m^{2}-m-12}$
27. $\frac{x^{3}+3 x^{2}-4 x}{x^{3}-16 x}$
28. $\frac{3 a^{2}-8 a+4}{9 a^{3}-4 a}$
29. $\frac{4 x^{3}-10 x^{2}+6 x}{2 x^{3}+x^{2}-3 x}$
30. $\frac{3 a^{3}-8 a^{2}+5 a}{4 a^{3}-5 a^{2}+1 a}$
31. $\frac{4 x^{2}-12 x+9}{4 x^{2}-9}$
32. $\frac{5 x^{2}+18 x-8}{5 x^{2}+13 x-6}$
33. $\frac{x+3}{x^{4}-81}$
34. $\frac{x^{2}+9}{x^{4}-81}$
35. $\frac{3 x^{2}+x-10}{x^{4}-16}$
36. $\frac{5 x^{2}-26 x+24}{x^{4}-64}$
37. $\frac{42 x^{3}-20 x^{2}-48 x}{6 x^{2}-5 x-4}$
38. $\frac{36 x^{3}+132 x^{2}-135 x}{6 x^{2}+25 x-9}$
39. $\frac{x^{3}-y^{3}}{x^{2}-y^{2}}$
40. $\frac{x^{3}+y^{3}}{x^{2}-y^{2}}$
41. $\frac{x^{3}+8}{x^{2}-4}$
42. $\frac{x^{3}-125}{x^{2}-25}$
43. $\frac{x^{3}+8}{x^{2}+x-2}$
44. $\frac{x^{2}-2 x-3}{x^{3}-27}$

To reduce each of the following rational expressions to lowest terms, you will have to use factoring by grouping. Be sure to factor each numerator and denominator completely before dividing out any common factors. (Remember, factoring by grouping takes two steps.)
45. $\frac{x y+3 x+2 y+6}{x y+3 x+5 y+15}$
46. $\frac{x y+7 x+4 y+28}{x y+3 x+4 y+12}$
47. $\frac{x^{2}-3 x+a x-3 a}{x^{2}-3 x+b x-3 b}$
48. $\frac{x^{2}-6 x+a x-6 a}{x^{2}-7 x+a x-7 a}$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working. Working these problems is an excellent way to get ready for a test or quiz
49. Work each problem according to the instructions given.
a. Add: $\left(x^{2}-4 x\right)+(4 x-16)$
b. Subtract: $\left(x^{2}-4 x\right)-(4 x-16)$
c. Multiply: $\left(x^{2}-4 x\right)(4 x-16)$
d. Reduce: $\frac{x^{2}-4 x}{4 x-16}$
50. Work each problem according to the instructions given.
a. Add: $\left(9 x^{2}-3 x\right)+(6 x-2)$
b. Subtract: $\left(9 x^{2}-3 x\right)-(6 x-2)$
c. Multiply: $\left(9 x^{2}-3 x\right)(6 x-2)$
d. Reduce: $\frac{9 x^{2}-3 x}{6 x-2}$

Write each ratio as a fraction in lowest terms.
51. 8 to 6
52. 6 to 8
53. 200 to 250
54. 250 to 200
55. 32 to 4
56. 4 to 32

## Applying the Concepts

57. Cost and Average Cost As we mentioned in the introduction to this chapter, if a bank charges $\$ 2.00$ per month and $\$ 0.15$ per check for one of its checking accounts, then the total monthly cost to write $x$ checks is $C=2.00+0.15 x$, and the average cost of each of the $x$ checks written is $A=\frac{2.00+0.15 x}{x}$. Compare these two formulas by filling in the following table. Round to the nearest cent.

| Checks <br> Written <br> $\boldsymbol{x}$ | Total Cost <br> $\mathbf{2 . 0 0}+\mathbf{0 . 1 5 x}$ | Cost per Check <br> $\mathbf{2 . 0 0 + \mathbf { 0 . 1 5 x }}$ |
| :---: | :---: | :---: |
| 0 |  |  |
| 5 |  |  |
| 10 |  |  |
| 15 |  |  |
| 20 |  |  |

58. Cost and Average Cost A rewritable CD-ROM drive for a computer costs $\$ 250$. An individual CD for the drive costs $\$ 5.00$ and can store 640 megabytes of information. The total cost of filling $x$ CDs with information is $C=250+5 x$ dollars. The average cost per megabyte of information is given by $A=\frac{5 x+250}{640 x}$. Compare the total cost and average cost per megabyte of storage by completing the following table. Round all answers to the nearest tenth of a cent.

| CDs <br> Purchased <br> $\boldsymbol{x}$ | Total Cost <br> $\mathbf{2 5 0 + 5 x}$ | Cost per Megabyte <br> $\frac{5 x+250}{640 x}$ |
| :---: | :---: | :---: |
| 0 |  |  |
| 5 |  |  |
| 10 |  |  |
| 15 |  |  |
| 20 |  |  |

59. Speed of a Car A car travels 122 miles in 3 hours. Find the average speed of the car in miles per hour. Round to the nearest tenth.
60. Speed of a Bullet A bullet fired from a gun travels a distance of 4,500 feet in 3 seconds. Find the average speed of the bullet in feet per second.
61. Ferris Wheel The first Ferris wheel was designed and built by George Ferris in 1893. It was a large wheel with a circumference of 785 feet. If one trip around the circumference of the wheel took 20 minutes, find the average speed of a rider in feet per minute.
62. Ferris Wheel In 1897 a large Ferris wheel was built in Vienna; it is still in operation today. Known as The Great Wheel, it has a circumference of 618 feet. If one trip around the wheel takes 15 minutes, find the average speed of a rider on this wheel in feet per minute.
63. Baseball The chart shown here appeared in USA Today in 2005. For the four pitchers mentioned in the chart, calculate the number of strikeouts per inning. Round to the nearest hundredth.


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64. Ferris Wheel A person riding a Ferris wheel travels once around the wheel, a distance of 188 feet, in 30 seconds. What is the average speed of the rider in feet per second? Round to the nearest tenth.

65. Average Speed Tina is training for a biathlon. As part of her training, she runs an 8-mile course, 2 miles of which is on level ground and 6 miles of which is downhill. It takes her 20 minutes to run the level part of the course and 40 minutes to run the downhill
part of the course. Find her average speed in minutes per mile and in miles per minute for each part of the course. Round to the nearest hundredth, if rounding is necessary.
66. Jogging A jogger covers a distance of 3 miles in 24 minutes. Find the average speed of the jogger in miles per minute.

67. Fuel Consumption An economy car travels 168 miles on 3.5 gallons of gas. Give the average fuel consumption of the car in miles per gallon.
68. Fuel Consumption A luxury car travels 100 miles on 8 gallons of gas. Give the average fuel consumption of the car in miles per gallon.
69. Comparing Expressions Replace $x$ with 5 and $y$ with 4 in the expression

$$
\frac{x^{2}-y^{2}}{x-y}
$$

and simplify the result. Is the result equal to $5-4$ or $5+4$ ?
70. Comparing Expressions Replace $x$ with 2 in the expression

$$
\frac{x^{3}-1}{x-1}
$$

and simplify the result. Your answer should be equal to what you would get if you replaced $x$ with 2 in $x^{2}+x+1$.
71. Comparing Expressions Complete the following table; then show why the table turns out as it does.

72. Comparing Expressions Complete the following table; then show why the table turns out as it does.

| $\boldsymbol{x}$ | $\frac{25-x^{2}}{x^{2}-25}$ |
| :--- | :--- |
| -4 |  |
| -2 |  |
| 0 |  |
| 2 |  |
| 4 |  |

73. Comparing Expressions You know from reading through Example 6 in this section that $\frac{x-5}{x^{2}-25}=\frac{1}{x+5}$. Compare these expressions by completing the following table. (Be careful-not all the rows have equal entries.)

| $\boldsymbol{x}$ | $\frac{x-5}{x^{2}-25}$ | $\frac{1}{x+5}$ |
| :--- | :--- | :--- |
| 0 |  |  |
| 2 |  |  |
| -2 |  |  |
| 5 |  |  |
| -5 |  |  |

74. Comparing Expressions You know from your work in this section that $\frac{x^{2}-6 x+9}{x^{2}-9}=\frac{x-3}{x+3}$. Compare these expressions by completing the following table. (Be careful-not all the rows have equal entries.)

| $x$ | $\frac{x^{2}-6 x+9}{x^{2}-9}$ | $\frac{x-3}{x+3}$ |
| :--- | :--- | :--- |
| -3 |  |  |
| -2 |  |  |
| -1 |  |  |
| 0 |  |  |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |

75. Stock Market One method of comparing stocks on the stock market is the price to earnings ratio, or P/E.

$$
\mathrm{P} / \mathrm{E}=\frac{\text { Current Stock Price }}{\text { Earnings per Share }}
$$

Most stocks have a P/E between 25 and 40. A stock with a P/E of less than 25 may be undervalued, while a stock with a P/E greater than 40 may be overvalued. Fill in the P/E for each stock listed in the table below. Based on your results, are any of the stocks undervalued?

| Stock | Price | Earnings per Share | P/E |
| :--- | :---: | :---: | :---: |
| Yahoo | 37.80 | 1.07 |  |
| Google | 381.24 | 4.51 |  |
| Disney | 24.96 | 1.34 |  |
| Nike | 85.46 | 4.88 |  |
| Ebay | 40.96 | 0.73 |  |

76. Improving Your Quantitative Literacy Each of the numbers in the chart is a rate with units of students per computer (students/computer). Use the information in the chart to answer the following questions.
a. For each of the categories in the chart, find the rate of computers per student. Round to the nearest hundredth.
b. If a senior high actually had 3.2 students per computer, and students had access to 550 computers, how many students would be enrolled in the school?
c. If an elementary school actually had 4.2 students per computer, and the total enrollment in the school was 651 students, how many computers would the students have access to?


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## Maintaining Your Skills

Simplify.
77. $\frac{27 x^{5}}{9 x^{2}}-\frac{45 x^{8}}{15 x^{5}}$
78. $\frac{36 x^{9}}{4 x}-\frac{45 x^{3}}{5 x^{-5}}$
79. $\frac{72 a^{3} b^{7}}{9 a b^{5}}+\frac{64 a^{5} b^{3}}{8 a^{3} b}$
80. $\frac{80 a^{5} b^{11}}{10 a^{2} b}+\frac{33 a^{6} b^{12}}{11 a^{3} b^{2}}$

Divide.
81. $\frac{38 x^{7}+42 x^{5}-84 x^{3}}{2 x^{3}}$
82. $\frac{49 x^{6}-63 x^{4}-35 x^{2}}{7 x^{2}}$
83. $\frac{28 a^{5} b^{5}+36 a b^{4}-44 a^{4} b}{4 a b}$
84. $\frac{30 a^{3} b-12 a^{2} b^{2}+6 a b^{3}}{6 a b}$

## Getting Ready for the Next Section

Perform the indicated operation.
85. $\frac{3}{4} \cdot \frac{10}{21}$
86. $\frac{2}{9} \cdot \frac{15}{22}$
87. $\frac{4}{5} \div \frac{8}{9}$
88. $\frac{3}{5} \div \frac{15}{7}$

Factor completely.
89. $x^{2}-9$
90. $x^{2}-25$
91. $3 x-9$
92. $2 x-4$
93. $x^{2}-x-20$
94. $x^{2}+7 x+12$
95. $a^{2}+5 a$
96. $a^{2}-4 a$

Reduce to lowest terms.
97. $\frac{a(a+5)(a-5)(a+4)}{a^{2}+5 a}$
98. $\frac{a(a+2)(a-4)(a+5)}{a^{2}-4 a}$

Multiply. Give the answers as decimals rounded to the nearest tenth.
99. $\frac{5,603}{11} \cdot \frac{1}{5,280} \cdot \frac{60}{1}$
100. $\frac{772}{2.2} \cdot \frac{1}{5,280} \cdot \frac{60}{1}$

## 6.2

## OBJECTIVES

A Multiply and divide rational expressions by factoring and then dividing out common factors.

B Convert between units using unit analysis.

## Multiplication and Division of Rational Expressions

Recall that to multiply two fractions we simply multiply numerators and multiply denominators and then reduce to lowest terms, if possible:

$$
\begin{aligned}
\frac{3}{4} \cdot \frac{10}{21} & =\frac{30}{84} & & \leftarrow \text { Multiply numerators } \\
& =\frac{5}{14} & & \leftarrow \text { Multiply denominators }
\end{aligned}
$$

Recall also that the same result can be achieved by factoring numerators and denominators first and then dividing out the factors they have in common:

$$
\begin{array}{rlrl}
\frac{3}{4} \cdot \frac{10}{21} & =\frac{3}{2 \cdot 2} \cdot \frac{2 \cdot 5}{3 \cdot 7} & & \text { Factor } \\
& =\frac{3 \cdot 2 \cdot 5}{2 \cdot 2 \cdot 3 \cdot 7} & & \text { Multiply numerators } \\
& =\frac{5}{14} & & \text { Multiply denominators } \\
& \text { Divide out common factors }
\end{array}
$$

We can apply the second process to the product of two rational expressions, as the following example illustrates.

## EXAMPLE 1 Multiply $\frac{x-2}{x+3} \cdot \frac{x^{2}-9}{2 x-4}$.

SOLUTION We begin by factoring numerators and denominators as much as possible. Then we multiply the numerators and denominators. The last step consists of dividing out all factors common to the numerator and denominator:

$$
\begin{aligned}
\frac{x-2}{x+3} \cdot \frac{x^{2}-9}{2 x-4} & =\frac{x-2}{x+3} \cdot \frac{(x-3)(x+3)}{2(x-2)} & & \text { Factor completely } \\
& =\frac{(x-2)(x-3)(x+3)}{(x+3)(2)(x-2)} & & \begin{array}{l}
\text { Multiply numerators } \\
\text { and denominators }
\end{array} \\
& =\frac{x-3}{2} & & \text { Divide out common factors }
\end{aligned}
$$

In Chapter 1 we defined division as the equivalent of multiplication by the reciprocal. This is how it looks with fractions:

$$
\left.\begin{array}{rl}
\frac{4}{5} \div \frac{8}{9} & =\frac{4}{5} \cdot \frac{9}{8} \quad \text { Division as multiplication by the reciprocal } \\
& =\frac{2 \cdot 2 \cdot 2 \cdot 3 \cdot 3}{5 \cdot 2 \cdot 2 \cdot 2} \\
& =\frac{9}{10}
\end{array}\right\} \text { Factor and divide out common factors }
$$

The same idea holds for division with rational expressions. The rational expression that follows the division symbol is called the divisor; to divide, we multiply by the reciprocal of the divisor.

EXAMPLE $2 \quad$ Divide $\frac{3 x-9}{x^{2}-x-20} \div \frac{x^{2}+2 x-15}{x^{2}-25}$.
SOLUTION We begin by taking the reciprocal of the divisor and writing the problem again in terms of multiplication. We then factor, multiply, and, finally, divide out all factors common to the numerator and denominator of the resulting expression. The complete solution looks like this:

$$
\begin{aligned}
\frac{3 x-9}{x^{2}-x-20} \div \frac{x^{2}+2 x-15}{x^{2}-25} & \\
=\frac{3 x-9}{x^{2}-x-20} \cdot \frac{x^{2}-25}{x^{2}+2 x-15} & \begin{array}{l}
\text { Multiply by the reciprocal of } \\
\text { the divisor }
\end{array} \\
=\frac{3(x-3)}{(x+4)(x-5)} \cdot \frac{(x-5)(x+5)}{(x+5)(x-3)} & \text { Factor } \\
=\frac{3(x-3)(x-5)(x+5)}{(x+4)(x-5)(x-5)(x-3)} & \text { Multiply } \\
=\frac{3}{x+4} & \text { Divide out common factors }
\end{aligned}
$$

As you can see, factoring is the single most important tool we use in working with rational expressions. Most of the work we have done or will do with rational expressions is accomplished most easily if the rational expressions are in factored form. Here are some more examples of multiplication and division with rational expressions.

## EXAMPLES

3. Multiply $\frac{3 a+6}{a^{2}} \cdot \frac{a}{2 a+4}$.

## SOLUTION

$$
\begin{array}{rlrl}
\frac{3 a+6}{a^{2}} \cdot \frac{a}{2 a+4} & & \\
& =\frac{3(a+2)}{a^{2}} \cdot \frac{a}{2(a+2)} & & \text { Factor completely } \\
& =\frac{3(a+2) d}{a^{2}(2)(a+2)} & & \text { Multiply } \\
& =\frac{3}{2 a} & & \begin{array}{l}
\text { Divide numerator and denom } \\
\text { by common factors } a(a+2)
\end{array}
\end{array}
$$

4. Divide $\frac{x^{2}+7 x+12}{x^{2}-16} \div \frac{x^{2}+6 x+9}{2 x-8}$.

## SOLUTION

$$
\begin{array}{rlr}
\frac{x^{2}+7 x+12}{x^{2}-16} \div \frac{x^{2}+6 x+9}{2 x-8} & \\
& =\frac{x^{2}+7 x+12}{x^{2}-16} \cdot \frac{2 x-8}{x^{2}+6 x+9} & \begin{array}{l}
\text { Division is multiplication } \\
\text { by the reciprocal }
\end{array} \\
& =\frac{(x+3)(x+4)(2)(x-4)}{(x-4)(x+4)(x)(x+3)} & \\
& =\frac{2}{x+3} & \text { Factor and multiply }
\end{array}
$$

In Example 4 we factored and multiplied the two expressions in a single step. This saves writing the problem one extra time.

## EXAMPLE 5 Multiply $\left(x^{2}-49\right)\left(\frac{x+4}{x+7}\right)$.

SOLUTION We can think of the polynomial $x^{2}-49$ as having a denominator of 1 . Thinking of $x^{2}-49$ in this way allows us to proceed as we did in previous examples:

$$
\begin{aligned}
\left(x^{2}-49\right)\left(\frac{x+4}{x+7}\right) & =\frac{x^{2}-49}{1} \cdot \frac{x+4}{x+7} & & \begin{array}{l}
\text { Write } x^{2}-49 \text { with } \\
\text { denominator } 1
\end{array} \\
& =\frac{(x+7)(x-7)(x+4)}{x+7} & & \text { Factor and multiply } \\
& =(x-7)(x+4) & & \text { Divide out common factors }
\end{aligned}
$$

We can leave the answer in this form or multiply to get $x^{2}-3 x-28$. In this section let's agree to leave our answers in factored form.

## EXAMPLE $6 \quad$ Multiply $a(a+5)(a-5)\left(\frac{a+4}{a^{2}+5 a}\right)$.

SOLUTION We can think of the expression $a(a+5)(a-5)$ as having a denominator of 1 :

$$
\begin{array}{rlr}
a(a+5)(a-5)\left(\frac{a+4}{a^{2}+5 a}\right) & \\
& =\frac{a(a+5)(a-5)}{1} \cdot \frac{a+4}{a^{2}+5 a} & \\
& =\frac{\not a(a+5)(a-5)(a+4)}{\not a(a+5)} & \text { Factor and multiply } \\
& =(a-5)(a+4) &
\end{array}
$$

## Unit Analysis

Unit analysis is a method of converting between units of measure by multiplying by the number 1. Here is our first illustration: Suppose you are flying in a commercial airliner and the pilot tells you the plane has reached its cruising altitude of 35,000 feet. How many miles is the plane above the ground?
If you know that 1 mile is 5,280 feet, then it is simply a matter of deciding what to do with the two numbers, 5,280 and 35,000. By using unit analysis, this decision is unnecessary:

$$
35,000 \text { feet }=\frac{35,000 \text { feet }}{1} \cdot \frac{1 \text { mile }}{5,280 \text { feet }}
$$

We treat the units common to the numerator and denominator in the same way we treat factors common to the numerator and denominator; common units can be divided out, just as common factors are. In the previous expression, we have feet common to the numerator and denominator. Dividing them out leaves us with miles only. Here is the complete solution:

$$
\begin{aligned}
35,000 \text { feet } & =\frac{35,000 \text { feet }}{1} \cdot \frac{1 \text { mile }}{5,280 \text { feet }} \\
& =\frac{35,000}{5,280} \text { miles } \\
& =6.6 \text { miles to the nearest tenth of a mile }
\end{aligned}
$$

The expression $\frac{1 \text { mile }}{5,280 \text { feet }}$ is called a conversion factor. It is simply the number 1 written in a convenient form. Because it is the number 1 , we can multiply any other number by it and always be sure we have not changed that number. The key to unit analysis is choosing the right conversion factors.

SOLUTION We are starting with acres and want to end up with square miles. We need to multiply by a conversion factor that will allow acres to divide out and leave us with square miles:

$$
\begin{aligned}
78 \text { acres } & =\frac{78 \text { acres }}{1} \cdot \frac{1 \text { square mile }}{640 \text { acres }} \\
& =\frac{78}{640} \text { square miles } \\
& =0.12 \text { square miles to the nearest hundredth }
\end{aligned}
$$



The next example is a continuation of Example 9 from Section 6.1.

## EXAMPLE 8

 The Forest chair lift at the Northstar ski resort in Lake Tahoe is 5,603 feet long. If a ride on this chair lift takes 11 minutes, what is the average speed of the lift in miles per hour?SOLUTION First, we find the speed of the lift in feet per minute, as we did in Example 9 of Section 6.1, by taking the ratio of distance to time.

$$
\begin{aligned}
\text { Rate } & =\frac{\text { distance }}{\text { time }}=\frac{5,603 \text { feet }}{11 \text { minutes }}=\frac{5,603}{11} \text { feet per minute } \\
& =509 \text { feet per minute }
\end{aligned}
$$

Next, we convert feet per minute to miles per hour. To do this, we need to know that

$$
\begin{aligned}
& 1 \text { mile }=5,280 \text { feet } \\
& 1 \text { hour }=60 \text { minutes } \\
& \begin{aligned}
\text { Speed } & =509 \text { feet per minute }=\frac{509 \text { feet }}{1 \text { minute }} \cdot \frac{1 \text { mile }}{5,280 \text { feet }} \cdot \frac{60 \text { minutes }}{1 \text { hour }} \\
& =\frac{509 \cdot 60}{5,280} \text { miles per hour } \\
& =5.8 \text { miles per hour to the nearest tenth }
\end{aligned}
\end{aligned}
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How do we multiply rational expressions?
2. Explain the steps used to divide rational expressions.
3. What part does factoring play in multiplying and dividing rational expressions?
4. Why are all conversion factors the same as the number 1?

Multiply or divide as indicated. Be sure to reduce all answers to lowest terms. (The numerator and denominator of the answer should not have any factors in common.)

1. $\frac{x+y}{3} \cdot \frac{6}{x+y}$
2. $\frac{x-1}{x+1} \cdot \frac{5}{x-1}$
3. $\frac{2 x+10}{x^{2}} \cdot \frac{x^{3}}{4 x+20}$
4. $\frac{3 x^{4}}{3 x-6} \cdot \frac{x-2}{x^{2}}$
5. $\frac{9}{2 a-8} \div \frac{3}{a-4}$
6. $\frac{8}{a^{2}-25} \div \frac{16}{a+5}$
7. $\frac{x+1}{x^{2}-9} \div \frac{2 x+2}{x+3}$
8. $\frac{11}{x-2} \div \frac{22}{2 x^{2}-8}$
9. $\frac{a^{2}+5 a}{7 a} \cdot \frac{4 a^{2}}{a^{2}+4 a}$
10. $\frac{4 a^{2}+4 a}{a^{2}-25} \cdot \frac{a^{2}-5 a}{8 a}$
11. $\frac{y^{2}-5 y+6}{2 y+4} \div \frac{2 y-6}{y+2}$
12. $\frac{y^{2}-7 y}{3 y^{2}-48} \div \frac{y^{2}-9}{y^{2}-7 y+12}$
13. $\frac{2 x-8}{x^{2}-4} \cdot \frac{x^{2}+6 x+8}{x-4}$
14. $\frac{x^{2}+5 x+1}{7 x-7} \cdot \frac{x-1}{x^{2}+5 x+1}$
15. $\frac{x-1}{x^{2}-x-6} \cdot \frac{x^{2}+5 x+6}{x^{2}-1}$
16. $\frac{x^{2}-3 x-10}{x^{2}-4 x+3} \cdot \frac{x^{2}-5 x+6}{x^{2}-3 x-10}$
17. $\frac{a^{2}+10 a+25}{a+5} \div \frac{a^{2}-25}{a-5}$
18. $\frac{a^{2}+a-2}{a^{2}+5 a+6} \div \frac{a-1}{a}$
19. $\frac{y^{3}-5 y^{2}}{y^{4}+3 y^{3}+2 y^{2}} \div \frac{y^{2}-5 y+6}{y^{2}-2 y-3}$
20. $\frac{y^{2}-5 y}{y^{2}+7 y+12} \div \frac{y^{3}-7 y^{2}+10 y}{y^{2}+9 y+18}$
21. $\frac{2 x^{2}+17 x+21}{x^{2}+2 x-35} \cdot \frac{x^{3}-125}{2 x^{2}-7 x-15}$
22. $\frac{x^{2}+x-42}{4 x^{2}+31 x+21} \cdot \frac{4 x^{2}-5 x-6}{x^{3}-8}$
23. $\frac{2 x^{2}+10 x+12}{4 x^{2}+24 x+32} \cdot \frac{2 x^{2}+18 x+40}{x^{2}+8 x+15}$
24. $\frac{3 x^{2}-3}{6 x^{2}+18 x+12} \cdot \frac{2 x^{2}-8}{x^{2}-3 x+2}$
25. $\frac{2 a^{2}+7 a+3}{a^{2}-16} \div \frac{4 a^{2}+8 a+3}{2 a^{2}-5 a-12}$
26. $\frac{3 a^{2}+7 a-20}{a^{2}+3 a-4} \div \frac{3 a^{2}-2 a-5}{a^{2}-2 a+1}$
27. $\frac{4 y^{2}-12 y+9}{y^{2}-36} \div \frac{2 y^{2}-5 y+3}{y^{2}+5 y-6}$
28. $\frac{5 y^{2}-6 y+1}{y^{2}-1} \div \frac{16 y^{2}-9}{4 y^{2}+7 y+3}$
29. $\frac{x^{2}-1}{6 x^{2}+42 x+60} \cdot \frac{7 x^{2}+17 x+6}{x^{3}+1} \cdot \frac{6 x+30}{7 x^{2}-11 x-6}$
30. $\frac{4 x^{2}-1}{3 x-15} \cdot \frac{4 x^{2}-17 x-15}{4 x^{2}-9 x-9} \cdot \frac{3 x-9}{8 x^{3}-1}$
31. $\frac{18 x^{3}+21 x^{2}-60 x}{21 x^{2}-25 x-4} \cdot \frac{28 x^{2}-17 x-3}{16 x^{3}+28 x^{2}-30 x}$
32. $\frac{56 x^{3}+54 x^{2}-20 x}{8 x^{2}-2 x-15} \cdot \frac{6 x^{2}+5 x-21}{63 x^{3}+129 x^{2}-42 x}$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working. Working these problems is an excellent way to get ready for a test or quiz.
33. Work each problem according to the instructions given.
a. Simplify: $\frac{9-1}{27-1}$
b. Reduce: $\frac{x^{2}-1}{x^{3}-1}$
c. Multiply: $\frac{x^{2}-1}{x^{3}-1} \cdot \frac{x-1}{x+1}$
d. Divide: $\frac{x^{2}-1}{x^{3}-1} \div \frac{x-1}{x^{2}+x+1}$
34. Work each problem according to the instructions given.
a. Simplify: $\frac{16-9}{16+24+9}$
b. Reduce: $\frac{4 x^{2}-9}{4 x^{2}+12 x+9}$
c. Multiply: $\frac{4 x^{2}-9}{4 x^{2}+12 x+9} \cdot \frac{2 x+3}{2 x-3}$
d. Divide: $\frac{4 x^{2}-9}{4 x^{2}+12 x+9} \div \frac{2 x+3}{2 x-3}$

Multiply the following expressions using the method shown in Examples 5 and 6 in this section.
35. $\left(x^{2}-9\right)\left(\frac{2}{x+3}\right)$
36. $\left(x^{2}-9\right)\left(\frac{-3}{x-3}\right)$
37. $\left(x^{2}-x-6\right)\left(\frac{x+1}{x-3}\right)$
38. $\left(x^{2}-2 x-8\right)\left(\frac{x+3}{x-4}\right)$
39. $\left(x^{2}-4 x-5\right)\left(\frac{-2 x}{x+1}\right)$
40. $\left(x^{2}-6 x+8\right)\left(\frac{4 x}{x-2}\right)$

Each of the following problems involves some factoring by grouping. Remember, before you can divide out factors common to the numerators and denominators of a product, you must factor completely.
41. $\frac{x^{2}-9}{x^{2}-3 x} \cdot \frac{2 x+10}{x y+5 x+3 y+15}$
42. $\frac{x^{2}-16}{x^{2}-4 x} \cdot \frac{3 x+18}{x y+6 x+4 y+24}$
43. $\frac{2 x^{2}+4 x}{x^{2}-y^{2}} \cdot \frac{x^{2}+3 x+x y+3 y}{x^{2}+5 x+6}$
44. $\frac{x^{2}-25}{3 x^{2}+3 x y} \cdot \frac{x^{2}+4 x+x y+4 y}{x^{2}+9 x+20}$
45. $\frac{x^{3}-3 x^{2}+4 x-12}{x^{4}-16} \cdot \frac{3 x^{2}+5 x-2}{3 x^{2}-10 x+3}$
46. $\frac{x^{3}-5 x^{2}+9 x-45}{x^{4}-81} \cdot \frac{5 x^{2}+18 x+9}{5 x^{2}-22 x-15}$

Simplify each expression. Work inside parentheses first, and then divide out common factors.
47. $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)$
48. $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)$

The dots in the following problems represent factors not written that are in the same pattern as the surrounding factors. Simplify.
49. $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right) \ldots\left(1-\frac{1}{99}\right)\left(1-\frac{1}{100}\right)$
50. $\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right) \ldots\left(1-\frac{1}{98}\right)\left(1-\frac{1}{99}\right)$

## Applying the Concepts

51. Mount Whitney The top of Mount Whitney, the highest point in California, is 14,494 feet above sea level. Give this height in miles to the nearest tenth of a mile.
52. Motor Displacement The relationship between liters and cubic inches, both of which are measures of volume, is 0.0164 liters $=1$ cubic inch. If a Ford Mustang has a motor with a displacement of 4.9 liters, what is the displacement in cubic inches? Round your answer to the nearest cubic inch.
53. Speed of Sound The speed of sound is 1,088 feet per second. Convert the speed of sound to miles per hour. Round your answer to the nearest whole number.
54. Average Speed A car travels 122 miles in 3 hours. Find the average speed of the car in feet per second. Round to the nearest whole number.
55. Ferris Wheel As we mentioned in Problem Set 6.1, the first Ferris wheel was built in 1893. It was a large wheel with a circumference of 785 feet. If one trip around the circumference of the wheel took 20 minutes, find the average speed of a rider in miles per hour. Round to the nearest hundredth.

56. Unit Analysis The photograph shows the Cone Nebula as seen by the Hubble telescope in April 2002. The distance across the photograph is about 2.5 lightyears. If we assume light travels 186,000 miles in 1 second, we can find the number of miles in 1 lightyear by converting 186,000 miles/second to miles/year. Find the number of miles in 1 light-year. Write your answer in expanded form and in scientific notation.

57. Ferris Wheel A Ferris wheel called Colossus has a circumference of 518 feet. If a trip around the circumference of Colossus takes 40 seconds, find the average speed of a rider in miles per hour. Round to the nearest tenth.
58. Average Speed Tina is training for a biathlon. As part of her training, she runs an 8-mile course, 2 miles of which is on level ground and 6 miles of which is downhill. It takes her 20 minutes to run the level part of the course and 40 minutes to run the downhill part of the course. Find her average speed in miles per hour for each part of the course.
59. Running Speed Have you ever wondered about the miles per hour speed of a fast runner? This problem will give you some insight into this question.


From USA Today. Copyright 2004. Reprinted with permission.
a. Calculate the average speed in kilometers per hour for Paul Koech.
b. If a 10 K race is approximately 6.2 miles, convert the answer from part a to miles per hour.
60. Improving Your Quantitative Literacy The guidelines for fitness now indicate that a person who walks 10,000 steps daily is physically fit. According to The Walking Site on the Internet, "The average person's stride length is approximately 2.5 feet long. That means it

takes just over 2,000 steps to walk one mile, and 10,000 steps is close to 5 miles." Use your knowledge of unit analysis to determine if these facts are correct.

## Maintaining Your Skills

Add the following fractions.
61. $\frac{1}{2}+\frac{5}{2}$
62. $\frac{2}{3}+\frac{8}{3}$
63. $2+\frac{3}{4}$
64. $1+\frac{4}{7}$

Simplify each term, then add.
65. $\frac{10 x^{4}}{2 x^{2}}+\frac{12 x^{6}}{3 x^{4}}$
66. $\frac{32 x^{8}}{8 x^{3}}+\frac{27 x^{7}}{3 x^{2}}$
67. $\frac{12 a^{2} b^{5}}{3 a b^{3}}+\frac{14 a^{4} b^{7}}{7 a^{3} b^{5}}$
68. $\frac{16 a^{3} b^{2}}{4 a b}+\frac{25 a^{6} b^{5}}{5 a^{4} b^{4}}$

## Getting Ready for the Next Section

Perform the indicated operation.
69. $\frac{1}{5}+\frac{3}{5}$
70. $\frac{1}{7}+\frac{5}{7}$
71. $\frac{1}{10}+\frac{3}{14}$
72. $\frac{1}{21}+\frac{4}{15}$
73. $\frac{1}{10}-\frac{3}{14}$
74. $\frac{1}{21}-\frac{4}{15}$

Multiply.
75. $2(x-3)$
76. $x(x+2)$
77. $(x+4)(x-5)$
78. $(x+3)(x-4)$

Reduce to lowest terms.
79. $\frac{x+3}{x^{2}-9}$
80. $\frac{x+7}{x^{2}-49}$
81. $\frac{x^{2}-x-30}{2(x+5)(x-5)}$
82. $\frac{x^{2}-x-20}{2(x+4)(x-4)}$

Simplify.
83. $(x+4)(x-5)-10$
84. $(x+3)(x-4)-8$

# Addition and Subtraction of Rational Expressions 

## OBJECTIVES

A
Add and subtract rational expressions that have the same denominators.

B Add and subtract rational expressions that have different denominators.

## Note

If you have had difficulty in the past with addition and subtraction of fractions with different denominators, this is the time to get it straightened out. Go over Example 3 as many times as is necessary for you to understand the process.

In Chapter 1 we combined fractions having the same denominator by combining their numerators and putting the result over the common denominator. We use the same process to add two rational expressions with the same denominator.

## EXAMPLES

1. $\operatorname{Add} \frac{5}{x}+\frac{3}{x}$.

SOLUTION Adding numerators, we have:

$$
\frac{5}{x}+\frac{3}{x}=\frac{8}{x}
$$

2. $\operatorname{Add} \frac{x}{x^{2}-9}+\frac{3}{x^{2}-9}$.

SOLUTION Since both expressions have the same denominator, we add numerators and reduce to lowest terms:

$$
\left.\begin{array}{rl}
\frac{x}{x^{2}-9}+\frac{3}{x^{2}-9} & =\frac{x+3}{x^{2}-9} \\
& =\frac{x+3}{(x+3)(x-3)} \\
& =\frac{1}{x-3}
\end{array}\right\} \begin{aligned}
& \text { Reduce to lowest terms by } \\
& \text { factoring the denominator } \\
& \text { and then dividing out the } \\
& \text { common factor } x+3
\end{aligned}
$$

Remember, it is the distributive property that allows us to add rational expressions by simply adding numerators. Because of this, we must begin all addition problems involving rational expressions by first making sure all the expressions have the same denominator.

DEFINITION The least common denominator (LCD) for a set of denominators is the simplest quantity that is exactly divisible by all the denominators.

## EXAMPLE $3 \quad$ Add $\frac{1}{10}+\frac{3}{14}$.

## SOLUTION

Step 1: Find the LCD for 10 and 14. To do so, we factor each denominator and build the LCD from the factors:

$$
\left.\begin{array}{l}
10=2 \cdot 5 \\
14=2 \cdot 7
\end{array}\right\} \quad \text { LCD }=\mathbf{2} \cdot \mathbf{5} \cdot \mathbf{7}=\mathbf{7 0}
$$

We know the LCD is divisible by 10 because it contains the factors 2 and 5. It is also divisible by 14 because it contains the factors 2 and 7.

Step 2: Change to equivalent fractions that each have denominator 70. To accomplish this task, we multiply the numerator and denominator of
each fraction by the factor of the LCD that is not also a factor of its denominator:

These Have the
Original
Denominators Multiply by Same Value as
Fractions Form to Obtain LCD Fractions

$$
\begin{aligned}
\frac{1}{10} & =\frac{1}{2 \cdot 5} \\
\frac{3}{14} & =\frac{1}{2 \cdot 5} \cdot \frac{7}{7}=\frac{7}{70} \\
2 \cdot 7 & =\frac{3}{2 \cdot 7} \cdot \frac{5}{5}=\frac{15}{70}
\end{aligned}
$$

The fraction $\frac{7}{70}$ has the same value as the fraction $\frac{1}{10}$. Likewise, the fractions $\frac{15}{70}$ and $\frac{3}{14}$ are equivalent; they have the same value.
Step 3: Add numerators and put the result over the LCD:

$$
\frac{7}{70}+\frac{15}{70}=\frac{7+15}{70}=\frac{22}{70}
$$

Step 4: Reduce to lowest terms:

$$
\frac{22}{70}=\frac{11}{35} \quad \text { Divide numerator and denominator by } 2
$$

The main idea in adding fractions is to write each fraction again with the LCD for a denominator. Once we have done that, we simply add numerators. The same process can be used to add rational expressions, as the next example illustrates.

## EXAMPLE 4 Subtract $\frac{3}{x}-\frac{1}{2}$.

## SOLUTION

Step 1: The LCD for $x$ and 2 is $2 x$. It is the smallest expression divisible by $x$ and by 2 .

Step 2: To change to equivalent expressions with the denominator $2 x$, we multiply the first fraction by $\frac{2}{2}$ and the second by $\frac{X}{X}$ :

$$
\begin{aligned}
& \frac{3}{x} \cdot \frac{\mathbf{2}}{\mathbf{2}}=\frac{6}{2 x} \\
& \frac{1}{2} \cdot \frac{\boldsymbol{x}}{\boldsymbol{x}}=\frac{x}{2 x}
\end{aligned}
$$

Step 3: Subtracting numerators of the rational expressions in step 2, we have:

$$
\frac{6}{2 x}-\frac{x}{2 x}=\frac{6-x}{2 x}
$$

Step 4: Since $6-x$ and $2 x$ do not have any factors in common, we cannot reduce any further. Here is the complete solution:

$$
\begin{aligned}
\frac{3}{x}-\frac{1}{2} & =\frac{3}{x} \cdot \frac{\mathbf{2}}{\mathbf{2}}-\frac{1}{2} \cdot \frac{\boldsymbol{x}}{\boldsymbol{x}} \\
& =\frac{6}{2 x}-\frac{x}{2 x} \\
& =\frac{6-x}{2 x}
\end{aligned}
$$

## Note

In the last step we reduced the rational expression to lowest terms by dividing out the common factor of $x+4$.

## EXAMPLE 5 Add $\frac{5}{2 x-6}+\frac{x}{x-3}$.

SOLUTION If we factor $2 x-6$, we have $2 x-6=2(x-3)$. We need only multiply the second rational expression in our problem by $\frac{2}{2}$ to have two expressions with the same denominator:

$$
\begin{aligned}
\frac{5}{2 x-6}+\frac{x}{x-3} & =\frac{5}{2(x-3)}+\frac{x}{x-3} \\
& =\frac{5}{2(x-3)}+\frac{\mathbf{2}}{\mathbf{2}}\left(\frac{x}{x-3}\right) \\
& =\frac{5}{2(x-3)}+\frac{2 x}{2(x-3)} \\
& =\frac{2 x+5}{2(x-3)}
\end{aligned}
$$

## EXAMPLE $6 \quad$ Add $\frac{1}{x+4}+\frac{8}{x^{2}-16}$.

SOLUTION After writing each denominator in factored form, we find that the least common denominator is $(x+4)(x-4)$. To change the first rational expression to an equivalent rational expression with the common denominator, we multiply its numerator and denominator by $x-4$ :

$$
\begin{array}{rlrl}
\frac{1}{x+4} & +\frac{8}{x^{2}-16} & & \\
& =\frac{1}{x+4}+\frac{8}{(x+4)(x-4)} & & \text { Factor each denominator } \\
& =\frac{1}{x+4} \cdot \frac{x-4}{x-4}+\frac{8}{(x+4)(x-4)} & \begin{array}{l}
\text { Change to equivalent } \\
\text { rational expressions }
\end{array} \\
& =\frac{x-4}{(x+4)(x-4)}+\frac{8}{(x+4)(x-4)} & & \text { Simplify } \\
& =\frac{x+4}{(x+4)(x-4)} & & \text { Add numerators } \\
& =\frac{1}{x-4} & \begin{array}{l}
\text { Divide out common } \\
\text { factor } x+4
\end{array}
\end{array}
$$

EXAMPLE $7 \quad$ Add $\frac{2}{x^{2}+5 x+6}+\frac{x}{x^{2}-9}$.

## SOLUTION

Step 1: We factor each denominator and build the LCD from the factors:

$$
\left.\begin{array}{rl}
x^{2}+5 x+6 & =(x+2)(x+3) \\
x^{2}-9 & =(x+3)(x-3)
\end{array}\right\} \mathbf{L C D}=(x+2)(x+3)(x-3)
$$

Step 2: Change to equivalent rational expressions:

$$
\begin{aligned}
& \frac{2}{x^{2}+5 x+6}=\frac{2}{(x+2)(x+3)} \cdot \frac{(x-3)}{(x-3)}=\frac{2 x-6}{(x+2)(x+3)(x-3)} \\
& \frac{x}{x^{2}-9}=\frac{x}{(x+3)(x-3)} \cdot \frac{(x+\mathbf{2})}{(x+2)}=\frac{x^{2}+2 x}{(x+2)(x+3)(x-3)}
\end{aligned}
$$

Step 3: Add numerators of the rational expressions produced in step 2:

$$
\begin{aligned}
& \frac{2 x-6}{(x+2)(x+3)(x-3)}+\frac{x^{2}+2 x}{(x+2)(x+3)(x-3)} \\
& \quad=\frac{x^{2}+4 x-6}{(x+2)(x+3)(x-3)}
\end{aligned}
$$

The numerator and denominator do not have any factors in common.

## Note

In the second step we replaced subtraction by addition of the opposite. There seems to be less chance for error when this is done on longer problems.

EXAMPLE 8 subtract $\frac{x+4}{2 x+10}-\frac{5}{x^{2}-25}$.
SOLUTION We begin by factoring each denominator:

$$
\frac{x+4}{2 x+10}-\frac{5}{x^{2}-25}=\frac{x+4}{2(x+5)}-\frac{5}{(x+5)(x-5)}
$$

The LCD is $2(x+5)(x-5)$. Completing the solution, we have:

$$
\begin{aligned}
& =\frac{x+4}{2(x+5)} \cdot \frac{(x-5)}{(x-5)}+\frac{-5}{(x+5)(x-5)} \cdot \frac{\mathbf{2}}{\mathbf{2}} \\
& =\frac{x^{2}-x-20}{2(x+5)(x-5)}+\frac{-10}{2(x+5)(x-5)} \\
& =\frac{x^{2}-x-30}{2(x+5)(x-5)}
\end{aligned}
$$

To see if this expression will reduce, we factor the numerator into $(x-6)(x+5)$ :

$$
\begin{aligned}
& =\frac{(x-6)(x+5)}{2(x+5)(x-5)} \\
& =\frac{x-6}{2(x-5)}
\end{aligned}
$$

EXAMPLE 9 Write an expression for the sum of a number and its reciprocal, and then simplify that expression.
SOLUTION If we let $x=$ the number, then its reciprocal is $\frac{1}{x}$. To find the sum of the number and its reciprocal, we add them:

$$
x+\frac{1}{x}
$$

The first term $x$ can be thought of as having a denominator of 1 . Since the denominators are 1 and $x$, the least common denominator is $x$.

$$
\begin{aligned}
X+\frac{1}{X} & =\frac{X}{1}+\frac{1}{X} & & \text { Write } x \text { as } \frac{x}{1} \\
& =\frac{X}{1} \cdot \frac{\boldsymbol{x}}{\boldsymbol{x}}+\frac{1}{X} & & \text { The LCD is } x \\
& =\frac{X^{2}}{X}+\frac{1}{X} & & \\
& =\frac{X^{2}+1}{X} & & \text { Add numerators }
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,2
B 4-9

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How do we add two rational expressions that have the same denominator?
2. What is the least common denominator for two fractions?
3. What role does factoring play in finding a least common denominator?
4. Explain how to find a common denominator for two rational expressions.

## Problem Set 6.3

29. $\frac{2}{x+5}-\frac{10}{x^{2}-25}$
30. $\frac{6}{x^{2}-1}+\frac{3}{x+1}$
31. $\frac{x-4}{x-3}+\frac{6}{x^{2}-9}$
32. $\frac{x+1}{x-1}-\frac{4}{x^{2}-1}$
33. $\frac{a-4}{a-3}+\frac{5}{a^{2}-a-6}$
34. $\frac{a+2}{a+1}+\frac{7}{a^{2}-5 a-6}$
35. $\frac{8}{x^{2}-16}-\frac{7}{x^{2}-x-12}$
36. $\frac{6}{x^{2}-9}-\frac{5}{x^{2}-x-6}$
37. $\frac{4 y}{y^{2}+6 y+5}-\frac{3 y}{y^{2}+5 y+4}$
38. $\frac{3 y}{y^{2}+7 y+10}-\frac{2 y}{y^{2}+6 y+8}$
39. $\frac{4 x+1}{x^{2}+5 x+4}-\frac{x+3}{x^{2}+4 x+3}$
40. $\frac{2 x-1}{x^{2}+x-6}-\frac{x+2}{x^{2}+5 x+6}$
41. $\frac{1}{x}+\frac{x}{3 x+9}-\frac{3}{x^{2}+3 x}$
42. $\frac{1}{x}+\frac{x}{2 x+4}-\frac{2}{x^{2}+2 x}$
43. Work each problem according to the instructions given.
a. Multiply: $\frac{4}{9} \cdot \frac{1}{6}$
b. Divide: $\frac{4}{9} \div \frac{1}{6}$
c. Add: $\frac{4}{9}+\frac{1}{6}$
d. Multiply: $\frac{x+2}{x-2} \cdot \frac{3 x+10}{x^{2}-4}$
e. Divide: $\frac{x+2}{x-2} \div \frac{3 x+10}{x^{2}-4}$
f. Subtract: $\frac{x+2}{x-2}-\frac{3 x+10}{x^{2}-4}$
44. Work each problem according to the instructions given.
a. Multiply: $\frac{9}{25} \cdot \frac{1}{15}$
b. Divide: $\frac{9}{25} \div \frac{1}{15}$
c. Subtract: $\frac{9}{25}-\frac{1}{15}$
d. Multiply: $\frac{3 x-2}{3 x+2} \cdot \frac{15 x+6}{9 x^{2}-4}$
e. Divide: $\frac{3 x-2}{3 x+2} \div \frac{15 x+6}{9 x^{2}-4}$
f. Subtract: $\frac{3 x+2}{3 x-2}-\frac{15 x+6}{9 x^{2}-4}$

Complete the following tables.
45.

| Number | Reciprocal | Sum | Sum |
| :---: | :---: | :---: | :---: |
| $x$ | $\frac{1}{x}$ | $\mathbf{1 + \frac { 1 } { x }}$ | $\frac{x+\mathbf{1}}{\boldsymbol{x}}$ |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

46. 

| Number | Reciprocal <br> $x$ | Difference <br> $\mathbf{1}-\frac{\mathbf{1}}{x}$ | Difference <br> $\frac{\boldsymbol{x - 1}}{x}$ |
| :---: | :---: | :---: | :---: |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

47. 

| $x$ | $x+\frac{4}{x}$ | $\frac{x^{2}+4}{x}$ | $x+4$ |
| :--- | :--- | :--- | :--- |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

48. 

| $x$ | $2 x+\frac{6}{x}$ | $\frac{2 x^{2}+6}{x}$ | $2 x+6$ |
| :--- | :--- | :--- | :--- |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

Add or subtract as indicated.
49. $1+\frac{1}{x+2}$
50. $1-\frac{1}{x+2}$
51. $1-\frac{1}{x+3}$
52. $1+\frac{1}{x+3}$

## Maintaining Your Skills

Solve each equation.
53. $2 x+3(x-3)=6$
54. $4 x-2(x-5)=6$
55. $x-3(x+3)=x-3$
56. $x-4(x+4)=x-4$
57. $7-2(3 x+1)=4 x+3$
58. $8-5(2 x-1)=2 x+4$

Solve each quadratic equation.
59. $x^{2}+5 x+6=0$
60. $x^{2}-5 x+6=0$
61. $x^{2}-x=6$
62. $x^{2}+x=6$
63. $x^{2}-5 x=0$
64. $x^{2}-6 x=0$

## Getting Ready for the Next Section

Simplify.
65. $6\left(\frac{1}{2}\right)$
66. $10\left(\frac{1}{5}\right)$
67. $\frac{0}{5}$
68. $\frac{0}{2}$
69. $\frac{5}{0}$
70. $\frac{2}{0}$
71. $1-\frac{5}{2}$
72. $1-\frac{5}{3}$

Use the distributive property to simplify.
73. $6\left(\frac{x}{3}+\frac{5}{2}\right)$
74. $10\left(\frac{x}{2}+\frac{3}{5}\right)$
75. $x^{2}\left(1-\frac{5}{x}\right)$
76. $x^{2}\left(1-\frac{3}{x}\right)$

Solve.
77. $2 x+15=3$
78. $15=3 x-3$
79. $-2 x-9=x-3$
80. $a^{2}-a-20=-2 a$

## Equations Involving Rational Expressions

The first step in solving an equation that contains one or more rational expressions is to find the LCD for all denominators in the equation. Once the LCD has been found, we multiply both sides of the equation by it. The resulting equation should be equivalent to the original one (unless we inadvertently multiplied by zero) and free from any denominators except the number 1.

EXAMPLE 1 Solve $\frac{x}{3}+\frac{5}{2}=\frac{1}{2}$ for $x$.
SOLUTION The LCD for 3 and 2 is 6 . If we multiply both sides by 6 , we have:

$$
\begin{aligned}
6\left(\frac{x}{3}+\frac{5}{2}\right) & =6\left(\frac{1}{2}\right) \quad \text { Multiply both sides by } 6 \\
6\left(\frac{x}{3}\right)+6\left(\frac{5}{2}\right) & =6\left(\frac{1}{2}\right) \quad \text { Distributive property } \\
2 x+15 & =3 \\
2 x & =-12 \\
x & =-6
\end{aligned}
$$

We can check our solution by replacing $x$ with -6 in the original equation:

$$
\begin{array}{r}
-\frac{6}{3}+\frac{5}{2} \stackrel{?}{=} \frac{1}{2} \\
\frac{1}{2}=\frac{1}{2}
\end{array}
$$

Multiplying both sides of an equation containing fractions by the LCD clears the equation of all denominators, because the LCD has the property that all denominators will divide it evenly.

EXAMPLE $2 \quad$ Solve for $x: \frac{3}{x-1}=\frac{3}{5}$.
SOLUTION The LCD for $(x-1)$ and 5 is $5(x-1)$. Multiplying both sides by $5(x-1)$, we have:

$$
\begin{aligned}
5(x-1) \cdot \frac{3}{x-1} & =5(x-1) \cdot \frac{3}{5} \\
5 \cdot 3 & =(x-1) \cdot 3 \\
15 & =3 x-3 \\
18 & =3 x \\
6 & =x
\end{aligned}
$$

If we substitute $x=6$ into the original equation, we have:

$$
\begin{array}{r}
\frac{3}{6-1} \stackrel{?}{=} \frac{3}{5} \\
\frac{3}{5}=\frac{3}{5}
\end{array}
$$

The solution set is $\{6\}$.

EXAMPLE 3 Solve $1-\frac{5}{x}=\frac{-6}{x^{2}}$.
SOLUTION The LCD is $x^{2}$. Multiplying both sides by $x^{2}$, we have

$$
\begin{aligned}
x^{2}\left(1-\frac{5}{x}\right) & =x^{2}\left(\frac{-6}{x^{2}}\right) & & \text { Multiply both sides by } \boldsymbol{x}^{2} \\
x^{2}(1)-x^{2}\left(\frac{5}{x}\right) & =x^{2}\left(\frac{-6}{x^{2}}\right) & & \begin{array}{l}
\text { Apply distributive property } \\
\text { to the left side }
\end{array} \\
x^{2}-5 x & =-6 & & \text { Simplify each side }
\end{aligned}
$$

We have a quadratic equation, which we write in standard form, factor, and solve as we did in Chapter 6.

$$
\begin{array}{rlrl}
x^{2}-5 x+6 & =0 & & \text { Standard form } \\
(x-2)(x-3) & =0 & & \text { Factor } \\
x-2=0 & \text { or } & x-3 & =0 \\
& \text { Set factors equal to } 0 \\
x=2 & \text { or } & x & =3
\end{array}
$$

The two possible solutions are 2 and 3. Checking each in the original equation, we find they both give true statements. They are both solutions to the original equation:

$$
\begin{array}{rlrl}
\text { Check } x & =2 & \text { Check } x & =3 \\
1-\frac{5}{2} \stackrel{?}{=} \frac{-6}{4} & 1-\frac{5}{3} \stackrel{?}{=} \frac{-6}{9} \\
\frac{2}{2}-\frac{5}{2} & =-\frac{3}{2} & \frac{3}{3}-\frac{5}{3}=-\frac{2}{3} \\
-\frac{3}{2} & =-\frac{3}{2} & -\frac{2}{3}=-\frac{2}{3}
\end{array}
$$

EXAMPLE 4 Solve $\frac{x}{x^{2}-9}-\frac{3}{x-3}=\frac{1}{x+3}$.
SOLUTION The factors of $x^{2}-9$ are $(x+3)(x-3)$. The LCD, then, is $(x+3)(x-3)$ :

$$
\begin{aligned}
(x+3)(x-3) \cdot \frac{x}{(x+3)(x-3)} & +(x+3)(x-3) \cdot \frac{-3}{x-3} \\
& =(x+3)(x-3) \cdot \frac{1}{x+3} \\
x+(x+3)(-3) & =(x-3) 1 \\
x+(-3 x)+(-9) & =x-3 \\
-2 x-9 & =x-3 \\
-3 x & =6 \\
x & =-2
\end{aligned}
$$

The solution is $x=-2$. It checks when substituted for $x$ in the original equation.

## EXAMPLE 5 Solve $\frac{x}{x-3}+\frac{3}{2}=\frac{3}{x-3}$.

SOLUTION We begin by multiplying each term on both sides of the equation by $2(x-3)$ :

$$
\begin{aligned}
2(x-3) \cdot \frac{x}{x-3}+2(x-3) \cdot \frac{3}{2} & =2(x-3) \cdot \frac{3}{x-3} \\
2 x+(x-3) \cdot 3 & =2 \cdot 3 \\
2 x+3 x-9 & =6 \\
5 x-9 & =6 \\
5 x & =15 \\
x & =3
\end{aligned}
$$

Our only possible solution is $x=3$. If we substitute $x=3$ into our original equation, we get:

$$
\begin{gathered}
\frac{3}{3-3}+\frac{3}{2} \stackrel{3}{=} \frac{3}{3-3} \\
\frac{3}{0}+\frac{3}{2}=\frac{3}{0}
\end{gathered}
$$

Two of the terms are undefined, so the equation is meaningless. What has happened is that we have multiplied both sides of the original equation by zero. The equation produced by doing this is not equivalent to our original equation. We always must check our solution when we multiply both sides of an equation by an expression containing the variable to make sure we have not multiplied both sides by zero.
Our original equation has no solution; that is, there is no real number $x$ such that:

$$
\frac{x}{x-3}+\frac{3}{2}=\frac{3}{x-3}
$$

The solution set is $\varnothing$.

When the proposed solution to an equation is not actually a solution, it is called an extraneous solution. In the previous example, $x=3$ is an extraneous solution.

## EXAMPLE 6 Solve $\frac{a+4}{a^{2}+5 a}=\frac{-2}{a^{2}-25}$ for $a$.

SOLUTION Factoring each denominator, we have:

$$
\begin{aligned}
& a^{2}+5 a=a(a+5) \\
& a^{2}-25=(a+5)(a-5)
\end{aligned}
$$

The LCD is $a(a+5)(a-5)$. Multiplying both sides of the equation by the LCD gives us:

$$
\begin{aligned}
\not d(a+5)(a-5) \cdot \frac{a+4}{d(a+5)} & =\frac{-2}{(a+5)(a-5)} \cdot a(a+5)(a-5) \\
(a-5)(a+4) & =-2 a \\
a^{2}-a-20 & =-2 a
\end{aligned}
$$

The result is a quadratic equation, which we write in standard form, factor, and solve:

$$
\begin{array}{rlrl}
a^{2}+a-20 & =0 & & \text { Add } 2 a \text { to both sides } \\
(a+5)(a-4) & =0 & & \text { Factor } \\
a+5=0 & \text { or } & a-4 & =0 \\
& \text { Set each factor to } 0 \\
a=-5 & \text { or } & a & =4 \\
& &
\end{array}
$$

The two possible solutions are -5 and 4 . There is no problem with the 4 . It checks when substituted for $a$ in the original equation. However, -5 is not a solution. Substituting -5 into the original equation gives:

$$
\begin{aligned}
\frac{-5+4}{(-5)^{2}+5(-5)} & \stackrel{?}{=} \frac{-2}{(-5)^{2}-25} \\
\frac{-1}{0} & =\frac{-2}{0}
\end{aligned}
$$

This indicates -5 is not a solution. The solution is 4 .

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-6

 and in complete sentences.1. What is the first step in solving an equation that contains rational expressions?
2. Explain how to find the LCD used to clear an equation of fractions.
3. When will a solution to an equation containing rational expressions be extraneous?
4. When do we check for extraneous solutions to an equation containing rational expressions?

Solve the following equations. Be sure to check each answer in the original equation if you multiply both sides by an expression that contains the variable.

1. $\frac{x}{3}+\frac{1}{2}=-\frac{1}{2}$
2. $\frac{x}{2}+\frac{4}{3}=-\frac{2}{3}$
3. $\frac{4}{a}=\frac{1}{5}$
4. $\frac{2}{3}=\frac{6}{a}$
5. $\frac{3}{x}+1=\frac{2}{x}$
6. $\frac{4}{x}+3=\frac{1}{x}$
7. $\frac{3}{a}-\frac{2}{a}=\frac{1}{5}$
8. $\frac{7}{a}+\frac{1}{a}=2$
9. $\frac{3}{x}+2=\frac{1}{2}$
10. $\frac{5}{x}+3=\frac{4}{3}$
11. $\frac{1}{y}-\frac{1}{2}=-\frac{1}{4}$
12. $\frac{3}{y}-\frac{4}{5}=-\frac{1}{5}$
13. $1-\frac{8}{x}=\frac{-15}{x^{2}}$
14. $1-\frac{3}{x}=\frac{-2}{x^{2}}$
15. $\frac{x}{2}-\frac{4}{x}=-\frac{7}{2}$
16. $\frac{x}{2}-\frac{5}{x}=-\frac{3}{2}$
17. $\frac{x-3}{2}+\frac{2 x}{3}=\frac{5}{6}$
18. $\frac{x-2}{3}+\frac{5 x}{2}=5$
19. $\frac{x+1}{3}+\frac{x-3}{4}=\frac{1}{6}$
20. $\frac{x+2}{3}+\frac{x-1}{5}=-\frac{3}{5}$
21. $\frac{6}{x+2}=\frac{3}{5}$
22. $\frac{4}{x+3}=\frac{1}{2}$
23. $\frac{3}{y-2}=\frac{2}{y-3}$
24. $\frac{5}{y+1}=\frac{4}{y+2}$
25. $\frac{x}{x-2}+\frac{2}{3}=\frac{2}{x-2}$
26. $\frac{x}{x-5}+\frac{1}{5}=\frac{5}{x-5}$
27. $\frac{x}{x-2}+\frac{3}{2}=\frac{9}{2(x-2)}$
28. $\frac{x}{x+1}+\frac{4}{5}=\frac{-14}{5(x+1)}$
29. $\frac{5}{x+2}+\frac{1}{x+3}=\frac{-1}{x^{2}+5 x+6}$
30. $\frac{3}{x-1}+\frac{2}{x+3}=\frac{-3}{x^{2}+2 x-3}$
31. $\frac{8}{x^{2}-4}+\frac{3}{x+2}=\frac{1}{x-2}$
32. $\frac{10}{x^{2}-25}-\frac{1}{x-5}=\frac{3}{x+5}$
33. $\frac{a}{2}+\frac{3}{a-3}=\frac{a}{a-3}$
34. $\frac{a}{2}+\frac{4}{a-4}=\frac{a}{a-4}$
35. $\frac{6}{y^{2}-4}=\frac{4}{y^{2}+2 y}$
36. $\frac{2}{y^{2}-9}=\frac{5}{y^{2}-3 y}$
37. $\frac{2}{a^{2}-9}=\frac{3}{a^{2}+a-12}$
38. $\frac{2}{a^{2}-1}=\frac{6}{a^{2}-2 a-3}$
39. $\frac{3 x}{x-5}-\frac{2 x}{x+1}=\frac{-42}{x^{2}-4 x-5}$
40. $\frac{4 x}{x-4}-\frac{3 x}{x-2}=\frac{-3}{x^{2}-6 x+8}$
41. $\frac{2 x}{x+2}=\frac{x}{x+3}-\frac{3}{x^{2}+5 x+6}$
42. $\frac{3 x}{x-4}=\frac{2 x}{x-3}+\frac{6}{x^{2}-7 x+12}$
43. Solve each equation.
a. $5 x-1=0$
b. $\frac{5}{x}-1=0$
c. $\frac{x}{5}-1=\frac{2}{3}$
d. $\frac{5}{x}-1=\frac{2}{3}$
e. $\frac{5}{x^{2}}+5=\frac{26}{x}$
44. Solve each equation.
a. $2 x-3=0$
b. $2-\frac{3}{x}=0$
c. $\frac{x}{3}-2=\frac{1}{2}$
d. $\frac{3}{x}-2=\frac{1}{2}$
e. $\frac{1}{x}+\frac{3}{x^{2}}=2$
45. Work each problem according to the instructions given.
a. Divide: $\frac{7}{a^{2}-5 a-6} \div \frac{a+2}{a+1}$
b. Add: $\frac{7}{a^{2}-5 a-6}+\frac{a+2}{a+1}$
c. Solve: $\frac{7}{a^{2}-5 a-6}+\frac{a+2}{a+1}=2$
46. Work each problem according to the instructions given.
a. Divide: $\frac{6}{x^{2}-9} \div \frac{x-4}{x-3}$
b. Add: $\frac{6}{x^{2}-9}+\frac{x-4}{x-3}$
c. Solve: $\frac{6}{x^{2}-9}+\frac{x-4}{x-3}=\frac{3}{4}$

## Maintaining Your Skills

Solve each word problem.
47. Number Problem If twice the difference of a number and 3 were decreased by 5 , the result would be 3 . Find the number.
48. Number Problem If 3 times the sum of a number and 2 were increased by 6 , the result would be 27 . Find the number.
49. Geometry The length of a rectangle is 5 more than twice the width. The perimeter is 34 inches. Find the length and width.
50. Geometry The length of a rectangle is 2 more than 3 times the width. The perimeter is 44 feet. Find the length and width.

Solve each problem. Be sure to show the equation that describes the situation.
51. Number Problem The product of two consecutive even integers is 48 . Find the two integers.
52. Number Problem The product of two consecutive odd integers is 35 . Find the two integers.
53. Geometry The hypotenuse (the longest side) of a right triangle is 10 inches, and the lengths of the two legs (the other two sides) are given by two consecutive even integers. Find the lengths of the two legs.
54. Geometry One leg of a right triangle is 2 more than twice the other. If the hypotenuse is 13 feet, find the lengths of the two legs.

## Getting Ready for the Next Section

Solve.
55. $\frac{1}{x}+\frac{1}{2 x}=\frac{9}{2}$
56. $\frac{50}{x+5}=\frac{30}{x-5}$
57. $\frac{1}{10}-\frac{1}{15}=\frac{1}{x}$
58. $\frac{15}{x}+\frac{15}{x+20}=2$

Find the value of $y=-\frac{6}{x}$ for the given value of $x$.
59. $x=-6$
60. $x=-3$
61. $x=2$
62. $x=1$

## 6.5 <br> Applications

## OBJECTIVES

A Solve applications whose solutions depend on solving an equation containing rational expressions.

B Graph an equation involving a rational expression.

In this section we will solve some word problems whose equations involve rational expressions. Like the other word problems we have encountered, the more you work with them, the easier they become.

Also in this section, we will look at the graphs of rational equations. As we did with the graphs of linear equations, we will first build a table by finding values of $x$ and $y$ that satisfy the given equation.

EXAMPLE 1 One number is twice another. The sum of their reciprocals is $\frac{9}{2}$. Find the two numbers.
SOLUTION Let $x=$ the smaller number. The larger then must be $2 x$. Their reciprocals are $\frac{1}{x}$ and $\frac{1}{2 x}$, respectively. An equation that describes the situation is:

$$
\frac{1}{x}+\frac{1}{2 x}=\frac{9}{2}
$$

We can multiply both sides by the LCD $2 x$ and then solve the resulting equation:

$$
\begin{aligned}
2 \times\left(\frac{1}{x}\right)+2 x\left(\frac{1}{2 x}\right) & =2 x\left(\frac{9}{2}\right) \\
2+1 & =9 x \\
3 & =9 x \\
x & =\frac{3}{9}=\frac{1}{3}
\end{aligned}
$$

The smaller number is $\frac{1}{3}$. The other number is twice as large, or $\frac{2}{3}$. If we add their reciprocals, we have:

$$
\frac{3}{1}+\frac{3}{2}=\frac{6}{2}+\frac{3}{2}=\frac{9}{2}
$$

The solutions check with the original problem.


EXAMPLE 2 A boat travels 30 miles up a river in the same amount of time it takes to travel 50 miles down the same river. If the current is 5 miles per hour, what is the speed of the boat in still water?

SOLUTION The easiest way to work a problem like this is with a table. The top row of the table is labeled with $d$ for distance, $r$ for rate, and $t$ for time. The left column of the table is labeled with the two trips: upstream and downstream. Here is what the table looks like:


The next step is to read the problem over again and fill in as much of the table as we can with the information in the problem. The distance the boat travels upstream is 30 miles and the distance downstream is 50 miles. Since we are asked for the speed of the boat in still water, we will let that be $x$. If the speed of the boat in still water is $x$, then its speed upstream (against the current) must be $x-5$, and its speed downstream (with the current) must be $x+5$. Putting these four quantities into the appropriate positions in the table, we have

|  | $\boldsymbol{d}$ | $\boldsymbol{r}$ | $\boldsymbol{t}$ |
| :--- | :---: | :---: | :---: |
| Upstream | 30 | $x-5$ |  |
| Downstream | 50 | $x+5$ |  |

## Note

There are two things to note about this problem. The first is that to solve the equation $d=r \cdot t$ for $t$, we divide each side by $r$, like this:

$$
\begin{gathered}
\frac{d}{r}=\frac{r \cdot t}{r} \\
\frac{d}{r}=t
\end{gathered}
$$

The second thing is this: The speed of the boat in still water is the rate at which it would be traveling if there were no current; that is, it is the speed of the boat through the water. Since the water itself is moving at 5 miles per hour, the boat is going 5 miles per hour slower when it travels against the current and 5 miles per hour faster when it travels with the current.


The last positions in the table are filled in by using the equation $t=\frac{d}{r}$.

|  | $\boldsymbol{d}$ | $\boldsymbol{r}$ | $\boldsymbol{t}$ |
| :--- | :---: | :---: | :---: |
| Upstream | 30 | $x-5$ | $\frac{30}{x-5}$ |
| Downstream | 50 | $x+5$ | $\frac{50}{x+5}$ |

Reading the problem again, we find that the time for the trip upstream is equal to the time for the trip downstream. Setting these two quantities equal to each other, we have our equation:

Time $($ downstream $)=$ time (upstream)

$$
\frac{50}{x+5}=\frac{30}{x-5}
$$

The LCD is $(x+5)(x-5)$. We multiply both sides of the equation by the LCD to clear it of all denominators. Here is the solution:

$$
\begin{aligned}
(x+5)(x-5) \cdot \frac{50}{x+5} & =(x+5)(x-5) \cdot \frac{30}{x-5} \\
50 x-250 & =30 x+150 \\
20 x & =400 \\
x & =20
\end{aligned}
$$

The speed of the boat in still water is 20 miles per hour.

## EXAMPLE 3 Tina is training for a biathlon. To train for the bicycle por-

 tion, she rides her bike 15 miles up a hill and then 15 miles back down the same hill. The complete trip takes her 2 hours. If her downhill speed is 20 miles per hour faster than her uphill speed, how fast does she ride uphill?SOLUTION Again, we make a table. As in the previous example, we label the top row with distance, rate, and time. We label the left column with the two trips, uphill and downhill.


Next, we fill in the table with as much information as we can from the problem. We know the distance traveled is 15 miles uphill and 15 miles downhill, which allows us to fill in the distance column. To fill in the rate column, we first note that she rides 20 miles per hour faster downhill than uphill. Therefore, if we let $x$ equal her rate uphill, then her rate downhill is $x+20$. Filling in the table with this information gives us

|  | $\boldsymbol{d}$ | $\boldsymbol{r}$ | $\boldsymbol{t}$ |
| :--- | :---: | :---: | :---: |
| Uphill | 15 | $X$ |  |
| Downhill | 15 | $x+20$ |  |

Since time is distance divided by rate, $t=d / r$, we can fill in the last column in the table.

|  | $\boldsymbol{d}$ | $\boldsymbol{r}$ | $\boldsymbol{t}$ |
| :--- | :---: | :---: | :---: |
| Uphill | 15 | $x$ | $\frac{15}{X}$ |
| Downhill | 15 | $x+20$ | $\frac{15}{x+20}$ |

Rereading the problem, we find that the total time (the time riding uphill plus the time riding downhill) is two hours. We write our equation as follows:

$$
\begin{aligned}
\text { Time }(\text { uphill })+\text { time }(\text { downhill }) & =2 \\
\frac{15}{x}+\frac{15}{x+20} & =2
\end{aligned}
$$

We solve this equation for $x$ by first finding the LCD and then multiplying each term in the equation by it to clear the equation of all denominators. Our LCD is $x(x+20)$. Here is our solution:

$$
\begin{aligned}
x(x+20) \frac{15}{x}+x(x+20) \frac{15}{x+20} & =2 \cdot[x(x+20)] \\
15(x+20)+15 x & =2 x(x+20) \\
15 x+300+15 x & =2 x^{2}+40 x \\
0 & =2 x^{2}+10 x-300 \\
0 & =x^{2}+5 x-150 \quad \text { Divide both sides by } 2 \\
0 & =(x+15)(x-10) \\
x=-15 \quad \text { or } \quad x & =10
\end{aligned}
$$

Since we cannot have a negative speed, our only solution is $x=10$. Tina rides her bike at a rate of 10 miles per hour when going uphill. (Her downhill speed is $x+20=30$ miles per hour.)

## EXAMPLE 4 An inlet pipe can fill a water tank in 10 hours, while an

Inlet pipe
10 hours to fill

Outlet pipe
15 hours to empty outlet pipe can empty the same tank in 15 hours. By mistake, both pipes are left open. How long will it take to fill the water tank with both pipes open?
SOLUTION Let $x=$ amount of time to fill the tank with both pipes open.
One method of solving this type of problem is to think in terms of how much of the job is done by a pipe in 1 hour.

1. If the inlet pipe fills the tank in 10 hours, then in 1 hour the inlet pipe fills $\frac{1}{10}$ of the tank.

## Note

In solving a problem of this type, we have to assume that the thing doing the work (whether it is a pipe, a person, or a machine) is working at a constant rate; that is, as much work gets done in the first hour as is done in the last hour and any other hour in between.
2. If the outlet pipe empties the tank in 15 hours, then in 1 hour the outlet pipe empties $\frac{1}{15}$ of the tank.
3. If it takes $x$ hours to fill the tank with both pipes open, then in 1 hour the tank is $\frac{1}{x}$ full.

Here is how we set up the equation. In 1 hour,


The LCD for our equation is $30 x$. We multiply both sides by the LCD and solve:

$$
\begin{aligned}
30 x\left(\frac{1}{10}\right)-30 x\left(\frac{1}{15}\right) & =30 x\left(\frac{1}{x}\right) \\
3 x-2 x & =30 \\
x & =30
\end{aligned}
$$

It takes 30 hours with both pipes open to fill the tank.

## EXAMPLE $5 \quad$ Graph the equation $y=\frac{1}{x}$.

SOLUTION Since this is the first time we have graphed an equation of this form, we will make a table of values for $x$ and $y$ that satisfy the equation. Before we do, let's make some generalizations about the graph (Figure 1).

First, notice that since $y$ is equal to 1 divided by $x, y$ will be positive when $x$ is positive. (The quotient of two positive numbers is a positive number.) Likewise, when $x$ is negative, $y$ will be negative. In other words, $x$ and $y$ always will have the same sign. Thus, our graph will appear in quadrants I and III only because in those quadrants $x$ and $y$ have the same sign.
Next, notice that the expression $\frac{1}{x}$ will be undefined when $x$ is 0 , meaning that there is no value of $y$ corresponding to $x=0$. Because of this, the graph will not cross the $y$-axis. Further, the graph will not cross the $x$-axis either. If we try to find the $x$-intercept by letting $y=0$, we have

$$
0=\frac{1}{X}
$$

But there is no value of $x$ to divide into 1 to obtain 0 . Therefore, since there is no solution to this equation, our graph will not cross the $x$-axis.
To summarize, we can expect to find the graph in quadrants I and III only, and the graph will cross neither axis.


## EXAMPLE $6 \quad$ Graph the equation $y=\frac{-6}{x}$.

SOLUTION Since $y$ is -6 divided by $x$, when $x$ is positive, $y$ will be negative (a negative divided by a positive is negative), and when $x$ is negative, $y$ will be positive (a negative divided by a negative). Thus, the graph (Figure 2) will appear in quadrants II and IV only. As was the case in Example 5, the graph will not cross either axis.

| $x$ | $y$ |
| :---: | :---: |
| -6 | 1 |
| -3 | 2 |
| -2 | 3 |
| -1 | 6 |
| 0 | Undefined |
| 1 | -6 |
| 2 | -3 |
| 3 | -2 |
| 6 | -1 |



FIGURE 2

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-4
B 5,6

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Write an application problem for which the solution depends on solving the equation $\frac{1}{2}+\frac{1}{3}=\frac{1}{x}$.
2. How does the current of a river affect the speed of a motor boat traveling against the current?
3. How does the current of a river affect the speed of a motor boat traveling in the same direction as the current?
4. What is the relationship between the total number of minutes it takes for a drain to empty a sink and the amount of water that drains out of the sink in 1 minute?

## Number Problems

1. One number is 3 times as large as another. The sum of their reciprocals is $\frac{16}{3}$. Find the two numbers.
2. If $\frac{3}{5}$ is added to twice the reciprocal of a number, the result is 1 . Find the number.
3. The sum of a number and its reciprocal is $\frac{13}{6}$. Find the number.
4. The sum of a number and 10 times its reciprocal is 7. Find the number.
5. If a certain number is added to both the numerator and denominator of the fraction $\frac{7}{9}$, the result is $\frac{5}{7}$. Find the number.
6. The numerator of a certain fraction is 2 more than the denominator. If $\frac{1}{3}$ is added to the fraction, the result is 2 . Find the fraction.
7. The sum of the reciprocals of two consecutive even integers is $\frac{5}{12}$. Find the integers.
8. The sum of the reciprocals of two consecutive integers is $\frac{7}{12}$. Find the two integers.

## Motion Problems

9. A boat travels 26 miles up the river in the same amount of time it takes to travel 38 miles down the same river. If the current is 3 miles per hour, what is the speed of the boat in still water?

10. A boat can travel 9 miles up a river in the same amount of time it takes to travel 11 miles down the same river. If the current is 2 miles per hour, what is the speed of the boat in still water?

11. An airplane flying against the wind travels 140 miles in the same amount of time it would take the same plane to travel 160 miles with the wind. If the wind speed is a constant 20 miles per hour, how fast would the plane travel in still air?
12. An airplane flying against the wind travels 500 miles in the same amount of time that it would take to travel 600 miles with the wind. If the speed of the wind is 50 miles per hour, what is the speed of the plane in still air?
13. One plane can travel 20 miles per hour faster than another. One of them goes 285 miles in the same time it takes the other to go 255 miles. What are their speeds?
14. One car travels 300 miles in the same amount of time it takes a second car traveling 5 miles per hour slower than the first to go 275 miles. What are the speeds of the cars?
15. Tina, whom we mentioned in Example 3 of this section, is training for a biathlon. To train for the running portion of the race, she runs 8 miles each day, over the same course. The first 2 miles of the course is on level ground, while the last 6 miles is downhill. She runs 3 miles per hour slower on level ground than she runs downhill. If the complete course takes 1 hour, how fast does she run on the downhill part of the course?
16. Jerri is training for the same biathlon as Tina (Example 3 and Problem 15). To train for the bicycle portion of the race, she rides 24 miles out a straight road, then turns around and rides 24 miles back. The trip out is against the wind, whereas the trip back is with the wind. If she rides 10 miles per hour faster with the wind then she does against the wind, and the complete trip out and back takes 2 hours, how fast does she ride when she rides against the wind?
17. To train for the running of a triathlon, Jerri jogs 1 hour each day over the same 9-mile course. Five miles of the course is downhill, whereas the other 4 miles is on level ground. Jerri figures that she runs 2 miles per hour faster downhill than she runs on level ground. Find the rate at which Jerri runs on level ground.
18. Travis paddles his kayak in the harbor at Morro Bay, California, where the incoming tide has caused a current in the water. From the point where he enters the water, he paddles 1 mile against the current, then turns around and paddles 1 mile back to where he started. His average speed when paddling with the current is 4 miles per hour faster than his speed against the current. If the complete trip (out and back) takes him 1.2 hours, find his average speed when he paddles against the current.

## Work Problems

19. An inlet pipe can fill a pool in 12 hours, while an outlet pipe can empty it in 15 hours. If both pipes are left open, how long will it take to fill the pool.
20. A water tank can be filled in 20 hours by an inlet pipe and emptied in 25 hours by an outlet pipe. How long will it take to fill the tank if both pipes are left open?
21. A bathtub can be filled by the cold water faucet in 10 minutes and by the hot water faucet in 12 minutes. How long does it take to fill the tub if both faucets are open?
22. A water faucet can fill a sink in 6 minutes, whereas the drain can empty it in 4 minutes. If the sink is full, how long will it take to empty if both the faucet and the drain are open?
23. A sink can be filled by the cold water faucet in 3 minutes. The drain can empty a full sink in 4 minutes. If the sink is empty and both the cold water faucet and the drain are open, how long will it take the sink to overflow?
24. A bathtub can be filled by the cold water faucet in 9 minutes and by the hot water faucet in 10 minutes. The drain can empty the tub in 5 minutes. Can the tub be filled if both faucets and the drain are open?

Graph each of the following equations.
25. $y=\frac{-4}{x}$
26. $y=\frac{4}{x}$
27. $y=\frac{8}{x}$
28. $y=\frac{-8}{x}$
29. Graph $y=\frac{3}{x}$ and $x+y=4$ on the same coordinate system. At what points do the two graphs intersect?
30. Graph $y=\frac{4}{x}$ and $x-y=3$ on the same coordinate system. At what points do the two graphs intersect?

## Maintaining Your Skills

31. Factor out the greatest common factor for $15 a^{3} b^{3}-20 a^{2} b-35 a b^{2}$.
32. Factor by grouping $3 a x-2 a+15 x-10$.

Factor completely.
33. $x^{2}-4 x-12$
34. $4 x^{2}-20 x y+25 y^{2}$
35. $x^{4}-16$
36. $2 x^{2}+x y-21 y^{2}$
37. $5 x^{3}-25 x^{2}-30 x$

Solve each equation.
38. $x^{2}-9 x+18=0$
39. $x^{2}-6 x=0$
40. $8 x^{2}=-2 x+15$
41. $x(x+2)=80$
42. Number Problem The product of two consecutive even integers is 4 more than twice their sum. Find the two integers.
43. Geometry The hypotenuse of a right triangle is 15 inches. One of the legs is 3 inches more than the other. Find the length of the two legs.

## Getting Ready for the Next Section

Simplify.
44. $\frac{1}{2} \div \frac{2}{3}$
45. $\frac{1}{3} \div \frac{3}{4}$
46. $1+\frac{1}{2}$
47. $1+\frac{2}{3}$
48. $y^{5} \cdot \frac{2 x^{3}}{y^{2}}$
49. $y^{7} \cdot \frac{3 x^{5}}{y^{4}}$
50. $\frac{2 x^{3}}{y^{2}} \cdot \frac{y^{5}}{4 x}$
51. $\frac{3 x^{5}}{y^{4}} \cdot \frac{y^{7}}{6 x^{2}}$

Factor.
52. $x^{2} y+x$
53. $x y^{2}+y$

Reduce.
54. $\frac{2 x^{3} y^{2}}{4 x}$
55. $\frac{3 x^{5} y^{3}}{6 x^{2}}$
56. $\frac{x^{2}-4}{x^{2}-x-6}$
57. $\frac{x^{2}-9}{x^{2}-5 x+6}$

### 6.6 Complex frations

## OBJECTIVES

A Simplify a complex fraction.

A complex fraction is a fraction or rational expression that contains other fractions in its numerator or denominator. Each of the following is a complex fraction:

$$
\frac{\frac{1}{2}}{\frac{2}{3}} \quad \frac{x+\frac{1}{y}}{y+\frac{1}{x}} \quad \frac{\frac{a+1}{a^{2}-9}}{\frac{2}{a+3}}
$$

We will begin this section by simplifying the first of these complex fractions. Before we do, though, let's agree on some vocabulary. So that we won't have to use phrases such as "the numerator of the denominator," let's call the numerator of a complex fraction the top and the denominator of a complex fraction the bottom.

## EXAMPLE $1 \quad$ Simplify $\frac{\frac{2}{2}}{\frac{2}{3}}$.

SOLUTION There are two methods we can use to solve this problem.
Method 1 We can multiply the top and bottom of this complex fraction by the LCD for both fractions. In this case the LCD is 6:

$$
\frac{\frac{1}{2}}{\frac{2}{3}}=\frac{6 \cdot \frac{1}{2}}{6 \cdot \frac{2}{3}}=\frac{3}{4}
$$

Method 2 We can treat this as a division problem. To divide by $\frac{2}{3}$, we multiply by its reciprocal $\frac{3}{2}$ :

$$
\frac{\frac{1}{2}}{\frac{2}{3}}=\frac{1}{2} \cdot \frac{3}{2}=\frac{3}{4}
$$

Using either method, we obtain the same result.

## EXAMPLE 2 simplify:

$$
\frac{\frac{2 x^{3}}{y^{2}}}{\frac{4 x}{y^{5}}}
$$

## SOLUTION

Method 1 The LCD for each rational expression is $y^{5}$. Multiplying the top and bottom of the complex fraction by $y^{5}$, we have:

$$
\frac{\frac{2 x^{3}}{y^{2}}}{\frac{4 x}{y^{5}}}=\frac{y^{5} \cdot \frac{2 x^{3}}{y^{2}}}{y^{5} \cdot \frac{4 x}{y^{5}}}=\frac{2 x^{3} y^{3}}{4 x}=\frac{x^{2} y^{3}}{2}
$$

Method 2 To divide by $\frac{4 x}{y^{5}}$ we multiply by its reciprocal, $\frac{y^{5}}{4 x}$ :

$$
\frac{\frac{2 x^{3}}{y^{2}}}{\frac{4 x}{y^{5}}}=\frac{2 x^{3}}{y^{2}} \cdot \frac{y^{5}}{4 x}=\frac{x^{2} y^{3}}{2}
$$

Again the result is the same, whether we use Method 1 or Method 2.

## EXAMPLE 3 Simplify:

$$
\frac{x+\frac{1}{y}}{y+\frac{1}{x}}
$$

SOLUTION To apply Method 2 as we did in the first two examples, we would have to simplify the top and bottom separately to obtain a single rational expression for both before we could multiply by the reciprocal. It is much easier, in this case, to multiply the top and bottom by the LCD xy:

$$
\begin{aligned}
\frac{x+\frac{1}{y}}{y+\frac{1}{x}} & =\frac{x y\left(x+\frac{1}{y}\right)}{x y\left(y+\frac{1}{x}\right)} & & \text { Multiply top and bottom by } x y \\
& =\frac{x y \cdot x+x y \cdot \frac{1}{y}}{x y \cdot y+x y \cdot \frac{1}{x}} & & \text { Distributive property } \\
& =\frac{x^{2} y+x}{x y^{2}+y} & & \text { Simplify }
\end{aligned}
$$

We can factor an $x$ from $x^{2} y+x$ and a $y$ from $x y^{2}+y$ and then reduce to lowest terms:

$$
\begin{aligned}
& =\frac{x(x y+1)}{y(x y+1)} \\
& =\frac{x}{y}
\end{aligned}
$$

## EXAMPLE 4 simplify:

$$
\frac{1-\frac{4}{x^{2}}}{1-\frac{1}{x}-\frac{6}{x^{2}}}
$$

SOLUTION The simplest way to simplify this complex fraction is to multiply the top and bottom by the LCD, $x^{2}$ :

$$
\frac{1-\frac{4}{x^{2}}}{1-\frac{1}{x}-\frac{6}{x^{2}}}=\frac{x^{2}\left(1-\frac{4}{x^{2}}\right)}{x^{2}\left(1-\frac{1}{x}-\frac{6}{x^{2}}\right)} \quad \text { Multiply top and bottom by } x^{2}
$$

$$
\begin{array}{ll}
=\frac{x^{2} \cdot 1-x^{2} \cdot \frac{4}{x^{2}}}{x^{2} \cdot 1-x^{2} \cdot \frac{1}{x}-x^{2} \cdot \frac{6}{x^{2}}} & \text { Distributive property } \\
=\frac{x^{2}-4}{x^{2}-x-6} & \text { Simplify } \\
=\frac{(x-2)(x+2)}{(x-3)(x+2)} & \text { Factor } \\
=\frac{x-2}{x-3} & \text { Reduce }
\end{array}
$$

In our next example, we find the relationship between a sequence of complex fractions and the numbers in the Fibonacci sequence.

EXAMPLE 5 Simplify each term in the following sequence, and then explain how this sequence is related to the Fibonacci sequence:

$$
1+\frac{1}{1+1}, 1+\frac{1}{1+\frac{1}{1+1}}, 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}, \ldots
$$

SOLUTION We can simplify our work somewhat if we notice that the first term $1+\frac{1}{1+1}$ is the larger denominator in the second term and that the second term is the largest denominator in the third term:

$$
\begin{aligned}
& \text { First term: } 1+\frac{1}{1+1}=1+\frac{1}{2}=\frac{2}{2}+\frac{1}{2}=\frac{3}{2} \\
& \text { Second term: } 1+\frac{1}{1+\frac{1}{1+1}}=1+\frac{1}{\frac{3}{2}}=1+\frac{2}{3}=\frac{3}{3}+\frac{2}{3}=\frac{5}{3} \\
& \text { Third term: } 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}=1+\frac{1}{\frac{5}{3}}=1+\frac{3}{5}=\frac{5}{5}+\frac{3}{5}=\frac{8}{5}
\end{aligned}
$$

Here are the simplified numbers for the first three terms in our sequence:

$$
\frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots
$$

Recall the Fibonacci sequence:

$$
1,1,2,3,5,8,13,21, \ldots
$$

As you can see, each term in the sequence we have simplified is the ratio of two consecutive numbers in the Fibonacci sequence. If the pattern continues in this manner, the next number in our sequence will be $\frac{13}{8}$.

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A ${ }^{1-5}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a complex fraction?
2. Explain one method of simplifying complex fractions.
3. How is a least common denominator used to simplify a complex fraction?
4. What types of complex fractions can be rewritten as division problems?

## Problem Set 6.6

Online support materials can be found at www.thomsonedu.com/login

Simplify each complex fraction.

1. $\frac{\frac{3}{4}}{\frac{1}{8}}$
2. $\frac{\frac{1}{3}}{\frac{5}{6}}$
3. $\frac{\frac{2}{3}}{4}$
4. $\frac{5}{\frac{1}{2}}$
5. $\frac{\frac{x^{2}}{y}}{\frac{x}{y^{3}}}$
6. $\frac{\frac{4 x^{3}}{y^{6}}}{\frac{8 x^{2}}{y^{7}}}$
7. $\frac{y+\frac{1}{x}}{x+\frac{1}{y}}$
8. $\frac{y-\frac{1}{x}}{x-\frac{1}{y}}$
9. $\frac{1+\frac{1}{a}}{1-\frac{1}{a}}$
10. $\frac{\frac{1}{a}-1}{\frac{1}{a}+1}$
11. $\frac{\frac{x+1}{x^{2}-9}}{\frac{2}{x+3}}$
12. $\frac{\frac{3}{x-5}}{\frac{x+1}{x^{2}-25}}$
13. $\frac{\frac{1}{a+2}}{\frac{1}{a^{2}-a-6}}$
14. $\frac{\frac{1}{a^{2}+5 a+6}}{\frac{1}{a+3}}$
15. $\frac{1-\frac{9}{y^{2}}}{1-\frac{1}{y}-\frac{6}{y^{2}}}$
16. $\frac{1-\frac{4}{y^{2}}}{1-\frac{2}{y}-\frac{8}{y^{2}}}$
17. $\frac{\frac{1}{y}+\frac{1}{x}}{\frac{1}{x y}}$
18. $\frac{\frac{1}{x y}}{\frac{1}{y}-\frac{1}{x}}$
19. $\frac{1-\frac{1}{a^{2}}}{1-\frac{1}{a}}$
20. $\frac{1+\frac{1}{a}}{1-\frac{1}{a^{2}}}$
21. $\frac{\frac{1}{10 x}-\frac{y}{10 x^{2}}}{\frac{1}{10}-\frac{y}{10 x}}$
22. $\frac{\frac{1}{2 x}+\frac{y}{2 x^{2}}}{\frac{1}{4}+\frac{y}{4 x}}$
23. $\frac{\frac{1}{a+1}+2}{\frac{1}{a+1}+3}$
24. $\frac{\frac{2}{a+1}+3}{\frac{3}{a+1}+4}$

Although the following problems do not contain complex fractions, they do involve more than one operation. Simplify inside the parentheses first, then multiply.
27. $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{x+1}\right)\left(1-\frac{1}{x+2}\right)$
28. $\left(1+\frac{1}{X}\right)\left(1+\frac{1}{X+1}\right)\left(1+\frac{1}{x+2}\right)$
29. $\left(1+\frac{1}{x+3}\right)\left(1+\frac{1}{x+2}\right)\left(1+\frac{1}{x+1}\right)$
30. $\left(1-\frac{1}{x+3}\right)\left(1-\frac{1}{x+2}\right)\left(1-\frac{1}{x+1}\right)$
31. Simplify each term in the following sequence.

$$
2+\frac{1}{2+1}, 2+\frac{1}{2+\frac{1}{2+1}}, 2+\frac{1}{2+\frac{1}{2+\frac{1}{2+1}}}, \cdots
$$

32. Simplify each term in the following sequence.

$$
2+\frac{3}{2+3}, 2+\frac{3}{2+\frac{3}{2+3}}, 2+\frac{3}{2+\frac{3}{2+\frac{3}{2+3}}}, \cdots
$$

Complete the following tables.
33.

| Number <br> $x$ | $\begin{gathered} \text { Reciprocal } \\ \frac{1}{x} \end{gathered}$ | Quotient $\frac{x}{\frac{1}{x}}$ | Square <br> $\boldsymbol{x}^{2}$ |
| :---: | :---: | :---: | :---: |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

34. 

| Number | Reciprocal | Quotient | Square |
| :---: | :---: | :---: | :---: |
| $x$ | $\frac{\mathbf{1}}{\boldsymbol{x}}$ | $\frac{\mathbf{x}}{\boldsymbol{x}}$ | $\boldsymbol{x}^{2}$ |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

35. 

| Number | Reciprocal | Sum | Quotient |
| :---: | :---: | :---: | :---: |
| $x$ | $\frac{1}{x}$ | $1+\frac{1}{x}$ | $\frac{1+\frac{1}{x}}{\frac{1}{x}}$ |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |

4
36.

| Number | Reciprocal | Difference | Quotient |
| :---: | :---: | :---: | :---: |
| $x$ | $\frac{1}{x}$ | $1-\frac{1}{x}$ | $\frac{1-\frac{1}{x}}{\frac{1}{x}}$ |
| 1 |  |  |  |
| 2 |  |  |  |
| 3 |  |  |  |
| 4 |  |  |  |

## Maintaining Your Skills

Solve each inequality.
37. $2 x+3<5$
38. $3 x-2>7$
39. $-3 x \leq 21$
40. $-5 x \geq-10$
41. $-2 x+8>-4$
42. $-4 x-1<11$
43. $4-2(x+1) \geq-2$
44. $6-2(x+3) \leq-8$

## Getting Ready for the Next Section

 Solve.45. $21=6 x$
46. $72=2 x$
47. $x^{2}+x=6$
48. $x^{2}+2 x=8$

## OBJECTIVES

A
Solve a proportion.
B Solve application problems involving proportions.

## Proportions

A proportion is two equal ratios; that is, if $\frac{a}{b}$ and $\frac{c}{d}$ are ratios, then:

$$
\frac{a}{b}=\frac{c}{d}
$$

is a proportion.
Each of the four numbers in a proportion is called a term of the proportion. We number the terms as follows:

$$
\begin{gathered}
\text { First term } \rightarrow \frac{a}{b}=\frac{c}{d} \leftarrow \text { Third term } \\
\text { Second term }
\end{gathered} \text { Fourth term }
$$

The first and fourth terms are called the extremes, and the second and third terms are called the means:

$$
\text { Means } \frac{a}{b}=\frac{c}{d}
$$

For example, in the proportion:

$$
\frac{3}{8}=\frac{12}{32}
$$

the extremes are 3 and 32 , and the means are 8 and 12 .

Means-Extremes Property If $a, b, c$, and $d$ are real numbers with $b \neq 0$
and $d \neq 0$, then

$$
\begin{array}{ll}
\text { if } & \frac{a}{b}=\frac{c}{d} \\
\text { then } & a d=b c
\end{array}
$$

In words: In any proportion, the product of the extremes is equal to the product of the means.

This property of proportions comes from the multiplication property of equality. We can use it to solve for a missing term in a proportion.

## EXAMPLE 1 Solve the proportion $\frac{3}{x}=\frac{6}{7}$ for $x$.

SOLUTION We could solve for $x$ by using the method developed in Section 6.4; that is, multiplying both sides by the LCD $7 x$. Instead, let's use our new meansextremes property:

$$
\begin{aligned}
\frac{3}{x} & =\frac{6}{7} & & \text { Extremes are } 3 \text { and 7; means are } x \text { and } 6 \\
21 & =6 x & & \text { Product of extremes }=\text { product of means } \\
\frac{21}{6} & =x & & \text { Divide both sides by } 6 \\
x & =\frac{7}{2} & & \text { Reduce to lowest terms }
\end{aligned}
$$

## EXAMPLE 2 Solve for $x: \frac{x+1}{2}=\frac{3}{x}$.

SOLUTION Again, we want to point out that we could solve for $x$ by using the method we used in Section 6.4. Using the means-extremes property is simply an alternative to the method developed in Section 6.4:

| $\frac{x+1}{2}$ | $=\frac{3}{x}$ | Extremes are $x+1$ and $x ;$ <br> means are 2 and 3 <br> $x^{2}+x$ |
| ---: | :--- | :--- |
| $=6$ | Product of extremes $=$ <br> product of means |  |
| $x^{2}+x-6$ | $=0$ | Standard form for a <br> quadratic equation |
| $(x+3)(x-2)=0$ | Factor |  |
| $x+3=0$ | or | $x-2=0$ |
| $x=-3$ | or | Set factors equal to 0 |
| $x$ | $=2$ |  |

This time we have two solutions: -3 and 2 .
EXAMPLE 3
A manufacturer knows that during a production run, 8 out of every 100 parts produced by a certain machine will be defective. If the machine produces 1,450 parts, how many can be expected to be defective?
SOLUTION The ratio of defective parts to total parts produced is $\frac{8}{100}$. If we let $x$ represent the numbers of defective parts out of the total of 1,450 parts, then we can write this ratio again as $\frac{x}{1,450}$. This gives us a proportion to solve:


Extremes are $x$ and 100; means are 1,450 and 8

Product of extremes $=$ product of means

$$
x=116
$$

The manufacturer can expect 116 defective parts out of the total of 1,450 parts if the machine usually produces 8 defective parts for every 100 parts it produces.

## EXAMPLE 4

The scale on a map indicates that 1 inch on the map corresponds to an actual distance of 85 miles. Two cities are 3.5 inches apart on the map. What is the actual distance between the two cities?


SOLUTION We let $x$ represent the actual distance between the two cities. The proportion is

$$
\begin{aligned}
& \begin{array}{c}
\text { Miles } \longrightarrow \\
\text { Inches } \longrightarrow
\end{array} \frac{x}{3.5}=\frac{85}{1} \longleftarrow \text { Miles } \\
& x \cdot 1=3.5(85) \\
& x=297.5 \text { miles }
\end{aligned}
$$

EXAMPLE 5 A woman drives her car 270 miles in 6 hours. If she continues at the same rate, how far will she travel in 10 hours?

SOLUTION We let $x$ represent the distance traveled in 10 hours. Using $x$, we translate the problem into the following proportion:

$$
\begin{aligned}
& \text { Miles } \longrightarrow \\
& \text { Hours } \longrightarrow \\
& 10
\end{aligned} \frac{x}{6} \longleftarrow \text { Mours }
$$

Notice that the two ratios in the proportion compare the same quantities. That is, both ratios compare miles to hours. In words this proportion says:

$x$ miles is to 10 hours as 270 miles is to 6 hours

| $\downarrow$ | $\downarrow$ | $\downarrow$ |
| :---: | :---: | :---: |
| $\frac{X}{10}$ | $=$ | $\frac{270}{6}$ |

Next, we solve the proportion.

$$
\begin{aligned}
x \cdot 6 & =10 \cdot 270 \\
x \cdot 6 & =2,700 \\
\frac{x \cdot 6}{6} & =\frac{2,700}{6} \\
x & =450 \text { miles }
\end{aligned}
$$

If the woman continues at the same rate, she will travel 450 miles in 10 hours.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1,2 <br> B 3-5

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a proportion?
2. What are the means and extremes of a proportion?
3. What is the relationship between the means and the extremes in a proportion? (It is called the means-extremes property of proportions.)
4. How are ratios and proportions related?

## Problem Set 6.7

Solve each of the following proportions.

1. $\frac{x}{2}=\frac{6}{12}$
2. $\frac{x}{4}=\frac{6}{8}$
3. $\frac{2}{5}=\frac{4}{x}$
4. $\frac{3}{8}=\frac{9}{x}$
5. $\frac{10}{20}=\frac{20}{x}$
6. $\frac{15}{60}=\frac{60}{x}$
7. $\frac{a}{3}=\frac{5}{12}$
8. $\frac{a}{2}=\frac{7}{20}$
9. $\frac{2}{x}=\frac{6}{7}$
10. $\frac{4}{x}=\frac{6}{7}$
11. $\frac{x+1}{3}=\frac{4}{x}$
12. $\frac{x+1}{6}=\frac{7}{x}$
13. $\frac{x}{2}=\frac{8}{x}$
14. $\frac{x}{9}=\frac{4}{x}$
15. $\frac{4}{a+2}=\frac{a}{2}$
16. $\frac{3}{a+2}=\frac{a}{5}$
17. $\frac{1}{x}=\frac{x-5}{6}$
18. $\frac{1}{x}=\frac{x-6}{7}$

## Applying the Concepts

19. Baseball A baseball player gets 6 hits in the first 18 games of the season. If he continues hitting at the same rate, how many hits will he get in the first 45 games?
20. Basketball A basketball player makes 8 of 12 free throws in the first game of the season. If she shoots with the same accuracy in the second game, how many of the 15 free throws she attempts will she make?
21. Mixture Problem A solution contains 12 milliliters of alcohol and 16 milliliters of water. If another solution is to have the same concentration of alcohol in water but is to contain 28 milliliters of water, how much alcohol must it contain?
22. Mixture Problem A solution contains 15 milliliters of HCl and 42 milliliters of water. If another solution is to have the same concentration of HCl in water but is to contain 140 milliliters of water, how much HCl must it contain?
23. Nutrition If 100 grams of ice cream contains 13 grams of fat, how much fat is in 350 grams of ice cream?
24. Nutrition A 6-ounce serving of grapefruit juice contains 159 grams of water. How many grams of water are in 20 ounces of grapefruit juice?
25. Map Reading A map is drawn so that every 3.5 inches on the map corresponds to an actual distance of 100 miles. If the actual distance between the two cities is 420 miles, how far apart are they on the map?
26. Map Reading The scale on a map indicates that 1 inch on the map corresponds to an actual distance of 105 miles. Two cities are 4.5 inches apart on the map. What is the actual distance between the two cities?

On the map shown here, 0.5 inches on the map is equal to 5 miles. Use the information from the map to work Problems 27 through 30.

27. Map Reading Suppose San Luis Obispo is 1.25 inches from Arroyo Grande on the map. How far apart are the two cities?
28. Map Reading Suppose San Luis Obispo is 3.4 inches from Paso Robles on the map. How far apart are the two cities?
29. Driving Time If Ava drives from Paso Robles to San Luis Obispo in 46 minutes, how long will it take her to drive from San Luis Obispo to Arroyo Grande, if she drives at the same speed? Round to the nearest minute.
30. Driving Time If Brooke drives from Arroyo Grande to San Luis Obispo in 15 minutes, how long will it take her to drive from San Luis Obispo to Paso Robles, if she drives at the same speed? Round to the nearest minute.
31. Distance A man drives his car 245 miles in 5 hours. At this rate, how far will he travel in 7 hours?
32. Distance An airplane flies 1,380 miles in 3 hours. How far will it fly in 5 hours?
33. Improving Your Quantitative Literacy Searching Google News for news articles that involve proportions produces a number of items involving proportions, the top three of which are shown below. Search the Internet for current news articles by searching on the word proportion. Then find an article that the material in this section helps you understand.

## Callahan Decries Commotion Over Gesture

... "I don't use that type of demeanor, and I never have. This is way blown out of proportion. I don't know where we get all these . . .
Rapid City Journal, SD, November 2, 2005

## Council to Look Again at Inn Plans

... meeting months ago. "On paper the size of the inn seemed to be all out of proportion with the rest of the park," he said. "It was . . .
Wairarapa Times Age, New Zealand, November 2, 2005

## The Dilemma of Halting Clinical Trials Early

... What's more, the authors found, a large proportion of the trials are funded by the drugs and devices industry, and involve drugs with large market potential in . . .
MedPage Today, NJ, November 1, 2005

## Maintaining Your Skills

Reduce to lowest terms.
34. $\frac{x^{2}-x-6}{x^{2}-9}$
35. $\frac{x y+5 x+3 y+15}{x^{2}+a x+3 x+3 a}$

Multiply or divide, as indicated.
36. $\frac{x^{2}-25}{x+4} \cdot \frac{2 x+8}{x^{2}-9 x+20}$
37. $\frac{3 x+6}{x^{2}+4 x+3} \div \frac{x^{2}+x-2}{x^{2}+2 x-3}$

Add or subtract, as indicated.
38. $\frac{x}{x^{2}-16}+\frac{4}{x^{2}-16}$
39. $\frac{2}{x^{2}-1}-\frac{5}{x^{2}+3 x-4}$

## Getting Ready for the Next Section

Use the formula $y=5 x$ to find $y$ when
40. $x=4$
41. $x=3$

Use the formula $y=\frac{20}{x}$ to find $y$ when
42. $x=10$
43. $x=5$

Use the formula $y=2 x^{2}$ to find $x$ when
44. $y=50$
45. $y=72$

Use the formula $y=K x$ to find $K$ when
46. $y=15$ and $x=3$
47. $y=72$ and $x=4$

Use the formula $y=K x^{2}$ to find $K$ when
48. $y=32$ and $x=4$
49. $y=45$ and $x=3$

\section*{| Chapter 6 SUMMARY\ |
| :---: | :---: |
| Rational Numbers [6.1] | <br> EXAMPLES}

1. We can reduce $\frac{6}{8}$ to lowest terms by dividing the numerator and denominator by their greatest common factor 2 :

$$
\frac{6}{8}=\frac{2 \cdot 3}{2 \cdot 4}=\frac{3}{4}
$$

2. We reduce rational expressions to lowest terms by factoring the numerator and denominator and then dividing out any factors they have in common:
$\frac{x-3}{x^{2}-9}=\frac{x-3}{(x-3)(x+3)}=\frac{1}{x+3}$

## Rational Expressions [6.1]

Any expression of the form $\frac{P}{Q}$, where $P$ and $Q$ are polynomials $(Q \neq 0)$, is a rational expression.

Multiplying or dividing the numerator and denominator of a rational expression by the same nonzero quantity always produces a rational expression equivalent to the original one.

## Multiplication [6.2]

3. $\frac{x-1}{x^{2}+2 x-3} \cdot \frac{x^{2}-9}{x-2}$
$=\frac{x-4}{(x+3)(x-1)} \cdot \frac{(x-3)(x+3)}{x-2}$
$=\frac{x-3}{x-2}$
To multiply two rational numbers or two rational expressions, multiply numerators, multiply denominators, and divide out any factors common to the numerator and denominator:

> For rational numbers $\frac{a}{b}$ and $\frac{c}{d}, \quad \frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}$
> For rational expressions $\frac{P}{Q}$ and $\frac{R}{S}, \quad \frac{P}{Q} \cdot \frac{R}{S}=\frac{P R}{Q S}$

## Division [6.2]

4. $\frac{2 x}{x^{2}-25} \div \frac{4}{x-5}$

$$
\begin{aligned}
& =\frac{2 x}{(x-5)(x+5)} \cdot \frac{(x-5)}{4} \\
& =\frac{x}{2(x+5)}
\end{aligned}
$$

To divide by a rational number or rational expression, simply multiply by its reciprocal:

For rational numbers $\frac{a}{b}$ and $\frac{c}{d^{\prime}}, \quad \frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}$
For rational expressions $\frac{P}{Q}$ and $\frac{R}{S}, \quad \frac{P}{Q} \div \frac{R}{S}=\frac{P}{Q} \cdot \frac{S}{R}$

## Addition [6.3]

5. $\frac{3}{x-1}+\frac{x}{2}$
$=\frac{3}{x-1} \cdot \frac{\mathbf{2}}{\mathbf{2}}+\frac{x}{2} \cdot \frac{\boldsymbol{x}-\mathbf{1}}{\boldsymbol{x}-\mathbf{1}}$
$=\frac{6}{2(x-1)}+\frac{x^{2}-x}{2(x-1)}$
$=\frac{x^{2}-x+6}{2(x-1)}$
To add two rational numbers or rational expressions, find a common denominator, change each expression to an equivalent expression having the common denominator, then add numerators and reduce if possible:

For rational numbers $\frac{a}{c}$ and $\frac{b}{c}, \quad \frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$
For rational expressions $\frac{P}{S}$ and $\frac{Q}{S}, \quad \frac{P}{S}+\frac{Q}{S}=\frac{P+Q}{S}$

## Subtraction [6.3]

6. $\frac{x}{x^{2}-4}-\frac{2}{x^{2}-4}$
$=\frac{x-2}{x^{2}-4}$
$=\frac{x-2}{(x-2)(x+2)}$
$=\frac{1}{x+2}$
To subtract a rational number or rational expression, simply add its opposite:
For rational numbers $\frac{a}{c}$ and $\frac{b}{c}, \quad \frac{a}{c}-\frac{b}{c}=\frac{a}{c}+\left(\frac{-b}{c}\right)$
For rational expressions $\frac{P}{S}$ and $\frac{Q}{S}, \quad \frac{P}{S}-\frac{Q}{S}=\frac{P}{S}+\left(\frac{-Q}{S}\right)$

## Equations [6.4]

7. Solve $\frac{1}{2}+\frac{3}{x}=5$.

$$
\begin{aligned}
2 x\left(\frac{1}{2}\right)+2 x\left(\frac{3}{x}\right) & =2 x(5) \\
x+6 & =10 x \\
6 & =9 x \\
x & =\frac{2}{3}
\end{aligned}
$$

To solve equations involving rational expressions, first find the least common denominator (LCD) for all denominators. Then multiply both sides by the LCD and solve as usual. Be sure to check all solutions in the original equation to be sure there are no undefined terms.

## Complex Fractions [6.6]

8. $\frac{1-\frac{4}{x}}{x-\frac{16}{x}}=\frac{x\left(1-\frac{4}{x}\right)}{x\left(x-\frac{16}{x}\right)}$

$$
\begin{aligned}
& =\frac{x-4}{x^{2}-16} \\
& =\frac{x-4}{(x-4)(x+4)} \\
& =\frac{1}{x+4}
\end{aligned}
$$

A rational expression that contains a fraction in its numerator or denominator is called a complex fraction. The most common method of simplifying a complex fraction is to multiply the top and bottom by the LCD for all denominators.

## Ratio and Proportion [6.1, 6.7]

9. Solve for $x: \frac{3}{x}=\frac{5}{20}$.

$$
\begin{aligned}
3 \cdot 20 & =5 \cdot x \\
60 & =5 x \\
x & =12
\end{aligned}
$$

The ratio of $a$ to $b$ is:

$$
\frac{a}{b}
$$

Two equal ratios form a proportion. In the proportion

$$
\frac{a}{b}=\frac{c}{d}
$$

$a$ and $d$ are the extremes, and $b$ and $c$ are the means. In any proportion the product of the extremes is equal to the product of the means.

## Chapter 6 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the odd-numbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the evennumbered problems for a second practice test.
The numbers in brackets refer to the sections of the text in which similar problems can be found.

Reduce to lowest terms. Also specify any restriction on the variable. [6.1]

1. $\frac{7}{14 x-28}$
2. $\frac{a+6}{a^{2}-36}$
3. $\frac{8 x-4}{4 x+12}$
4. $\frac{x+4}{x^{2}+8 x+16}$
5. $\frac{3 x^{3}+16 x^{2}-12 x}{2 x^{3}+9 x^{2}-18 x}$
6. $\frac{x+2}{x^{4}-16}$
7. $\frac{x^{2}+5 x-14}{x+7}$
8. $\frac{a^{2}+16 a+64}{a+8}$
9. $\frac{x y+b x+a y+a b}{x y+5 x+a y+5 a}$

Multiply or divide as indicated. [6.2]
10. $\frac{3 x+9}{x^{2}} \cdot \frac{x^{3}}{6 x+18}$
11. $\frac{x^{2}+8 x+16}{x^{2}+x-12} \div \frac{x^{2}-16}{x^{2}-x-6}$
12. $\left(a^{2}-4 a-12\right)\left(\frac{a-6}{a+2}\right)$
13. $\frac{3 x^{2}-2 x-1}{x^{2}+6 x+8} \div \frac{3 x^{2}+13 x+4}{x^{2}+8 x+16}$

Find the following sums and differences. [6.3]
14. $\frac{2 x}{2 x+3}+\frac{3}{2 x+3}$
15. $\frac{x^{2}}{x-9}-\frac{18 x-81}{x-9}$
16. $\frac{a+4}{a+8}-\frac{a-9}{a+8}$
17. $\frac{x}{x+9}+\frac{5}{x}$
18. $\frac{5}{4 x+20}+\frac{x}{x+5}$
19. $\frac{3}{x^{2}-36}-\frac{2}{x^{2}-4 x-12}$
20. $\frac{3 a}{a^{2}+8 a+15}-\frac{2}{a+5}$

Solve each equation. [6.4]
21. $\frac{3}{x}+\frac{1}{2}=\frac{5}{x}$
22. $\frac{a}{a-3}=\frac{3}{2}$
23. $1-\frac{7}{x}=\frac{-6}{x^{2}}$
24. $\frac{3}{x+6}-\frac{1}{x-2}=\frac{-8}{x^{2}+4 x-12}$
25. $\frac{2}{y^{2}-16}=\frac{10}{y^{2}+4 y}$
26. Number Problem The sum of a number and 7 times its reciprocal is $\frac{16}{3}$. Find the number. [6.5]
27. Distance, Rate, and Time A boat travels 48 miles up a river in the same amount of time it takes to travel 72 miles down the same river. If the current is 3 miles per hour, what is the speed of the boat in still water? [6.5]
28. Filling a Pool An inlet pipe can fill a pool in 21 hours, whereas an outlet pipe can empty it in 28 hours. If both pipes are left open, how long will it take to fill the pool? [6.5]

Simplify each complex fraction. [6.6]
29. $\frac{\frac{x+4}{x^{2}-16}}{\frac{2}{x-4}}$
30. $\frac{1-\frac{9}{y^{2}}}{1+\frac{4}{y}-\frac{21}{y^{2}}}$
31. $\frac{\frac{1}{a-2}+4}{\frac{1}{a-2}+1}$
32. Write the ratio of 40 to 100 as a fraction in lowest terms. [6.7]
33. If there are 60 seconds in 1 minute, what is the ratio of 40 seconds to 3 minutes? [6.7]

Solve each proportion. [6.7]
34. $\frac{x}{9}=\frac{4}{3}$
35. $\frac{a}{3}=\frac{12}{a}$
36. $\frac{8}{x-2}=\frac{x}{6}$

## Chapter 6 Projects

## GROUP PROJECT Rational Expressions

## Number of People 3

Time Needed 10-15 minutes
Equipment Pencil and paper
Procedure The four problems shown here all involve the same rational expressions. Often, students who have worked problems successfully on their homework have trouble when they take a test on rational expressions because the problems are mixed up and do not have similar instructions. Noticing similarities and differences between the types of problems involving rational expressions can help with this situation.

Problem 1: Add: $\frac{-2}{x^{2}-2 x-3}+\frac{3}{x^{2}-9}$
Problem 2: Divide: $\frac{-2}{x^{2}-2 x-3} \div \frac{3}{x^{2}-9}$
Problem 3: Solve: $\frac{-2}{x^{2}-2 x-3}+\frac{3}{x^{2}-9}=-1$
Problem 4: Simplify: $\frac{\frac{-2}{x^{2}-2 x-3}}{\frac{3}{x^{2}-9}}$

1. Which problems here do not require the use of a least common denominator?
2. Which two problems involve multiplying by the least common denominator?
3. Which of the problems will have an answer that is one or two numbers but no variables?
4. Work each of the four problems.

## Ferris Wheel and The Third Man



Among the large Ferris wheels built around the turn of the twentieth century was one built in Vienna in 1897. It is the only one of those large wheels that is still in operation today. Known as the Riesenrad, it has a diameter of 197 feet and can carry a total of 800 people. A brochure that gives some statistics associated with the Riesenrad indicates that passengers riding it travel at 2 feet 6 inches per second. You can check the accuracy of this number by watching the movie The Third Man. In the movie, Orson Welles rides the Riesenrad through one complete revolution. Play The Third Man on a VCR so you can view the Riesenrad in operation. Use the pause button and the timer on the VCR to time how long it takes Orson Welles to ride once around the wheel. Then calculate his average speed during the ride. Use your results to either prove or disprove the claim that passengers travel at 2 feet 6 inches per second on the Riesenrad. When you have finished, write your procedures and results in essay form.

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## Transitions

## 7


uppose you decide to buy a cellular phone and are trying to decide between two rate plans. Plan A is $\$ 18.95$ per month plus $\$ 0.48$ for each minute, or fraction of a minute, that you use the phone. Plan B is $\$ 34.95$ per month plus $\$ 0.36$ for each minute, or fraction of a minute. The monthly $\operatorname{cost} C(x)$ for each plan can be represented with a linear equation in two variables:

$$
\begin{array}{ll}
\text { Plan A: } & C(x)=0.48 x+18.95 \\
\text { Plan B: } & C(x)=0.36 x+34.95
\end{array}
$$

To compare the two plans, we use the table and graph shown below.

| Monthly Cellular Phone Charges <br> Number of <br> Num <br> Minutes $\boldsymbol{x}$ | Monthly Cost |  |
| :--- | :---: | :---: |
| Plan A (\$) | Plan B (\$) |  |
| 0 | 18.95 | 34.95 |
| 40 | 38.15 | 49.35 |
| 80 | 57.35 | 63.75 |
| 120 | 76.55 | 78.15 |
| 160 | 95.75 | 92.55 |
| 200 | 114.95 | 106.95 |
| 240 | 134.15 | 121.35 |



FIGURE 1
The point of intersection of the two lines in Figure 1 is the point at which the monthly costs of the two plans are equal. Among the topics in this chapter is a review of the methods of finding that point of intersection.

## CHAPTER OUTLINE

7.1 Review of Solving Equations
7.2 Equations with Absolute Value
7.3 Compound Inequalities and Interval Notation
7.4 Inequalities Involving Absolute Value
7.5 Review of Systems of Linear Equations in Two Variables
7.6 Systems of Linear Equations in Three Variables
7.7 Linear Inequalities in Two Variables
7.8 Systems of Linear Inequalities

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## Review of Solving Equations

## OBJECTIVES

A Solve a linear equation in one variable.

B Solve equations by factoring.

C Apply the Blueprint for Problem Solving to solve application problems whose solutions involve quadratic equations

This chapter marks the transition from introductory algebra to intermediate algebra. Some of the material here is a review of material we covered earlier, and some of the material here is new. If you cover all the sections in this chapter, you will review all the important points contained in the first six chapters of the book. So, it is a good idea to put some extra time and effort into this chapter to insure that you get a good start with the rest of the course. Let's begin by reviewing the methods we use to solve equations.

## Linear Equations in One Variable

A linear equation in one variable is any equation that can be put in the form

$$
a x+b=c
$$

where $a, b$, and $c$ are constants and $a \neq 0$. For example, each of the equations

$$
5 x+3=2 \quad 2 x=7 \quad 2 x+5=0
$$

are linear because they can be put in the form $a x+b=c$. In the first equation, $5 x, 3$, and 2 are called terms of the equation: $5 x$ is a variable term; 3 and 2 are constant terms.

DEFINITION The solution set for an equation is the set of all numbers that, when used in place of the variable, make the equation a true statement.

DEFINITION Two or more equations with the same solution set are called
equivalent equations.

EXAMPLE 1 The equations $2 x-5=9, x-1=6$, and $x=7$ are all equivalent equations because the solution set for each is $\{7\}$.

## Properties of Equality

The first property states that adding the same quantity to both sides of an equation preserves equality. Or, more importantly, adding the same amount to both sides of an equation never changes the solution set. This property is called the addition property of equality and is stated in symbols as follows.

Addition Property of Equality For any three algebraic expressions, $A, B$, and $C$,

$$
\begin{array}{llrl}
\text { if } & & A & =B \\
& \text { then } & A+C & =B+C
\end{array}
$$

In words: Adding the same quantity to both sides of an equation will not change the solution set.

## Note

Because subtraction is defined in terms of addition and division is defined in terms of multiplication, we do not need to introduce separate properties for subtraction and division. The solution set for an equation will never be changed by subtracting the same amount from both sides or by dividing both sides by the same nonzero quantity.

Our second new property is called the multiplication property of equality and is stated as follows.

Multiplication Property of Equality For any three algebraic expressions $A, B$, and $C$, where $C \neq 0$,

|  | if | $A$ | $=B$ |
| ---: | :--- | ---: | :--- |
|  | then | $A C$ | $=B C$ |

In words: Multiplying both sides of an equation by the same nonzero quantity will not change the solution set.

## EXAMPLE 2 Find the solution set for $3 a-5=-6 a+1$.

SOLUTION To solve for $a$ we must isolate it on one side of the equation. Let's decide to isolate $a$ on the left side by adding $6 a$ to both sides of the equation.

$$
\begin{aligned}
3 a-5 & =-6 a+1 & & \\
3 a+\mathbf{6 a - 5} & =-6 a+\mathbf{6 a + 1} & & \text { Add } \mathbf{6} \boldsymbol{a} \text { to both sides } \\
9 a-5 & =1 & & \\
9 a-5+\mathbf{5} & =1+\mathbf{5} & & \text { Add } 5 \text { to both sides } \\
9 a & =6 & & \\
\frac{\mathbf{1}}{\mathbf{9}}(9 a) & =\frac{\mathbf{1}}{\mathbf{9}}(6) & & \text { Multiply both sides by } \frac{1}{9} \\
a & =\frac{2}{3} & & \frac{1}{9}(6)=\frac{6}{9}=\frac{2}{3}
\end{aligned}
$$

The solution set is $\left\{\frac{2}{3}\right\}$.
We can check our solution in Example 2 by replacing $a$ in the original equation with $\frac{2}{3}$.

When

$$
a=\frac{2}{3}
$$

the equation

$$
3 a-5=-6 a+1
$$

becomes

$$
\begin{aligned}
3\left(\frac{2}{3}\right)-5 & =-6\left(\frac{2}{3}\right)+1 \\
2-5 & =-4+1 \\
-3 & =-3 \quad \text { A true statement }
\end{aligned}
$$

There will be times when we solve equations and end up with a negative sign in front of the variable. The next example shows how to handle this situation.

## EXAMPLE 3 Solve each equation.

a. $-x=4$
b. $-y=-8$

SOLUTION Neither equation can be considered solved because of the negative sign in front of the variable. To eliminate the negative signs we simply multiply both sides of each equation by -1 .
a. $-x=4$
$-\mathbf{1}(-x)=-\mathbf{1}(4)$
$x=-4$
b. $-y=-8$
$-\mathbf{1}(-y)=\mathbf{- 1}(-8)$
Multiply each side by -1
$y=8$

EXAMPLE 4 Solve $\frac{2}{3} x+\frac{1}{2}=-\frac{3}{8}$.
SOLUTION We can solve this equation by applying our properties and working with fractions, or we can begin by eliminating the fractions. Let's use both methods.

Method 1 Working with the fractions.

$$
\begin{aligned}
\frac{2}{3} x+\frac{1}{2}+\left(-\frac{\mathbf{1}}{\mathbf{2}}\right) & =-\frac{3}{8}+\left(-\frac{\mathbf{1}}{\mathbf{2}}\right) & & \text { Add }-\frac{1}{2} \text { to each side } \\
\frac{2}{3} x & =-\frac{7}{8} & & -\frac{3}{8}+\left(-\frac{1}{2}\right)=-\frac{3}{8}+\left(-\frac{4}{8}\right) \\
\frac{\mathbf{3}}{\mathbf{2}}\left(\frac{2}{3} x\right) & =\frac{\mathbf{3}}{\mathbf{2}}\left(-\frac{7}{8}\right) & & \text { Multiply each side by } \frac{3}{2} \\
x & =-\frac{21}{16} & &
\end{aligned}
$$

Method 2 Eliminating the fractions in the beginning.
Our original equation has denominators of 3,2 , and 8 . The least common denominator, abbreviated LCD, for these three denominators is 24 , and it has the property that all three denominators will divide it evenly. If we multiply both sides of our equation by 24 , each denominator will divide into 24 , and we will be left with an equation that does not contain any denominators other than 1.

$$
\begin{aligned}
24\left(\frac{2}{3} x+\frac{1}{2}\right) & =\mathbf{2 4}\left(-\frac{3}{8}\right) & & \text { Multiply each side by the LCD } 24 \\
24\left(\frac{2}{3} x\right)+24\left(\frac{1}{2}\right) & =24\left(-\frac{3}{8}\right) & & \text { Distributive property on the left side } \\
16 x+12 & =-9 & & \text { Multiply } \\
16 x & =-21 & & \text { Add }-12 \text { to each side } \\
x & =-\frac{21}{16} & & \text { Multiply each side by } \frac{1}{16}
\end{aligned}
$$

## Note

We are placing a question mark over the equal sign because we don't know yet if the expression on the left will be equal to the expression on the right.

Check To check our solution, we substitute $x=-\frac{21}{16}$ back into our original equation to obtain

$$
\begin{aligned}
\frac{2}{3}\left(-\frac{21}{16}\right)+\frac{1}{2} & \stackrel{?}{=}-\frac{3}{8} \\
-\frac{7}{8}+\frac{1}{2} & \stackrel{?}{=}-\frac{3}{8} \\
-\frac{7}{8}+\frac{4}{8} & \stackrel{?}{=}-\frac{3}{8} \\
-\frac{3}{8} & =-\frac{3}{8} \quad \text { A true statement }
\end{aligned}
$$

## EXAMPLE 5 Solve the equation $0.06 x+0.05(10,000-x)=560$.

SOLUTION We can solve the equation in its original form by working with the decimals, or we can eliminate the decimals first by using the multiplication property of equality and solve the resulting equation. Here are both methods.

Method 1 Working with the decimals.

$$
\begin{aligned}
0.06 x+0.05(10,000-x) & =560 & & \text { Original equation } \\
0.06 x+0.05(10,000)-0.05 x & =560 & & \text { Distributive property } \\
0.01 x+500 & =560 & & \text { Simplify the left side } \\
0.01 x+500+(-\mathbf{5 0 0}) & =560+(-\mathbf{5 0 0}) & & \text { Add }-500 \text { to each side } \\
0.01 x & =60 & & \\
\frac{0.01 x}{\mathbf{0 . 0 1}} & =\frac{60}{\mathbf{0 . 0 1}} & & \text { Divide each side } \\
x & =6,000 & & \text { by } 0.01
\end{aligned}
$$

Method 2 Eliminating the decimals in the beginning: To move the decimal point two places to the right in $0.06 x$ and 0.05 , we multiply each side of the equation by 100 .

$$
\begin{aligned}
0.06 x+0.05(10,000-x) & =560 & & \text { Original equation } \\
0.06 x+500-0.05 x & =560 & & \text { Distributive property } \\
\mathbf{1 0 0}(0.06 x)+\mathbf{1 0 0}(500)-\mathbf{1 0 0}(0.05 x) & =\mathbf{1 0 0}(560) & & \text { Multiply each side } \\
6 x+50,000-5 x & =56,000 & & \text { by } 100 \\
x+50,000 & =56,000 & & \text { Simplify the left side } \\
x & =6,000 & & \begin{array}{l}
\text { Add }-50,000 \text { to each } \\
\text { side }
\end{array}
\end{aligned}
$$

Using either method, the solution to our equation is 6,000.
Check We check our work (to be sure we have not made a mistake in applying the properties or in arithmetic) by substituting 6,000 into our original equation and simplifying each side of the result separately, as the following shows.

$$
\begin{aligned}
0.06(\mathbf{6 , 0 0 0})+0.05(10,000-\mathbf{6 , 0 0 0}) & \stackrel{?}{=} 560 \\
0.06(6,000)+0.05(4,000) & \stackrel{?}{=} 560 \\
360+200 & \stackrel{?}{=} 560 \\
560 & =560 \quad \text { A true statement }
\end{aligned}
$$

## Note

It would be a mistake to subtract 3 from 8 first because the rule for order of operations indicates we are to do multiplication before subtraction.

Here is a list of steps to use as a guideline for solving linear equations in one variable.

## Strategy for Solving Linear Equations in One Variable

Step 1: a. Use the distributive property to separate terms, if necessary.
b. If fractions are present, consider multiplying both sides by the LCD to eliminate the fractions. If decimals are present, consider multiplying both sides by a power of 10 to clear the equation of decimals.
c. Combine similar terms on each side of the equation.

Step 2: Use the addition property of equality to get all variable terms on one side of the equation and all constant terms on the other side. A variable term is a term that contains the variable. A constant term is a term that does not contain the variable (the number 3, for example).

Step 3: Use the multiplication property of equality to get the variable by itself on one side of the equation.

Step 4: Check your solution in the original equation to be sure that you have not made a mistake in the solution process.

As you will see as you work through the problems in the problem set, it is not always necessary to use all four steps when solving equations. The number of steps used depends on the equation. In Example 6 there are no fractions or decimals in the original equation, so Step 1 b will not be used.

## EXAMPLE 6 Solve the equation $8-3(4 x-2)+5 x=35$.

SOLUTION We must begin by distributing the -3 across the quantity $4 x-2$.

| Step 1: | a. | Original equation Distributive property |
| :---: | :---: | :---: |
|  | c. $\quad-7 x+14=35$ | Simplify |
| Step 2: | $-7 x=21$ | Add -14 to each side |
| Step 3: | $x=-3$ | Multiply by $-\frac{1}{7}$ |

Step 4: When $x$ is replaced by -3 in the original equation, a true statement results. Therefore, -3 is the solution to our equation.

## Identities and Equations with No Solution

Two special cases are associated with solving linear equations in one variable, each of which is illustrated in the following examples.

## EXAMPLE 7 Solve for $x: 2(3 x-4)=3+6 x$.

SOLUTION Applying the distributive property to the left side gives us

$$
6 x-8=3+6 x \quad \text { Distributive property }
$$

Now, if we add $-6 x$ to each side, we are left with the following

$$
-8=3
$$

which is a false statement. This means that there is no solution to our equation. Any number we substitute for $x$ in the original equation will lead to a similar false statement.

## EXAMPLE 8 Solve for $x:-15+3 x=3(x-5)$.

SOLUTION We start by applying the distributive property to the right side.

$$
-15+3 x=3 x-15 \quad \text { Distributive property }
$$

If we add $-3 x$ to each side, we are left with the true statement

$$
-15=-15
$$

In this case, our result tells us that any number we use in place of $x$ in the original equation will lead to a true statement. Therefore, all real numbers are solutions to our equation. We say the original equation is an identity because the left side is always identically equal to the right side.

## Solving Equations by Factoring

Next we will use our knowledge of factoring to solve equations. Most of the equations we will see are quadratic equations. Here is the definition of a quadratic equation.

DEFINITION Any equation that can be written in the form

$$
a x^{2}+b x+c=0
$$

where $a, b$, and $c$ are constants and $a$ is not $0(a \neq 0)$ is called a quadratic equation. The form $a x^{2}+b x+c=0$ is called standard form for quadratic equations.

Each of the following is a quadratic equation:

$$
2 x^{2}=5 x+3 \quad 5 x^{2}=75 \quad 4 x^{2}-3 x+2=0
$$

Notation For a quadratic equation written in standard form, the first term $a x^{2}$ is called the quadratic term; the second term bx is the linear term; and the last term $c$ is called the constant term.

In the past we have noticed that the number 0 is a special number. There is another property of 0 that is the key to solving quadratic equations. It is called the zero-factor property.

Zero-Factor Property For all real numbers $r$ and $s$,
$r \cdot s=0 \quad$ if and only if $\quad r=0 \quad$ or $\quad s=0$ (or both)

## EXAMPLE 9 Solve $x^{2}-2 x-24=0$.

## Note

What the zero-factor property says in words is that we can't multiply and get 0 without multiplying by 0 ; that is, if we multiply two numbers and get 0 , then one or both of the original two numbers we multiplied must have been 0 .

## Note

We are placing a question mark over the equal sign because we don't know yet if the expression on the left will be equal to the expression on the right.

SOLUTION We begin by factoring the left side as $(x-6)(x+4)$ and get

$$
(x-6)(x+4)=0
$$

Now both $(x-6)$ and $(x+4)$ represent real numbers. We notice that their product is 0 . By the zero-factor property, one or both of them must be 0 :

$$
x-6=0 \quad \text { or } \quad x+4=0
$$

We have used factoring and the zero-factor property to rewrite our original second-degree equation as two first-degree equations connected by the word or. Completing the solution, we solve the two first-degree equations:

$$
\begin{array}{rlrlrl}
x-6 & =0 & \text { or } & & x+4 & =0 \\
x & =6 & \text { or } & x & =-4
\end{array}
$$

We check our solutions in the original equation as follows:

$$
\begin{array}{rlrl}
\text { Check } x & =6 & \text { Check } x & =-4 \\
62-2(6)-24 & \stackrel{?}{=} 0 & (-4)^{2}-2(-4)-24 & \stackrel{?}{=} 0 \\
36-12-24 & \stackrel{?}{=} 0 & 16+8-24 & \stackrel{?}{=} 0 \\
0 & =0 & 0 & =0
\end{array}
$$

In both cases the result is a true statement, which means that both 6 and -4 are solutions to the original equation.

Although the next equation is not quadratic, the method we use is similar.

## EXAMPLE 10 Solve $\frac{1}{3} x^{3}=\frac{5}{6} x^{2}+\frac{1}{2} x$.

SOLUTION We can simplify our work if we clear the equation of fractions. Multiplying both sides by the LCD, 6, we have

$$
\begin{aligned}
\mathbf{6} \cdot \frac{1}{3} x^{3} & =\mathbf{6} \cdot \frac{5}{6} x^{2}+\mathbf{6} \cdot \frac{1}{2} x \\
2 x^{3} & =5 x^{2}+3 x
\end{aligned}
$$

Next we add $-5 x^{2}$ and $-3 x$ to each side so that the right side will become 0 .

$$
2 x^{3}-5 x^{2}-3 x=0 \quad \text { Standard form }
$$

We factor the left side and then use the zero-factor property to set each factor to 0 .

$$
\begin{array}{rll}
x\left(2 x^{2}-5 x-3\right)=0 & \begin{array}{l}
\text { Factor out the greatest } \\
\text { common factor }
\end{array} \\
x(2 x+1)(x-3)=0 & & \text { Continue factoring } \\
x=0 \quad \text { or } \quad 2 x+1=0 & \text { or } \quad x-3=0 & \text { Zero-factor property }
\end{array}
$$

Solving each of the resulting equations, we have

$$
x=0 \quad \text { or } \quad x=-\frac{1}{2} \quad \text { or } \quad x=3
$$

To generalize the preceding example, here are the steps used in solving a quadratic equation by factoring.

## To Solve an Equation by Factoring

Step 1: Write the equation in standard form.
Step 2: Factor the left side.
Step 3: Use the zero-factor property to set each factor equal to 0.
Step 4: Solve the resulting linear equations.

EXAMPLE 11 Solve $100 x^{2}=300 x$.
SOLUTION We begin by writing the equation in standard form and factoring:

$$
\begin{array}{rlrl}
100 x^{2} & =300 x \\
100 x^{2}-300 x & =0 & & \text { Standard form } \\
100 x(x-3) & =0 & & \text { Factor }
\end{array}
$$

Using the zero-factor property to set each factor to 0 , we have

$$
\begin{array}{rlrrr}
100 x=0 & \text { or } & x-3=0 \\
x=0 & \text { or } & x=3
\end{array}
$$

The two solutions are 0 and 3 .

EXAMPLE 12 Solve $(x-2)(x+1)=4$.
SOLUTION We begin by multiplying the two factors on the left side. (Notice that it would be incorrect to set each of the factors on the left side equal to 4 . The fact that the product is 4 does not imply that either of the factors must be 4.)

$$
\begin{array}{rlrlrl}
(x-2)(x+1) & =4 & & \\
x^{2}-x-2 & =4 & & \text { Multiply the left side } \\
x^{2}-x-6 & =0 & & \text { Standard form } \\
(x-3)(x+2) & =0 & & \text { Factor } \\
x-3=0 & & \text { or } \quad x+2 & =0 & & \text { Zero-factor property } \\
x=3 & & \text { or } & x & =-2 &
\end{array}
$$

EXAMPLE 13 Solve for $x: x^{3}+2 x^{2}-9 x-18=0$.
SOLUTION We start with factoring by grouping.

$$
\begin{array}{rlrlrl}
x^{3}+2 x^{2}-9 x-18 & =0 & & \\
x^{2}(x+2)-9(x+2) & =0 & & \\
(x+2)\left(x^{2}-9\right) & =0 & & \\
(x+2)(x-3)(x+3) & =0 & & \text { The difference of two squares } \\
x+2=0 \quad \text { or } \quad x-3 & =0 & \text { or } & x+3 & =0 \quad \text { Set factors to } 0 \\
x=-2 & \text { or } \quad x & =3 & \text { or } & x & =-3
\end{array}
$$

We have three solutions: $-2,3$, and -3 .

## Application

Recall that the method, or strategy, that we use to solve application problems is called the Blueprint for Problem Solving.

## BLUEPRINT FOR PROBLEM SOLVING

STEP 1: Read the problem, and then mentally list the items that are known and the items that are unknown.

STEP 2: Assign a variable to one of the unknown items. (In most cases this will amount to letting $x=$ the item that is asked for in the problem.) Then translate the other information in the problem to expressions involving the variable.

STEP 3: Reread the problem, and then write an equation, using the items and variable listed in steps 1 and 2 , that describes the situation.

STEP 4: Solve the equation found in step 3.
STEP 5: Write your answer using a complete sentence.
STEP 6: Reread the problem, and check your solution with the original words in the problem.

EXAMPLE 14 The sum of the squares of two consecutive integers is 25 .
Find the two integers.
SOLUTION We apply the Blueprint for Problem Solving to solve this application problem. Remember, step 1 in the blueprint is done mentally.

Step 1: Read and list.
Known items: Two consecutive integers. If we add their squares, the result is 25 .
Unknown items: The two integers
Step 2: Assign a variable and translate information.
Let $x=$ the first integer; then $x+1=$ the next consecutive integer.
Step 3: Reread and write an equation.
Since the sum of the squares of the two integers is 25 , the equation that describes the situation is

$$
x^{2}+(x+1)^{2}=25
$$

Step 4: Solve the equation.

$$
\begin{aligned}
x^{2}+(x+1)^{2} & =25 \\
x^{2}+\left(x^{2}+2 x+1\right) & =25 \\
2 x^{2}+2 x-24 & =0 \\
x^{2}+x-12 & =0 \quad \text { Divide each side by } 2 \\
(x+4)(x-3) & =0 \\
x=-4 \quad \text { or } \quad x & =3
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $2-8$
B $\quad 9-13$
C 14
Step 5: Write the answer.
If $x=-4$, then $x+1=-3$. If $x=3$, then $x+1=4$. The two integers are -4 and -3 , or the two integers are 3 and 4.

## Step 6: Reread and check.

The two integers in each pair are consecutive integers, and the sum of the squares of either pair is 25 .

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a solution to an equation?
2. What are equivalent equations?
3. Describe how to eliminate fractions in an equation.
4. What is the zero-factor property?

## Problem Set 7.1

Solve each of the following equations.

1. $2 x-4=6$
2. $3 x-5=4$
3. $-300 y+100=500$
4. $-20 y+80=30$
5. $-x=2$
6. $-x=\frac{1}{2}$
7. $-a=-\frac{3}{4}$
8. $-a=-5$
9. $-\frac{3}{5} a+2=8$
10. $-\frac{5}{3} a+3=23$
11. $2 x-5=3 x+2$
12. $5 x-1=4 x+3$
13. $5-2 x=3 x+1$
14. $7-3 x=8 x-4$
15. $5(y+2)-4(y+1)=3$
16. $6(y-3)-5(y+2)=8$
17. $6-7(m-3)=-1$
18. $3-5(2 m-5)=-2$
19. $7+3(x+2)=4(x-1)$
20. $5+2(4 x-4)=3(2 x-1)$
21. $\frac{1}{2} x+\frac{1}{4}=\frac{1}{3} x+\frac{5}{4}$
22. $\frac{2}{3} x-\frac{3}{4}=\frac{1}{6} x+\frac{21}{4}$
23. $\frac{1}{2} x+\frac{1}{3} x+\frac{1}{4} x=13$
24. $\frac{1}{3} x+\frac{1}{4} x+\frac{1}{5} x=47$
25. $0.08 x+0.09(9,000-x)=750$
26. $0.08 x+0.09(9,000-x)=500$
27. $0.35 x-0.2=0.15 x+0.1$
28. $0.25 x-0.05=0.2 x+0.15$
29. $x^{2}-5 x-6=0$
30. $x^{2}+5 x-6=0$
31. $x^{3}-5 x^{2}+6 x=0$
32. $x^{3}+5 x^{2}+6 x=0$
33. $3 y^{2}+11 y-4=0$
34. $3 y^{2}-y-4=0$
35. $\frac{1}{10} t^{2}-\frac{5}{2}=0$
36. $\frac{2}{7} t^{2}-\frac{7}{2}=0$
37. $\frac{1}{5} y^{2}-2=-\frac{3}{10} y$
38. $\frac{1}{2} y^{2}+\frac{5}{3}=\frac{17}{6} y$
39. $9 x^{2}-12 x=0$
40. $4 X^{2}+4 X=0$
41. $0.02 r+0.01=0.15 r^{2}$
42. $0.02 r-0.01=-0.08 r^{2}$
43. $9 a^{3}=16 a$
44. $16 a^{3}=25 a$
45. $-100 x=10 x^{2}$
46. $800 x=100 x^{2}$
47. $(x+6)(x-2)=-7$
48. $(x-7)(x+5)=-20$
49. $(x+1)^{2}=3 x+7$
50. $(x+2)^{2}=9 x$
51. $x^{3}+3 x^{2}-4 x-12=0$
52. $x^{3}+5 x^{2}-4 x-20=0$
53. $2 x^{3}+3 x^{2}-8 x-12=0$
54. $3 x^{3}+2 x^{2}-27 x-18=0$
55. $3 x-6=3(x+4)$
56. $7 x-14=7(x-2)$
57. $4 y+2-3 y+5=3+y+4$
58. $7 y+5-2 y-3=6+5 y-4$
59. $2(4 t-1)+3=5 t+4+3 t$
60. $5(2 t-1)+1=2 t-4+8 t$

Now that you have practiced solving a variety of equations, we can turn our attention to the type of equation you will see as you progress through the book. Each equation appears later in the book exactly as you see it below.

Solve each equation.
61. $0=6400 a+70$
62. $0=6400 a+60$
63. $x+2=2 x$
64. $x+2=7 x$
65. . $07 x=1.4$
66. $.02 x=0.3$
67. $5(2 x+1)=12$
68. $4(3 x-2)=21$
69. $50=\frac{K}{48}$
70. $50=\frac{K}{24}$
71. $100 P=2,400$
72. $3.5 d=16(3.5)^{2}$
73. $x+(3 x+2)=26$
74. $2(1)+y=4$
75. $2 x-3(3 x-5)=-6$
76. $2(2 y+6)+3 y=5$
77. $5\left(-\frac{19}{15}\right)+5 y=9 \quad$ 78. $5\left(-\frac{19}{15}\right)-2 y=4$
79. $2\left(-\frac{29}{22}\right)-3 y=4$
80. $2 x-3\left(-\frac{5}{11}\right)=4$
81. $3 x^{2}+x=10$
82. $y^{2}+y-20=2 y$
83. $12(x+3)+12(x-3)=3\left(x^{2}-9\right)$
84. $8(x+2)+8(x-2)=3\left(x^{2}-4\right)$

## Applying the Concepts

85. Rectangle A rectangle is twice as long as it is wide. The perimeter is 60 feet. Find the dimensions.
86. Rectangle The length of a rectangle is 5 times the width. The perimeter is 48 inches. Find the dimensions.
87. Livestock Pen A livestock pen is built in the shape of a rectangle that is twice as long as it is wide. The perimeter is 48 feet. If the material used to build the pen is $\$ 1.75$ per foot for the longer sides and $\$ 2.25$ per foot for the shorter sides (the shorter sides have gates, which increase the cost per foot), find the cost to build the pen.

88. Garden A garden is in the shape of a square with a perimeter of 42 feet. The garden is surrounded by two fences. One fence is around the perimeter of the garden, whereas the second fence is 3 feet from the first fence, as Figure 2 indicates. If the material used to build the two fences is $\$ 1.28$ per foot, what was the total cost of the fences?

89. Sales Tax A woman owns a small, cash-only business in a state that requires her to charge $6 \%$ sales tax on each item she sells. At the beginning of the day, she has $\$ 250$ in the cash register. At the end of the day, she has $\$ 1,204$ in the register. How much money should she send to the state government for the sales tax she collected?
90. Sales Tax A store is located in a state that requires $6 \%$ tax on all items sold. If the store brings in a total of \$3,392 in one day, how much of that total was sales tax?

Maximum Heart Rate In exercise physiology, a person's maximum heart rate, in beats per minute, is found by subtracting his age in years from 220 . So, if $A$ represents your age in years, then your maximum heart rate is

$$
M=220-A
$$

Use this formula to complete the following tables.
91.

| Age <br> (years) | Maximum Heart Rate <br> (beats per minute) |
| :---: | :---: |
| 18 |  |
| 19 |  |
| 20 |  |
| 21 |  |
| 22 |  |
| 23 |  |

92. 

| Age <br> (years) | Maximum Heart Rate <br> (beats per minute) |
| :---: | :---: |
| 15 |  |
| 20 |  |
| 25 |  |
| 30 |  |
| 35 |  |
| 40 |  |

Training Heart Rate A person's training heart rate, in beats per minute, is the person's resting heart rate plus $60 \%$ of the difference between maximum heart rate and
their resting heart rate. If resting heart rate is $R$ and maximum heart rate is $M$, then the formula that gives training heart rate is

$$
T=R+0.6(M-R)
$$

Use this formula along with the results of Problems 91 and 92 to fill in the following two tables.
93. For a 20-year-old person

| Resting Heart Rate <br> (beats per minute) | Training Heart Rate <br> (beats per minute) |
| :---: | :---: |
| 60 |  |
| 62 |  |
| 64 |  |
| 70 |  |
| 72 |  |

94. For a 40-year-old person

| Resting Heart Rate <br> (beats per minute) | Training Heart Rate <br> (beats per minute) |
| :---: | :---: |
| 60 |  |
| 62 |  |
| 64 |  |
| 70 |  |
| 72 |  |

95. Consecutive Integers The sum of the squares of two consecutive odd integers is 34 . Find the two integers.
96. Consecutive Integers The sum of the squares of two consecutive even integers is 100 . Find the two integers.

## Maintaining Your Skills

Simplify.
97. $|-3|$
98. $|3|$
99. $-|-3|$
100. $-(-3)$
101. Give a definition for the absolute value of $x$ that involves the number line. (This is the geometric definition.)
102. Give a definition of the absolute value of $x$ that does not involve the number line. (This is the algebraic definition.)

## Getting Ready for the Next Section

Solve each equation.
103. $2 a-1=-7$
104. $3 x-6=9$
105. $\frac{2}{3} x-3=7$
106. $\frac{2}{3} x-3=-7$
107. $x-5=x-7$
108. $x+3=x+8$
109. $x-5=-x-7$
110. $x+3=-x+8$

## 7.2

## OBJECTIVES

A Solve equations with absolute value symbols.

## Equations with Absolute Value

Previously we defined the absolute value of $x,|x|$, to be the distance between $x$ and 0 on the number line. The absolute value of a number measures its distance from 0 .

## EXAMPLE 1 Solve for $x:|x|=5$.

SOLUTION Using the definition of absolute value, we can read the equation as, "The distance between $x$ and 0 on the number line is 5 ." If $x$ is 5 units from 0 , then $x$ can be 5 or -5 :

$$
\text { If }|x|=5 \quad \text { then } x=5 \quad \text { or } \quad x=-5
$$

In general, then, we can see that any equation of the form $|a|=b$ is equivalent to the equations $a=b$ or $a=-b$, as long as $b>0$.

## EXAMPLE $2 \quad$ Solve $|2 a-1|=7$.

SOLUTION We can read this question as " $2 a-1$ is 7 units from 0 on the number line." The quantity $2 a-1$ must be equal to 7 or -7 :

$$
\begin{gathered}
|2 a-1|=7 \\
2 a-1=7 \text { or } 2 a-1=-7
\end{gathered}
$$

We have transformed our absolute value equation into two equations that do not involve absolute value. We can solve each equation separately.

$$
\begin{array}{rlrlrl}
2 a-1 & =7 & \text { or } & 2 a-1 & =-7 & \\
2 a & =8 & \text { or } & 2 a & =-6 & \\
\text { Add } 1 \text { to both sides } \\
a & =4 & \text { or } & & a & =-3
\end{array} \begin{array}{ll}
\text { Multiply by } \frac{1}{2}
\end{array}
$$

Our solution set is $\{4,-3\}$.

To check our solutions, we put them into the original absolute value equation:

| When | $a=4$ | When | $a=-3$ |
| :---: | :---: | :---: | :---: |
| the equation | $\|2 a-1\|=7$ | the equation | $\|2 a-1\|=7$ |
| becomes | $\|2(4)-1\|=7$ | becomes | $\|2(-3)-1\|=7$ |
|  | $\|7\|=7$ |  | $\|-7\|=7$ |
|  | $7=7$ |  | $7=7$ |

## EXAMPLE 3 Solve $\left|\frac{2}{3} x-3\right|+5=12$.

SOLUTION To use the definition of absolute value to solve this equation, we must isolate the absolute value on the left side of the equal sign. To do so, we add -5 to both sides of the equation to obtain

$$
\left|\frac{2}{3} x-3\right|=7
$$

Now that the equation is in the correct form, we can write

$$
\begin{aligned}
& \frac{2}{3} x-3=7 \quad \text { or } \quad \frac{2}{3} x-3=-7 \\
& \frac{2}{3} x=10 \quad \text { or } \quad \frac{2}{3} x=-4 \\
& x=15 \quad \text { or } \quad x=-6 \quad \text { Multiply by } \frac{3}{2}
\end{aligned}
$$

The solution set is $\{15,-6\}$.

## EXAMPLE $4 \quad$ Solve $|3 a-6|=-4$.

## Note

Recall that $\varnothing$ is the symbol we use to denote the empty set. When we use it to indicate the solutions to an equation, then we are saying the equation has no solution.

SOLUTION The solution set is $\varnothing$ because the left side cannot be negative and the right side is negative. No matter what we try to substitute for the variable $a$, the quantity $|3 a-6|$ will always be positive or zero. It can never be -4 .

Consider the statement $|a|=|b|$. What can we say about $a$ and $b$ ? We know they are equal in absolute value. By the definition of absolute value, they are the same distance from 0 on the number line. They must be equal to each other or opposites of each other. In symbols, we write

| $\|a\|=\|b\|$ | $a=b$ | or | $a=-b$ |
| :---: | :---: | :---: | :---: |
| $\uparrow$$\uparrow$ <br> Equal in <br> absolute value | Equals |  | or |
| $\uparrow$ |  | Opposites |  |

## EXAMPLE $5 \quad$ Solve $|x-5|=|x-7|$.

SOLUTION The quantities $x-5$ and $x-7$ must be equal or they must be opposites because their absolute values are equal:

## Equals <br> Opposites

$$
\begin{aligned}
x-5 & =x-7 & \text { or } & x-5
\end{aligned}=-(x-7) ~ 子 \begin{aligned}
x-5 & =-7 \\
& \text { No solution here } \\
x-5 & =-x+7 \\
2 x-5 & =7 \\
2 x & =12 \\
x & \\
x & =6
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-5

Because the first equation leads to a false statement, it will not give us a solution. (If either of the two equations were to reduce to a true statement, it would mean all real numbers would satisfy the original equation.) In this case, our only solution is $x=6$.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Why do some of the equations in this section have two solutions instead of one?
2. Translate $|x|=6$ into words using the definition of absolute value.
3. Explain in words what the equation $|x-3|=4$ means with respect to distance on the number line.
4. When is the statement $|x|=x$ true?

## Problem Set 7.2

Use the definition of absolute value to solve each of the following equations.

1. $|x|=4$
2. $|x|=7$
3. $2=|a|$
4. $5=|a|$
5. $|x|=-3$
6. $|x|=-4$
7. $|a|+2=3$
8. $|a|-5=2$
9. $|y|+4=3$
10. $|y|+3=1$
11. $4=|x|-2$
12. $3=|x|-5$
13. $|x-2|=5$
14. $|x+1|=2$
15. $|a-4|=\frac{5}{3}$
16. $|a+2|=\frac{7}{5}$
17. $1=|3-x|$
18. $2=|4-x|$
19. $\left|\frac{3}{5} a+\frac{1}{2}\right|=1$
20. $\left|\frac{2}{7} a+\frac{3}{4}\right|=1$
21. $60=|20 x-40|$
22. $800=|400 x-200|$
23. $|2 x+1|=-3$
24. $|2 x-5|=-7$
25. $\left|\frac{3}{4} x-6\right|=9$
26. $\left|\frac{4}{5} x-5\right|=15$
27. $\left|1-\frac{1}{2} a\right|=3$

Solve each equation.
29. $|3 x+4|+1=7$
30. $|5 x-3|-4=3$
31. $|3-2 y|+4=3$
32. $|8-7 y|+9=1$
33. $3+|4 t-1|=8$
34. $2+|2 t-6|=10$
35. $\left|9-\frac{3}{5} x\right|+6=12$
36. $\left|4-\frac{2}{7} x\right|+2=14$
37. $5=\left|\frac{2 x}{7}+\frac{4}{7}\right|-3$
38. $7=\left|\frac{3 x}{5}+\frac{1}{5}\right|+2$
39. $2=-8+\left|4-\frac{1}{2} y\right|$
40. $1=-3+\left|2-\frac{1}{4} y\right|$
41. $|3 a+1|=|2 a-4|$
42. $|5 a+2|=|4 a+7|$
43. $\left|x-\frac{1}{3}\right|=\left|\frac{1}{2} x+\frac{1}{6}\right|$
44. $\left|\frac{1}{10} x-\frac{1}{2}\right|=\left|\frac{1}{5} x+\frac{1}{10}\right|$

[^4]- = Online student support materials available at www.thomsonedu.com/login

45. $|y-2|=|y+3|$ 46. $|y-5|=|y-4|$
46. $|3 x-1|=|3 x+1|$
47. $|5 x-8|=|5 x+8|$
48. $|3-m|=|m+4|$
49. $|5-m|=|m+8|$
50. $|0.03-0.01 x|=|0.04+0.05 x|$
51. $|0.07-0.01 x|=|0.08-0.02 x|$
52. $|x-2|=|2-x|$
53. $|x-4|=|4-x|$
54. $\left|\frac{x}{5}-1\right|=\left|1-\frac{x}{5}\right|$
55. $\left|\frac{x}{3}-1\right|=\left|1-\frac{x}{3}\right|$
56. Work each problem according to the instructions given.
a. Solve: $4 x-5=0$
b. Solve: $|4 x-5|=0$
c. Solve: $4 x-5=3$
d. Solve: $|4 x-5|=3$
e. Solve: $|4 x-5|=|2 x+3|$
57. Work each problem according to the instructions given.
a. Solve: $3 x+6=0$
b. Solve: $|3 x+6|=0$
c. Solve: $3 x+6=4$
d. Solve: $|3 x+6|=4$
e. Solve: $|3 x+6|=|7 x+4|$

## Applying the Concepts

59. Amtrak Amtrak's annual passenger revenue for the years 1985-1995 is modeled approximately by the formula

$$
R=-60|x-11|+962
$$

where $R$ is the annual revenue in millions of dollars and $x$ is the number of years since January 1,1980 (Association of American Railroads, Washington, DC, Railroad Facts, Statistics of Railroads of Class 1, annual). In what years was the passenger revenue $\$ 722$ million?
60. Corporate Profits The corporate profits for various U.S. industries vary from year to year. An approximate model for profits of U.S. "communications
companies" during a given year between 1990 and 1997 is given by

$$
P=-3,400|x-5.5|+36,000
$$

where $P$ is the annual profits (in millions of dollars) and $x$ is the number of years since January 1, 1990 (U.S. Bureau of Economic Analysis, Income and Product Accounts of the U.S. (1929-1994), Survey of Current Business, September 1998). Use the model to determine the years in which profits of "communication companies" were $\$ 31.5$ billion ( $\$ 31,500$ million).

## Maintaining Your Skills

Simplify each expression.
61. $\frac{38}{30}$
62. $\frac{10}{25}$
63. $\frac{240}{6}$
64. $\frac{39}{13}$
65. $\frac{0+6}{0-3}$
66. $\frac{6+6}{6-3}$
67. $\frac{4-4}{4-2}$
68. $\frac{3+6}{3-3}$

## Getting Ready for the Next Section

Solve each equation.
69. $-2 x-3=7$
70. $3 x+3=2 x-1$
71. $3(2 x-4)-7 x=-3 x$
72. $3(2 x+5)=-3 x$

## Extending the Concepts

Solve each formula for $x$. (Assume $a, b$, and $c$ are positive.)
73. $|x-a|=b$
74. $|x+a|-b=0$
75. $|a x+b|=c$
76. $|a x-b|-c=0$
77. $\left|\frac{x}{a}+\frac{y}{b}\right|=1$
78. $\left|\frac{x}{a}+\frac{y}{b}\right|=c$

## OBJECTIVES

A Solve a linear inequality in one variable and graph the solution set.

B Solve a compound inequality and graph the solution set.

C Write solutions to inequalities using interval notation.

D Solve application problems using inequalities

The instrument panel on most cars includes a temperature gauge. The one shown below indicates that the normal operating temperature for the engine is from $50^{\circ} \mathrm{F}$ to $270^{\circ} \mathrm{F}$.


We can represent the same situation with an inequality by writing $50 \leq F \leq 270$, where $F$ is the temperature in degrees Fahrenheit. This inequality is a compound inequality. In this section we present the notation and definitions associated with compound inequalities.
Let's begin with an example reviewing how we solved inequalities in Chapter 2.

## EXAMPLE 1 <br> Solve $3 x+3<2 x-1$, and graph the solution.

SOLUTION We use the addition property for inequalities to write all the variable terms on one side and all constant terms on the other side:

$$
\begin{aligned}
3 x+3 & <2 x-1 & & \\
3 x+(-\mathbf{2 x})+3 & <2 x+(-\mathbf{2 x})-1 & & \text { Add }-\mathbf{2 x} \text { to each side } \\
x+3 & <-1 & & \\
x+3+(-\mathbf{3}) & <-1+(-\mathbf{3}) & & \text { Add }-\mathbf{3} \text { to each side } \\
x & <-4 & &
\end{aligned}
$$

The solution set is all real numbers that are less than -4 . To show this we can use set notation and write

$$
\{x \mid x<-4\}
$$

We can graph the solution set on the number line using an open circle at -4 to show that -4 is not part of the solution set. This is the format you used when graphing inequalities in Chapter 2.


Here is an equivalent graph that uses a parenthesis opening left, instead of an open circle, to represent the end point of the graph.


## Note

The English mathematician John Wallis (1616-1703) was the first person to use the $\infty$ symbol to represent infinity. When we encounter the interval $(3, \infty)$, we read it as "the interval from 3 to infinity," and we mean the set of real numbers that are greater than three. Likewise, the interval $(-\infty,-4)$ is read "the interval from negative infinity to $-4, "$ which is all real numbers less than -4 .

This graph gives rise to the following notation, called interval notation, that is an alternative way to write the solution set.

$$
(-\infty,-4)
$$

The preceding expression indicates that the solution set is all real numbers from negative infinity up to, but not including, -4 .
We have three equivalent representations for the solution set to our original inequality. Here are all three together.


## Interval Notation and Graphing

The table below shows the connection between inequalities, interval notation, and number line graphs. We have included the graphs with open and closed circles for those of you who have used this type of graph previously. In this book, we will continue to show our graphs using the parentheses/brackets method.


Recall that the multiplication property for inequalities allows us to multiply both sides of an inequality by any nonzero number we choose. If the number we multiply by happens to be negative, then we must also reverse the direction of the inequality.

EXAMPLE 2 Find the solution set for $-2 y-3 \leq 7$.
SOLUTION We begin by adding 3 to each side of the inequality:

$$
\begin{array}{rlrl}
-2 y-3 & \leq 7 & \\
-2 y & \leq 10 & & \text { Add } 3 \text { to both sides } \\
& & \\
-\frac{\mathbf{1}}{\mathbf{2}}(-2 y) & \geq-\frac{\mathbf{1}}{\mathbf{2}}(10) & & \begin{array}{l}
\text { Multiply by }-\frac{1}{2} \text { and reverse the } \\
\text { direction of the inequality symb }
\end{array} \\
y & \geq-5 & &
\end{array}
$$

The solution set is all real numbers that are greater than or equal to -5 . Below are three equivalent ways to represent this solution set.


Notice how a bracket is used with interval notation to show that -5 is part of the solution set.

## Compound Inequalities

The union of two sets $A$ and $B$ is the set of all elements that are in $A$ or in $B$. The word or is the key word in the definition. The intersection of two sets $A$ and $B$ is the set of elements contained in both $A$ and $B$. The key word in this definition is and. We can put the words and and or together with our methods of graphing inequalities to find the solution sets for compound inequalities.

DEFINITION A compound inequality is two or more inequalities connected by the word and or or.

EXAMPLE 3 Graph the solution set for the compound inequality

$$
x<-1 \quad \text { or } \quad x \geq 3
$$

SOLUTION Graphing each inequality separately, we have


Because the two inequalities are connected by or, we want to graph their union; that is, we graph all points that are on either the first graph or the second graph. Essentially, we put the two graphs together on the same number line.


To represent this set of numbers with interval notation, we use two intervals connected with the symbol for the union of two sets. Here is the equivalent set of numbers described with interval notation:

$$
(-\infty,-1) \cup[3, \infty)
$$

EXAMPLE 4 Graph the solution set for the compound inequality

$$
x>-2 \quad \text { and } \quad x<3
$$

SOLUTION Graphing each inequality separately, we have


Because the two inequalities are connected by the word and, we will graph their intersection, which consists of all points that are common to both graphs; that is, we graph the region where the two graphs overlap.


This graph corresponds to the interval $(-2,3)$, which is called an open interval because neither end point is included in the interval.

Notation: Sometimes compound inequalities that use the word and as the connecting word can be written in a shorter form. For example, the compound inequality $-3 \leq x$ and $x \leq 4$ can be written $-3 \leq x \leq 4$. The word and does not appear when an inequality is written in this form. It is implied. Inequalities of the form $-3 \leq x \leq 4$ are called continued inequalities. This new notation is useful because it takes fewer symbols to write it. The graph of $-3 \leq x \leq 4$ is


The corresponding interval is $[-3,4]$, which is a closed interval because both endpoints are included in the interval.

## More Interval Notation and Graphing

The table below shows the connection between interval notation and number line graphs for a variety of continued inequalities. We have included the graphs with open and closed circles for those of you who have used this type of graph previously.

| Inequality notation | Interval notation | Graph using parentheses/brackets | Graph using open and closed circles |
| :---: | :---: | :---: | :---: |
| $-3<x<2$ | $(-3,2)$ |  |  |
| $-3 \leq x \leq 2$ | [-3, 2] |  | $-3$ $2$ |
| $-3 \leq x<2$ | $[-3,2)$ |  |  |
| $-3<x \leq 2$ | $(-3,2]$ |  |  |

## EXAMPLE 5 Solve and graph $-3 \leq 2 x-5 \leq 3$.

SOLUTION We can extend our properties for addition and multiplication to cover this situation. If we add a number to the middle expression, we must add the same number to the outside expressions. If we multiply the center expression by a number, we must do the same to the outside expressions, remembering to reverse the direction of the inequality symbols if we multiply by a negative number. We begin by adding 5 to all three parts of the inequality:

$$
\begin{array}{rlrl}
-3 & \leq 2 x-5 & \leq 3 & \\
2 & \leq 2 x & \leq 8 & \\
\text { Add } 5 \text { to all three members } \\
1 & \leq x \quad \leq 4 & & \text { Multiply through by } \frac{1}{2}
\end{array}
$$

Here are three ways to write this solution set:


EXAMPLE 6 Solve the compound inequality.

$$
3 t+7 \leq-4 \quad \text { or } \quad 3 t+7 \geq 4
$$

SOLUTION We solve each half of the compound inequality separately, then we graph the solution set:

$$
\begin{array}{rlrlrl}
3 t+7 & \leq-4 & \text { or } & 3 t+7 & \geq 4 & \\
3 t & \leq-11 & \text { or } & 3 t & \geq-3 & \\
\text { Add }-7 \\
t & \leq-\frac{11}{3} & \text { or } & t & \geq-1 & \text { Multiply by } \frac{-1}{3}
\end{array}
$$

The solution set can be written in any of the following ways:


EXAMPLE 7
A company that manufactures ink cartridges for printers
finds that they can sell $x$ cartridges each week at a price of $p$ dollars each, according to the formula $x=1,300-100 p$. What price should they charge for each cartridge if they want to sell at least 300 cartridges a week?

SOLUTION Because $x$ is the number of cartridges they sell each week, an inequality that corresponds to selling at least 300 cartridges a week is

$$
x \geq 300
$$

Substituting 1,300-100p for $x$ gives us an inequality in the variable $p$.

$$
\begin{array}{rlrl}
1,300-100 p & \geq 300 & & \\
-100 p & \geq-1,000 & & \text { Add }-1,300 \text { to each side } \\
& & \\
p & \leq 10 & & \begin{array}{l}
\text { Divide each side by }-\mathbf{1 0 0}, \text { and reverse } \\
\text { the direction of the inequality symbol }
\end{array}
\end{array}
$$

To sell at least 300 cartridges each week, the price per cartridge should be no more than $\$ 10$; that is, selling the cartridges for $\$ 10$ or less will produce weekly sales of 300 or more cartridges.

EXAMPLE 8 The formula $F=\frac{9}{5} C+32$ gives the relationship between the Celsius and Fahrenheit temperature scales. If the temperature range on a certain day is $86^{\circ}$ to $104^{\circ}$ Fahrenheit, what is the temperature range in degrees Celsius?
SOLUTION From the given information we can write $86 \leq F \leq 104$. But, because $F$ is equal to $\frac{9}{5} C+32$, we can also write


$$
\begin{aligned}
& 86 \leq \frac{9}{5} C+32 \\
& 54 \leq 104 \\
& \leq \frac{9}{5} C \quad \leq 72 \quad \text { Add }-32 \text { to each member } \\
& \frac{5}{9}(54) \leq \frac{5}{9}\left(\frac{9}{5} C\right) \\
& 30 \leq \frac{5}{9}(72) \\
& C \quad \leq 40
\end{aligned}
$$

A temperature range of $86^{\circ}$ to $104^{\circ}$ Fahrenheit corresponds to a temperature range of $30^{\circ}$ to $40^{\circ}$ Celsius.

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,2
B 5,6
C ${ }_{1-6}$
D 7,8

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the addition property for inequalities?
2. When we use interval notation to denote a section of the real number line, when do we use parentheses ( ) and when do we use brackets [ ]?
3. Explain the difference between the multiplication property of equality and the multiplication property for inequalities.
4. When solving an inequality, when do we change the direction of the inequality symbol?

Solve each of the following inequalities and graph each solution.

1. $2 x \leq 3$
2. $5 x \geq-115$
3. $\frac{1}{2} x>2$
4. $\frac{1}{3} x>4$
5. $-5 x \leq 25$
6. $-7 x \geq 35$
7. $-\frac{3}{2} x>-6$
8. $-\frac{2}{3} x<-8$
9. $-12 \leq 2 x$
10. $-20 \geq 4 x$
11. $-1 \geq-\frac{1}{4} x$
12. $-1 \leq-\frac{1}{5} x$
13. $-3 x+1>10$
14. $-2 x-5 \leq 15$
15. $\frac{1}{2}-\frac{m}{12} \leq \frac{7}{12}$
16. $\frac{1}{2}-\frac{m}{10}>-\frac{1}{5}$
17. $\frac{1}{2} \geq-\frac{1}{6}-\frac{2}{9} x$
18. $\frac{9}{5}>-\frac{1}{5}-\frac{1}{2} x$
19. $-40 \leq 30-20 y$
20. $-20>50-30 y$
21. $\frac{2}{3} x-3<1$
22. $\frac{3}{4} x-2>7$
23. $10-\frac{1}{2} y \leq 36$
24. $8-\frac{1}{3} y \geq 20$

Simplify each side first, then solve the following inequalities. Write your answers with interval notation.
25. $2(3 y+1) \leq-10$
26. $3(2 y-4)>0$
27. $-(a+1)-4 a \leq 2 a-8$
28. $-(a-2)-5 a \leq 3 a+7$
29. $\frac{1}{3} t-\frac{1}{2}(5-t)<0$
30. $\frac{1}{4} t-\frac{1}{3}(2 t-5)<0$
31. $-2 \leq 5-7(2 a+3)$
32. $1<3-4(3 a-1)$
33. $-\frac{1}{3}(x+5) \leq-\frac{2}{9}(x-1)$
34. $-\frac{1}{2}(2 x+1) \leq-\frac{3}{8}(x+2)$

Solve each inequality. Write your answer using inequality notation.
35. $20 x+9,300>18,000$
36. $20 x+4,800>18,000$

Solve the following continued inequalities. Use both a line graph and interval notation to write each solution set.
37. $-2 \leq m-5 \leq 2$
38. $-3 \leq m+1 \leq 3$
39. $-60<20 a+20<60$
40. $-60<50 a-40<60$
41. $0.5 \leq 0.3 a-0.7 \leq 1.1$
42. $0.1 \leq 0.4 a+0.1 \leq 0.3$
43. $3<\frac{1}{2} x+5<6$
44. $5<\frac{1}{4} x+1<9$
45. $4<6+\frac{2}{3} x<8$
46. $3<7+\frac{4}{5} x<15$

Graph the solution sets for the following compound inequalities. Then write each solution set using interval notation.
47. $x+5 \leq-2$ or $x+5 \geq 2$
48. $3 x+2<-3$ or $3 x+2>3$
49. $5 y+1 \leq-4$ or $5 y+1 \geq 4$
50. $7 y-5 \leq-2$ or $7 y-5 \geq 2$
51. $2 x+5<3 x-1$ or $x-4>2 x+6$
52. $3 x-1>2 x+4$ or $5 x-2<3 x+4$

Translate each of the following phrases into an equivalent inequality statement.
53. $x$ is greater than -2 and at most 4 .
54. $x$ is less than 9 and at least -3 .
55. $x$ is less than -4 or at least 1 .
56. $x$ is at most 1 or more than 6 .
57. Write each statement using inequality notation.
a. $x$ is always positive.
b. $x$ is never negative.
c. $x$ is greater than or equal to 0 .
58. Match each expression on the left with a phrase on the right.
a. $X^{2} \geq 0$
e. never true
b. $x^{2}<0$
f. sometimes true
c. $x^{2} \leq 0$
g. always true

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[^5]Solve each inequality by inspection, without showing any work.
59. $x^{2}<0$
60. $x^{2} \leq 0$
61. $x^{2} \geq 0$
62. $\frac{1}{x^{2}} \geq 0$
63. $\frac{1}{x^{2}}<0$
64. $\frac{1}{x^{2}}=0$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working.
65. Work each problem according to the instructions given.
a. Evaluate when $x=0:-\frac{1}{2} x+1$
b. Solve: $-\frac{1}{2} x+1=-7$
c. Is 0 a solution to $-\frac{1}{2} x+1<-7$ ?
d. Solve: $-\frac{1}{2} x+1<-7$
66. Work each problem according to the instructions given.
a. Evaluate when $x=0:-\frac{2}{3} x-5$
b. Solve: $-\frac{2}{3} x-5=1$
c. Is 0 a solution to $-\frac{2}{3} x-5>1$ ?
d. Solve: $-\frac{2}{3} x-5>1$

## Applying the Concepts

A store selling art supplies finds that they can sell $x$ sketch pads each week at a price of $p$ dollars each, according to the formula $x=900-300 p$. What price should they charge if they want to sell
67. at least 300 pads each week?
68. more than 600 pads each week?
69. less than 525 pads each week?
70. at most 375 pads each week?
71. Amtrak The average number of passengers carried by Amtrak declined each year for the years 1990-1996 (American Association of Railroads, Washington, DC, Railroad Facts, Statistics of Railroads of Class 1.) The linear model for the number of passengers carried each year by Amtrak is given by $P=22,419-399 x$, where $P$ is the number of passengers, in millions, and $x$ ปै
 is the number of years after January 1, 1990. In what years did Amtrak have more than 20,500 million passengers?
72. Student Loan When considering how much debt to incur in student loans, you learn that it is wise to keep your student loan payment to $8 \%$ or less of your starting monthly income. Suppose you anticipate a starting annual salary of $\$ 24,000$. Set up and solve an inequality that represents the amount of monthly debt for student loans that would be considered manageable.
73. Here is what the U.S. Geological Survey has to say about the survival rates of the Apapane, one of the endemic birds of Hawaii.

> Annual survival rates based on 1,584 recaptures of 429 banded individuals $0.72 \pm 0.11$ for adults and $0.13 \pm 0.07$ for juveniles.

Write the survival rates using inequalities. Then give the survival rates in terms of percent.

74. Survival Rates for Sea Gulls Here is part of a report concerning the survival rates of Western Gulls that appeared on the website of Cornell University.

> Survival of eggs to hatching is 70\%-80\%; of hatched chicks to fledglings $50 \%-70 \%$; of fledglings to age of first breeding $<50 \%$.

Write the survival rates using inequalities without percent.

75. Temperature Each of the following temperature ranges is in degrees Fahrenheit. Use the formula $F=\frac{9}{5} C+32$ to find the corresponding temperature range in degrees Celsius.
a. $95^{\circ}$ to $113^{\circ}$
b. $68^{\circ}$ to $86^{\circ}$
c. $-13^{\circ}$ to $14^{\circ}$
d. $-4^{\circ}$ to $23^{\circ}$
76. Improving Your Quantitative Literacy As you can see, the percents shown in the graph are from a survey of 770 families.

## USA TODAY Snapshots ${ }^{\text {© }}$



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a. How many families responded that luxury is happiness and satisfaction from being with friends and family and enjoying good times?
b. If $x$ represents the annual income of any of the families surveyed, write an inequality using $x$ that indicates their level of income.
c. The margin of error for the survey is $\pm 4 \%$. What does this mean in relation to the $19 \%$ category?

## Maintaining Your Skills

Reduce each rational expression to lowest terms.
77. $\frac{x^{2}-16}{x+4}$
78. $\frac{x-5}{x^{2}-25}$
79. $\frac{10 a+20}{5 a^{2}-20}$
80. $\frac{8 a-16}{4 a^{2}-16}$
81. $\frac{2 x^{2}-5 x-3}{x^{2}-3 x}$
82. $\frac{x^{2}-5 x}{3 x^{2}-13 x-10}$
83. $\frac{x y+3 x+2 y+6}{x y+3 x+a y+3 a}$
84. $\frac{x y+5 x+4 y+20}{x^{2}+b x+4 x+4 b}$

## Getting Ready for the Next Section

To understand all of the explanations and examples in the next section, you must be able to work the problems below.

Solve each inequality. Do not graph the solution set.
85. $2 x-5<3$
86. $-3<2 x-5$
87. $-4 \leq 3 a+7$
88. $3 a+7 \leq 4$
89. $4 t-3 \leq-9$
90. $4 t-3 \geq 9$

## Inequalities Involving Absolute Value

## OBJECTIVES

A Solve inequalities with absolute value and graph the solution set.

In this section we will again apply the definition of absolute value to solve inequalities involving absolute value. Again, the absolute value of $x$, which is $|x|$, represents the distance that $x$ is from 0 on the number line. We will begin by considering three absolute value equations/inequalities and their English translations:

Expression
In Words
$\begin{array}{ll}|x|=7 & x \text { is exactly } 7 \text { units from } 0 \text { on the number line } \\ |a|<5 & a \text { is less than } 5 \text { units from } 0 \text { on the number line } \\ |y| \geq 4 & y \text { is greater than or equal to } 4 \text { units from } 0 \text { on the number line }\end{array}$
Once we have translated the expression into words, we can use the translation to graph the original equation or inequality. The graph then is used to write a final equation or inequality that does not involve absolute value.


Although we will not always write out the English translation of an absolute value inequality, it is important that we understand the translation. Our second expression, $|a|<5$, means $a$ is within 5 units of 0 on the number line. The graph of this relationship is

which can be written with the following continued inequality:

$$
-5<a<5
$$

We can follow this same kind of reasoning to solve more complicated absolute value inequalities.

## EXAMPLE 1 Graph the solution set: $|2 x-5|<3$.

SOLUTION The absolute value of $2 x-5$ is the distance that $2 x-5$ is from 0 on the number line. We can translate the inequality as " $2 x-5$ is less than 3 units from 0 on the number line"; that is, $2 x-5$ must appear between -3 and 3 on the number line.

A picture of this relationship is


Using the picture, we can write an inequality without absolute value that describes the situation:

$$
-3<2 x-5<3
$$

Next, we solve the continued inequality by first adding 5 to all three members and then multiplying all three by $\frac{1}{2}$.

$$
\begin{aligned}
-3 & <2 x-5<3 & & \\
2 & <2 x<8 & & \text { Add } 5 \text { to all three members } \\
1 & <x<4 & & \text { Multiply each member by } \frac{1}{2}
\end{aligned}
$$

The graph of the solution set is


We can see from the solution that for the absolute value of $2 x-5$ to be within 3 units of 0 on the number line, $x$ must be between 1 and 4 .

## EXAMPLE $2 \quad$ Solve and graph $|3 a+7| \leq 4$.

SOLUTION We can read the inequality as, "The distance between $3 a+7$ and 0 is less than or equal to 4 ." Or, " $3 a+7$ is within 4 units of 0 on the number line." This relationship can be written without absolute value as

$$
-4 \leq 3 a+7 \leq 4
$$

Solving as usual, we have

$$
\begin{array}{rlr}
-4 & \leq 3 a+7 & \leq 4 \\
-11 & \leq \quad 3 a \quad<-3 & \text { Add -7 to all three members } \\
-\frac{11}{3} & \leq a \quad \leq-1 & \text { Multiply each member by } \frac{1}{3} \\
\longleftrightarrow &
\end{array}
$$

We can see from Examples 1 and 2 that to solve an inequality involving absolute value, we must be able to write an equivalent expression that does not involve absolute value.

EXAMPLE 3 Solve $|x-3|>5$ and graph the solution.
SOLUTION We interpret the absolute value inequality to mean that $x-3$ is more than 5 units from 0 on the number line. The quantity $x-3$ must be either above +5 or below -5 . Here is a picture of the relationship:


An inequality without absolute value that also describes this situation is

$$
x-3<-5 \quad \text { or } \quad x-3>5
$$

Adding 3 to both sides of each inequality we have

$$
x<-2 \quad \text { or } \quad x>8
$$

The graph of which is


EXAMPLE $4 \quad$ Graph the solution set: $|4 t-3| \geq 9$.
SOLUTION The quantity $4 t-3$ is greater than or equal to 9 units from 0 . It must be either above +9 or below -9 .

$$
\begin{array}{rlrlr}
4 t-3 & \text { or } & 4 t-3 & \geq 9 & \\
4 t \leq-6 & \text { or } & 4 t \geq 12 & \text { Add } 3 \\
t \leq-\frac{6}{4} & \text { or } & t \geq \frac{12}{4} & \text { Multiply by } \frac{1}{4} \\
t \leq-\frac{3}{2} & \text { or } & t \geq 3 & \\
\hdashline
\end{array}
$$

We can use the results of our first few examples and the material in the previous section to summarize the information we have related to absolute value equations and inequalities.

Rewriting Absolute Value Equations and Inequalities If $c$ is a positive real number, then each of the following statements on the left is equivalent to the corresponding statement on the right.
With Absolute Value
Without Absolute Value

\[

\]

## EXAMPLE $5 \quad$ Solve and graph $|2 x+3|+4<9$.

SOLUTION Before we can apply the method of solution we used in the previous examples, we must isolate the absolute value on one side of the inequality. To do so, we add $\mathbf{- 4}$ to each side.

$$
\begin{aligned}
|2 x+3|+4 & <9 \\
|2 x+3|+4+(-4) & <9+(-4) \\
|2 x+3| & <5
\end{aligned}
$$

From this last line we know that $2 x+3$ must be between -5 and +5 .

$$
\begin{aligned}
-5<2 x+3<5 & \\
-8<2 x<2 & \text { Add }-3 \text { to each member } \\
-4<x<1 \quad & \text { Multiply each member by } \frac{1}{2}
\end{aligned}
$$

The graph is


## EXAMPLE $6 \quad$ Solve and graph $|4-2 t|>2$.

SOLUTION The inequality indicates that $4-2 t$ is less than -2 or greater than +2 . Writing this without absolute value symbols, we have

$$
4-2 t<-2 \quad \text { or } \quad 4-2 t>2
$$

To solve these inequalities we begin by adding $\mathbf{- 4}$ to each side.

$$
\begin{array}{clrl}
4+(-\mathbf{4})-2 t<-2+(-\mathbf{4}) & \text { or } & 4+(-\mathbf{4})-2 t>2+(-\mathbf{4}) \\
-2 t<-6 & \text { or } & -2 t>-2
\end{array}
$$

## Note

Remember, the multiplication property for inequalities requires that we reverse the direction of the inequality symbol every time we multiply both sides of an inequality by a negative number.

Next we must multiply both sides of each inequality by $-\frac{\mathbf{1}}{\mathbf{2}}$. When we do so, we must also reverse the direction of each inequality symbol.

$$
\begin{array}{rlrc}
-2 t & <-6 & \text { or } & -2 t>-2 \\
-\frac{\mathbf{1}}{\mathbf{2}}(-2 t) & >-\frac{\mathbf{1}}{\mathbf{2}}(-6) & \text { or } & -\frac{\mathbf{1}}{\mathbf{2}}(-2 t)<-\frac{\mathbf{1}}{\mathbf{2}}(-2) \\
t & >3 & \text { or } & t<1
\end{array}
$$

Although in situations like this we are used to seeing the "less than" symbol written first, the meaning of the solution is clear. We want to graph all real numbers that are either greater than 3 or less than 1. Here is the graph.


Because absolute value always results in a nonnegative quantity, we sometimes come across special solution sets when a negative number appears on the right side of an absolute value inequality.

## EXAMPLE 7 Solve $|7 y-1|<-2$.

SOLUTION The left side is never negative because it is an absolute value. The right side is negative. We have a positive quantity (or zero) less than a negative quantity, which is impossible. The solution set is the empty set, $\varnothing$. There is no real number to substitute for $y$ to make this inequality a true statement.

## EXAMPLE 8 Solve $|6 x+2|>-5$.

SOLUTION This is the opposite case from that in Example 7. No matter what real number we use for $x$ on the left side, the result will always be positive, or zero. The right side is negative. We have a positive quantity (or zero) greater than a negative quantity. Every real number we choose for $x$ gives us a true statement. The solution set is the set of all real numbers.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A ${ }^{1-8}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Write an inequality containing absolute value, the solution to which is all the numbers between -5 and 5 on the number line.
2. Translate $|x| \geq 3$ into words using the definition of absolute value.
3. Explain in words what the inequality $|x-5|<2$ means with respect to distance on the number line.
4. Why is there no solution to the inequality $|2 x-3|<0$ ?

## Problem Set 7.4

Solve each of the following inequalities using the definition of absolute value. Graph the solution set in each case.

1. $|x|<3$
2. $|x| \leq 7$
3. $|x| \geq 2$
4. $|x|>4$
5. $|x|+2<5$
6. $|x|-3<-1$
7. $|t|-3>4$
8. $|t|+5>8$
9. $|y|<-5$
10. $|y|>-3$
11. $|x| \geq-2$
12. $|x| \leq-4$
13. $|x-3|<7$
14. $|x+4|<2$
15. $|a+5| \geq 4$
16. $|a-6| \geq 3$

Solve each inequality and graph the solution set.
17. $|a-1|<-3$
18. $|a+2| \geq-5$
19. $|2 x-4|<6$
20. $|2 x+6|<2$
21. $|3 y+9| \geq 6$
22. $|5 y-1| \geq 4$
23. $|2 k+3| \geq 7$
24. $|2 k-5| \geq 3$
25. $|x-3|+2<6$
26. $|x+4|-3<-1$
27. $|2 a+1|+4 \geq 7$
28. $|2 a-6|-1 \geq 2$
29. $|3 x+5|-8<5$
30. $|6 x-1|-4 \leq 2$

Solve each inequality, and graph the solution set. Keep in mind that if you multiply or divide both sides of an inequality by a negative number, you must reverse the inequality sign.
31. $|5-x|>3$
32. $|7-x|>2$
33. $\left|3-\frac{2}{3} x\right| \geq 5$
34. $\left|3-\frac{3}{4} x\right| \geq 9$
35. $\left|2-\frac{1}{2} X\right|>1$
36. $\left|3-\frac{1}{3} x\right|>1$
$=$ Videos available by instructor request
> = Online student support materials available at www.thomsonedu.com/login

Solve each inequality.
37. $|x-1|<0.01$
38. $|x+1|<0.01$
39. $|2 x+1| \geq \frac{1}{5}$
40. $|2 x-1| \geq \frac{1}{8}$
41. $\left|\frac{3 x-2}{5}\right| \leq \frac{1}{2}$
42. $\left|\frac{4 x-3}{2}\right| \leq \frac{1}{3}$
43. $\left|2 x-\frac{1}{5}\right|<0.3$
44. $\left|3 x-\frac{3}{5}\right|<0.2$
45. Write the continued inequality $-4 \leq x \leq 4$ as a single inequality involving absolute value.
46. Write the continued inequality $-8 \leq x \leq 8$ as a single inequality involving absolute value.
47. Write $-1 \leq x-5 \leq 1$ as a single inequality involving absolute value.
48. Write $-3 \leq x+2 \leq 3$ as a single inequality involving absolute value.
49. Work each problem according to the instructions given.
a. Evaluate when $x=0:|5 x+3|$
b. Solve: $|5 x+3|=7$
c. Is 0 a solution to $|5 x+3|>7$ ?
d. Solve: $|5 x+3|>7$
50. Work each problem according to the instructions given.
a. Evaluate when $x=0:|-2 x-5|$
b. Solve: $|-2 x-5|=1$
c. Is 0 a solution to $|-2 x-5|>1$ ?
d. Solve: $|-2 x-5|>1$

## Applying the Concepts

51. Speed Limits The interstate speed limit for cars is 75 miles per hour in Nebraska, Nevada, New Mexico, Oklahoma, South Dakota, Utah, and Wyoming and is the highest in the United States. To discourage
passing, minimum speeds are also posted, so that the difference between the fastest and slowest moving traffic is no more than 20 miles per hour. Therefore, the speed $x$ of a car must satisfy the inequality $55 \leq x \leq 75$. Write this inequality as an absolute value inequality.
52. Wavelengths of Light When white light from the sun passes through a prism, it is broken down into bands of light that form colors. The wavelength, $v$, (in nanometers) of some common colors are:

Blue:
$424<v<491$
Green: $\quad 491<v<575$
Yellow: $\quad 575<v<585$
Orange: $\quad 585<v<647$
Red: $\quad 647<v<700$
When a fireworks display made of copper is burned, it lets out light with wavelengths, $v$, that satisfy the relationship $|v-455|<23$. Write this inequality without absolute values; find the range of possible values for $v$; and then, using the preceding list of wavelengths, determine the color of that copper fireworks display.


## Maintaining Your Skills

Multiply or divide as indicated.
53. $\frac{8 x}{x^{2}-5 x} \cdot \frac{x^{2}-25}{4 x^{2}+4 x}$
54. $\frac{x^{2}+4 x}{4 x^{2}} \cdot \frac{7 x}{x^{2}+5 x}$
55. $\frac{x^{2}+3 x-4}{3 x^{2}+7 x-20} \div \frac{x^{2}-2 x+1}{3 x^{2}-2 x-5}$
56. $\frac{x^{2}-16}{2 x^{2}+7 x+3} \div \frac{2 x^{2}-5 x-12}{4 x^{2}+8 x+3}$
57. $\left(x^{2}-36\right)\left(\frac{x+3}{x-6}\right)$
58. $\left(x^{2}-49\right)\left(\frac{x+5}{x+7}\right)$

## Getting Ready for the Next Section

Solve each system by graphing.
59. $x-y=-2$
$x+y=4$
60. $2 x+4 y=20$
$3 x-2 y=6$

Solve each system by the elimination method.
61. $x+y=4$
$x-y=-2$
62. $5 x+2 y=11$
$7 x+8 y=18$

Solve each system by the substitution method.
63. $x+y=2$
$y=2 x-1$
64. $4 x+2 y=3$
$x=4 y-3$

## 7.5

## OBJECTIVES

A
Solve systems of linear equations in two variables by graphing.

B Solve systems of linear equations in two variables by the addition method.

## C Solve systems of linear equation in two variables by the substitution method.



FIGURE 1

## Note

It is important that you write solutions to systems of equations in two variables as ordered pairs, like $(4,3)$; that is, you should always enclose the ordered pairs in parentheses.

## Review of Systems of Linear Equations in Two Variables

Previously we found that two linear equations considered together form a linear system of equations. For example,

$$
\begin{aligned}
& 3 x-2 y=6 \\
& 2 x+4 y=20
\end{aligned}
$$

is a linear system. The solution set to the system is the set of all ordered pairs that satisfy both equations. If we graph each equation on the same set of axes, we can see the solution set (Figure 1).

The point $(4,3)$ lies on both lines and therefore must satisfy both equations. It is obvious from the graph that it is the only point that does so. The solution set for the system is $\{(4,3)\}$.

More generally, if $a_{1} x+b_{1} y=c_{1}$ and $a_{2} x+b_{2} y=c_{2}$ are linear equations, then the solution set for the system

$$
\begin{aligned}
& a_{1} x+b_{1} y=c_{1} \\
& a_{2} x+b_{2} y=c_{2}
\end{aligned}
$$

can be illustrated through one of the graphs in Figure 2.


Case I
(One solution)


Case II
(No solutions)


Case III
(Infinite number of solutions)

FIGURE 2

Case I The two lines intersect at one and only one point. The coordinates of the point give the solution to the system. This is what usually happens.

Case II The lines are parallel and therefore have no points in common. The solution set to the system is the empty set, $\varnothing$. In this case, we say the system is inconsistent.
Case III The lines coincide; that is, their graphs represent the same line. The solution set consists of all ordered pairs that satisfy either equation. In this case, the equations are said to be dependent.

In the beginning of this section we found the solution set for the system

$$
\begin{aligned}
& 3 x-2 y=6 \\
& 2 x+4 y=20
\end{aligned}
$$

by graphing each equation and then reading the solution set from the graph. Solving a system of linear equations by graphing is the least accurate method. If the coordinates of the point of intersection are not integers, it can be very difficult to read the solution set from the graph.

## The Elimination Method

## EXAMPLE 1 Solve the system.

$$
\begin{aligned}
4 x+3 y & =10 \\
2 x+y & =4
\end{aligned}
$$

SOLUTION If we multiply the bottom equation by -3 , the coefficients of $y$ in the resulting equation and the top equation will be opposites:

$$
\begin{aligned}
& 4 x+3 y=10 \xrightarrow{\text { No change }} 4 x+3 y=10 \\
& 2 x+y=4 \xrightarrow[\text { Multiply by }-3]{ }-6 x-3 y=-12
\end{aligned}
$$

Adding the left and right sides of the resulting equations, we have

$$
\begin{array}{rr}
4 x+3 y= & 10 \\
-6 x-3 y & =-12 \\
\hline-2 x \quad & =-2
\end{array}
$$

The result is a linear equation in one variable. We have eliminated the variable $y$ from the equations by addition. (It is for this reason we call this method of solving a linear system the elimination method.) Solving $-2 x=-2$ for $x$, we have

$$
x=1
$$

This is the $x$-coordinate of the solution to our system. To find the $y$-coordinate, we substitute $x=1$ into any of the equations containing both the variables $x$ and $y$. Let's try the second equation in our original system:

$$
\begin{array}{r}
2(1)+y=4 \\
2+y=4 \\
y=2
\end{array}
$$

This is the $y$-coordinate of the solution to our system. The ordered pair $(1,2)$ is the solution to the system.

Checking Solutions We can check our solution by substituting it into both of our equations.

Substituting $x=1$ and $y=2$ into $4 x+3 y=10$, we have

$$
\begin{aligned}
4(1)+3(2) & \stackrel{?}{=} 10 \\
4+6 & \stackrel{?}{=} 10 \\
10 & =10 \quad \text { A true statement }
\end{aligned}
$$

Substituting $x=1$ and $y=2$ into $2 x+y=4$, we have

$$
\begin{array}{r}
2(1)+2 \stackrel{?}{=} 4 \\
2+2 \stackrel{?}{=} 4
\end{array}
$$

$4=4 \quad$ A true statement

Our solution satisfies both equations; therefore, it is a solution to our system of equations.

EXAMPLE 2 Solve the system.

$$
\begin{aligned}
& 3 x-5 y=-2 \\
& 2 x-3 y=1
\end{aligned}
$$

SOLUTION We can eliminate either variable. Let's decide to eliminate the variable $x$. We can do so by multiplying the top equation by 2 and the bottom equation by -3 , and then adding the left and right sides of the resulting equations:

$$
\begin{aligned}
& 3 x-5 y=-2 \xrightarrow{\text { Multiply by } 2} \begin{array}{r}
6 x-10 y
\end{array}=-4 \\
& 2 x-3 y=1 \xrightarrow[\text { Multiply by }-3]{ } \begin{array}{r}
-6 x+9 y \\
=-3
\end{array} \\
&-y=-7 \\
& y=7
\end{aligned}
$$

The $y$-coordinate of the solution to the system is 7 . Substituting this value of $y$ into any of the equations with both $x$ - and $y$-variables gives $x=11$. The solution to the system is $(11,7)$. It is the only ordered pair that satisfies both equations.

Checking Solutions Checking $(11,7)$ in each equation looks like this
Substituting $x=11$ and $y=7$ into Substituting $x=11$ and $y=7$ into $3 x-5 y=-2$, we have
$2 x-3 y=1$, we have

$$
\begin{aligned}
3(11)-5(7) & \stackrel{?}{=}-2 \\
33-35 & \stackrel{?}{=}-2 \\
-2 & =-2 \quad \text { A true statement }
\end{aligned}
$$

$$
\begin{aligned}
2(11)-3(7) & \stackrel{?}{=} 1 \\
22-21 & \stackrel{?}{=} 1 \\
1 & =1 \quad \text { A true statement }
\end{aligned}
$$

Our solution satisfies both equations; therefore, $(11,7)$ is a solution to our system.

## EXAMPLE 3 Solve the system.

$$
\begin{aligned}
& 2 x-3 y=4 \\
& 4 x+5 y=3
\end{aligned}
$$

SOLUTION We can eliminate $x$ by multiplying the top equation by -2 and adding it to the bottom equation:

$$
\begin{aligned}
& 2 x-3 y=4 \xrightarrow{\text { Multiply by }-2}-4 x+6 y=-8 \\
& 4 x+5 y=3 \xrightarrow[\text { No change }]{ } \begin{aligned}
4 x+5 y & =3 \\
11 y & =-5 \\
y & =-\frac{5}{11}
\end{aligned}
\end{aligned}
$$

The $y$-coordinate of our solution is $-\frac{5}{11}$. If we were to substitute this value of $y$ back into either of our original equations, we would find the arithmetic necessary to solve for $x$ cumbersome. For this reason, it is probably best to go back to the original system and solve it a second time-for $x$ instead of $y$. Here is how we do that:

$$
\begin{aligned}
& 2 x-3 y=4 \xrightarrow{\text { Multiply by } 5} 10 x-15 y=20 \\
& 4 x+5 y=3 \xrightarrow[\text { Multiply by } 3]{ } \frac{12 x+15 y}{}=9 \\
& 22 x=29 \\
& x=\frac{29}{22}
\end{aligned}
$$

The solution to our system is $\left(\frac{29}{22},-\frac{5}{11}\right)$.
The main idea in solving a system of linear equations by the elimination method is to use the multiplication property of equality on one or both of the original equations, if necessary, to make the coefficients of either variable opposites. The following box shows some steps to follow when solving a system of linear equations by the elimination method.

## Strategy for Solving a System of Linear Equations

 by the Elimination MethodStep 1: Decide which variable to eliminate. (In some cases, one variable will be easier to eliminate than the other. With some practice, you will notice which one it is.)
Step 2: Use the multiplication property of equality on each equation separately to make the coefficients of the variable that is to be eliminated opposites.
Step 3: Add the respective left and right sides of the system together.
Step 4: Solve for the remaining variable.
Step 5: Substitute the value of the variable from step 4 into an equation containing both variables and solve for the other variable. (Or repeat steps $2-4$ to eliminate the other variable.)
Step 6: Check your solution in both equations, if necessary.

## EXAMPLE 4 Solve the system.

$$
\begin{array}{r}
5 x-2 y=5 \\
-10 x+4 y=15
\end{array}
$$

SOLUTION We can eliminate $y$ by multiplying the first equation by 2 and adding the result to the second equation:

$$
\begin{aligned}
& 5 x-2 y=5 \xrightarrow{\text { Multiply by } 2} 10 x-4 y=10 \\
& -10 x+4 y=15 \xrightarrow[\text { No change }]{-10 x+4 y=15} \begin{array}{l}
0=25
\end{array}
\end{aligned}
$$

The result is the false statement $0=25$, which indicates there is no solution to the system. If we were to graph the two lines, we would find that they are parallel. In a case like this, we say the system is inconsistent. Whenever both variables have been eliminated and the resulting statement is false, the solution set for the system will be the empty set, $\varnothing$.

## EXAMPLE 5 Solve the system.

$$
\begin{aligned}
& 4 x+3 y=2 \\
& 8 x+6 y=4
\end{aligned}
$$

SOLUTION Multiplying the top equation by -2 and adding, we can eliminate the variable $x$ :

$$
\begin{aligned}
& 4 x+3 y=2 \xrightarrow{\text { Multiply by }-2}-8 x-6 y=-4 \\
& 8 x+6 y=4 \xrightarrow[\text { No change }]{ } \begin{array}{r}
8 x+6 y=4 \\
0=
\end{array}
\end{aligned}
$$

Both variables have been eliminated and the resulting statement $0=0$ is true. In this case, the lines coincide and the equations are said to be dependent. The solution set consists of all ordered pairs that satisfy either equation. We can write the solution set as $\{(x, y) \mid 4 x+3 y=2\}$ or $\{(x, y) \mid 8 x+6 y=4\}$.

## Special Cases

The previous two examples illustrate the two special cases in which the graphs of the equations in the system either coincide or are parallel. In both cases the left-hand sides of the equations were multiples of each other. In the case of the dependent equations the right-hand sides were also multiples. We can generalize these observations for the system

$$
\begin{aligned}
& a_{1} x+b_{1} y=c_{1} \\
& a_{2} x+b_{2} y=c_{2}
\end{aligned}
$$

## Inconsistent System

What Happens
Both variables are eliminated, and the resulting statement is false.

Geometric Interpretation
The lines are parallel, and there is no solution to the system.

Algebraic Interpretation

$$
\frac{a_{1}}{a_{2}}=\frac{b_{1}}{b_{2}} \neq \frac{c_{1}}{c_{2}}
$$

## Dependent Equations

What Happens
Both variables are eliminated, and the resulting statement is true.

Geometric Interpretation
The lines coincide, and there are an infinite number of solutions to the system.

Algebraic Interpretation

$$
\frac{a_{1}}{a_{2}}=\frac{b_{1}}{b_{2}}=\frac{c_{1}}{c_{2}}
$$

EXAMPLE 6 Solve the system.

$$
\begin{aligned}
& \frac{1}{2} x-\frac{1}{3} y=2 \\
& \frac{1}{4} x+\frac{2}{3} y=6
\end{aligned}
$$

SOLUTION Although we could solve this system without clearing the equations of fractions, there is probably less chance for error if we have only integer coefficients to work with. So let's begin by multiplying both sides of the top equation by 6 and both sides of the bottom equation by 12 to clear each equation of fractions:

$$
\begin{aligned}
& \frac{1}{2} x-\frac{1}{3} y=2 \xrightarrow{\text { Multiply by } 6} 3 x-2 y=12 \\
& \frac{1}{4} x+\frac{2}{3} y=6 \xrightarrow[\text { Multiply by } 12]{ } 3 x+8 y=72
\end{aligned}
$$

Now we can eliminate $x$ by multiplying the top equation by -1 and leaving the bottom equation unchanged:

$$
\begin{aligned}
3 x-2 y=12 \xrightarrow{\text { Multiply by }-1}-3 x+2 y & =-12 \\
3 x+8 y=72 \longrightarrow \text { No change } & 3 x+8 y \\
= & 72 \\
10 y & =60 \\
y & =
\end{aligned}
$$

We can substitute $y=6$ into any equation that contains both $x$ and $y$. Let's use $3 x-2 y=12$.

$$
\begin{aligned}
3 x-2(6) & =12 \\
3 x-12 & =12 \\
3 x & =24 \\
x & =8
\end{aligned}
$$

The solution to the system is $(8,6)$.

## The Substitution Method

EXAMPLE 7 Solve the system.

$$
\begin{aligned}
2 x-3 y & =-6 \\
y & =3 x-5
\end{aligned}
$$

SOLUTION The second equation tells us $y$ is $3 x-5$. Substituting the expression $3 x-5$ for $y$ in the first equation, we have

$$
2 x-3(3 x-5)=-6
$$

The result of the substitution is the elimination of the variable $y$. Solving the resulting linear equation in $x$ as usual, we have

$$
\begin{aligned}
2 x-9 x+15 & =-6 \\
-7 x+15 & =-6 \\
-7 x & =-21 \\
x & =3
\end{aligned}
$$

Putting $x=3$ into the second equation in the original system, we have

$$
\begin{aligned}
y & =3(3)-5 \\
& =9-5 \\
& =4
\end{aligned}
$$

The solution to the system is $(3,4)$.
Checking Solutions Checking $(3,4)$ in each equation looks like this

Substituting $x=3$ and $y=4$ into
$2 x-3 y=-6$, we have

$$
\begin{aligned}
2(3)-3(4) & \stackrel{?}{=}-6 \\
6-12 & \stackrel{?}{=}-6 \\
-6 & =-6 \quad \text { A true statement }
\end{aligned}
$$

Substituting $x=3$ and $y=4$ into $y=3 x-5$, we have
$4 \stackrel{?}{=} 3(3)-5$
$4 \stackrel{?}{=} 9-5$
$4=4 \quad$ A true statement

Our solution satisfies both equations; therefore, $(3,4)$ is a solution to our system.

## Strategy for Solving a System of Equations by the Substitution Method

Step 1: Solve either one of the equations for $x$ or $y$. (This step is not necessary if one of the equations is already in the correct form, as in Example 7.)
Step 2: Substitute the expression for the variable obtained in step 1 into the other equation and solve it.

Step 3: Substitute the solution for step 2 into any equation in the system that contains both variables and solve it.
Step 4: Check your results, if necessary. method because solving the second equation for $x$ is a one-step process.

> her. A system like the one in Example 2 is easier to solve using the elimination method because solving for one of the variables would lead to an expression involving fractions. The system in Example 8 could be solved easily by either

## Note

Both the substitution method and the elimination method can be used to solve any system of linear equations in two variables. Systems like the one in Example 7, however, are easier to solve using the substitution method because one of the variables is already written in terms of the other. A

EXAMPLE 8 Solve by substitution.

$$
\begin{array}{r}
2 x+3 y=5 \\
x-2 y=6
\end{array}
$$

SOLUTION To use the substitution method, we must solve one of the two equations for $x$ or $y$. We can solve for $x$ in the second equation by adding $2 y$ to both sides:

$$
\begin{aligned}
x-2 y & =6 \\
x & =2 y+6 \quad \text { Add } 2 y \text { to both sides. }
\end{aligned}
$$

Substituting the expression $2 y+6$ for $x$ in the first equation of our system, we have

$$
\begin{aligned}
2(2 y+6)+3 y & =5 \\
4 y+12+3 y & =5 \\
7 y+12 & =5 \\
7 y & =-7 \\
y & =-1
\end{aligned}
$$

Using $y=-1$ in either equation in the original system, we get $x=4$. The solution is $(4,-1)$.

## Application

To review, here is our Blueprint for Problem Solving, modified to fit the application problems that depend on a system of equations.

## BLUEPRINT FOR PROBLEM SOLVING USING A SYSTEM OF EQUATIONS

STEP 1: Read the problem, and then mentally list the items that are known and the items that are unknown.

STEP 2: Assign variables to each of the unknown items; that is, let $x=$ one of the unknown items and $y=$ the other unknown item (and $z=$ the third unknown item, if there is a third one). Then translate the other information in the problem to expressions involving the two (or three) variables.
STEP 3: Reread the problem, and then write a system of equations, using the items and variables listed in steps 1 and 2, that describes the situation.
STEP 4: Solve the system found in step 3.
STEP 5: Write your answers using complete sentences.
STEP 6: Reread the problem, and check your solution with the original words in the problem.

## EXAMPLE 9

It takes 2 hours for a boat to travel 28 miles downstream (with the current). The same boat can travel 18 miles upstream (against the current) in 3 hours. What is the speed of the boat in still water, and what is the speed of the current of the river?

## SOLUTION

## Step 1: Read and list.

A boat travels 18 miles upstream and 28 miles downstream. The trip upstream takes 3 hours. The trip downstream takes 2 hours. We don't know the speed of the boat or the speed of the current.

Step 2: Assign variables and translate information.
Let $x=$ the speed of the boat in still water and let $y=$ the speed of the current. The average speed (rate) of the boat upstream is $x-y$ because it is traveling against the current. The rate of the boat downstream is $x+y$ because the boat is traveling with the current.


## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A None

B $1-6$
C 7,8

## Step 3: Write a system of equations.

Putting the information into a table, we have

|  | $\boldsymbol{d}$ <br> (Distance, miles) | $\boldsymbol{r}$ <br> (Rate, $\mathbf{m p h})$ | $\boldsymbol{t}$ <br> (Time, h) |
| :--- | :---: | :---: | :---: |
| Upstream | 18 | $x-y$ | 3 |
| Downstream | 28 | $x+y$ | 2 |

The formula for the relationship between distance $d$, rate $r$, and time $t$ is $d=r t$ (the rate equation). Because $d=r \cdot t$, the system we need to solve the problem is

$$
\begin{aligned}
& 18=(x-y) \cdot 3 \\
& 28=(x+y) \cdot 2
\end{aligned}
$$

which is equivalent to

$$
\begin{array}{r}
6=x-y \\
14=x+y
\end{array}
$$

Step 4: Solve the system.
Adding the two equations, we have

$$
\begin{aligned}
20 & =2 x \\
x & =10
\end{aligned}
$$

Substituting $x=10$ into $14=x+y$, we see that

$$
y=4
$$

## Step 5: Write answers.

The speed of the boat in still water is 10 miles per hour; the speed of the current is 4 miles per hour.

## Step 6: Reread and check.

The boat travels at $10+4=14$ miles per hour downstream, so in 2 hours it will travel $14 \cdot 2=28$ miles. The boat travels at $10-4=6$ miles per hour upstream, so in 3 hours it will travel $6 \cdot 3=18$ miles.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Two linear equations, each with the same two variables, form a system of equations. How do we define a solution to this system? That is, what form will a solution have, and what properties does a solution possess?
2. When would substitution be more efficient than the elimination method in solving two linear equations?
3. Explain what an inconsistent system of linear equations looks like graphically and what would result algebraically when attempting to solve the system.
4. When might the graphing method of solving a system of equations be more desirable than the other techniques, and when might it be less desirable?

## Problem Set 7.5

Solve each system by graphing both equations on the same set of axes and then reading the solution from the graph.

1. $3 x-2 y=6$
$x-y=1$
2. $y=\frac{3}{5} x-3$
$2 x-y=-4$
3. $y=\frac{1}{2} x$
$y=-\frac{3}{4} x+5$
4. $3 x+3 y=-2$

$$
y=-x+4
$$

9. $2 x-y=5$

$$
y=2 x-5
$$

2. $5 x-2 y=10$
$x-y=-1$
3. $y=\frac{1}{2} x-2$

$$
2 x-y=-1
$$

6. $y=\frac{2}{3} x$

$$
y=-\frac{1}{3} x+6
$$

8. $2 x-2 y=6$

$$
y=x-3
$$

10. $x+2 y=5$

$$
y=-\frac{1}{2} x+3
$$

Solve each of the following systems by the elimination method.
11. $x+y=5$
$3 x-y=3$
13. $3 x+y=4$
$4 x+y=5$
15. $3 x-2 y=6$
$6 x-4 y=12$
12. $x-y=4$
$-x+2 y=-3$
14. $6 x-2 y=-10$ $6 x+3 y=-15$
16. $\begin{aligned} 4 x+5 y & =-3 \\ -8 x-10 y & =3\end{aligned}$
18. $x+3 y=3$ $2 x-9 y=1$
19. $2 x-5 y=16$ $4 x-3 y=11$
20. $5 x-3 y=-11$ $7 x+6 y=-12$
21. $6 x+3 y=-1$ $9 x+5 y=1$
22. $5 x+4 y=-1$ $7 x+6 y=-2$
23. $\begin{aligned} 4 x+3 y & =14 \\ 9 x-2 y & =14\end{aligned}$ $9 x-2 y=14$
24. $7 x-6 y=13$ $6 x-5 y=11$
25. $2 x-5 y=3$ $-4 x+10 y=3$
26. $3 x-2 y=1$ $-6 x+4 y=-2$
27. $\frac{1}{4} x-\frac{1}{6} y=-2$
28. $-\frac{1}{3} x+\frac{1}{4} y=0$ $-\frac{1}{6} x+\frac{1}{5} y=4$
$\frac{1}{5} x-\frac{1}{10} y=1$
29. $\frac{1}{2} x+\frac{1}{3} y=13$
$\frac{2}{5} x+\frac{1}{4} y=10$

Solve each of the following systems by the substitution method.
31. $7 x-y=24$
$x=2 y+9$
32. $3 x-y=-8$
$y=6 x+3$
33. $6 x-y=10$

$$
y=-\frac{3}{4} x-1
$$

34. $2 x-y=6$

$$
y=-\frac{4}{3} x+1
$$

35. $3 y+4 z=23$
36. $2 x-y=650$ $3.5 x-y=1,400$
37. $\begin{aligned} y & =3 x-2 \\ y & =4 x-4\end{aligned}$
38. $y=5 x-2$
$y=-2 x+5$
39. $2 x-y=5$
$4 x-2 y=10$
40. $-10 x+8 y=-6$

$$
y=\frac{5}{4} x
$$

41. $\begin{aligned} \frac{1}{3} x-\frac{1}{2} y & =0 \\ x & =\frac{3}{2} y\end{aligned}$
42. $\begin{aligned} \frac{2}{5} x-\frac{2}{3} y & =0 \\ y & =\frac{3}{5} x\end{aligned}$

You may want to read Example 3 again before solving the systems that follow.
43. $4 x-7 y=3$
$5 x+2 y=-3$
44. $3 x-4 y=7$
$6 x-3 y=5$
45. $9 x-8 y=4$
$2 x+3 y=6$
46. $4 x-7 y=10$
$-3 x+2 y=-9$
47. $3 x-5 y=2$
$7 x+2 y=1$
48. $4 x-3 y=-1$
$5 x+8 y=2$

Solve each of the following systems by using either the elimination or substitution method. Choose the method that is most appropriate for the problem.
49. $x-3 y=7$
$2 x+y=-6$
50. $2 x-y=9$
$x+2 y=-11$
51. $y=\frac{1}{2} x+\frac{1}{3}$
$y=-\frac{1}{3} x+2$
52. $y=\frac{3}{4} x-\frac{4}{5}$
$y=\frac{1}{2} x-\frac{1}{2}$
53. $3 x-4 y=12$
$x=\frac{2}{3} y-4$
54. $-5 x+3 y=-15$

$$
x=\frac{4}{5} y-2
$$

55. $4 x-3 y=-7$
$-8 x+6 y=-11$
56. $3 x-4 y=8$

$$
y=\frac{3}{4} x-2
$$

57. $3 y+z=17$
$5 y+20 z=65$
58. $x+y=850$
$1.5 x+y=1,100$
59. $\frac{3}{4} x-\frac{1}{3} y=1$
60. $-\frac{2}{3} x+\frac{1}{2} y=-1$

$$
y=\frac{1}{4} x
$$

61. $\frac{1}{4} x-\frac{1}{2} y=\frac{1}{3}$
$\frac{1}{3} x-\frac{1}{4} y=-\frac{2}{3}$
62. $\frac{1}{5} x-\frac{1}{10} y=-\frac{1}{5}$
$\frac{2}{3} x-\frac{1}{2} y=-\frac{1}{6}$

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working.
63. Work each problem according to the instructions given.
a. Simplify: $(3 x-4 y)-3(x-y)$
b. Find $y$ when $x$ is 0 in $3 x-4 y=8$.
c. Find the $y$-intercept: $3 x-4 y=8$
d. Graph: $3 x-4 y=8$
e. Find the point where the graphs of $3 x-4 y=8$ and $x-y=2$ cross.
64. Work each problem according to the instructions given.
a. Solve: $4 x-5=20$
b. Solve for $y: 4 x-5 y=20$
c. Solve for $x$ : $x-y=5$
d. Solve the system: $4 x-5 y=20$

$$
x-y=5
$$

## Applying the Concepts

65. Number Problem One number is 3 more than twice another. The sum of the numbers is 18 . Find the two numbers.
66. Number Problem The sum of two numbers is 32 . One of the numbers is 4 less than 5 times the other. Find the two numbers.
67. Ticket Problem A total of 925 tickets were sold for a game for a total of $\$ 1,150$. If adult tickets sold for $\$ 2.00$ and children's tickets sold for $\$ 1.00$, how many of each kind of ticket were sold?
68. Internet Problem If tickets for a show cost $\$ 2.00$ for adults and $\$ 1.50$ for children, how many of each
kind of ticket were sold if a total of 300 tickets were sold for $\$ 525$ ?
69. Mixture Problem A mixture of $16 \%$ disinfectant solution is to be made from $20 \%$ and $14 \%$ disinfectant solutions. How much of each solution should be used if 15 gallons of the $16 \%$ solution are needed?
70. Mixture Problem Paul mixes nuts worth $\$ 1.55$ per pound with oats worth $\$ 1.35$ per pound to get 25 pounds of trail mix worth $\$ 1.45$ per pound. How many pounds of nuts and how many pounds of oats did he use?
71. Rate Problem It takes about 2 hours to travel 24 miles downstream and 3 hours to travel 18 miles upstream. What is the speed of the boat in still water? What is the speed of the current of the river?

72. Rate Problem A boat on a river travels 20 miles downstream in only 2 hours. It takes the same boat 6 hours to travel 12 miles upstream. What are the speed of the boat and the speed of the current?
73. Rate Problem An airplane flying with the wind can cover a certain distance in 2 hours. The return trip against the wind takes $2 \frac{1}{2}$ hours. How fast is the plane and what is the speed of the air, if the distance is 600 miles?

74. Rate Problem An airplane covers a distance of 1,500 miles in 3 hours when it flies with the wind and $3 \frac{1}{3}$ hours when it flies against the wind. What is the speed of the plane in still air?

## Maintaining Your Skills

Add or subtract as indicated.
75. $\frac{x^{2}}{x+5}+\frac{10 x+25}{x+5}$
76. $\frac{x^{2}}{x-3}-\frac{9}{x-3}$
77. $\frac{a}{3}+\frac{2}{5}$
78. $\frac{4}{a}+\frac{2}{3}$
79. $\frac{6}{a^{2}-9}-\frac{5}{a^{2}-a-6}$
80. $\frac{4 a}{a^{2}+6 a+5}-\frac{3 a}{a^{2}+5 a+4}$

## Getting Ready for the Next Section

Simplify.
81. $(x+3 y)-1(x-2 z)$
82. $(x+y+z)+(2 x-y+z)$

Solve.
83. $-9 y=-9$
84. $30 x=38$
85. $3(1)+2 z=9$
86. $4\left(\frac{19}{15}\right)-2 y=4$

Apply the distributive property, then simplify if possible.
87. $2(5 x-z)$
88. $-1(x-2 z)$
89. $3(3 x+y-2 z)$
90. $2(2 x-y+z)$


## OBJECTIVES

A Solve systems of linear equations in three variables.

## Systems of Linear Equations in Three Variables

A solution to an equation in three variables such as

$$
2 x+y-3 z=6
$$

Is an ordered triple of numbers $(x, y, z)$. For example, the ordered triples $(0,0,-2),(2,2,0)$, and $(0,9,1)$ are solutions to the equation $2 x+y-3 z=6$ since they produce a true statement when their coordinates are substituted for $x, y$, and $z$ in the equation.

DEFINITION The solution set for a system of three linear equations in three variables is the set of ordered triples that satisfy all three equations.

## EXAMPLE 1 Solve the system.

$$
\begin{align*}
x+y+z & =6 \\
2 x-y+z & =3  \tag{2}\\
x+2 y-3 z & =-4 \tag{3}
\end{align*}
$$

SOLUTION We want to find the ordered triple ( $x, y, z$ ) that satisfies all three equations. We have numbered the equations so it will be easier to keep track of where they are and what we are doing.

There are many ways to proceed. The main idea is to take two different pairs of equations and eliminate the same variable from each pair. We begin by
adding equations (1) and (2) to eliminate the $y$-variable. The resulting equation is numbered (4):

$$
\begin{array}{r}
x+y+z=6 \\
2 x-y+z=3 \\
\hline 3 x+2 z=9 \tag{4}
\end{array}
$$

Adding twice equation (2) to equation (3) will also eliminate the variable $y$. The resulting equation is numbered (5):

$$
\begin{align*}
4 x-2 y+2 z & =6 \\
x+2 y-3 z & =-4  \tag{3}\\
\hline 5 x-z & =2
\end{align*}
$$

Equations (4) and (5) form a linear system in two variables. By multiplying equation (5) by 2 and adding the result to equation (4), we succeed in eliminating the variable $z$ from the new pair of equations:

$$
\begin{aligned}
3 x+2 z & =9 \\
10 x-2 z & =4 \\
\hline 13 x & \\
\hline 13 & \\
& \text { Twice (5) } \\
X & =1
\end{aligned}
$$

Substituting $x=1$ into equation (4), we have

$$
\begin{aligned}
3(1)+2 z & =9 \\
2 z & =6 \\
z & =3
\end{aligned}
$$

Using $x=1$ and $z=3$ in equation (1) gives us

$$
\begin{array}{r}
1+y+3=6 \\
y+4=6 \\
y=2
\end{array}
$$

The solution is the ordered triple $(1,2,3)$.

EXAMPLE 2 Solve the system.

$$
\begin{align*}
& 2 x+y-z=3 \\
& 3 x+4 y+z=6 \\
& 2 x-3 y+z=1 \tag{3}
\end{align*}
$$

SOLUTION It is easiest to eliminate $z$ from the equations. The equation produced by adding (1) and (2) is

$$
5 x+5 y=9
$$

The equation that results from adding (1) and (3) is

$$
\begin{equation*}
4 x-2 y=4 \tag{5}
\end{equation*}
$$

Equations (4) and (5) form a linear system in two variables. We can eliminate the variable $y$ from this system as follows:

$$
\begin{aligned}
5 x+5 y=9 \xrightarrow[\text { Multiply by } 2]{ } 10 x+10 y & =18 \\
4 x-2 y=4 \xrightarrow[\text { Multiply by } 5]{ } \frac{20 x-10 y}{30 x} & =20 \\
& =38 \\
x & =\frac{38}{30} \\
& =\frac{19}{15}
\end{aligned}
$$

Substituting $x=\frac{19}{15}$ into equation (5) or equation (4) and solving for $y$ gives

$$
y=\frac{8}{15}
$$

Using $x=\frac{19}{15}$ and $y=\frac{8}{15}$ in equation (1),(2), or (3) and solving for $z$ results in

$$
z=\frac{1}{15}
$$

The ordered triple that satisfies all three equations is $\left(\frac{19}{15}, \frac{8}{15}, \frac{1}{15}\right)$.

## EXAMPLE 3 Solve the system.

$$
\begin{array}{r}
2 x+3 y-z=5 \\
4 x+6 y-2 z=10 \\
x-4 y+3 z=5 \tag{3}
\end{array}
$$

SOLUTION Multiplying equation (1) by -2 and adding the result to equation (2) looks like this:

$$
\begin{aligned}
-4 x-6 y+2 z & =-10 \\
4 x+6 y-2 z & =10 \\
\hline 0 & =0
\end{aligned}
$$

## Note

On the last page of this section, you will find a discussion of the geometric interpretations associated with systems of equations in three variables.

All three variables have been eliminated, and we are left with a true statement. As was the case in the previous section, this implies that the two equations are dependent. With a system of three equations in three variables, however, a system such as this one can have no solution or an infinite number of solutions. In either case, we have no unique solution, meaning there is no single ordered triple that is the only solution to the system.

EXAMPLE 4 Solve the system.

$$
\begin{array}{r}
x-5 y+4 z=8 \\
3 x+y-2 z=7  \tag{2}\\
-9 x-3 y+6 z=5
\end{array}
$$

SOLUTION Multiplying equation (2) by 3 and adding the result to equation (3) produces

$$
\begin{align*}
9 x+3 y-6 z & =21 \\
-9 x-3 y+6 z & =5  \tag{3}\\
\hline 0 & =26
\end{align*}
$$

In this case all three variables have been eliminated, and we are left with a false statement. The system is inconsistent: there are no ordered triples that satisfy both equations. The solution set for the system is the empty set, $\varnothing$. If equations (2) and (3) have no ordered triples in common, then certainly (1), (2), and (3) do not either.

EXAMPLE 5 Solve the system.

$$
\begin{array}{rr}
x+3 y= & 5 \\
6 y+z= & 12 \\
x-2 z=-10 \tag{3}
\end{array}
$$

SOLUTION It may be helpful to rewrite the system as

$$
\begin{aligned}
x+3 y & =5 \\
6 y+z & =12 \\
x-\quad 2 z & =-10
\end{aligned}
$$

Equation (2) does not contain the variable $x$. If we multiply equation (3) by -1 and add the result to equation (1), we will be left with another equation that does not contain the variable $x$ :

$$
\begin{align*}
& x+3 y=5 \\
&-x \\
&-2 z=10
\end{aligned} \quad \begin{aligned}
& \text { (1) }  \tag{4}\\
& -1 \text { times (3) }
\end{align*}
$$

Equations (2) and (4) form a linear system in two variables. Multiplying equation (2) by -2 and adding the result to equation (4) eliminates the variable $z$ :

$$
\begin{aligned}
& 6 y+z=12 \xrightarrow{\text { Multiply by }-2}-12 y-2 z=-24 \\
& 3 y+2 z=15 \xrightarrow[\text { No change }]{ } \frac{3 y+2 z}{}=15 \\
& \hline-9 y=-9 \\
& y=1
\end{aligned}
$$

Using $y=1$ in equation (4) and solving for $z$, we have

$$
z=6
$$

Substituting $y=1$ into equation (1) gives

$$
x=2
$$

The ordered triple that satisfies all three equations is $(2,1,6)$.

EXAMPLE 6 A coin collection consists of 14 coins with a total value of $\$ 1.35$. If the coins are nickels, dimes, and quarters, and the number of nickels is 3 less than twice the number of dimes, how many of each coin is there in the collection?

SOLUTION This problem will require three variables and three equations.

## Step 1: Read and list.

We have 14 coins with a total value of $\$ 1.35$. The coins are nickels, dimes, and quarters. The number of nickels is 3 less than twice the number of dimes. We do not know how many of each coin we have.

Step 2: Assign variables and translate information.
Because we have three types of coins, we will have to use three variables. Let's let $x=$ the number of nickels, $y=$ the number of dimes, and $z=$ the number of quarters.

Step 3: Write a system of equations.
Because the total number of coins is 14 , our first equation is

$$
x+y+z=14
$$

Because the number of nickels is 3 less than twice the number of dimes, a second equation is
$x=2 y-3 \quad$ which is equivalent to $\quad x-2 y=-3$
Our last equation is obtained by considering the value of each coin and the total value of the collection. Let's write the equation in terms of cents, so we won't have to clear it of decimals later.

$$
5 x+10 y+25 z=135
$$

Here is our system, with the equations numbered for reference:

$$
\begin{align*}
x+y+z & =14  \tag{1}\\
x-2 y & =-3  \tag{2}\\
5 x+10 y+25 z & =135 \tag{3}
\end{align*}
$$

## Step 4: Solve the system.

Let's begin by eliminating $x$ from the first and second equations and the first and third equations. Adding -1 times the second equation to the first equation gives us an equation in only $y$ and $z$. We call this equation (4).

$$
\begin{equation*}
3 y+z=17 \tag{4}
\end{equation*}
$$

Adding - 5 times equation (1) to equation (3) gives us

$$
\begin{equation*}
5 y+20 z=65 \tag{5}
\end{equation*}
$$

We can eliminate $z$ from equations (4) and (5) by adding -20 times (4) to (5). Here is the result:

$$
\begin{aligned}
-55 y & =-275 \\
y & =5
\end{aligned}
$$

Substituting $y=5$ into equation (4) gives us $z=2$. Substituting $y=5$ and $z=2$ into equation (1) gives us $x=7$.


Case 1 The three planes have exactly one point in common. In this case we get one solution to our system, as in Examples 1, 2, and 5.


Case 2 The three planes have no points in common because they are all parallel to each other. The system they represent is an inconsistent system.


Case 3 The three planes intersect in a line. Any point on the line is a solution to the system.

Step 5: Write answers.
The collection consists of 7 nickels, 5 dimes, and 2 quarters.

## Step 6: Reread and check.

The total number of coins is $7+5+2=14$. The number of nickels, 7 , is 3 less than twice the number of dimes, 5. To find the total value of the collection, we have

The value of the 7 nickels is $7(0.05)=\$ 0.35$
The value of the 5 dimes is $5(0.10)=\$ 0.50$
The value of the 2 quarters is $2(0.25)=\$ 0.50$
The total value of the collection is $\$ 1.35$

## The Geometry Behind Equations in Three Variables

We can graph an ordered triple on a coordinate system with three axes. The graph will be a point in space. The coordinate system is drawn in perspective; you have to imagine that the $x$-axis comes out of the paper and is perpendicular to both the $y$-axis and the $z$-axis. To graph the point $(3,4,5)$, we move 3 units in the $x$-direction, 4 units in the $y$-direction, and then 5 units in the $z$-direction, as shown in Figure 1.


FIGURE 1
Although in actual practice it is sometimes difficult to graph equations in three variables, if we have to graph a linear equation in three variables, we would find that the graph was a plane in space. A system of three equations in three variables is represented by three planes in space.

There are a number of possible ways in which these three planes can intersect, some of which are shown in the margin on this page. There are still other possibilities that are not among those shown in the margin.

In Example 3 we found that equations 1 and 2 were dependent equations. They represent the same plane; that is, they have all their points in common. But the system of equations that they came from has either no solution or an infinite number of solutions. It all depends on the third plane. If the third plane coincides with the first two, then the solution to the system is a plane. If the third plane is distinct from but parallel to the first two, then there is no solution to the system. And, finally, if the third plane intersects the first two, but does not coincide with them, then the solution to the system is that line of intersection.

In Example 4 we found that trying to eliminate a variable from the second and third equations resulted in a false statement. This means that the two planes represented by these equations are parallel. It makes no difference where the


Case 4 In this case the three planes have no points in common. There is no solution to the system; it is an inconsistent system.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

third plane is; there is no solution to the system in Example 4. (If we were to graph the three planes from Example 4, we would obtain a diagram similar to Case 4 in the margin.)
If, in the process of solving a system of linear equations in three variables, we eliminate all the variables from a pair of equations and are left with a false statement, we will say the system is inconsistent. If we eliminate all the variables and are left with a true statement, then we will say the system has no unique solution.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is an ordered triple of numbers?
2. Explain what it means for $(1,2,3)$ to be a solution to a system of linear equations in three variables.
3. Explain in a general way the procedure you would use to solve a system of three linear equations in three variables.
4. How do you know when a system of linear equations in three variables has no solution?

## Problem Set 7.6

Solve the following systems.

1. $x+y+z=4$
$x-y+2 z=1$
$x-y-3 z=-4$
2. $x+y+z=6$
$x-y+2 z=7$
$2 x-y-4 z=-9$
3. $x+2 y+z=3$
$2 x-y+2 z=6$
$3 x+y-z=5$
4. $\begin{aligned} x-y-2 z= & -1 \\ x+y+z= & 6 \\ x+y-z= & 4\end{aligned}$
5. $x+y+z=0$
$x+y-z=6$
$x-y+2 z=-7$

$$
\text { 6. } \begin{aligned}
2 x+y-3 z= & -14 \\
x-3 y+4 z= & 22 \\
3 x+2 y+z= & 0
\end{aligned}
$$

8. $\begin{aligned} 4 x+y-2 z & =0 \\ 2 x-3 y+3 z & =9 \\ -6 x-2 y+z & =0\end{aligned}$
9. $4 x+6 y-8 z=1$

$$
\begin{array}{r}
-6 x-9 y+12 z=0 \\
x-2 y-2 z=3
\end{array}
$$

$$
\text { 7. } \begin{aligned}
2 x+3 y-2 z= & 4 \\
x+3 y-3 z= & 4 \\
3 x-6 y+z= & -3
\end{aligned}
$$

9. $-x+4 y-3 z=2$
$2 x-8 y+6 z=1$
$3 x-y+z=0$
10. $\frac{1}{2} x-y+z=0$
11. $\frac{1}{3} x+\frac{1}{2} y+z=-1$
$\begin{aligned} 2 x+\frac{1}{3} y+z & =2 \\ x+y+z & =-4\end{aligned}$
$x-y+\frac{1}{5} z=1$
$x+y+z=5$
12. $2 x-y-3 z=1$
$x+2 y+4 z=3$
$4 x-2 y-6 z=2$
13. $3 x+2 y+z=3$

$$
\begin{array}{r}
x-3 y+z=4 \\
-6 x-4 y-2 z=1
\end{array}
$$

15. $2 x-y+3 z=4$
$x+2 y-z=-3$
$4 x+3 y+2 z=-5$
16. $6 x-2 y+z=5$
$3 x+y+3 z=7$
$x+4 y-z=4$
17. $x+y=9$
$y+z=7$
$x-z=2$
18. $2 x+y=2$
$y+z=3$
$4 x-z=0$
19. $x-y=-3$
$x+z=2$
$y-z=7$
20. $2 x+y=6$
$3 y-2 z=-8$ $x+z=5$
= Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login

$$
\text { 21. } \begin{aligned}
2 x-3 y & =0 \\
6 y-4 z & =1 \\
x+2 z & =1
\end{aligned}
$$

22. $3 x+2 y=3$
$y+2 z=2$
$6 x-4 z=1$
23. $x+y-z=2$
$2 x+y+3 z=4$
$x-2 y+2 z=6$
24. $2 x+3 y=-\frac{1}{2}$
$4 x+8 z=2$
$3 y+2 z=-\frac{3}{4}$
25. $\frac{1}{3} x+\frac{1}{2} y-\frac{1}{6} z=4$
$\frac{1}{4} x-\frac{3}{4} y+\frac{1}{2} z=\frac{3}{2}$
$\frac{1}{2} x-\frac{2}{3} y-\frac{1}{4} z=-\frac{16}{3}$
26. $x-\frac{1}{2} y-\frac{1}{3} z=-\frac{4}{3}$
$\frac{1}{3} x+y-\frac{1}{2} z=5$ $-\frac{1}{4} x+\frac{2}{3} y-z=-\frac{3}{4}$
27. $\frac{1}{2} x+\frac{2}{3} y=\frac{5}{2}$
$\frac{1}{5} X-\frac{1}{2} Z=-\frac{3}{10}$
$\frac{1}{3} y-\frac{1}{4} z=\frac{3}{4}$
28. $\frac{1}{2} x-\frac{1}{4} y+\frac{1}{2} z=-2$
$\frac{1}{4} x-\frac{1}{12} y-\frac{1}{3} z=\frac{1}{4}$
$\frac{1}{6} x+\frac{1}{3} y-\frac{1}{2} z=\frac{3}{2}$
29. $-\frac{1}{4} x+\frac{3}{8} y+\frac{1}{2} z=-1$
$\frac{2}{3} x-\frac{1}{6} y-\frac{1}{2} z=2$
$\frac{2}{3} x-\frac{1}{6} y-\frac{1}{2} z=2$
$\frac{3}{4} x-\frac{1}{2} y-\frac{1}{8} z=1$
30. $x+2 y-2 z=4$
$3 x+4 y-z=-2$
$2 x+3 y-3 z=-5$
31. $3 x-5 y=2$
$4 x+6 z=\frac{1}{3}$
$5 y-7 z=\frac{1}{6}$
32. $x+\frac{1}{3} y-\frac{1}{2} z=-\frac{3}{2}$
$\frac{1}{2} x-y+\frac{1}{3} z=8$
$\frac{1}{3} x-\frac{1}{4} y-z=-\frac{5}{6}$
33. $\frac{1}{2} x-\frac{1}{3} y=\frac{1}{6}$
$\frac{1}{3} y-\frac{1}{3} z=1$
$\frac{1}{5} X-\frac{1}{2} Z=-\frac{4}{5}$
34. $\frac{1}{2} x+\frac{1}{2} y+\quad z=\frac{1}{2}$
$\frac{1}{2} x-\frac{1}{4} y-\frac{1}{4} z=0$
$\frac{1}{4} x+\frac{1}{12} y+\frac{1}{6} z=\frac{1}{6}$

## Applying the Concepts

35. Electric Current In the following diagram of an electrical circuit, $x, y$, and $z$ represent the amount of current (in amperes) flowing across the 5 -ohm, 20 -ohm, and 10 -ohm resistors, respectively. (In circuit dia-
grams resistors are represented by $-W$ and potential differences by $-\vdash$.)


The system of equations used to find the three currents $x, y$, and $z$ is

$$
\begin{aligned}
x-y-z & =0 \\
5 x+20 y & =80 \\
20 y-10 z & =50
\end{aligned}
$$

Solve the system for all variables.
36. Cost of a Rental Car If a car rental company charges $\$ 10$ a day and $8 \$$ a mile to rent one of its cars, then the cost $z$, in dollars, to rent a car for $x$ days and drive $y$ miles can be found from the equation

$$
z=10 x+0.08 y
$$

a. How much does it cost to rent a car for 2 days and drive it 200 miles under these conditions?
b. A second company charges $\$ 12$ a day and $6 \$$ a mile for the same car. Write an equation that gives the cost $z$, in dollars, to rent a car from this company for $x$ days and drive it $y$ miles.
c. A car is rented from each of the companies mentioned in (a) and (b) for 2 days. To find the mileage at which the cost of renting the cars from each of the two companies will be equal, solve the following system for $y$ :

$$
\begin{aligned}
& z=10 x+0.08 y \\
& z=12 x+0.06 y \\
& x=2
\end{aligned}
$$

37. Interest A man invests $\$ 2,200$ in three accounts that pay $6 \%, 8 \%$, and $9 \%$ in annual interst, respectively. He has three times as much invested at $9 \%$ as he does at $6 \%$. If his total interest for the year is $\$ 178$, how much is invested at each rate?
38. Interest A student has money in three accounts that pays $5 \%, 7 \%$, and $8 \%$ in annual interest. She has three times as much invested at $8 \%$ as she does at $5 \%$. If the total amount she has invested is $\$ 1,600$ and her interest for the year comes to $\$ 115$, how much money does she have in each account?
39. Coin Problem A collection of nickels, dimes, and quarters consists of 9 coin with a total value of $\$ 1.20$. If the number of dimes is equal to the number of nickels, find the number of each type of coin.
40. Coin Problem A coin collection consists of 12 coins with a total value of $\$ 1.20$. If the collection consists only of nickels, dimes, and quarters, and the number of dimes is two more than twice the number of nickels, how many of each type of coin are in the collection?

## Maintaining Your Skills

Simplify each expression.
41. $-|-5|$
42. $\left(-\frac{2}{3}\right)^{3}$
43. $-3-4(-2)$
44. $2^{4}+3^{3} \div 9-4^{2}$
45. $5|3-8|-6|2-5|$
46. $7-3(2-6)$
47. $5-2[-3(5-7)-8]$
48. $\frac{5+3(7-2)}{2(-3)-4}$
49. Find the difference of -3 and -9 .
50. If you add -4 to the product of -3 and 5 , what number results?

## Getting Ready for the Next Section

51. Which of the following are solutions to $x+y \leq 4$ ?
$(0,0)$
$(4,0)$
$(2,3)$
52. Which of the following are solutions to $y<2 x-3$ ?
$(0,0) \quad(3,-2) \quad(-3,2)$
53. Which of the following are solutions to $y \leq \frac{1}{2} x$ ?
$(0,0) \quad(2,0) \quad(-2,0)$
54. Which of the following are solutions to $y>-2 x$ ?
$(0,0) \quad(2,0) \quad(-2,0)$

## Linear Inequalities in Two Variables

## OBJECTIVES

A Graph linear inequalities in two variables.

A small movie theater holds 100 people. The owner charges more for adults than for children, so it is important to know the different combinations of adults and children that can be seated at one time. The shaded region in Figure 1 contains all the seating combinations. The line $x+y=100$ shows the combinations for a full theater: The $y$-intercept corresponds to a theater full of adults, and the $x$-intercept corresponds to a theater full of children. In the shaded region below the line $x+y=100$ are the combinations that occur if the theater is not full.

Shaded regions like the one shown in Figure 1 are produced by linear inequalities in two variables, which is the topic of this section.


A linear inequality in two variables is any expression that can be put in the form

$$
a x+b y<c
$$

where $a, b$, and $c$ are real numbers ( $a$ and $b$ not both 0 ). The inequality symbol can be any one of the following four: $<, \leq,>, \geq$.

Some examples of linear inequalities are

$$
2 x+3 y<6 \quad y \geq 2 x+1 \quad x-y \leq 0
$$

Although not all of these examples have the form $a x+b y<c$, each one can be put in that form.

The solution set for a linear inequality is a section of the coordinate plane. The boundary for the section is found by replacing the inequality symbol with an equal sign and graphing the resulting equation. The boundary is included in the solution set (and represented with a solid line) if the inequality symbol used originally is $\leq$ or $\geq$. The boundary is not included (and is represented with a broken line) if the original symbol is $<$ or $>$.

## EXAMPLE 1 Graph the solution set for $x+y \leq 4$.

SOLUTION The boundary for the graph is the graph of $x+y=4$. The boundary is included in the solution set because the inequality symbol is $\leq$.

Figure 2 is the graph of the boundary:


FIGURE 2
The boundary separates the coordinate plane into two regions: the region above the boundary and the region below it. The solution set for $x+y \leq 4$ is one of these two regions along with the boundary. To find the correct region, we simply choose any convenient point that is not on the boundary. We then substitute the coordinates of the point into the original inequality $x+y \leq 4$. If the point we choose satisfies the inequality, then it is a member of the solution set, and we can assume that all points on the same side of the boundary as the chosen point are also in the solution set. If the coordinates of our point do not satisfy the original inequality, then the solution set lies on the other side of the boundary.

In this example, a convenient point that is not on the boundary is the origin.

$$
\begin{array}{lr}
\text { Substituting } & (0,0) \\
\text { into } & x+y \leq 4 \\
\text { gives us } & 0+0 \leq 4 \\
& 0 \leq 4 \quad \text { A true statement }
\end{array}
$$

Because the origin is a solution to the inequality $x+y \leq 4$ and the origin is below the boundary, all other points below the boundary are also solutions.

Figure 3 is the graph of $x+y \leq 4$.


FIGURE 3

The region above the boundary is described by the inequality $x+y>4$.
Here is a list of steps to follow when graphing the solution set for a linear inequality in two variables.

To Graph a Linear Inequality in Two Variables
Step 1: Replace the inequality symbol with an equal sign. The resulting equation represents the boundary for the solution set.

Step 2: Graph the boundary found in step 1 using a solid line if the boundary is included in the solution set (that is, if the original inequality symbol was either $\leq$ or $\geq$ ). Use a broken line to graph the boundary if it is not included in the solution set. (It is not included if the original inequality was either $<$ or $>$.)

Step 3: Choose any convenient point not on the boundary and substitute the coordinates into the original inequality. If the resulting statement is true, the graph lies on the same side of the boundary as the chosen point. If the resulting statement is false, the solution set lies on the opposite side of the boundary.

EXAMPLE 2 Graph the solution set for $y<2 x-3$.
SOLUTION The boundary is the graph of $y=2 x-3$, a line with slope 2 and $y$-intercept -3 . The boundary is not included because the original inequality symbol is $<$. We therefore use a broken line to represent the boundary in Figure 4. A convenient test point is again the origin:

| Using | $(0,0)$ |
| :--- | :--- |
| in | $y<2 x-3$ |
| we have | $0<2(0)-3$ |
|  | $0<-3 \quad$ A false statement |



FIGURE 4
Because our test point gives us a false statement and it lies above the boundary, the solution set must lie on the other side of the boundary (Figure 5).


FIGURE 5

## USING TECHNOLOGY

## Graphing Calculators

Most graphing calculators have a Shade command that allows a portion of a graphing screen to be shaded. With this command we can visualize the solution sets to linear inequalities in two variables. Because most graphing calculators cannot draw a dotted line, however, we are not actually "graphing" the solution set, only visualizing it.

## Strategy for Visualizing a Linear Inequality in Two Variables on a Graphing Calculator

Step 1: Solve the inequality for $y$.
Step 2: Replace the inequality symbol with an equal sign. The resulting equation represents the boundary for the solution set.

Step 3: Graph the equation in an appropriate viewing window.
Step 4: Use the Shade command to indicate the solution set:
For inequalities having the $<$ or $\leq \operatorname{sign}$, use Shade(Ymin, $\mathrm{Y}_{1}$ ).
For inequalities having the $>$ or $\geq$ sign, use Shade( $\mathrm{Y}_{1}$, Ymax).
Note: On the TI-83, step 4 can be done by manipulating the icons in the left column in the list of Y variables.


FIGURE $6 \quad \mathrm{Y}_{1}=2 \mathrm{X}-3$
Figures 6 and 7 show the graphing calculator screens that help us visualize the solution set to the inequality $y<2 x-3$ that we graphed in Example 2.


FIGURE 7 Shade (Ymin, $\mathrm{Y}_{1}$ )

## EXAMPLE $3 \quad$ Graph the solution set for $x \leq 5$.

SOLUTION The boundary is $x=5$, which is a vertical line. All points in Figure 8 to the left of the boundary have $x$-coordinates less than 5 and all points to the right have $x$-coordinates greater than 5 .


FIGURE 8
EXAMPLE $4 \quad$ Graph the solution set for $y>\frac{1}{4} x$.
SOLUTION The boundary is the line $y=\frac{1}{4} x$, which has a slope of $\frac{1}{4}$ and passes through the origin. The graph of the boundary line is shown in Figure 9. Since the boundary passes through the origin, we cannot use the origin as our test point. Remember, the test point cannot be on the boundary line. Let's use the point $(0,-4)$ as our test point. It lies below the boundary line. When we substitute the coordinates into our original inequality, the result is the false statement $-4>0$. This tells us that the solution set is on the other side of the boundary line. The solution set for our original inequality is shown in Figure 10.


FIGURE 9


FIGURE 10

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $1-4$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. When graphing a linear inequality in two variables, how do you find the equation of the boundary line?
2. What is the significance of a broken line in the graph of an inequality?
3. When graphing a linear inequality in two variables, how do you know which side of the boundary line to shade?
4. Does the graph of $x+y<4$ include the boundary line? Explain.

## Problem Set 7.7

Use a coordinate system to graph the solution set for each of the following.

1. $x+y<5$
2. $x+y \leq 5$
3. $x-y \geq-3$
4. $x-y>-3$
5. $2 x+3 y<6$
6. $2 x-3 y>-6$
7. $-x+2 y>-4$
8. $-x-2 y<4$
9. $2 x+y<5$
10. $2 x+y<-5$
11. $y<2 x-1$
12. $y \leq 2 x-1$
13. $3 x-4 y<12$
14. $-2 x+3 y<6$
15. $-5 x+2 y \leq 10$
16. $4 x-2 y \leq 8$
17. $x \geq 3$
18. $x>-2$
19. $y \leq 4$
20. $y>-5$
21. $y<2 x$
22. $y>-3 x$
23. $y \geq \frac{1}{2} x$
24. $y \leq \frac{1}{3} x$
25. $y \geq \frac{3}{4} x-2$
26. $y>-\frac{2}{3} x+3$
27. $\frac{x}{3}+\frac{y}{2}>1$
28. $\frac{x}{5}+\frac{y}{4}<1$
29. $\frac{x}{3}-\frac{y}{2}>1$
30. $-\frac{x}{4}+\frac{y}{3}>1$
31. $y \leq-\frac{2}{3} x$
32. $y \geq \frac{1}{4} x$
33. $5 x-3 y<0$
34. $2 x+3 y>0$
35. $\frac{x}{4}+\frac{y}{5} \leq 1$
36. $\frac{x}{2}+\frac{y}{3}<1$

For each graph shown here, name the linear inequality in two variables that is represented by the shaded region.
37.

38.

39.

40.


The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working.
41. Work each problem according to the instructions given.
a. Solve: $\frac{1}{3}+\frac{y}{2}<1$
b. Solve: $\frac{1}{3}-\frac{y}{2}<1$
c. Solve for $y: \frac{x}{3}+\frac{y}{2}=1$
d. Graph: $y<-\frac{4}{3} x+4$
42. Work each problem according to the instructions given.
a. Solve: $3 x+4 \geq-8$
b. Solve: $-3 x+4 \geq-8$
c. Solve for $y$ : $-3 x+4 y=-8$
d. Graph: $y \geq \frac{3}{4} x-2$

## Applying the Concepts

43. Number of People in a Dance Club A dance club holds a maximum of 200 people. The club charges one price for students and a higher price for nonstudents. If the number of students in the club at any time is $x$ and the number of nonstudents is $y$, shade the region in the first quadrant that contains all combinations of students and nonstudents that are in the club at any time.
44. Many Perimeters Suppose you have 500 feet of fencing that you will use to build a rectangular livestock pen. Let $x$ represent the length of the pen and $y$ represent the width. Shade the region in the first quadrant that contains all possible values of $x$ and $y$ that will give you a rectangle from 500 feet of fencing. (You don't have to use all of the fencing, so the perimeter of the pen could be less than 500 feet.)
45. Gas Mileage You have two cars. The first car travels an average of 12 miles on a gallon of gasoline, and the second averages 22 miles per gallon. Suppose you can afford to buy up to 30 gallons of gasoline this month. If the first car is driven $x$ miles this month, and the second car is driven $y$ miles this month, shade the region in the first quadrant that gives all the possible values of $x$ and $y$ that will keep you from buying more than 30 gallons of gasoline this month.
46. Student Loan Payments When considering how much debt to incur in student loans, it is advisable to keep your student loan payment after graduation to $8 \%$ or less of your starting monthly income. Let $x$ represent your starting monthly salary and let $y$ represent your monthly student loan payment, and write an inequality that describes this situation. Shade the region in the first quadrant that is a solution to your inequality.

## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in the previous sections and chapters.

Solve each of the following inequalities.
47. $\frac{1}{3}+\frac{y}{5} \leq \frac{26}{15}$
48. $-\frac{1}{3} \geq \frac{1}{6}-\frac{y}{2}$
49. $5 t-4>3 t-8$
50. $-3(t-2)<6-5(t+1)$
51. $-9<-4+5 t<6$
52. $-3<2 t+1<3$

## Getting Ready for the Next Section

53. Does the graph of $x+y<4$ include the boundary line?
54. Does the graph of $-x+y \leq 3$ include the boundary line?
55. Where do the graphs of the lines $x+y=4$ and $x-2 y=4$ intersect?
56. Where do the graphs of the line $x=-1$ and $x-2 y=4$ intersect?

Solve.
57. $20 x+9,300>18,000$
58. $20 x+4,800>18,000$


## OBJECTIVES

A Graph the solution to a system of linear inequalities in two variables.

## Systems of Linear Inequalities

Figure 1 shows the graph of the inequality $x+y<4$. Note that the boundary is not included in the solution set and is therefore drawn with a broken line. Figure 2 shows the graph of $-x+y \leq 3$. Note that the boundary is drawn with a solid line because it is part of the solution set.



FIGURE 2

If we form a system of inequalities with the two inequalities, the solution set will be all the points common to both solution sets shown in the two figures above: It is the intersection of the two solution sets. Therefore, the solution set for the system of inequalities

$$
\begin{array}{r}
x+y<4 \\
-x+y \leq 3
\end{array}
$$

is all the ordered pairs that satisfy both inequalities. It is the set of points that are below the line $x+y=4$ and also below (and including) the line $-x+y=3$. The graph of the solution set to this system is shown in Figure 3. We have written the system in Figure 3 with the word and just to remind you that the solution
set to a system of equations or inequalities is all the points that satisfy both equations or inequalities.


FIGURE 3

EXAMPLE 1
Graph the solution to the system of linear inequalities.

$$
\begin{aligned}
& y<\frac{1}{2} x+3 \\
& y \geq \frac{1}{2} x-2
\end{aligned}
$$

SOLUTION Figures 4 and 5 show the solution set for each of the inequalities separately.


FIGURE 4


FIGURE 5

Figure 6 is the solution set to the system of inequalities. It is the region consisting of points whose coordinates satisfy both inequalities.

EXAMPLE 2 Graph the solution to the system of linear inequalities.

$$
\begin{aligned}
x+y & <4 \\
x & \geq 0 \\
y & \geq 0
\end{aligned}
$$

SOLUTION We graphed the first inequality, $x+y<4$, in Figure 1 at the beginning of this section. The solution set to the inequality $x \geq 0$, shown in Figure 7, is all the points to the right of the $y$-axis; that is, all the points with $x$-coordinates that are greater than or equal to 0 . Figure 8 shows the graph of $y \geq 0$. It consists of all points with $y$-coordinates greater than or equal to 0 ; that is, all points from the $x$-axis up.


The regions shown in Figures 7 and 8 overlap in the first quadrant. Therefore, putting all three regions together we have the points in the first quadrant that are below the line $x+y=4$. This region is shown in Figure 9, and it is the solution to our system of inequalities.


FIGURE 9

Extending the discussion in Example 2 we can name the points in each of the four quadrants using systems of inequalities.


FIGURE 10


FIGURE 12


FIGURE 11


FIGURE 13

EXAMPLE 3 Graph the solution to the system of linear inequalities.

$$
\begin{aligned}
& x \leq 4 \\
& y \geq-3
\end{aligned}
$$

SOLUTION The solution to this system will consist of all points to the left of and including the vertical line $x=4$ that intersect with all points above and including the horizontal line $y=-3$. The solution set is shown in Figure 14.


FIGURE 14

## EXAMPLE 4 Graph the solution set for the following system.

$$
\begin{aligned}
x-2 y & \leq 4 \\
x+y & \leq 4 \\
x & \geq-1
\end{aligned}
$$

SOLUTION We have three linear inequalities, representing three sections of the coordinate plane. The graph of the solution set for this system will be the intersection of these three sections. The graph of $x-2 y \leq 4$ is the section above and including the boundary $x-2 y=4$. The graph of $x+y \leq 4$ is the section below and including the boundary line $x+y=4$. The graph of $x \geq-1$ is all the points to the right of, and including, the vertical line $x=-1$. The intersection of these three graphs is shown in Figure 15.


FIGURE 15

EXAMPLE 5 A college basketball arena plans on charging $\$ 20$ for certain seats and $\$ 15$ for others. They want to bring in more than $\$ 18,000$ from all ticket sales and have reserved at least 500 tickets at the $\$ 15$ rate. Find a system of inequalities describing all possibilities and sketch the graph. If 620 tickets are sold for $\$ 15$, at least how many tickets are sold for $\$ 20$ ?


SOLUTION Let $x=$ the number of $\$ 20$ tickets and $y=$ the number of $\$ 15$ tickets. We need to write a list of inequalities that describe this situation. That list
will form our system of inequalities. First of all, we note that we cannot use negative numbers for either $x$ or $y$. So, we have our first inequalities:

$$
\begin{aligned}
& x \geq 0 \\
& y \geq 0
\end{aligned}
$$

Next, we note that they are selling at least 500 tickets for $\$ 15$, so we can replace our second inequality with $y \geq 500$. Now our system is

$$
\begin{aligned}
& x \geq 0 \\
& y \geq 500
\end{aligned}
$$

Now, the amount of money brought in by selling $\$ 20$ tickets is $20 x$, and the amount of money brought in by selling $\$ 15$ tickets is $15 y$. If the total income from ticket sales is to be more than $\$ 18,000$, then $20 x+15 y$ must be greater than 18,000 . This gives us our last inequality and completes our system.

$$
\begin{aligned}
20 x+15 y & >18,000 \\
x & \geq 0 \\
y & \geq 500
\end{aligned}
$$

We have used all the information in the problem to arrive at this system of inequalities. The solution set contains all the values of $x$ and $y$ that satisfy all the conditions given in the problem. Here is the graph of the solution set.


FIGURE 16

If 620 tickets are sold for $\$ 15$, then we substitute 620 for $y$ in our first inequality to obtain

$$
\begin{aligned}
20 x+15(620) & >18,000 & & \text { Substitute } \mathbf{6 2 0} \text { for } y \\
20 x+9,300 & >18,000 & & \text { Multiply } \\
20 x & >8,700 & & \text { Add }-\mathbf{9 , 3 0 0} \text { to each side } \\
x & >435 & & \text { Divide each side by } \mathbf{2 0}
\end{aligned}
$$

If they sell 620 tickets for $\$ 15$ each, then they need to sell more than 435 tickets at $\$ 20$ each to bring in more than $\$ 18,000$.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-5

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the solution set to a system of inequalities?
2. When graphing a system of linear inequalities, how do you find the equations of the boundary lines?
3. For the boundary lines of a system of linear inequalities, when do you use a dotted line rather than a solid line?
4. Once you have graphed the solution set for each inequality in a system, how do you determine the region to shade for the solution to the system of inequalities?

## Problem Set 7.8

Graph the solution set for each system of linear inequalities.
2. $x+y<5$
$2 x-y<4$
3. $y<\frac{1}{3} x+4$
$y \geq \frac{1}{3} x-3$
5. $x \geq-3$
$y<2$
7. $1 \leq x \leq 3$
$2 \leq y \leq 4$
9
9. $x+y \leq 4$
$x \geq 0$
$y \geq 0$
11. $x+y \leq 3$
$x-3 y \leq 3$
$x \geq-2$
13. $x+y \leq 2$
$-x+y \leq 2$

$$
y \geq-2
$$

4. $y<2 x+4$
$y \geq 2 x-3$
5. $x \leq 4$
$y>-2$
6. $-4 \leq x \leq-2$
$1 \leq y \leq 3$
7. $x-y \leq 2$
$x \geq 0$
$y \leq 0$
8. $x-y \leq 4$
$x+2 y \leq 4$

$$
x \geq-1
$$

14. $x-y \leq 3$
$-x-y \leq 3$
$y \leq-1$
15. $x+y<5$
$y>x$
$y \geq 0$
16. $x+y<5$
$y>x$

$$
x \geq 0
$$

17. $2 x+3 y \leq 6$

$$
\begin{aligned}
& x \geq 0 \\
& y \geq 0
\end{aligned}
$$

18. $x+2 y \leq 10$
$3 x+y \leq 12$

$$
x \geq 0
$$

For each figure below, find a system of inequalities that describes the shaded region.
19.


FIGURE 17
21.


FIGURE 19
20.


FIGURE 18


FIGURE 20

- Videos available by instructor request
- = Online student support materials available at www.thomsonedu.com/login


## Applying the Concepts

23. Office Supplies An office worker wants to purchase some $\$ 0.55$ postage stamps and also some $\$ 0.65$ postage stamps totaling no more than $\$ 40$. It also is desired to have at least twice as many $\$ 0.55$ stamps and more than $15 \$ 0.55$ stamps.
a. Find a system of inequalities describing all the possibilities and sketch the graph.
b. If he purchases $20 \$ 0.55$ stamps, what is the maximum number of $\$ 0.65$ stamps he can purchase?
24. Inventory A store sells two brands of VCRs. Customer demand indicates that it is necessary to stock at least twice as many VCRs of brand A as of brand $B$. At least 30 of brand $A$ and 15 of brand B must be on hand. There is room for not more than 100 VCRs in the store.
a. Find a system of inequalities describing all possibilities, then sketch the graph.
b. If there are 35 VCRs of brand A , what is the maximum number of brand $B$ VCRs on hand?

## Maintaining Your Skills

25. Reduce to lowest terms $\frac{x^{2}-x-6}{x^{2}-9}$.
26. Divide using long division $\frac{x^{2}-2 x+6}{x-4}$.

Perform the indicated operations.
27. $\frac{x^{2}-25}{x+4} \cdot \frac{2 x+8}{x^{2}-9 x+20}$
28. $\frac{3 x+6}{x^{2}+4 x+3} \div \frac{x^{2}+x-2}{x^{2}+2 x-3}$
29. $\frac{x}{x^{2}-16}+\frac{4}{x^{2}-16}$
30. $\frac{2}{x^{2}-1}-\frac{5}{x^{2}+3 x-4}$
31. $\frac{1-\frac{25}{X^{2}}}{1-\frac{8}{X}+\frac{15}{X^{2}}}$

Solve each equation.
32. $\frac{x}{2}-\frac{5}{x}=-\frac{3}{2}$
33. $\frac{x}{x^{2}-9}-\frac{3}{x-3}=\frac{1}{x+3}$

## Chapter 7 SUMMARY

## EXAMPLES

1. Solve: $2(x+3)=10$.
$2 x+6=10$
$2 x+6+(-6)=10+(-6)$
$2 x=4$
$\frac{\mathbf{1}}{\mathbf{2}}(2 x)=\frac{\mathbf{1}}{\mathbf{2}}(4)$
$x=2$

## Strategy for Solving Linear Equations in One Variable [7.1]

Step 1a: Use the distributive property to separate terms, if necessary.
$\mathbf{1 b}$ : If fractions are present, consider multiplying both sides by the LCD to eliminate the fractions. If decimals are present, consider multiplying both sides by a power of 10 to clear the equation of decimals.
$1 \boldsymbol{c}$ : Combine similar terms on each side of the equation.
Step 2: Use the addition property of equality to get all variable terms on one side of the equation and all constant terms on the other side. A variable term is a term that contains the variable (for example, $5 x$ ). A constant term is a term that does not contain the variable (the number 3, for example.)
Step 3: Use the multiplication property of equality to get $x$ (that is, $1 x$ ) by itself on one side of the equation.
Step 4: Check your solution in the original equation to be sure that you have not made a mistake in the solution process.

## To Solve a Quadratic Equation by Factoring [7.1]

Step 1: Write the equation in standard form.
Step 2: Factor the left side.
Step 3: Use the zero-factor property to set each factor equal to 0.
Step 4: Solve the resulting linear equations.
Step 5: Check the solutions in the original equation.

## Blueprint for Problem Solving [7.1, 7.5]

Step 1: Read the problem, and then mentally list the items that are known and the items that are unknown.
Step 2: Assign a variable to one of the unknown items. (If you are using a system of equations, assign variables to each unknown item.) Then translate the other information in the problem to expressions involving the variable, or variables.
Step 3: Reread the problem, and then write an equation or system of equations, using the items and variables listed in steps 1 and 2 , that describes the situation.
Step 4: Solve the equation or system of equations found in step 3.
Step 5: Write your answer using a complete sentence.
Step 6: Reread the problem, and check your solution with the original words in the problem.
3. To solve

$$
|2 x-1|+2=7
$$

we first isolate the absolute value on the left side by adding -2 to each side to obtain

$$
\begin{array}{rlrlrl} 
& |2 x-1| & & \\
2 x-1 & =5 & \text { or } & 2 x-1 & =-5 \\
2 x & =6 & \text { or } & 2 x & =-4 \\
x & =3 & \text { or } & x & =-2
\end{array}
$$

## Absolute Value Equations [7.2]

To solve an equation that involves absolute value, we isolate the absolute value on one side of the equation and then rewrite the absolute value equation as two separate equations that do not involve absolute value. In general, if $b$ is a positive real number, then

$$
|a|=b \quad \text { is equivalent to } a=b \text { or } a=-b
$$

## Compound Inequalities [7.3]

Two inequalities connected by the word and or or form a compound inequality. If the connecting word is or, we graph all points that are on either graph. If the connecting word is and, we graph only those points that are common to both graphs. The inequality $-2 \leq x \leq 3$ is equivalent to the compound inequality $-2 \leq x$ and $x \leq 3$.

## Absolute Value Inequalities [7.4]

To solve an inequality that involves absolute value, we first isolate the absolute value on the left side of the inequality symbol. Then we rewrite the absolute value inequality as an equivalent continued or compound inequality that does not contain absolute value symbols. In general, if $b$ is a positive real number, then

$$
|a|<b \quad \text { is equivalent to } \quad-b<a<b
$$

and

$$
|a|>b \quad \text { is equivalent to } \quad a<-b \quad \text { or } \quad a>b
$$

## To Solve a System by the Elimination Method [7.5]

Step 1: Look the system over to decide which variable will be easier to eliminate.
Step 2: Use the multiplication property of equality on each equation separately, if necessary, to ensure that the coefficients of the variable to be eliminated are opposites.
Step 3: Add the left and right sides of the system produced in step 2, and solve the resulting equation.

Substituting $x=2$ into either of the original two equations gives $y=1$. The solution is $(2,1)$.
7. We can apply the substitution method to the system in Example 6 by first solving the second equation for $x$ to get

$$
x=y+1
$$

Substituting this expression for $x$ into the first equation we have

$$
\begin{array}{r}
y+1+2 y=4 \\
3 y+1=4 \\
3 y=3 \\
y=1
\end{array}
$$

Using $y=1$ in either of the original equations gives $x=2$.
8. If the two lines are parallel, then the system will be inconsistent and the solution is $\varnothing$. If the two lines coincide, then the equations are dependent.
9. The graph of


Step 4: Substitute the solution from step 3 back into any equation with both $x$ - and $y$-variables, and solve.
Step 5: Check your solution in both equations if necessary.

## To Solve a System by the Substitution Method [7.5]

Step 1: Solve either of the equations for one of the variables (this step is not necessary if one of the equations has the correct form already).
Step 2: Substitute the results of step 1 into the other equation, and solve.
Step 3: Substitute the results of step 2 into an equation with both $x$ - and $y$ variables, and solve. (The equation produced in step 1 is usually a good one to use.)
Step 4: Check your solution if necessary.

## Inconsistent and Dependent Equations [7.5]

Two linear equations that have no solutions in common are said to be an inconsistent system, whereas two linear equations that have all their solutions in common are said to be dependent equations.

## Linear Inequalities in Two Variables [7.7]

An inequality of the form $a x+b y<c$ is a linear inequality in two variables. The equation for the boundary of the solution set is given by $a x+b y=c$. (This equation is found by simply replacing the inequality symbol with an equal sign.)

To graph a linear inequality, first graph the boundary, using a solid line if the boundary is included in the solution set and a broken line if the boundary is not included in the solution set. Next, choose any point not on the boundary and substitute its coordinates into the original inequality. If the resulting statement is true, the graph lies on the same side of the boundary as the test point. A false statement indicates that the solution set lies on the other side of the boundary.

## Systems of Linear Inequalities [7.8]

10. The solution set for the system

$$
\begin{array}{r}
x+y<4 \\
-x+y \leq 3
\end{array}
$$

is shown below.


A system of linear inequalities is two or more linear inequalities considered at the same time. To find the solution set to the system, we graph each of the inequalities on the same coordinate system. The solution set is the region that is common to all the regions graphed.

## Chapter 7 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.
The numbers in brackets refer to the sections of the text in which similar problems can be found.

Solve each equation. [7.1]

1. $4 x-2=7 x+7$
2. $\frac{3 y}{4}-\frac{1}{2}+\frac{3 y}{2}=2-y$
3. $8-3(2 t+1)=5(t+2)$
4. $6+4(1-3 t)=-3(t-4)+2$
5. $2 x^{2}-5 x=12$
6. $81 a^{2}=1$
7. $(x-2)(x-3)=2$
8. $9 x^{3}+18 x^{2}-4 x-8=0$
9. Geometry The length of a rectangle is 3 times the width. The perimeter is 32 feet. Find the length and width. [7.1]
10. Geometry The lengths of the three sides of a right triangle are given by three consecutive integers. Find the three sides. [7.1]

Solve each equation. [7.2]
11. $|x-3|=1$
12. $|x-2|=3$
13. $|2 y-3|=5$
14. $|3 y-2|=7$
15. $|4 x-3|+2=11$
16. $|6 x-2|+4=16$

Solve each inequality. Write your answer using interval notation. [7.3]
17. $\frac{3}{4} x+1 \leq 10$
18. $600-300 x<900$
19. $\frac{1}{3} \leq \frac{1}{6} x \leq 1$
20. $-\frac{1}{2} \leq \frac{1}{6} x \leq \frac{1}{3}$
21. $5 t+1 \leq 3 t-2$ or $-7 t \leq-21$
22. $6 t-3 \leq t+1$ or $-8 t \leq-16$

Solve each inequality and graph the solution set. [7.4]
23. $|y-2|<3$
24. $|5 x-1|>3$
25. $|2 t+1|-3<2$
26. $|2 t+1|-1<5$
27. $|5+8 t|+4 \leq 1$
28. $|5 x-8| \geq-2$

Solve each system using the elimination method. [7.5]
29. $\begin{aligned} 6 x-5 y & =-5 \\ 3 x+y & =1\end{aligned}$
30. $6 x+4 y=8$
$9 x+6 y=12$
31. $-7 x+4 y=-1$
$5 x-3 y=0$
32. $\frac{1}{2} x-\frac{3}{4} y=-4$
$\frac{1}{4} x+\frac{3}{2} y=13$

Solve each system by using the substition method. [7.5]
33. $x+y=2$
$y=x-1$
34. $2 x-3 y=5$

$$
y=2 x-7
$$

35. $3 x+7 y=6$

$$
x=-3 y+4
$$

36. $5 x-y=4$

$$
y=5 x-3
$$

Solve each system. [7.7]
37. $x+y+z=6$
$x-y-3 z=-8$
$x+y-2 z=-6$
39. $5 x-2 y+z=6$
$\begin{aligned}-3 x+4 y-z= & 2 \\ 6 x-8 y+2 z= & -4\end{aligned}$
38. $3 x+2 y+z=4$
$2 x-4 y+z=-1$
40. $4 x-6 y+8 z=4$
$5 x+y-2 z=4$
$6 x-9 y+12 z=6$
41. $2 x-y=5$
42. $x-y=2$
$3 x-2 z=-2$
$y-z=-3$
$5 y+z=-1$
$x-z=-1$
Graph each linear inequality. [7.7]
43. $y \leq 2 x-3$
44. $x \geq-1$

Graph the solution set for each system of linear inequalities. [7.8]
45. $3 x+4 y<12$

$$
-3 x+2 y \leq 6
$$

46. $3 x+4 y<12$
$-3 x+2 y \leq 6$

$$
y \geq 0
$$

47. $3 x+4 y<12$

$$
\begin{aligned}
& x \geq 0 \\
& y \geq 0
\end{aligned}
$$

48. $x>-3$
$y>-2$

## Chapter 7 Projects

## GROUP PROJECT Systems of Linear Equations and Inequalities

Number of People 2 or 3
Time Needed 10-15 minutes
Equipment Pencil and paper
Background The break-even point for a company occurs when the revenue from sales of a product equals the cost of producing the product. This group project is designed to give you more insight into revenue, cost, and break-even point.

Procedure A company is planning to open a factory to manufacture calculators.

1. It costs them $\$ 120,000$ to open the factory, and it will cost $\$ 10$ for each calculator they make. What is the expression for $C(x)$, the cost of making $x$ calculators?
2. They can sell the calculators for $\$ 50$ each. What is the expression for $R(x)$, their revenue from selling $x$ calculators? Remember that $R=p x$, where $p$ is the price per calculator.
3. Graph both the cost equation $C(x)$ and the revenue equation $R(X)$ on a coordinate system like the one below.

4. The break-even point is the value of $x$ (the number of calculators) for which the revenue is equal to the cost. Where is the break-even point on the graph you produced in Part 3? Estimate the break-even point from the graph.
5. Set the cost equal to the revenue and solve to find $x$ to find the exact value of the break-even point. How many calculators do they need to make and sell to exactly break even? What will be their revenue and their cost for that many calculators?
6. Write an inequality that gives the values of $x$ that will produce a profit for the company. (A profit occurs when the revenue is larger than the cost.)
7. Write an inequality that gives the values of $x$ that will produce a loss for the company. (A loss occurs when the cost is larger than the revenue.)
8. Profit is the difference between revenue and cost, or $P(x)=R(x)-C(x)$. Write the equation for profit and then graph it on a coordinate system like the one below.

9. How do you recognize the break-even point and the regions of loss and profit on the graph you produced above?

## RESEARCH PROJECT

## Maria Gaetana Agnesi (1718-1799)



theanuary 9, 1999, was the 200th anniversary of the death of Maria Agnesi, the author of Instituzioni analitiche ad uso della gioventu italiana (1748), a calculus textbook considered to be the best book of its time and the first surviving mathematical work written by a woman. Maria Agnesi is also famous for a curve that is named for her. The curve is called the Witch of Agnesi in English, but its actual translation is the Locus of Agnesi. The foundation of the curve is shown in the figure. Research the Witch of Agnesi and then explain, in essay form, how the diagram in the figure is used to produce the Witch of Agnesi. Include a rough sketch of the curve, starting with a circle of diameter $a$ as shown in the figure. Then, for comparison, sketch the curve again, starting with a circle of diameter $2 a$.


## Equations and Functions


student is heating water in a chemistry lab. As the water heats, she records the temperature readings from two thermometers, one giving temperature in degrees Fahrenheit and the other in degrees Celsius. The table below shows some of the data she collects. Figure 1 is a scatter diagram that gives a visual representation of the data in the table.

| Corresponding Temperatures <br> in Degrees <br> Fahrenheit | in Degrees <br> Celsius |
| :---: | :---: |
| 77 | 25 |
| 95 | 35 |
| 167 | 75 |
| 212 | 100 |



The exact relationship between the Fahrenheit and Celsius temperature scales is given by the formula

$$
C=\frac{5}{9}(F-32)
$$

We have three ways to describe the relationship between the two temperature scales: a table, a graph, and an equation. But, most important to us, we don't need to accept this formula on faith, because the material we will present in this chapter gives us the ability to derive the formula from the data in the table above.

## CHAPTER OUTLINE

8.1 The Slope of a Line
8.2 The Equation of a Line
8.3 Introduction to Functions
8.4 Function Notation
8.5 Algebra and Composition with Functions

### 8.6 Variation

- Improve your grade and save time! Go online to www.thomsonedu.com/login where you can
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## 8.1 <br> The Slope of a Line

## OBJECTIVES

A
Graph a linear equation from its intercepts.

B Find the slope of a line from its graph.

C Find the slope of a line given two points on the line.

We begin this section with a review of the definition for a linear equation in two variables and how we graph a linear equation from its intercepts.

DEFINITION Any equation that can be put in the form $a x+b y=c$, where $a, b$, and $c$ are real numbers and $a$ and $b$ are not both 0 , is called a linear equation in two variables. The graph of any equation of this form is a straight line (that is why these equations are called "linear"). The form $a x+b y=c$ is called standard form.

## Intercepts

DEFINITION The $x$-intercept of the graph of an equation is the $x$ coordinate of the point where the graph crosses the $x$-axis. The $y$-intercept is defined similarly.

Because any point on the $x$-axis has a $y$-coordinate of 0 , we can find the $x$-intercept by letting $y=0$ and solving the equation for $x$. We find the $y$-intercept by letting $x=0$ and solving for $y$.

## EXAMPLE 1

Find the $x$ - and $y$-intercepts for $2 x+3 y=6$; then graph the solution set.

SOLUTION To find the $y$-intercept we let $x=0$.

$$
\text { When } \quad \begin{aligned}
x & =0 \\
\text { we have } \quad 2(0)+3 y & =6 \\
3 y & =6 \\
y & =2
\end{aligned}
$$

The $y$-intercept is 2 , and the graph crosses the $y$-axis at the point $(0,2)$.

$$
\text { When } \quad \begin{aligned}
y & =0 \\
\text { we have } \quad 2 x+3(0) & =6 \\
2 x & =6 \\
x & =3
\end{aligned}
$$

The $x$-intercept is 3 , so the graph crosses the $x$-axis at the point $(3,0)$. We use these results to graph the solution set for $2 x+3 y=6$. The graph is shown in Figure 1.


## USING TECHNOLOGY

## Graphing with Trace and Zoom

All graphing calculators have the ability to graph equations and then trace over the points on the graph, giving their coordinates. Furthermore, all graphing calculators can zoom in and out on a graph that has been drawn. To graph a linear equation on a graphing calculator, we first set the graph window. Most calculators call the smallest value of $x$ Xmin and the largest value of $x$ Xmax. The counterpart values of $y$ are Ymin and Ymax. We will use the notation

$$
\text { Window: } \quad \mathrm{X} \text { from }-5 \text { to } 4, \mathrm{Y} \text { from }-3 \text { to } 2
$$

to stand for a window in which

$$
\begin{array}{ll}
\mathrm{X} \min =-5 & \mathrm{Y} \min =-3 \\
\mathrm{X} \max =4 & \mathrm{Y} \max =2
\end{array}
$$

Set your calculator with the following window:
Window: X from -10 to $10, \mathrm{Y}$ from -10 to 10
Graph the equation $Y=-X+8$. On the TI-83/84, you use the $Y=$ key to enter the equation; you enter a negative sign with the $(-)$ key, and a subtraction sign with the $\square-$ key. The graph will be similar to the one shown in Figure 2.

Use the Trace feature of your calculator to name three points on the graph. Next, use the Zoom feature of your calculator to zoom out so your window is twice as large.

## Solving for $y$ First

To graph the equation from Example $1,2 x+3 y=6$, on a graphing calculator, you must first solve it


FIGURE 2 for $y$. When you do so, you will get $y=-\frac{2}{3} x+2$, which you enter into your calculator as $Y=-(2 / 3) X+2$. Graph this equation in the window described here, and compare your results with the graph in Figure 1.

$$
\text { Window: } \quad X \text { from }-6 \text { to } 6, Y \text { from }-6 \text { to } 6
$$

## Hint on Tracing

If you are going to use the Trace feature and you want the $x$-coordinates to be exact numbers, set your window so that the range of $X$ inputs is a multiple of the number of horizontal pixels on your calculator screen. On the TI-82/83, the screen has 94 pixels across. Here are a few convenient trace windows:

$$
\begin{array}{ll}
\text { X from }-4.7 \text { to } 4.7 & \text { To trace to the nearest tenth } \\
\text { X from }-47 \text { to } 47 & \text { To trace to the nearest integer } \\
\text { X from } 0 \text { to } 9.4 & \text { To trace to the nearest tenth } \\
\text { X from } 0 \text { to } 94 & \text { To trace to the nearest integer } \\
\text { X from }-94 \text { to } 94 & \text { To trace to the nearest even integer }
\end{array}
$$

## The Slope of a Line

A highway sign tells us we are approaching a 6\% downgrade. As we drive down this hill, each 100 feet we travel horizontally is accompanied by a 6-foot drop in elevation.


In mathematics we say the slope of the highway is $-0.06=-\frac{6}{100}=-\frac{3}{50}$. The slope is the ratio of the vertical change to the accompanying horizontal change.

In defining the slope of a straight line, we are looking for a number to associate with a straight line that does two things. First, we want the slope of a line to measure the "steepness" of the line; that is, in comparing two lines, the slope of the steeper line should have the larger numerical value. Second, we want a line that rises going from left to right to have a positive slope. We want a line that falls going from left to right to have a negative slope. (A line that neither rises nor falls going from left to right must, therefore, have 0 slope.)


Geometrically, we can define the slope of a line as the ratio of the vertical change to the horizontal change encountered when moving from one point to another on the line. The vertical change is sometimes called the rise. The horizontal change is called the run.

## EXAMPLE 2 Find the slope of the line $y=2 x-3$.

SOLUTION To use our geometric definition, we first graph $y=2 x-3$ (Figure 3). We then pick any two convenient points and find the ratio of rise to run. By convenient points we mean points with integer coordinates. If we let $x=2$ in the equation, then $y=1$. Likewise if we let $x=4$, then $y$ is 5 .


FIGURE 3
Our line has a slope of 2 .
Notice that we can measure the vertical change (rise) by subtracting the $y$-coordinates of the two points shown in Figure 3: $5-1=4$. The horizontal change (run) is the difference of the $x$-coordinates: $4-2=2$. This gives us a second way of defining the slope of a line.

DEFINITION The slope of the line between two points $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ is given by

$$
\begin{gathered}
\text { Slope }=m=\frac{\text { Rise }}{\text { Run }}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}} \\
\text { Geometric Form }
\end{gathered}
$$



EXAMPLE 3
Find the slope of the line through $(-2,-3)$ and $(-5,1)$.

## SOLUTION

$$
m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{1-(-3)}{-5-(-2)}=\frac{4}{-3}=-\frac{4}{3}
$$



FIGURE 5

Looking at the graph of the line between the two points (Figure 5), we can see our geometric approach does not conflict with our algebraic approach.

We should note here that it does not matter which ordered pair we call $\left(x_{1}, y_{1}\right)$ and which we call $\left(x_{2}, y_{2}\right)$. If we were to reverse the order of subtraction of both the $x$ - and $y$-coordinates in the preceding example, we would have

$$
m=\frac{-3-1}{-2-(-5)}=\frac{-4}{3}=-\frac{4}{3}
$$

which is the same as our previous result.
Note: The two most common mistakes students make when first working with the formula for the slope of a line are

1. Putting the difference of the $x$-coordinates over the difference of the $y$-coordinates.
2. Subtracting in one order in the numerator and then subtracting in the opposite order in the denominator. You would make this mistake in Example 3 if you wrote $1-(-3)$ in the numerator and then $-2-(-5)$ in the denominator.

## EXAMPLE 4 Find the slope of the line containing $(3,-1)$ and $(3,4)$.

SOLUTION Using the definition for slope, we have

$$
m=\frac{-1-4}{3-3}=\frac{-5}{0}
$$

The expression $\frac{-5}{0}$ is undefined; that is, there is no real number to associate with it. In this case, we say the line has no defined slope; the slope is undefined.


FIGURE 6

The graph of our line is shown in Figure 6. All vertical lines have undefined slope. (And all horizontal lines, as we mentioned earlier, have 0 slope.)

## Slopes of Parallel and Perpendicular Lines

In geometry we call lines in the same plane that never intersect parallel. For two lines to be nonintersecting, they must rise or fall at the same rate. In other words, two lines are parallel if and only if they have the same slope.

Although it is not as obvious, it is also true that two nonvertical lines are perpendicular if and only if the product of their slopes is -1 . This is the same as saying their slopes are negative reciprocals.

We can state these facts with symbols as follows: If line $l_{1}$ has slope $m_{1}$ and line $l_{2}$ has slope $m_{2}$, then

$$
l_{1} \text { is parallel to } l_{2} \Leftrightarrow m_{1}=m_{2}
$$

and

$$
I_{1} \text { is perpendicular to } l_{2} \Leftrightarrow m_{1} \cdot m_{2}=-1\left(\text { or } m_{1}=\frac{-1}{m_{2}}\right)
$$

For example, if a line has a slope of $\frac{2}{3}$, then any line parallel to it has a slope of $\frac{2}{3}$. Any line perpendicular to it has a slope of $-\frac{3}{2}$ (the negative reciprocal of $\frac{2}{3}$ ).

Although we cannot give a formal proof of the relationship between the slopes of perpendicular lines at this level of mathematics, we can offer some justification for the relationship. Figure 7 shows the graphs of two lines. One of the lines has a slope of $\frac{2}{3}$; the other has a slope of $-\frac{3}{2}$. As you can see, the lines are perpendicular.


FIGURE 7

## Slope and Rate of Change

So far, the slopes we have worked with represent the ratio of the change in $y$ to a corresponding change in $x$, or, on the graph of the line, the ratio of vertical change to horizontal change in moving from one point on the line to another. However, when our variables represent quantities from the world around us, slope can have additional interpretations.

EXAMPLE 5 On the chart below, find the slope of the line connecting the first point $(1955,0.29)$ with the last point $(2005,2.93)$. Explain the significance of the result.


SOLUTION The slope of the line connecting the first point (1955, 0.29) with the last point $(2005,2.93)$, is

$$
m=\frac{2.93-0.29}{2005-1955}=\frac{2.64}{50}=0.0528
$$

The units are dollars/year. If we write this in terms of cents we have

$$
m=5.28 \text { cents/year }
$$

which is the average change in the price of a gallon of gasoline over a 50 -year period of time.

For the chart in Example 5, if we connect the points (1995, 1.10) and $(2005,2.93)$, the line that results has a slope of

$$
m=\frac{2.93-1.10}{2005-1995}=\frac{1.83}{10}=0.183 \text { dollars } / \text { year }=18.3 \text { cents } / \text { year }
$$

which is the average change in the price of a gallon of gasoline over a 10-year period. As you can imagine by looking at the chart, the line connecting the first and last point is not as steep as the line connecting the points from 1995 and 2005, and this is what we are seeing numerically with our slope calculations. If we were summarizing this information for an article in the newspaper, we could say: Although the price of a gallon of gasoline has increased only 5.28 cents per year over the last 50 years, for the last 10 years the average annual rate of increase is more than triple that at 18.3 cents per year.

## Slope and Average Speed

Previously we introduced the rate equation $d=r t$. Suppose that a boat is traveling at a constant speed of 15 miles per hour in still water. The following table shows the distance the boat will have traveled in the specified number of hours.

The graph of these data is shown in Figure 8. Notice that the points all lie along a line.
$\left.\begin{array}{cc}\boldsymbol{t} \\ \text { (Hours) }\end{array} \quad \begin{array}{c}\boldsymbol{d} \\ \text { (Miles) }\end{array}\right]$


FIGURE 8
We can calculate the slope of this line using any two points from the table. Notice we have graphed the data with $t$ on the horizontal axis and $d$ on the vertical axis. Using the points $(2,30)$ and $(3,45)$, the slope will be

$$
m=\frac{\text { rise }}{\text { run }}=\frac{45-30}{3-2}=\frac{15}{1}=15
$$

The units of the rise are miles and the units of the run are hours, so the slope will be in units of miles per hour. We see that the slope is simply the change in distance divided by the change in time, which is how we compute the average speed. Since the speed is constant, the slope of the line represents the speed of 15 miles per hour.

## EXAMPLE 6

A car is traveling at a constant speed. A graph of the distance the car has traveled over time is shown in Figure 9. Use the graph to find the speed of the car.


FIGURE 9
SOLUTION Using the second and third points, we see the rise is $240-120=$ 120 miles, and the run is $4-2=2$ hours. The speed is given by the slope, which is

$$
\begin{aligned}
m & =\frac{\text { rise }}{\text { run }} \\
& =\frac{120 \text { miles }}{2 \text { hours }} \\
& =60 \text { miles per hour }
\end{aligned}
$$



## USING TECHNOLOGY

## Families of Curves

We can use a graphing calculator to investigate the effects of the numbers $a$ and $b$ on the graph of $y=a x+b$. To see how the number $b$ affects the graph, we can hold $a$ constant and let $b$ vary. Doing so will give us a family of curves. Suppose we set $a=1$ and then let $b$ take on integer values from -3 to 3 . The equations we obtain are

$$
\begin{aligned}
& y=x-3 \\
& y=x-2 \\
& y=x-1 \\
& y=x \\
& y=x+1 \\
& y=x+2 \\
& y=x+3
\end{aligned}
$$

We will give three methods of graphing this set of equations on a graphing calculator.

## Method 1: Y-Variables List

To use the Y -variables list, enter each equation at one of the Y variables, set the graph window, then graph. The calculator will graph the equations in order, starting with $Y_{1}$ and ending with $Y_{7}$. Following is the $Y$-variables list, an appropriate window, and a sample of the type of graph obtained (Figure 10).
$Y_{1}=X-3$
$\mathrm{Y}_{2}=\mathrm{X}-2$
$Y_{3}=X-1$
$Y_{4}=X$
$Y_{5}=X+1$
$\mathrm{Y}_{6}=\mathrm{X}+2$
$\mathrm{Y}_{7}=\mathrm{X}+3$


FIGURE 10

Window: X from -4 to $4, \mathrm{Y}$ from -4 to 4

## Method 2: Programming

The same result can be obtained by programming your calculator to graph $y=x+b$ for $b=-3,-2,-1,0,1,2$, and 3 . Here is an outline of a program that will do this. Check the manual that came with your calculator to find the commands for your calculator.

Step 1: Clear screen
Step 2: Set window for $X$ from -4 to 4 and $Y$ from -4 to 4
Step 3: $-3 \rightarrow B$
Step 4: Label 1
Step 5: Graph $Y=X+B$
Step 6: $B+1 \rightarrow B$
Step 7: If $B<4$, go to 1
Step 8: End

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1
B 2,6
C ${ }^{3-5}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. If you were looking at a graph that described the performance of a stock you had purchased, why would it be better if the slope of the line were positive, rather than negative?
2. Describe the behavior of a line with a negative slope.
3. Would you rather climb a hill with a slope of $\frac{1}{2}$ or a slope of 3 ? Explain why.
4. Describe how to obtain the slope of a line if you know the coordinates of two points on the line.

## Problem Set 8.1

Find the slope of each of the following lines from the given graph.
1.

2.

= Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
3.

4.

5.

6.


Find the slope of the line through each of the following pairs of points. Then, plot each pair of points, draw a line through them, and indicate the rise and run in the graph in the manner shown in Example 3.
7. $(2,1),(4,4)$
8. $(3,1),(5,4)$
9. $(1,4),(5,2)$
10. $(1,3),(5,2)$
11. $(1,-3),(4,2)$
12. $(2,-3),(5,2)$
13. $(-3,-2),(1,3)$
14. $(-3,-1),(1,4)$
15. $(-3,2),(3,-2)$
16. $(-3,3),(3,-1)$
17. $(2,-5),(3,-2)$
18. $(2,-4),(3,-1)$

Solve for the indicated variable if the line through the two given points has the given slope.
19. $(5,2)$ and $(x,-2), m=2$.
20. $(-1,-4)$ and $(x, 5), m=3$.
21. $(a, 3)$ and $(2,6), m=-1$.
22. $(a,-2)$ and $(4,-6), m=-3$.
23. $(2, b)$ and $(-1,4 b), m=-2$.
24. $(-4, y)$ and $(-1,6 y), m=2$.

For each equation below, complete the table, and then use the results to find the slope of the graph of the equation.
25. $2 x+3 y=6$

26. $3 x-2 y=6$

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :--- | :--- |
| 0 | 0 |

0
$27 y=\frac{2}{3} x-5$

| $x$ | $y$ |
| :--- | :--- |
| 0 |  |
| 3 |  |

28. $y=-\frac{3}{4} x+2$

29. Finding Slope from Intercepts Graph the line with $x$-intercept 4 and $y$-intercept 2 . What is the slope of this line?
30. Finding Slope from Intercepts Graph the line with $x$-intercept -4 and $y$-intercept -2 . What is the slope of this line?
31. Parallel Lines Find the slope of any line parallel to the line through $(2,3)$ and $(-8,1)$.
32. Parallel Lines Find the slope of any line parallel to the line through $(2,5)$ and $(5,-3)$.
33. Perpendicular Lines Line $l$ contains the points $(5,-6)$ and $(5,2)$. Give the slope of any line perpendicular to 1 .
34. Perpendicular Lines Line $l$ contains the points $(3,4)$ and $(-3,1)$. Give the slope of any line perpendicular to 1 .
35. Parallel Lines Line $l$ contains the points $(-2,1)$ and $(4,-5)$. Find the slope of any line parallel to 1 .
36. Parallel Lines Line $l$ contains the points $(3,-4)$ and $(-2,-6)$. Find the slope of any line parallel to $l$.
37. Perpendicular Lines Line $l$ contains the points $(-2,-5)$ and $(1,-3)$. Find the slope of any line perpendicular to 1 .
38. Perpendicular Lines Line $l$ contains the points $(6,-3)$ and $(-2,7)$. Find the slope of any line perpendicular to 1 .
39. Determine if each of the following tables could represent ordered pairs from an equation of a line.
a.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| ---: | ---: |
| 0 | 5 |
| 1 | 7 |
| 2 | 9 |
| 3 | 11 |

b.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| ---: | ---: |
| -2 | -5 |
| 0 | -2 |
| 2 | 0 |
| 4 | 1 |

40. The following lines have slope $2, \frac{1}{2}, 0$, and -1 . Match each line to its slope value.


## Applying the Concepts

41. An object is traveling at a constant speed. The distance and time data are shown on the given graph. Use the graph to find the speed of the object
a.

b.

c.

d.

42. Improving Your Quantitative Literacy A cyclist is traveling at a constant speed. The graph shows the distance the cyclist travels over time when there is no wind present, when she travels against the wind, and when she travels with the wind.

a. Use the concept of slope to find the speed of the cyclist under each of the three conditions.
b. Compare the speed of the cyclist when she is traveling without any wind to when she is riding against the wind. How do the two speeds differ?
c. Compare the speed of the cyclist when she is traveling without any wind to when she is riding with the wind. How do the two speeds differ?
d. What is the speed of the wind?
43. Non-Camera Phone Sales The table and line graph here each show the projected non-camera phone sales each year through 2010. Find the slope of each of the three line segments, $A, B$, and $C$.

| Year | 2006 | 2007 | 2008 | 2009 | 2010 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Sales <br> (in millions) | 300 | 250 | 175 | 150 | 125 |

Projected Non-Camera Phone Sales

44. Camera Phone Sales The table and line graph here each show the projected camera phone sales each year through 2010. Find the slope of each of the three line segments, $A, B$, and $C$.


Projected Camera Phone Sales

45. Slope of a Highway A sign at the top of the Cuesta Grade, outside of San Luis Obispo, reads " $7 \%$ downgrade next 3 miles." The diagram shown here is a model of the Cuesta Grade that takes into account the information on that sign.

a. At point $B$, the graph crosses the $y$-axis at 1,106 feet. How far is it from the origin to point $A$ ?
b. What is the slope of the Cuesta Grade?
46. Heating a Block of Ice A block of ice with an initial temperature of $-20^{\circ} \mathrm{C}$ is heated at a steady rate. The graph shows how the temperature changes as the ice melts to become water and the water boils to become steam and water.

a. How long does it take all the ice to melt?
b. From the time the heat is applied to the block of ice, how long is it before the water boils?
c. Find the slope of the line segment labeled A. What units would you attach to this number?
d. Find the slope of the line segment labeled C. Be sure to attach units to your answer.
e. Is the temperature changing faster during the 1 st minute or the 16 th minute?
47. Slope and Rate of Change Find the slope of the line connecting the first and last points on the chart. Explain in words what the slope represents.

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48. Bridal Books Find the slope of the line that connects the points $(1999,118)$ and $(2002,75)$. Round to the nearest tenth. Explain in words what the slope represents.

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49. Health Care Use the chart to work the following problems involving slope.

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a. Find the slope of the line from 1969 to 1979, and then give a written explanation of the significance of that number.
b. Find the slope of the line from 1979 to 1989, and then give a written explanation of the significance of that number.
c. What is the slope of the line connecting $(1989,9)$ and $(2004,9)$ ? What does this tell us about German measles?
50. Triathlon Entries Use the chart to work the following problems.


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a. Find the slope of the line that connects the points $(1978,0)$ and $(2005,1,793)$. Round your answer to the nearest tenth. Explain in words what the slope represents.
b. Of the line segments shown in the chart, one has a larger slope than the rest. Find that slope and give an interpretation of its meaning.

## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in previous sections and chapters.
51. If $3 x+2 y=12$, find $y$ when $x$ is 4 .
52. If $y=3 x-1$, find $x$ when $y$ is 0 .
53. Solve the formula $3 x+2 y=12$ for $y$.
54. Solve the formula $y=3 x-1$ for $x$.
55. Solve the formula $A=P+\operatorname{Prt}$ for $t$.
56. Solve the formula $S=\pi r^{2}+2 \pi r h$ for $h$.

## Getting Ready for the Next Section

Simplify.
57. $2\left(-\frac{1}{2}\right)$
58. $\frac{3-(-1)}{-3-3}$
59. $\frac{5-(-3)}{2-6}$
60. $3\left(-\frac{2}{3} x+1\right)$

Solve for $y$.
61. $\frac{y-b}{x-0}=m$
62. $2 x+3 y=6$
63. $y-3=-2(x+4)$
64. $y+1=-\frac{2}{3}(x-3)$
65. If $y=-\frac{4}{3} x+5$, find $y$ when $x$ is 0 .
66. If $y=-\frac{4}{3} x+5$, find $y$ when $x$ is 3 .

## Extending the Concepts

67. Find a point $P$ on the line $y=3 x-2$ such that the slope of the line through $(1,2)$ and $P$ is equal to -1 .
68. Find a point $P$ on the line $y=-2 x+3$ such that the slope of the line through $(2,3)$ and $P$ is equal to 2 .
69. Use your Y-variables list or write a program to graph the family of curves $\mathrm{Y}=-2 \mathrm{X}+\mathrm{B}$ for $\mathrm{B}=-3,-2$, $-1,0,1,2$, and 3.
70. Use your Y-variables list or write a program to graph the family of curves $\mathrm{Y}=-2 \mathrm{X}-\mathrm{B}$ for $\mathrm{B}=-3,-2$, $-1,0,1,2$, and 3.
71. Use your Y-variables list or write a program to graph the family of curves $\mathrm{Y}=\mathrm{AX}$ for $\mathrm{A}=-3,-2$, $-1,0,1,2$, and 3 .
72. Use your Y-variables list or write a program to graph the family of curves $\mathrm{Y}=\mathrm{AX}$ for $\mathrm{A}=\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, 1,2 , and 3 .
73. Use your Y-variables list or write a program to graph the family of curves $\mathrm{Y}=\mathrm{AX}+2$ for $\mathrm{A}=-3$, $-2,-1,0,1,2$, and 3 .
74. Use your Y-variables list or write a program to graph the family of curves $\mathrm{Y}=\mathrm{AX}-2$ for $\mathrm{A}=\frac{1}{4}, \frac{1}{3}$, $\frac{1}{2}, 1,2$, and 3 .

## The Equation of a Line

## OBJECTIVES

A Find the equation of a line given its slope and $y$-intercept.

B Find the slope and $y$-intercept from the equation of a line.

C Find the equation of a line given the slope and a point on the line.

D Find the equation of a line given two points on the line.

The table and illustrations show some corresponding temperatures on the Fahrenheit and Celsius temperature scales. For example, water freezes at $32^{\circ} \mathrm{F}$ and $0^{\circ} \mathrm{C}$, and boils at $212^{\circ} \mathrm{F}$ and $100^{\circ} \mathrm{C}$.


If we plot all the points in the table using the $x$-axis for temperatures on the Celsius scale and the $y$-axis for temperatures on the Fahrenheit scale, we see that they line up in a straight line (Figure 1). This means that a linear equation in two variables will give a perfect description of the relationship between the two scales. That equation is

$$
F=\frac{9}{5} C+32
$$

The techniques we use to find the equation of a line from a set of points is what this section is all about.


FIGURE 1
Suppose line $l$ has slope $m$ and $y$-intercept $b$. What is the equation of $l$ ? Because the $y$-intercept is $b$, we know the point $(0, b)$ is on the line. If $(x, y)$ is any other point on $l$, then using the definition for slope, we have

$$
\begin{aligned}
\frac{y-b}{x-0} & =m & & \text { Definition of slope } \\
y-b & =m x & & \text { Multiply both sides by } x \\
y & =m x+b & & \text { Add } b \text { to both sides }
\end{aligned}
$$

This last equation is known as the slope-intercept form of the equation of a straight line.

Slope-Intercept Form of the Equation of a Line The equation of any line with slope $m$ and $y$-intercept $b$ is given by

$$
\begin{aligned}
& y=m x+\underset{\nearrow}{b} \\
& \text { Slope } y \text {-intercept }
\end{aligned}
$$

When the equation is in this form, the slope of the line is always the coefficent of $x$, and the $y$-intercept is always the constant term.

EXAMPLE 1 Find the equation of the line with slope $-\frac{4}{3}$ and $y$-intercept 5. Then graph the line.
SOLUTION Substituting $m=-\frac{4}{3}$ and $b=5$ into the equation $y=m x+b$, we have

$$
y=-\frac{4}{3} x+5
$$

Finding the equation from the slope and $y$-intercept is just that easy. If the slope is $m$ and the $y$-intercept is $b$, then the equation is always $y=m x+b$. Now, let's graph the line.
Because the $y$-intercept is 5 , the graph goes through the point $(0,5)$. To find a second point on the graph, we start at $(0,5)$ and move 4 units down (that's a rise of -4 ) and 3 units to the right (a run of 3 ). The point we end up at is $(3,1)$. Drawing a line that passes through $(0,5)$ and $(3,1)$, we have the graph of our equation. (Note that we could also let the rise $=4$ and the run $=-3$ and obtain the same graph.) The graph is shown in Figure 2.


FIGURE 2
EXAMPLE 2
Give the slope and $y$-intercept for the line $2 x-3 y=5$.
SOLUTION To use the slope-intercept form, we must solve the equation for $y$ in terms of $x$.

$$
\begin{array}{rlrl}
2 x-3 y & =5 & \\
-3 y & =-2 x+5 & \text { Add }-\mathbf{2} x \text { to } \mathbf{b} \\
y & =\frac{2}{3} x-\frac{5}{3} & & \text { Divide by }-\mathbf{3}
\end{array}
$$

The form $y=m x+b$. The slope must be $m=\frac{2}{3}$, and the $y$-intercept is $b=-\frac{5}{3}$.

EXAMPLE 3 Graph the equation $2 x+3 y=6$ using the slope and $y$-intercept.
SOLUTION Although we could graph this equation by finding ordered pairs that are solutions to the equation and drawing a line through their graphs, it is sometimes easier to graph a line using the slope-intercept form of the equation.

Solving the equation for $y$, we have,

$$
\begin{array}{rlrl}
2 x+3 y & =6 & \\
3 y & =-2 x+6 & & \text { Add }-\mathbf{2 x} \text { to both sides } \\
y & =-\frac{2}{3} x+2 & & \text { Divide by } \mathbf{3}
\end{array}
$$

The slope is $m=-\frac{2}{3}$ and the $y$-intercept is $b=2$. Therefore, the point $(0,2)$ is on the graph and the ratio of rise to run going from $(0,2)$ to any other point on the line is $-\frac{2}{3}$. If we start at $(0,2)$ and move 2 units up (that's a rise of 2 ) and 3 units to the left (a run of -3 ), we will be at another point on the graph. (We could also go down 2 units and right 3 units and still be assured of ending up at another point on the line because $\frac{2}{-3}$ is the same as $\frac{-2}{3}$.)


FIGURE 3
A second useful form of the equation of a straight line is the point-slope form.
Let line $I$ contain the point $\left(x_{1}, y_{1}\right)$ and have slope $m$. If $(x, y)$ is any other point on $l$, then by the definition of slope we have

$$
\frac{y-y_{1}}{x-x_{1}}=m
$$

Multiplying both sides by $\left(x-x_{1}\right)$ gives us

$$
\begin{aligned}
\left(\boldsymbol{x}-\boldsymbol{x}_{\mathbf{1}}\right) \cdot \frac{y-y_{1}}{x-x_{1}} & =m\left(\boldsymbol{x}-\boldsymbol{x}_{\mathbf{1}}\right) \\
y-y_{1} & =m\left(x-x_{1}\right)
\end{aligned}
$$

This last equation is known as the point-slope form of the equation of a straight line.

Point-Slope Form of the Equation of a Line The equation of the line through $\left(x_{1}, y_{1}\right)$ with slope $m$ is given by

$$
y-y_{1}=m\left(x-x_{1}\right)
$$

This form of the equation of a straight line is used to find the equation of a line, either given one point on the line and the slope, or given two points on the line.

## EXAMPLE 4 Find the equation of the line with slope -2 that contains

 the point $(-4,3)$. Write the answer in slope-intercept form.
## SOLUTION

$$
\begin{array}{lrl}
\text { Using }\left(x_{1}, y_{1}\right) & =(-4,3) \text { and } m=-2 & \\
\text { in } & \begin{aligned}
y-y_{1} & =m\left(x-x_{1}\right) \\
\text { gives us } & y-3
\end{aligned}=-2(x+4) & \\
& & \text { Point-slope form } x-(-4)=x+4 \\
y-3 & =-2 x-8 & \\
\text { Multiply out right side } \\
y & =-2 x-5 & \\
\text { Add } 3 \text { to each side }
\end{array}
$$

Figure 4 is the graph of the line that contains $(-4,3)$ and has a slope of -2 . Notice that the $y$-intercept on the graph matches that of the equation we found.


FIGURE 4

EXAMPLE 5 Find the equation of the line that passes through the points $(-3,3)$ and $(3,-1)$.

SOLUTION We begin by finding the slope of the line:

$$
m=\frac{3-(-1)}{-3-3}=\frac{4}{-6}=-\frac{2}{3}
$$

Using $\left(x_{1}, y_{1}\right)=(3,-1)$ and $m=-\frac{2}{3}$ in $y-y_{1}=m\left(x-x_{1}\right)$ yields

$$
\begin{aligned}
y+1 & =-\frac{2}{3}(x-3) & \\
y+1 & =-\frac{2}{3} x+2 & \text { Multiply out right side } \\
y & =-\frac{2}{3} x+1 & \text { Add }-\mathbf{1} \text { to each side }
\end{aligned}
$$

## Note

In Example 5 we could have used the point $(-3,3)$ instead of $(3,-1)$ and obtained the same equation; that is, using $\left(x_{1}, y_{1}\right)=(-3,3)$ and $m=-\frac{2}{3}$ in $y-y_{1}=m\left(x-x_{1}\right)$ gives us

$$
\begin{aligned}
y-3 & =-\frac{2}{3}(x+3) \\
y-3 & =-\frac{2}{3} x-2 \\
y & =-\frac{2}{3} x+1
\end{aligned}
$$

which is the same result we obtained using $(3,-1)$.

Figure 5 shows the graph of the line that passes through the points $(-3,3)$ and $(3,-1)$. As you can see, the slope and $y$-intercept are $-\frac{2}{3}$ and 1 , respectively.


FIGURE 5
The last form of the equation of a line that we will consider in this section is called the standard form. It is used mainly to write equations in a form that is free of fractions and is easy to compare with other equations.

Standard Form for the Equation of a Line If $a, b$, and $c$ are integers, then the equation of a line is in standard form when it has the form

$$
a x+b y=c
$$

If we were to write the equation

$$
y=-\frac{2}{3} x+1
$$

in standard form, we would first multiply both sides by 3 to obtain

$$
3 y=-2 x+3
$$

Then we would add $2 x$ to each side, yielding

$$
2 x+3 y=3
$$

which is a linear equation in standard form.

EXAMPLE 6 Give the equation of the line through $(-1,4)$ whose graph is perpendicular to the graph of $2 x-y=-3$. Write the answer in standard form.

SOLUTION To find the slope of $2 x-y=-3$, we solve for $y$ :

$$
\begin{aligned}
2 x-y & =-3 \\
y & =2 x+3
\end{aligned}
$$

The slope of this line is 2 . The line we are interested in is perpendicular to the line with slope 2 and must, therefore, have a slope of $-\frac{1}{2}$.

Using $\left(x_{1}, y_{1}\right)=(-1,4)$ and $m=-\frac{1}{2}$, we have

$$
\begin{aligned}
& y-y_{1}=m\left(x-x_{1}\right) \\
& y-4=-\frac{1}{2}(x+1)
\end{aligned}
$$

Because we want our answer in standard form, we multiply each side by 2.

$$
\begin{aligned}
2 y-8 & =-1(x+1) \\
2 y-8 & =-x-1 \\
x+2 y-8 & =-1 \\
x+2 y & =7
\end{aligned}
$$

The last equation is in standard form.
As a final note, the summary reminds us that all horizontal lines have equations of the form $y=b$ and slopes of 0 . Because they cross the $y$-axis at $b$, the $y$-intercept is $b$; there is no $x$-intercept. Vertical lines have undefined slope and equations of the form $x=a$. Each will have an $x$-intercept at $a$ and no $y$-intercept. Finally, equations of the form $y=m x$ have graphs that pass through the origin. The slope is always $m$ and both the $x$-intercept and the $y$-intercept are 0 .

## FACTS FROM GEOMETRY

Special Equations and Their Graphs, Slopes, and Intercepts For the equations below, $m, a$, and $b$ are real numbers.

| Through the Origin |  |  |
| :--- | :--- | :--- |
| Equation: $y=m x$ |  |  |
| Slope $=m$ |  |  |
| $x$-intercept $=0$ | Vertical Line |  |
| $y$-intercept $=0$ | Equation: $x=a$ <br> Slope is undefined <br> $x$-intercept $=a$ <br> No $y$-intercept | Horizontal Line <br> Equation: $y=b$ <br> Slope $=0$ |
| No $x$-intercept |  |  |
| $y$-intercept $=b$ |  |  |

## USING TECHNOLOGY

## Graphing Calculators

One advantage of using a graphing calculator to graph lines is that a calculator does not care whether the equation has been simplified or not. To illustrate, in Example 5 we found that the equation of the line with slope $-\frac{2}{3}$ that passes through the point $(3,-1)$ is

$$
y+1=-\frac{2}{3}(x-3)
$$

Normally, to graph this equation we would simplify it first. With a graphing calculator, we add -1 to each side and enter the equation this way:

$$
\mathrm{Y}_{1}=-(2 / 3)(\mathrm{X}-3)-1
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1
B 2,3
C 4,6
D 5

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How would you graph the line $y=\frac{1}{2} x+3$ ?
2. What is the slope-intercept form of the equation of a line?
3. Describe how you would find the equation of a line if you knew the slope and the $y$-intercept of the line.
4. If you had the graph of a line, how would you use it to find the equation of the line?

## Problem Set 8.2

Write the equation of the line with the given slope and $y$-intercept.

1. $m=2, b=3$
2. $m=-4, b=2$
3. $m=1, b=-5$
4. $m=-5, b=-3$
5. $m=\frac{1}{2}, b=\frac{3}{2}$
6. $m=\frac{2}{3}, b=\frac{5}{6}$
7. $m=0, b=4$
8. $m=0, b=-2$

Find the slope of a line (a) parallel and (b) perpendicular to the given line.
9. $y=3 x-4$
10. $y=-4 x+1$
11. $3 x+y=-2$
12. $2 x-y=-4$
13. $2 x+5 y=-11$
14. $3 x-5 y=-4$

Give the slope and $y$-intercept for each of the following equations. Sketch the graph using the slope and $y$-intercept. Give the slope of any line perpendicular to the given line.
15. $y=3 x-2$
16. $y=2 x+3$
17. $2 x-3 y=12$
18. $3 x-2 y=12$
19. $4 x+5 y=20$
20. $5 x-4 y=20$

For each of the following lines, name the slope and $y$-intercept. Then write the equation of the line in slope-intercept form.



For each of the following problems, the slope and one point on a line are given. In each case, find the equation of that line. (Write the equation for each line in slope-intercept form.)
25. $(-2,-5) ; m=2$
26. $(-1,-5) ; m=2$
27. $(-4,1) ; m=-\frac{1}{2}$
28. $(-2,1) ; m=-\frac{1}{2}$
29. $\left(-\frac{1}{3}, 2\right) ; m=-3$
30. $\left(-\frac{2}{3}, 5\right) ; m=-3$
31. $(-4,-2) ; m=\frac{2}{3}$
32. $(3,4) ; m=-\frac{1}{3}$
33. $(-5,2) ; m=-\frac{1}{4}$
34. $(-4,3) ; m=-\frac{1}{6}$

Find the equation of the line that passes through each pair of points. Write your answers in standard form
35. $(-2,-4),(1,-1)$
36. $(2,4),(-3,-1)$
37. $(-1,-5),(2,1)$
38. $(-1,6),(1,2)$
39. $\left(\frac{1}{3},-\frac{1}{5}\right),\left(-\frac{1}{3},-1\right)$
40. $\left(-\frac{1}{2},-\frac{1}{2}\right),\left(\frac{1}{2}, \frac{1}{10}\right)$
41. The equation $3 x-2 y=10$ is a linear equation in standard form. From this equation, answer the following:
a. Find the $x$ - and $y$-intercepts.
b. Find a solution to this equation other than the intercepts in part a.
c. Write this equation in slope-intercept form.
d. Is the point $(2,2)$ a solution to the equation?
42. The equation $4 x+3 y=8$ is a linear equation in standard form. From this equation, answer the following:
a. Find the $x$ - and $y$-intercepts.
b. Find a solution to this equation other than the intercepts in part a.
c. Write this equation in slope-intercept form.
d. Is the point $(-3,2)$ a solution to the equation?
43. The equation $\frac{3 x}{4}-\frac{y}{2}=1$ is a linear equation. From this equation, answer the following:
a. Find the $x$ - and $y$-intercepts.
b. Find a solution to this equation other than the intercepts in part a.
c. Write this equation in slope-intercept form.
d. Is the point $(1,2)$ a solution to the equation?
44. The equation $\frac{3 x}{5}+\frac{2 y}{3}=1$ is a linear equation. From this equation, answer the following:
a. Find the $x$ - and $y$-intercepts.
b. Find a solution to this equation other than the intercepts in part a.
c. Write this equation in slope-intercept form.
d. Is the point $(-5,3)$ a solution to the equation?

The next two problems are intended to give you practice reading, and paying attention to, the instructions that accompany the problems you are working.
Working these problems is an excellent way to get ready for a test or a quiz.
45. Work each problem according to the instructions given.
a. Solve: $-2 x+1=-3$
b. Write in slope-intercept form:
$-2 x+y=-3$
c. Find the $y$-intercept: $-2 x+y=-3$
d. Find the slope: $-2 x+y=-3$
e. Graph: $-2 x+y=-3$
46. Work each problem according to the instructions given.
a. Solve: $\frac{x}{3}+\frac{1}{4}=1$
b. Write in slope-intercept form: $\frac{x}{3}+\frac{y}{4}=1$
c. Find the $y$-intercept: $\frac{x}{3}+\frac{y}{4}=1$
d. Find the slope: $\frac{x}{3}+\frac{y}{4}=1$
e. Graph: $\frac{x}{3}+\frac{y}{4}=1$

For each of the following lines, name the coordinates of any two points on the line. Then use those two points to find the equation of the line.

## 47. <br> 

48. 


50.

51. Give the slope and $y$-intercept of $y=-2$. Sketch the graph.
52. For the line $x=-3$, sketch the graph, give the slope, and name any intercepts.
53. Find the equation of the line parallel to the graph of $3 x-y=5$ that contains the point $(-1,4)$.
54. Find the equation of the line parallel to the graph of $2 x-4 y=5$ that contains the point $(0,3)$.
55. Line $l$ is perpendicular to the graph of the equation $2 x-5 y=10$ and contains the point $(-4,-3)$. Find the equation for $l$.
56. Line $l$ is perpendicular to the graph of the equation $-3 x-5 y=2$ and contains the point $(2,-6)$. Find the equation for 1 .
57. Give the equation of the line perpendicular to the graph of $y=-4 x+2$ that has an $x$-intercept of -1 .
58. Write the equation of the line parallel to the graph of $7 x-2 y=14$ that has an $x$-intercept of 5 .
59. Find the equation of the line with $x$-intercept 3 and $y$-intercept 2.
60. Find the equation of the line with $x$-intercept 2 and $y$-intercept 3 .

## Applying the Concepts

61. Deriving the Temperature Equation The table from the introduction to this section is repeated here. The rows of the table give us ordered pairs ( $C, F$ ).

| Degrees <br> Celsius | Degrees <br> Fahrenheit <br> $\boldsymbol{F}$ |
| :---: | :---: |
| 0 | 32 |
| 25 | 77 |
| 50 | 122 |
| 75 | 167 |
| 100 | 212 |
|  |  |

a. Use any two of the ordered pairs from the table to derive the equation $F=\frac{9}{5} C+32$.
b. Use the equation from part (a) to find the Fahrenheit temperature that corresponds to a Celsius temperature of $30^{\circ}$.
62. Maximum Heart Rate The table gives the maximum heart rate for adults $30,40,50$, and 60 years old. Each row of the table gives us an ordered pair $(A, M)$.

| Age <br> (years) <br> $\boldsymbol{A}$ | Maximum Heart Rate <br> (beats per minute) <br> $\boldsymbol{M}$ |
| :---: | :---: |
| 30 | 190 |
| 40 | 180 |
| 50 | 170 |
| 60 | 160 |

a. Use any two of the ordered pairs from the table to derive the equation $M=220-A$, which gives the maximum heart rate $M$ for an adult whose age is $A$.
b. Use the equation from part (a) to find the maximum heart rate for a 25 -year-old adult.
63. Textbook Cost To produce this textbook, suppose the publisher spent $\$ 125,000$ for typesetting and $\$ 6.50$ per book for printing and binding. The total cost to produce and print $n$ books can be written as

a. Suppose the number of books printed in the first printing is 10,000 . What is the total cost?
b. If the average cost is the total cost divided by the number of books printed, find the average cost of producing 10,000 textbooks.
c. Find the cost to produce one more textbook when you have already produced 10,000 textbooks.

64. Exercise Heart Rate In an aerobics class, the instructor indicates that her students' exercise heart rate is $60 \%$ of their maximum heart rate, where maximum heart rate is 220 minus their age.
a. Determine the equation that gives exercise heart rate $E$ in terms of age $A$.
b. Use the equation to find the exercise heart rate of a 22-year-old student.
65. Bridal Books Use the chart to work the following problems. Assume 75 books in 2002.
a. Find the equation of the line that runs from 2002 to 2004.
b. Use your answer to part a to estimate the number of bridal books produced in 2006.


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66. Triathlon Entries Use the chart to work the following problems.


From USA Today. Copyright 2005. Reprinted with permission.
a. Find the equation of the line that connects the points $(1978,0)$ and $(2005,1,793)$.
b. Use your answer to part a to estimate the number of entries in the 2006 Ironman.
c. Do you think the actual number of entries was more or less than your answer to part b? Explain why.
67. Improving Your Quantitative Literacy Use the information in the chart to work the following problems.

USA TODAY Snapshots ${ }^{\circledR}$

a. Find the equation of the line connecting the first and last points on the chart.
b. Use the equation from part a to estimate the percent of adults accessing the Internet in 2006.
c. What happens when we use our equation to estimate the number of adults online in 2010, and why is this result unacceptable?
68. Definitions
a. Use an online dictionary to find definitions for the following words:

interpolation and extrapolation
b. Which of the two words describes what you did in problem 67 parts b and c above.

## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in previous sections and chapters.
69. The length of a rectangle is 3 inches more than 4 times the width. The perimeter is 56 inches. Find the length and width.
70. One angle is 10 degrees less than four times another. Find the measure of each angle if:
a. The two angles are complementary.
b. The two angles are supplementary.
71. The cash register in a candy shop contains $\$ 66$ at the beginning of the day. At the end of the day, it contains $\$ 732.50$. If the sales tax rate is $7.5 \%$, how much of the total is sales tax?
72. The third angle in an isosceles triangle is 20 degrees less than twice as large as each of the two base angles. Find the measure of each angle.

## Getting Ready for the Next Section

Complete each table using the given equation.
73. $y=7.5 x$

75. $x=y^{2}$

74. $h=32 t-16 t^{2}$

76. $y=|x|$

## Extending the Concepts

The midpoint $M$ of two points $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ is defined to be the average of each of their coordinates, so

$$
M=\left(\frac{x_{1}+x_{2}}{2}, \frac{y_{1}+y_{2}}{2}\right)
$$

For example, the midpoint of $(-2,3)$ and $(6,8)$ is given by

$$
\left(\frac{-2+6}{2}, \frac{3+8}{2}\right)=\left(2, \frac{11}{2}\right)
$$

For each given pair of points, find the equation of the line that is perpendicular to the line through these points and that passes through their midpoint. Answer using slope-intercept form.
Note: This line is called the perpendicular bisector of the line segment connecting the two points.
77. $(1,4)$ and $(7,8)$
78. $(-2,1)$ and $(6,7)$
79. $(-5,1)$ and $(-1,4)$
80. $(-6,-2)$ and $(2,1)$

## 8.3

## Introduction to Functions

## OBJECTIVES

A Construct a table or a graph from a function rule.

B Identify the domain and range of a function or a relation.

C Determine whether a relation is also a function.

The ad shown here appeared in the help wanted section of the local newspaper the day I was writing this section of the book. We can use the information in the ad to start an informal discussion of our next topic: functions.

## An Informal Look at Functions

To begin with, suppose you have a job that pays $\$ 7.50$ per hour and that you work anywhere from 0 to 40 hours
per week. The amount of money you make in one week

## 312 Help Wanted

YARD PERSON
Full-time 40 hrs. with weekend work required. Cleaning \& loading trucks. \$7.50/hr. Valid CDL with clean record \& drug screen required. Submit current MVR to KCI, 225 Suburban Rd., SLO. 805-555-3304. depends on the number of hours you work that week. In mathematics we say that your weekly earnings are a function of the number of hours you work. If we let the variable $x$ represent hours and the variable $y$ represent the money you make, then the relationship between $x$ and $y$ can be written as

$$
y=7.5 x \text { for } 0 \leq x \leq 40
$$

## EXAMPLE 1 Construct a table and graph for the function

$$
y=7.5 x \text { for } 0 \leq x \leq 40
$$

SOLUTION Table 1 gives some of the paired data that satisfy the equation $y=7.5 x$. Figure 1 is the graph of the equation with the restriction $0 \leq x \leq 40$.


FIGURE 1 Weekly wages at $\$ 7.50$ per hour.

The equation $y=7.5 x$ with the restriction $0 \leq x \leq 40$, Table 1, and Figure 1 are three ways to describe the same relationship between the number of hours you work in 1 week and your gross pay for that week. In all three, we input values of $x$, and then use the function rule to output values of $y$.

## Domain and Range of a Function

We began this discussion by saying that the number of hours worked during the week was from 0 to 40, so these are the values that $x$ can assume. From the line graph in Figure 1, we see that the values of $y$ range from 0 to 300. We call the complete set of values that $x$ can assume the domain of the function. The values that are assigned to $y$ are called the range of the function.


## EXAMPLE 2 State the domain and range for the function

$$
y=7.5 x, \quad 0 \leq x \leq 40
$$

SOLUTION From the previous discussion we have

$$
\begin{aligned}
\text { Domain } & =\{x \mid 0 \leq x \leq 40\} \\
\text { Range } & =\{y \mid 0 \leq y \leq 300\}
\end{aligned}
$$

## Function Maps

Another way to visualize the relationship between $x$ and $y$ is with the diagram in Figure 2, which we call a function map:


FIGURE 2 A FUNCTION MAP

Although Figure 2 does not show all the values that $x$ and $y$ can assume, it does give us a visual description of how $x$ and $y$ are related. It shows that values of $y$ in the range come from values of $x$ in the domain according to a specific rule (multiply by 7.5 each time).

## A Formal Look at Functions

What is apparent from the preceding discussion is that we are working with paired data. The solutions to the equation $y=7.5 x$ are pairs of numbers; the points on the line graph in Figure 1 come from paired data; and the diagram in Figure 2 pairs numbers in the domain with numbers in the range. We are now ready for the formal definition of a function.

DEFINITION A function is a rule that pairs each element in one set, called the domain, with exactly one element from a second set, called the range.

In other words, a function is a rule for which each input is paired with exactly one output.

## Functions as Ordered Pairs

The function rule $y=7.5 x$ from Example 1 produces ordered pairs of numbers $(x, y)$. The same thing happens with all functions: The function rule produces ordered pairs of numbers. We use this result to write an alternative definition for a function.

> ALTERNATIVE DEFINITION A function is a set of ordered pairs in which no two different ordered pairs have the same first coordinate. The set of all first coordinates is called the domain of the function. The set of all second coordinates is called the range of the function.

The restriction on first coordinates in the alternative definition keeps us from assigning a number in the domain to more than one number in the range.

## A Relationship That Is Not a Function

You may be wondering if any sets of paired data fail to qualify as functions. The answer is yes, as the next example reveals.

EXAMPLE 3 Table 2 shows the prices of used Ford Mustangs that were listed in the local newspaper. The diagram in Figure 3 is called a scatter diagram. It gives a visual representation of the data in Table 2. Why is this data not a function?

| TABLE 2 |  |
| :---: | :---: |
| Used Mustang Prices |  |
| $\begin{gathered} \text { Year } \\ \quad x \\ \hline \end{gathered}$ | $\begin{gathered} \text { Price }(\$) \\ y \\ \hline \end{gathered}$ |
| 1997 | 13,925 |
| 1997 | 11,850 |
| 1997 | 9,995 |
| 1996 | 10,200 |
| 1996 | 9,600 |
| 1995 | 9,525 |
| 1994 | 8,675 |
| 1994 | 7,900 |
| 1993 | 6,975 |



SOLUTION In Table 2, the year 1997 is paired with three different prices: $\$ 13,925, \$ 11,850$, and $\$ 9,995$. That is enough to disqualify the data from belonging to a function. For a set of paired data to be considered a function, each number in the domain must be paired with exactly one number in the range.

Still, there is a relationship between the first coordinates and second coordinates in the used-car data. It is not a function relationship, but it is a relationship. To classify all relationships specified by ordered pairs, whether they are functions or not, we include the following two definitions.

DEFINITION A relation is a rule that pairs each element in one set, called the domain, with one or more elements from a second set, called the range.

ALTERNATIVE DEFINITION A relation is a set of ordered pairs. The set of all first coordinates is the domain of the relation. The set of all second coordinates is the range of the relation.

Here are some facts that will help clarify the distinction between relations and functions.

1. Any rule that assigns numbers from one set to numbers in another set is a relation. If that rule makes the assignment so that no input has more than one output, then it is also a function.
2. Any set of ordered pairs is a relation. If none of the first coordinates of those ordered pairs is repeated, the set of ordered pairs is also a function.
3. Every function is a relation.
4. Not every relation is a function.

## Graphing Relations and Functions

To give ourselves a wider perspective on functions and relations, we consider some equations whose graphs are not straight lines.

EXAMPLE 4 Kendra is tossing a softball into the air with an underhand motion. The distance of the ball above her hand at any time is given by the function

$$
h=32 t-16 t^{2} \text { for } 0 \leq t \leq 2
$$

where $h$ is the height of the ball in feet and $t$ is the time in seconds. Construct a table that gives the height of the ball at quarter-second intervals, starting with $t=0$ and ending with $t=2$. Construct a line graph from the table.
SOLUTION We construct Table 3 using the following values of $t: 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$, $\frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2$. The values of $h$ come from substituting these values of $t$ into the equation $h=32 t-16 t^{2}$. (This equation comes from physics. If you take a physics class, you will learn how to derive this equation.) Then we construct the graph in Figure 4 from the table. The graph appears only in the first quadrant because neither $t$ nor $h$ can be negative.

TABLE 3
Tossing a Softball into the Air

| Time (sec) | Function Rule | Distance (ft) |
| :---: | :---: | :---: |
| $\boldsymbol{t}$ | $\boldsymbol{h}=\mathbf{3 2 t}-\mathbf{1 6 t ^ { \mathbf { 2 } }}$ | $\boldsymbol{h}$ |
| 0 | $h=32(0)-16(0)^{2}=0-0=0$ | 0 |
| $\frac{1}{4}$ | $h=32\left(\frac{1}{4}\right)-16\left(\frac{1}{4}\right)^{2}=8-1=7$ | 7 |
| $\frac{1}{2}$ | $h=32\left(\frac{1}{2}\right)-16\left(\frac{1}{2}\right)^{2}=16-4=12$ | 12 |
| $\frac{3}{4}$ | $h=32\left(\frac{3}{4}\right)-16\left(\frac{3}{4}\right)^{2}=24-9=15$ | 15 |
| 1 | $h=32(1)-16(1)^{2}=32-16=16$ | 16 |
| $\frac{5}{4}$ | $h=32\left(\frac{5}{4}\right)-16\left(\frac{5}{4}\right)^{2}=40-25=15$ | 15 |
| $\frac{3}{2}$ | $h=32\left(\frac{3}{2}\right)-16\left(\frac{3}{2}\right)^{2}=48-36=12$ | 12 |
| $\frac{7}{4}$ | $h=32\left(\frac{7}{4}\right)-16\left(\frac{7}{4}\right)^{2}=56-49=7$ | 7 |
| 2 | $h=32(2)-16(2)^{2}=64-64=0$ | 0 |



Here is a summary of what we know about functions as it applies to this example: We input values of $t$ and output values of $h$ according to the function rule

$$
h=32 t-16 t^{2} \quad \text { for } \quad 0 \leq t \leq 2
$$

The domain is given by the inequality that follows the equation; it is

$$
\text { Domain }=\{t \mid 0 \leq t \leq 2\}
$$

The range is the set of all outputs that are possible by substituting the values of $t$ from the domain into the equation. From our table and graph, it seems that the range is

$$
\text { Range }=\{h \mid 0 \leq h \leq 16\}
$$

## USING TECHNOLOGY

## More About Example 4

Most graphing calculators can easily produce the information in Table 3. Simply set $Y_{1}$ equal to $32 \mathrm{X}-16 \mathrm{X}^{2}$. Then set up the table so it starts at 0 and increases by an increment of 0.25 each time. (On a TI-82/83, use the TBLSET key to set up the table.)

| Table Setup | Y Variables Setup |
| :--- | :--- |
| Table minimum $=0$ | $\mathrm{Y}_{1}=32 \mathrm{X}-16 \mathrm{X}^{2}$ |
| Table increment $=.25$ |  |
| Dependent variable: Auto |  |
| Independent variable: Auto |  |

The table will look like this:

| $\mathbf{X}$ | $\mathbf{Y}_{\mathbf{1}}$ |
| :---: | ---: |
| 0.00 | 0 |
| 0.25 | 7 |
| 0.50 | 12 |
| 0.75 | 15 |
| 1.00 | 16 |
| 1.25 | 15 |
| 1.50 | 12 |

## EXAMPLE 5 Sketch the graph of $x=y^{2}$.

SOLUTION Without going into much detail, we graph the equation $x=y^{2}$ by finding a number of ordered pairs that satisfy the equation, plotting these points, then drawing a smooth curve that connects them. A table of values for $x$ and $y$ that satisfy the equation follows, along with the graph of $x=y^{2}$ shown in Figure 5.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :--- | ---: |
| 0 | 0 |
| 1 | 1 |
| 1 | -1 |
| 4 | 2 |
| 4 | -2 |
| 9 | 3 |
| 9 | -3 |



FIGURE 5

As you can see from looking at the table and the graph in Figure 5, several ordered pairs whose graphs lie on the curve have repeated first coordinates. For instance, $(1,1)$ and $(1,-1),(4,2)$ and $(4,-2)$, and $(9,3)$ and $(9,-3)$. The graph is therefore not the graph of a function.

## Vertical Line Test

Look back at the scatter diagram for used Mustang prices shown in Figure 3. Notice that some of the points on the diagram lie above and below each other along vertical lines. This is an indication that the data do not constitute a function. Two data points that lie on the same vertical line must have come from two ordered pairs with the same first coordinates.
Now, look at the graph shown in Figure 5. The reason this graph is the graph of a relation, but not of a function, is that some points on the graph have the same first coordinates-for example, the points $(4,2)$ and $(4,-2)$. Furthermore, any time two points on a graph have the same first coordinates, those points must lie on a vertical line. [To convince yourself, connect the points $(4,2)$ and $(4,-2)$ with a straight line. You will see that it must be a vertical line.] This allows us to write the following test that uses the graph to determine whether a relation is also a function.

Vertical Line Test If a vertical line crosses the graph of a relation in more than one place, the relation cannot be a function. If no vertical line can be found that crosses a graph in more than one place, then the graph is the graph of a function.

If we look back to the graph of $h=32 t-16 t^{2}$ as shown in Figure 4, we see that no vertical line can be found that crosses this graph in more than one place. The graph shown in Figure 4 is therefore the graph of a function.

## EXAMPLE 6 Graph $y=|x|$. Use the graph to determine whether we

 have the graph of a function. State the domain and range.SOLUTION We let $x$ take on values of $-4,-3,-2,-1,0,1,2,3$, and 4 . The corresponding values of $y$ are shown in the table. The graph is shown in Figure 6.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| ---: | ---: |
| -4 | 4 |
| -3 | 3 |
| -2 | 2 |
| -1 | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |



FIGURE 6

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,4
B 2,4
C 3-6

Because no vertical line can be found that crosses the graph in more than one place, $y=|x|$ is a function. The domain is all real numbers. The range is $\{y \mid y \geq 0\}$.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a function?
2. What is the vertical line test?
3. Is every line the graph of a function? Explain.
4. Which variable is usually associated with the domain of a function?

## Problem Set 8.3

Online support materials can be found at www.thomsonedu.com/login

For each of the following relations, give the domain and range, and indicate which are also functions.

1. $\{(1,3),(2,5),(4,1)\}$
2. $\{(3,1),(5,7),(2,3)\}$
3. $\{(-1,3),(1,3),(2,-5)\}$
4. $\{(3,-4),(-1,5),(3,2)\}$
5. $\{(7,-1),(3,-1),(7,4)\}$
6. $\{(5,-2),(3,-2),(5,-1)\}$
7. $\{(a, 3),(b, 4),(c, 3),(d, 5)\}$
8. $\{(a, 5),(b, 5),(c, 4),(d, 5)\}$
9. $\{(a, 1),(a, 2),(a, 3),(a, 4)\}$
10. $\{(a, 1),(b, 1),(c, 1),(d, 1)\}$
11. 

State whether each of the following graphs represents a function.
11.

12.


= Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
14.

15.

16.

17.

18.

19.

20.


Determine the domain and range of the following functions. Assume the entire function is shown.

- 21. 


22.

23.

24.


Graph each of the following relations. In each case, use the graph to find the domain and range, and indicate whether the graph is the graph of a function.
25. $y=x^{2}-1$
26. $y=x^{2}+1$
27. $y=x^{2}+4$
28. $y=x^{2}-9$
29. $x=y^{2}-1$
30. $x=y^{2}+1$
31. $x=y^{2}+4$
32. $x=y^{2}-9$
33. $y=|x-2|$
34. $y=|x+2|$
35. $y=|x|-2$
36. $y=|x|+2$

## Applying the Concepts

37. Weekly Wages Suppose you have a job that pays $\$ 8.50$ per hour and you work anywhere from 10 to 40 hours per week.
a. Write an equation, with a restriction on the variable $x$, that gives the amount of money, $y$, you will earn for working $x$ hours in one week.
b. Use the function rule to complete the table.

## TABLE 4

Weekly Wages

| Hours Worked | Function Rule <br> $\boldsymbol{x}$ | Gross Pay (\$) <br> $\mathbf{y} .5 \boldsymbol{x}$ |
| :---: | :---: | :---: |
| 10 |  | $\boldsymbol{y}$ |
| 20 |  |  |
| 30 |  |  |
| 40 |  |  |

c. Use the data from the table to graph the function.
d. State the domain and range of this function.
e. What is the minimum amount you can earn in a week with this job? What is the maximum amount?
38. Weekly Wages The ad shown here was in the local newspaper. Suppose you are hired for the job described in the ad.
a. If $x$ is the number of hours you work per week and $y$ is your weekly gross pay, write the equation for $y$. (Be sure to include any restrictions on the variable $x$ that are given in the ad.)
b. Use the function rule to complete the table.

## TABLE 5

Weekly Wages

| Hours Worked <br> $\boldsymbol{x}$ | Function Rule <br> $\boldsymbol{y}=5.25 \boldsymbol{x}$ | Gross Pay (\$) <br> $\boldsymbol{y}$ |
| :---: | :---: | :---: |
| 15 |  |  |
| 20 |  |  |
| 25 |  |  |
| 30 |  |  |

c. Use the data from the table to graph the function.
d. State the domain and range of this function.
e. What is the minimum amount you can earn in a week with this job? What is the maximum amount?
39. Tossing a Coin Hali is tossing a quarter into the air with an underhand motion. The distance the quarter is above her hand at any time is given by the function

$$
h=16 t-16 t^{2} \quad \text { for } \quad 0 \leq t \leq 1
$$

where $h$ is the height of the quarter in feet, and $t$ is the time in seconds.
a. Fill in the table.

| Time (sec) | Function Rule <br> $\boldsymbol{h}=\mathbf{1 6 t}-\mathbf{1 6 t} \boldsymbol{t}^{\mathbf{2}}$ | Distance (ft) <br> $\boldsymbol{h}$ |
| :---: | :---: | :---: |
| 0 |  |  |
| 0.1 |  |  |
| 0.2 |  |  |
| 0.3 |  |  |
| 0.4 |  |  |
| 0.5 |  |  |
| 0.6 |  |  |
| 0.7 |  |  |
| 0.8 |  |  |
| 0.9 |  |  |
| 1 |  |  |

b. State the domain and range of this function.
c. Use the data from the table to graph the function.
40. Intensity of Light The formula below gives the intensity of light that falls on a surface at various distances from a 100-watt light bulb:

$$
I=\frac{120}{d^{2}} \quad \text { for } \quad d>0
$$

where $I$ is the intensity of light (in lumens per square foot), and $d$ is the distance (in feet) from the light bulb to the surface.
a. Fill in the table.

| Distance (ft) | Function Rule | Intensity |
| :---: | ---: | :---: |
| $\boldsymbol{D}$ | $\boldsymbol{I}=\frac{\mathbf{1 2 0}}{\boldsymbol{d}^{\mathbf{2}}}$ | $\boldsymbol{I}$ |
| 1 |  |  |
| 2 |  |  |
| 3 |  |  |
| 4 |  |  |
| 6 |  |  |

b. Use the data from the table to graph the function.
41. Area of a Circle The formula for the area $A$ of a circle with radius $r$ is given by $A=\pi r^{2}$. The formula shows that $A$ is a function of $r$.
a. Graph the function $A=\pi r^{2}$ for $0 \leq r \leq 3$. (On the graph, let the horizontal axis be the $r$-axis, and let the vertical axis be the $A$-axis.)
b. State the domain and range of the function $A=\pi r^{2}, 0 \leq r \leq 3$.
42. Area and Perimeter of a Rectangle A rectangle is 2 inches longer than it is wide. Let $x=$ the width, $P=$ the perimeter, and $A=$ the area of the rectangle.
a. Write an equation that will give the perimeter $P$ in terms of the width $x$ of the rectangle. Are there any restrictions on the values that $x$ can assume?
b. Graph the relationship between $P$ and $x$.
43. Tossing a Ball A ball is thrown straight up into the air from ground level. The relationship between the height $h$ of the ball at any time $t$ is illustrated by the following graph:


The horizontal axis represents time $t$, and the vertical axis represents height $h$.
a. Is this graph the graph of a function?
b. State the domain and range.
c. At what time does the ball reach its maximum height?
d. What is the maximum height of the ball?
e. At what time does the ball hit the ground?
44. Company Profits The amount of profit a company earns is based on the number of items it sells. The relationship between the profit $P$ and number of items it sells $x$, is illustrated by the following graph:


The horizontal axis represents items sold, $x$, and the vertical axis represents the profit, $P$.
a. Is this graph the graph of a function?
b. State the domain and range.
c. How many items must the company sell to make their maximum profit?
d. What is their maximum profit?
45. Profits Match each of the following statements to the appropriate graph indicated by labels I-IV.
a. Sarah works 25 hours to earn $\$ 250$.
b. Justin works 35 hours to earn $\$ 560$.
c. Rosemary works 30 hours to earn $\$ 360$.
d. Marcus works 40 hours to earn $\$ 320$.

46. Find an equation for each of the functions shown in Problem 45. Show dollars earned, $E$, as a function of hours worked, $t$. Then, indicate the domain and range of each function.
a. Graph I: $E=$

$$
\begin{array}{ll}
\text { Domain }=\{t \mid & \} \\
\text { Range }=\{E \mid & \}
\end{array}
$$

b. Graph II: $E=$

$$
\begin{array}{ll}
\text { Domain }=\{t \mid & \} \\
\text { Range }=\{E \mid & \}
\end{array}
$$

c. Graph III: $E=$

$$
\begin{array}{ll}
\text { Domain }=\{t \mid & \} \\
\text { Range }=\{E \mid & \}
\end{array}
$$

d. Graph IV: $E=$

$$
\begin{array}{ll}
\text { Domain }=\{t \mid & \} \\
\text { Range }=\{E \mid & \}
\end{array}
$$

## Maintaining Your Skills

The problems that follow review some of the more important skills you have learned in previous sections and chapters.

For the equation $y=3 x-2$ :
47. Find $y$ if $x$ is 4 .
48. Find $y$ if $x$ is 0 .
49. Find $y$ if $x$ is -4 .
50. Find $y$ if $x$ is -2 .

For the equation $y=x^{2}-3$ :
51. Find $y$ if $x$ is 2 .
52. Find $y$ if $x$ is -2 .
53. Find $y$ if $x$ is 0 .
54. Find $y$ if $x$ is -4 .
55. If $x-2 y=4$ and $x=\frac{8}{5}$, find $y$.
56. If $5 x-10 y=15$, find $y$ when $x$ is 3 .
57. Let $x=0$ and $y=0$ in $y=a(x-80)^{2}+70$ and solve for $a$.
58. Find $R$ if $p=2.5$ and $R=(900-300 p) p$.

## Getting Ready for the Next Section

Simplify. Round to the nearest whole number if necessary.
59. 7.5(20)
60. $60 \div 7.5$
61. $4(3.14)(9)$
62. $\frac{4}{3}(3.14) \cdot 3^{3}$
63. $4(-2)-1$
64. $3(3)^{2}+2(3)-1$
65. If $s=\frac{60}{t}$, find $s$ when
a. $t=10$
b. $t=8$
66. If $y=3 x^{2}+2 x-1$, find $y$ when
a. $x=0$
b. $x=-2$
67. Find the value of $x^{2}+2$ for
a. $x=5$
b. $x=-2$
68. Find the value of $125 \cdot 2^{t}$ for
a. $t=0$
b. $t=1$

## Extending the Concepts

Graph each of the following relations. In each case, use the graph to find the domain and range, and indicate whether the graph is the graph of a function.
69. $y=5-|x|$
70. $y=|x|-3$
71. $x=|y|+3$
72. $x=2-|y|$
73. $|x|+|y|=4$
74. $2|x|+|y|=6$

## Function Notation

## OBJECTIVES

A Use function notation to find the value of a function for a given value of the variable.

$f(x)$
Some students like to think of functions as machines. Values of $x$ are put into the machine, which transforms them into values of $f(x)$, which then are output by the machine.

Let's return to the discussion that introduced us to functions. If a job pays $\$ 7.50$ per hour for working from 0 to 40 hours a week, then the amount of money $y$ earned in 1 week is a function of the number of hours worked, $x$. The exact relationship between $x$ and $y$ is written

$$
y=7.5 x \text { for } 0 \leq x \leq 40
$$

Because the amount of money earned $y$ depends on the number of hours worked $x$, we call $y$ the dependent variable and $x$ the independent variable. Furthermore, if we

## 312 Help Wanted

YARD PERSON Full-time 40 hrs. with weekend work required. Cleaning \& loading trucks. \$7.50/hr. Valid CDL with clean record \& drug screen required. Submit current MVR to KCI, 225 Suburban Rd., SLO. 805-555-3304. let $f$ represent all the ordered pairs produced by the equation, then we can write

$$
f=\{(x, y) \mid y=7.5 x \text { and } 0 \leq x \leq 40\}
$$

Once we have named a function with a letter, we can use an alternative notation to represent the dependent variable $y$. The alternative notation for $y$ is $f(x)$. It is read " $f$ of $x$ " and can be used instead of the variable $y$ when working with functions. The notation $y$ and the notation $f(x)$ are equivalent-that is,

$$
y=7.5 x \Leftrightarrow f(x)=7.5 x
$$

When we use the notation $f(x)$ we are using function notation. The benefit of using function notation is that we can write more information with fewer symbols than we can by using just the variable $y$. For example, asking how much money a person will make for working 20 hours is simply a matter of asking for $f(20)$. Without function notation, we would have to say, "Find the value of $y$ that corresponds to a value of $x=20$." To illustrate further, using the variable $y$, we can say, " $y$ is 150 when $x$ is 20 ." Using the notation $f(x)$, we simply say, " $f(20)=$ 150." Each expression indicates that you will earn $\$ 150$ for working 20 hours.

## EXAMPLE 1 If $f(x)=7.5 x$, find $f(0), f(10)$, and $f(20)$.

SOLUTION To find $f(0)$ we substitute 0 for $x$ in the expression $7.5 x$ and simplify. We find $f(10)$ and $f(20)$ in a similar manner-by substitution.

If $\quad f(x)=7.5 x$
then $\quad f(\mathbf{0})=7.5(\mathbf{0})=0$
$f(\mathbf{1 0})=7.5(\mathbf{1 0})=75$
$f(\mathbf{2 0})=7.5(\mathbf{2 0})=150$
If we changed the example in the discussion that opened this section so that the hourly wage was $\$ 6.50$ per hour, we would have a new equation to work with:

$$
y=6.5 x \quad \text { for } \quad 0 \leq x \leq 40
$$

Suppose we name this new function with the letter $g$. Then

$$
g=\{(x, y) \mid y=6.5 x \text { and } 0 \leq x \leq 40\}
$$

and

$$
g(x)=6.5 x
$$

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If we want to talk about both functions in the same discussion, having two different letters, $f$ and $g$, makes it easy to distinguish between them. For example, because $f(x)=7.5 x$ and $g(x)=6.5 x$, asking how much money a person makes for working 20 hours is simply a matter of asking for $f(20)$ or $g(20)$, avoiding any confusion over which hourly wage we are talking about.

The diagrams shown in Figure 1 further illustrate the similarities and differences between the two functions we have been discussing.


FIGURE 1 Function maps

## Function Notation and Graphs

We can visualize the relationship between $x$ and $f(x)$ on the graph of the function. Figure 2 shows the graph of $f(x)=7.5 x$ along with two additional line segments. The horizontal line segment corresponds to $x=20$, and the vertical line segment corresponds to $f(20)$. (Note that the domain is restricted to $0 \leq x \leq 40$.)


We can use functions and function notation to talk about numbers in the chart on gasoline prices. Let's let $x$ represent one of the years in the chart.

## USA TODAY Snapshots ${ }^{\circledR}$



If the function $f$ pairs each year in the chart with the average price of regular gasoline for that year, then each statement below is true:

$$
\begin{aligned}
& f(1955)=\$ 0.29 \\
& \text { The domain of } f=\{1955,1965,1975,1985,1995,2005\} \\
& f(1985)>f(1995)
\end{aligned}
$$

In general, when we refer to the function $f$ we are referring to the domain, the range, and the rule that takes elements in the domain and outputs elements in the range. When we talk about $f(x)$ we are talking about the rule itself, or an element in the range, or the variable $y$.

$\overbrace{$|  Domain of $f$ |
| :---: |
|  Inputs  |}$^{$| $y=f(x)$ |
| :---: |
|  Rule  |$}$| Range of $f$ <br> Outputs |
| :---: |
| The function $f$ |

## Using Function Notation

The remaining examples in this section show a variety of ways to use and interpret function notation.

EXAMPLE 2 If it takes Lorena $t$ minutes to run a mile, then her average speed $s(t)$ in miles per hour is given by the formula

$$
s(t)=\frac{60}{t} \text { for } t>0
$$

Find $s(10)$ and $s(8)$, and then explain what they mean.
SOLUTION To find $s(10)$, we substitute 10 for $t$ in the equation and simplify:

$$
s(\mathbf{1 0})=\frac{60}{\mathbf{1 0}}=6
$$

In words: When Lorena runs a mile in 10 minutes, her average speed is 6 miles per hour.
We calculate $s(8)$ by substituting 8 for $t$ in the equation. Doing so gives us

$$
s(\boldsymbol{8})=\frac{60}{\mathbf{8}}=7.5
$$

In words: Running a mile in 8 minutes is running at a rate of 7.5 miles per hour.

## EXAMPLE 3 A painting is purchased as an investment for $\$ 125$. If its

 value increases continuously so that it doubles every 5 years, then its value is given by the function$$
V(t)=125 \cdot 2^{1 / 5} \text { for } t \geq 0
$$

where $t$ is the number of years since the painting was purchased, and $V(t)$ is its value (in dollars) at time $t$. Find $V(5)$ and $V(10)$, and explain what they mean.

SOLUTION The expression $V(5)$ is the value of the painting when $t=5$ (5 years after it is purchased). We calculate $V(5)$ by substituting 5 for $t$ in the equation $V(t)=125 \cdot 2^{t / 5}$. Here is our work:

$$
V(\mathbf{5})=125 \cdot 2^{5 / 5}=125 \cdot 2^{1}=125 \cdot 2=250
$$

In words: After 5 years, the painting is worth $\$ 250$.
The expression $V(10)$ is the value of the painting after 10 years. To find this number, we substitute 10 for $t$ in the equation:

$$
V(\mathbf{1 0})=125 \cdot 2^{\mathbf{1 0} / 5}=125 \cdot 2^{2}=125 \cdot 4=500
$$

In words: The value of the painting 10 years after it is purchased is $\$ 500$.
EXAMPLE 4 A balloon has the shape of a sphere with a radius of 3 inches. Use the following formulas to find the volume and surface area of the balloon.

$$
V(r)=\frac{4}{3} \pi r^{3} \quad S(r)=4 \pi r^{2}
$$

SOLUTION As you can see, we have used function notation to write the two formulas for volume and surface area because each quantity is a function of the radius. To find these quantities when the radius is 3 inches, we evaluate $V(3)$ and $S(3)$ :
$V(\mathbf{3})=\frac{4}{3} \pi \mathbf{3}^{3}=\frac{4}{3} \pi 27=36 \pi$ cubic inches, or 113 cubic inches
$S(\mathbf{3})=4 \pi \mathbf{3}^{2}=36 \pi$ square inches, or 113 square inches
To the nearest whole number
To the nearest whole number

The fact that $V(3)=36 \pi$ means that the ordered pair $(3,36 \pi)$ belongs to the function $V$. Likewise, the fact that $S(3)=36 \pi$ tells us that the ordered pair $(3,36 \pi)$ is a member of function $S$.

We can generalize the discussion at the end of Example 4 this way:

$$
(a, b) \in f \quad \text { if and only if } \quad f(a)=b
$$

## USING TECHNOLOGY

## More About Example 4

If we look back at Example 4, we see that when the radius of a sphere is 3, the numerical values of the volume and surface area are equal. How unusual is this? Are there other values of $r$ for which $V(r)$ and $S(r)$ are equal? We can answer this question by looking at the graphs of both $V$ and $S$.

To graph the function $V(r)=\frac{4}{3} \pi r^{3}$, set $\mathrm{Y}_{1}=4 \pi \mathrm{X}^{3} / 3$. To graph $S(r)=4 \pi r^{2}$, set $Y_{2}=4 \pi X^{2}$. Graph the two functions in each of the following windows:

Window 1: X from -4 to $4, \mathrm{Y}$ from -2 to 10
Window 2: X from 0 to 4, Y from 0 to 50
Window 3: X from 0 to $4, \mathrm{Y}$ from 0 to 150
Then use the Trace and Zoom features of your calculator to locate the point in the first quadrant where the two graphs intersect. How do the coordinates of this point compare with the results in Example 4?

## EXAMPLE $5 \quad$ If $f(x)=3 x^{2}+2 x-1$, find $f(0), f(3)$, and $f(-2)$.

SOLUTION Because $f(x)=3 x^{2}+2 x-1$, we have

$$
\begin{aligned}
f(\mathbf{0}) & =3(\mathbf{0})^{2}+2(\mathbf{0})-1=0+0-1=-1 \\
f(\mathbf{3}) & =3(\mathbf{3})^{2}+2(\mathbf{3})-1=27+6-1=32 \\
f(-\mathbf{2}) & =3(-\mathbf{2})^{2}+2(-\mathbf{2})-1=12-4-1=7
\end{aligned}
$$

In Example 5, the function $f$ is defined by the equation $f(x)=3 x^{2}+2 x-1$. We could just as easily have said $y=3 x^{2}+2 x-1$; that is, $y=f(x)$. Saying $f(-2)=7$ is exactly the same as saying $y$ is 7 when $x$ is -2 .

## EXAMPLE $6 \quad$ If $f(x)=4 x-1$ and $g(x)=x^{2}+2$, then

$$
\begin{aligned}
f(\mathbf{5}) & =4(\mathbf{5})-1=19 & & \text { and } & g(\mathbf{5}) & =\mathbf{5}^{2}+2=27 \\
f(-\mathbf{2}) & =4(-\mathbf{2})-1=-9 & & \text { and } & g(-\mathbf{2}) & =(-\mathbf{2})^{2}+2=6 \\
f(\mathbf{0}) & =4(\mathbf{0})-1=-1 & & \text { and } & g(\mathbf{0}) & =\mathbf{0}^{2}+2=2 \\
f(\mathbf{z}) & =4 \boldsymbol{z}-1 & & \text { and } & g(\boldsymbol{z}) & =\boldsymbol{z}^{2}+2 \\
f(\boldsymbol{a}) & =4 \boldsymbol{a}-1 & & \text { and } & g(\boldsymbol{a}) & =\boldsymbol{a}^{2}+2
\end{aligned}
$$

## USING TECHNOLOGY

## More About Example 6

Most graphing calculators can use tables to evaluate functions. To work Example 6 using a graphing calculator table, set $Y_{1}$ equal to $4 \mathrm{X}-1$ and $\mathrm{Y}_{2}$ equal to $X^{2}+2$. Then set the independent variable in the table to Ask instead of Auto. Go to your table and input $5,-2$, and 0 . Under $Y_{1}$ in the table, you will find $f(5), f(-2)$, and $f(0)$. Under $\mathrm{Y}_{2}$, you will find $g(5), g(-2)$, and $g(0)$.

Table Setup
Table minimum $=0$
Y Variables Setup
$\mathrm{Y}_{1}=4 \mathrm{X}-1$
Table increment $=1$
$Y_{2}=X^{2}+2$
Independent variable: Ask
Dependent variable: Ask
The table will look like this:

| $\mathbf{X}$ | $\mathbf{Y}_{\mathbf{1}}$ | $\mathbf{Y}_{\mathbf{2}}$ |
| ---: | ---: | ---: |
| 5 | 19 | 27 |
| -2 | -9 | 6 |
| 0 | -1 | 2 |

Although the calculator asks us for a table increment, the increment doesn't matter since we are inputting the X -values ourselves.

EXAMPLE 7 If the function $f$ is given by

$$
f=\{(-2,0),(3,-1),(2,4),(7,5)\}
$$

then $f(-2)=0, f(3)=-1, f(2)=4$, and $f(7)=5$.

## EXAMPLE $8 \quad$ If $f(x)=2 x^{2}$ and $g(x)=3 x-1$, find

a. $f[g(2)]$
b. $g[f(2)]$

SOLUTION The expression $f[g(2)]$ is read " $f$ of $g$ of 2 ."
a. Since $g(2)=3(2)-1=5$,

$$
f[g(2)]=f(5)=2(5)^{2}=50
$$

b. Since $f(2)=2(2)^{2}=8$,

$$
g[f(2)]=g(8)=3(8)-1=23
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-8

1. Explain what you are calculating when you find $f(2)$ for a given function $f$.
2. If $s(t)=\frac{60}{t}$, how do you find $s(10)$ ?
3. If $f(2)=3$ for a function $f$, what is the relationship between the numbers 2 and 3 and the graph of $f$ ?
4. If $f(6)=O$ for a particular function $f$, then you can immediately graph one of the intercepts. Explain.

## Problem Set 8.4

## Online support materials can be found at www.thomsonedu.com/login

Let $f(x)=2 x-5$ and $g(x)=x^{2}+3 x+4$. Evaluate the following.

1. $f(2)$
2. $f(3)$
3. $f(-3)$
4. $g(-2)$
5. $g(-1)$
6. $f(-4)$
7. $g(-3)$
8. $g(2)$
9. $g(4)+f(4)$
10. $f(2)-g(3)$
11. $f(3)-g(2)$
12. $g(-1)+f(-1)$

Let $f(x)=3 x^{2}-4 x+1$ and $g(x)=2 x-1$. Evaluate the following.
13. $f(0)$
14. $g(0)$
15. $g(-4)$
16. $f(1)$
17. $f(-1)$
18. $g(-1)$
19. $g(10)$
20. $f(10)$
21. $f(3)$
22. $g(3)$
23. $g\left(\frac{1}{2}\right)$
24. $g\left(\frac{1}{4}\right)$
25. $f(a)$
26. $g(b)$

If $f=\left\{(1,4),(-2,0),\left(3, \frac{1}{2}\right),(\pi, 0)\right\}$ and $g=\{(1,1),(-2,2)$, $\left.\left(\frac{1}{2}, 0\right)\right\}$, find each of the following values of $f$ and $g$.
27. $f(1)$
28. $g(1)$
29. $g\left(\frac{1}{2}\right)$
30. $f(3)$
31. $g(-2)$
32. $f(\pi)$

Let $f(x)=2 x^{2}-8$ and $g(x)=\frac{1}{2} x+1$. Evaluate each of the following.
33. $f(0)$
34. $g(0)$
35. $g(-4)$
36. $f(1)$
37. $f(a)$
38. $g(z)$
39. $f(b)$
40. $g(t)$
41. $f[g(2)]$
42. $g[f(2)]$
43. $g[f(-1)]$
44. $f[g(-2)]$
45. $g[f(0)]$
46. $f[g(0)]$
47. Graph the function $f(x)=\frac{1}{2} x+2$. Then draw and label the line segments that represent $x=4$ and $f(4)$.
48. Graph the function $f(x)=-\frac{1}{2} x+6$. Then draw and label the line segments that represent $x=4$ and $f(4)$.
49. For the function $f(x)=\frac{1}{2} x+2$, find the value of $x$ for which $f(x)=x$.
50. For the function $f(x)=-\frac{1}{2} x+6$, find the value of $x$ for which $f(x)=x$.
51. Graph the function $f(x)=x^{2}$. Then draw and label the line segments that represent $x=1$ and $f(1)$, $x=2$ and $f(2)$, and, finally, $x=3$ and $f(3)$.
52. Graph the function $f(x)=x^{2}-2$. Then draw and label the line segments that represent $x=2$ and $f(2)$, and the line segments corresponding to $x=3$ and $f(3)$.

## Applying the Concepts

53. Investing in Art A painting is purchased as an investment for $\$ 150$. If its value increases continuously so that it doubles every 3 years, then its value is given by the function

$$
V(t)=150 \cdot 2^{t / 3} \text { for } t \geq 0
$$

where $t$ is the number of years since the painting was purchased, and $V(t)$ is its value (in dollars) at time $t$. Find $V(3)$ and $V(6)$, and then explain what they mean.
54. Average Speed If it takes Minke $t$ minutes to run a mile, then her average speed $s(t)$, in miles per hour, is given by the formula

$$
s(t)=\frac{60}{t} \quad \text { for } t>0
$$

Find $s(4)$ and $s(5)$, and then explain what they mean.
55. Antidepressant Sales Suppose $x$ represents one of the years in the chart. Suppose further that we have three functions $f, g$, and $h$ that do the following:
$f$ pairs each year with the total sales of Zoloft in billions of dollars for that year.
$g$ pairs each year with the total sales of Effexor in billions of dollars for that year.
$h$ pairs each year with the total sales of Wellbutrin in billions of dollars for that year.


For each statement below, indicate whether the statement is true or false.
a. The domain of $g$ is $\{2003,2004,2005\}$.
b. $f(2003)<f(2004)$
c. $f(2004)>g(2004)$
d. $h(2005)>1.5$
e. $h(2005)>h(2004)>h(2003)$
56. Mobile Phone Sales Suppose $x$ represents one of the years in the chart. Suppose further that we have three functions $f, g$, and $h$ that do the following: $f$ pairs each year with the number of camera phones sold that year.
$g$ pairs each year with the number of noncamera phones sold that year.
$h$ is such that $h(x)=f(x)+g(x)$.

## Got Camera Phone?

Estimates for worldwide mobile phone and camera phone shipments in millions. In 2005, 741 million handsets are sold, $50 \%$ are camera phones. By 2010, 1,034 million handset are expected to sell, $87 \%$ being camera phones.


Source: www.InfoTrends.com Estimates result of interviews of 4,782 people in U.S., U.K.,
France, Germany, Spain, Japan and China.

For each statement below, indicate whether the statement is true or false.
a. The domain of $f$ is $\{2004,2005,2006,2007$, 2008, 2009, 2010\}.
b. $h(2005)=741,000,000$
c. $f(2009)>g(2009)$
d. $f(2004)<f(2005)$
e. $h(2010)>h(2007)>h(2004)$
57. Value of a Copy Machine The function $V(t)=-3,300 t+$ 18,000 , where $V$ is value and $t$ is time in years, can be used to find the value of a large copy machine during the first 5 years of use.
a. What is the value of the copier after 3 years and 9 months?

b. What is the salvage value of this copier if it is replaced after 5 years?
c. State the domain of this function.
d. Sketch the graph of this function.
e. What is the range of this function?
f. After how many years will the copier be worth only $\$ 10,000$ ?
58. Value of a Forklift The function $V=-16,500 t+125,000$, where $V$ is value and $t$ is time in years, can be used to find the value of an electric forklift during the first 6 years of use.
a. What is the value of the forklift after 2 years and 3 months?
b. What is the salvage value of this forklift if it is replaced after 6 years?
c. State the domain of this function.
d. Sketch the graph of this function.
e. What is the range of this function?
f. After how many years will the forklift be worth only $\$ 45,000$ ?


## Maintaining Your Skills

Solve each equation.
59. $|3 x-5|=7$
60. $|0.04-0.03 x|=0.02$
61. $|4 y+2|-8=-2$
62. $4=|3-2 y|-5$
63. $5+|6 t+2|=3$
64. $7+\left|3-\frac{3}{4} t\right|=10$

## Getting Ready for the Next Section

Simplify.
65. $\left(35 x-0.1 x^{2}\right)-(8 x+500)$
66. $70+0.6(m-70)$
67. $\left(4 x^{2}+3 x+2\right)+\left(2 x^{2}-5 x-6\right)$
68. $(4 x-3)+\left(4 x^{2}-7 x+3\right)$
69. $\left(4 x^{2}+3 x+2\right)-\left(2 x^{2}-5 x-6\right)$
70. $(x+5)^{2}-2(x+5)$
71. $0.6(m-70)$
72. $x(35-0.1 x)$
73. $(4 x-3)(x-1)$
74. $(4 x-3)\left(4 x^{2}-7 x+3\right)$

## Extending the Concepts

The graphs of two functions are shown in Figures 3 and 4. Use the graphs to find the following.
75. a. $f(2)$
b. $f(-4)$
76. a. $g(0)$
b. $g(3)$


FIGURE 3


FIGURE 4
77. Step Function Figure 5 shows the graph of the step function $C$ that was used to calculate the first-class postage on a letter weighing $x$ ounces in 2006. Use this graph to answer questions (a) through (d).


FIGURE 5 The graph of $C(x)$
a. Fill in the following table:
$\begin{array}{lllllllll}\text { Weight (ounces) } & 0.6 & 1.0 & 1.1 & 2.5 & 3.0 & 4.8 & 5.0 & 5.3\end{array}$
Cost (cents)
b. If a letter costs 87 cents to mail, how much does it weigh? State your answer in words. State your answer as an inequality.
c. If the entire function is shown in Figure 5, state the domain.
d. State the range of the function shown in Figure 5.
78. Step Function A taxi ride in Boston at the time I am writing this problem is $\$ 1.50$ for the first $\frac{1}{4}$ mile, and then $\$ 0.25$ for each additional $\frac{1}{8}$ of a mile. The following graph shows how much you will pay for a taxi ride of 1 mile or less.

a. What is the most you will pay for this taxi ride?
b. How much does it cost to ride the taxi for $\frac{8}{10}$ of a mile?
c. Find the values of A and B on the horizontal axis.
d. If a taxi ride costs $\$ 2.50$, what distance was the ride?
e. If the complete function is shown in Figure 6, find the domain and range of the function.

## 8.5

## Algebra and Composition with Functions

## OBJECTIVES

A
Find the sum, difference, product, and quotient of two functions.

B Find the composition of two functions.

A company produces and sells copies of an accounting program for home computers. The price they charge for the program is related to the number of copies sold by the demand function

$$
p(x)=35-0.1 x
$$

We find the revenue for this business by multiplying the number of items sold by the price per item. When we do so, we are forming a new function by combining two existing functions; that is, if $n(x)=x$ is the number of items sold and $p(x)=$ $35-0.1 x$ is the price per item, then revenue is

$$
R(x)=n(x) \cdot p(x)=x(35-0.1 x)=35 x-0.1 x^{2}
$$

In this case, the revenue function is the product of two functions. When we combine functions in this manner, we are applying our rules for algebra to functions.

To carry this situation further, we know the profit function is the difference between two functions. If the cost function for producing $x$ copies of the accounting program is $C(x)=8 x+500$, then the profit function is

$$
P(x)=R(x)-C(x)=\left(35 x-0.1 x^{2}\right)-(8 x+500)=-500+27 x-0.1 x^{2}
$$

The relationship between these last three functions is shown visually in Figure 1.


Again, when we combine functions in the manner shown, we are applying our rules for algebra to functions. To begin this section, we take a formal look at addition, subtraction, multiplication, and division with functions.

If we are given two functions $f$ and $g$ with a common domain, we can define four other functions as follows.

## DEFINITION

$(f+g)(x)=f(x)+g(x)$
The function $f+g$ is the sum of the functions $f$ and $g$.
$(f-g)(x)=f(x)-g(x)$
The function $f-g$ is the difference of the functions $f$ and $g$.
$(f g)(x)=f(x) g(x) \quad$ The function $f g$ is the product of the functions $f$ and $g$. $\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)} \quad \begin{aligned} & \text { The function } f / g \text { is the quotient of the functions } f \\ & \text { and } g \text {, where } g(x) \neq 0 .\end{aligned}$

EXAMPLE $1 \quad$ If $f(x)=4 x^{2}+3 x$ and $g(x)=-5 x-6$, write the formula for the functions $f+g$ and $f-g$.
SOLUTION The function $f+g$ is defined by

$$
\begin{aligned}
(f+g)(x) & =f(x)+g(x) \\
& =\left(4 x^{2}+3 x\right)+(-5 x-6) \\
& =4 x^{2}-2 x-6
\end{aligned}
$$

The function $f-g$ is defined by

$$
\begin{aligned}
(f-g)(x) & =f(x)-g(x) \\
& =\left(4 x^{2}+3 x\right)-(-5 x-6) \\
& =4 x^{2}+3 x+5 x+6 \\
& =4 x^{2}+8 x+6
\end{aligned}
$$

## EXAMPLE $2 \operatorname{Let} f(x)=4 x-3$ and $g(x)=4 x^{2}$. Find $f+g, f g$, and $g / f$.

SOLUTION The function $f+g$, the sum of functions $f$ and $g$, is defined by

$$
\begin{aligned}
(f+g)(x) & =f(x)+g(x) \\
& =(4 x-3)+4 x^{2} \\
& =4 x^{2}+4 x-3
\end{aligned}
$$

The product of the functions $f$ and $g, f g$, is given by

$$
\begin{aligned}
(f g)(x) & =f(x) g(x) \\
& =(4 x-3) 4 x^{2} \\
& =16 x^{3}-12 x^{2}
\end{aligned}
$$

The quotient of the functions $g$ and $f, g / f$, is defined as

$$
\begin{aligned}
\left(\frac{g}{f}\right)(x) & =\frac{g(x)}{f(x)} \\
& =\frac{4 x^{2}}{4 x-3}
\end{aligned}
$$

EXAMPLE 3 Let $f(x)=x+3, g(x)=2 x-1$, and $h(x)=2 x^{2}+5 x-3$.
Evaluate $(f g)(2), h(2), \frac{h(1)}{g(1)}$, and $f(1)$
SOLUTION

$$
\begin{aligned}
(f g)(2) & =f(2) g(2) \\
& =5 \cdot 3 \\
& =15 \\
h(2) & =2 \cdot 2^{2}+5 \cdot 2-3 \\
& =8+10-3 \\
& =15
\end{aligned}
$$

$$
\begin{aligned}
\frac{h(1)}{g(1)} & =\frac{2 \cdot 1^{2}+5 \cdot 1-3}{2 \cdot 1-1} \\
& =\frac{2+5-3}{2-1} \\
& =\frac{4}{1} \\
& =4 \\
f(1) & =1+3 \\
& =4
\end{aligned}
$$

## Composition of Functions

In addition to the four operations used to combine functions shown so far in this section, there is a fifth way to combine two functions to obtain a new function. It is called composition of functions. To illustrate the concept, recall the definition of training heart rate: Training heart rate, in beats per minute, is resting heart rate plus $60 \%$ of the difference between maximum heart rate and resting heart rate. If your resting heart rate is 70 beats per minute, then your training heart rate is a function of your maximum heart rate $M$ :

$$
T(M)=70+0.6(M-70)=70+0.6 M-42=28+0.6 M
$$

But your maximum heart rate is found by subtracting your age in years from 220. So, if $x$ represents your age in years, then your maximum heart rate is

$$
M(x)=220-x
$$

Therefore, if your resting heart rate is 70 beats per minute and your age in years is $x$, then your training heart rate can be written as a function of $x$.

$$
T(x)=28+0.6(220-x)
$$

This last line is the composition of functions $T$ and $M$. We input $x$ into function $M$, which outputs $M(x)$. Then, we input $M(x)$ into function $T$, which outputs $T(M(x)$ ), which is the training heart rate as a function of age $x$. Here is a diagram of the situation, which is called a function map:


FIGURE 2
Now let's generalize the preceding ideas into a formal development of composition of functions. To find the composition of two functions $f$ and $g$, we first require that the range of $g$ have numbers in common with the domain of $f$. Then the composition of $f$ with $g, f \circ g$, is defined this way:

$$
(f \circ g)(x)=f(g(x))
$$

To understand this new function, we begin with a number $x$, and we operate on it with $g$, giving us $g(x)$. Then we take $g(x)$ and operate on it with $f$, giving us $f(g(x))$. The only numbers we can use for the domain of the composition of $f$ with
$g$ are numbers $x$ in the domain of $g$, for which $g(x)$ is in the domain of $f$. The diagrams in Figure 3 illustrate the composition of $f$ with $g$.


FIGURE 3
Composition of functions is not commutative. The composition of $f$ with $g$, $f \circ g$, may therefore be different from the composition of $g$ with $f, g \circ f$.

$$
(g \circ f)(x)=g(f(x))
$$

Again, the only numbers we can use for the domain of the composition of $g$ with $f$ are numbers in the domain of $f$, for which $f(x)$ is in the domain of $g$. The diagrams in Figure 4 illustrate the composition of $g$ with $f$.


FIGURE 4

## EXAMPLE $4 \quad$ If $f(x)=x+5$ and $g(x)=2 x$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

SOLUTION The composition of $f$ with $g$ is

$$
\begin{aligned}
(f \circ g)(x) & =f[g(x)] \\
& =f(2 x) \\
& =2 x+5
\end{aligned}
$$

The composition of $g$ with $f$ is

$$
\begin{aligned}
(g \circ f)(x) & =g[f(x)] \\
& =g(x+5) \\
& =2(x+5) \\
& =2 x+10
\end{aligned}
$$

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A ${ }^{1-3}$
B 4

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How are profit, revenue, and cost related?
2. How do you find the maximum heart rate?
3. For functions $f$ and $g$, how do you find the composition of $f$ with $g$ ?
4. For functions $f$ and $g$, how do you find the composition of $g$ with $f$ ?

## Problem Set 8.5

Online support materials can be found at www.thomsonedu.com/login

Let $f(x)=4 x-3$ and $g(x)=2 x+5$. Write a formula for each of the following functions.

1. $f+g$
2. $f-g$
3. $g-f$
4. $g+f$

If the functions $f, g$, and $h$ are defined by $f(x)=3 x-5$, $g(x)=x-2$, and $h(x)=3 x^{2}$, write a formula for each of the following functions.

| 5. $g+f$ | 6. $f+h$ |
| :--- | :--- |
| 7. $g+h$ | 8. $f-g$ |
| 9. $g-f$ | 10. $h-g$ |
| 11. $f h$ | 12. $g h$ |
| 13. $h / f$ | $14 . h / g$ |
| 15. $f / h$ | $16 . g / h$ |
| 17. $f+g+h$ | $18 . h-g+f$ |

Let $f(x)=2 x+1, g(x)=4 x+2$, and $h(x)=4 x^{2}+4 x+1$, and find the following.
19. $(f+g)(2)$
20. $(f-g)(-1)$
21. $(f g)(3)$
22. $(f / g)(-3)$
23. $(h / g)(1)$
24. (hg)(1)
25. $(f h)(0)$
26. $(h-g)(-4)$
27. $(f+g+h)(2)$
28. $(h-f+g)(0)$
29. $(h+f g)(3)$
30. $(h-f g)(5)$
31. Let $f(x)=x^{2}$ and $g(x)=x+4$, and find
a. $(f \circ g)(5)$
b. $(g \circ f)(5)$
c. $(f \circ g)(x)$
d. $(g \circ f)(x)$
32. Let $f(x)=3-x$ and $g(x)=x^{3}-1$, and find
a. $(f \circ g)(0)$
b. $(g \circ f)(0)$
c. $(f \circ g)(x)$
33. Let $f(x)=x^{2}+3 x$ and $g(x)=4 x-1$, and find
a. $(f \circ g)(0)$
b. $(g \circ f)(0)$
c. $(g \circ f)(x)$
34. Let $f(x)=(x-2)^{2}$ and $g(x)=x+1$, and find the following
a. $(f \circ g)(-1)$
b. $(g \circ f)(-1)$

For each of the following pairs of functions $f$ and $g$, show that $(f \circ g)(x)=(g \circ f)(x)=x$.
35. $f(x)=5 x-4$ and $g(x)=\frac{x+4}{5}$
36. $f(x)=\frac{x}{6}-2$ and $g(x)=6 x+12$

Use the graph to answer problems 37-44.


Evaluate.
37. $f(2)+5$
38. $g(-2)-5$
39. $f(-3)+g(-3)$
40. $f(5)-g(5)$
41. $(f \circ g)(0)$
42. $(g \circ f)(0)$
43. Find $x$ if $f(x)=-3$.
44. Find $x$ if $g(x)=1$.

Use the graph to answer problems 45-52.


Evaluate.
45. $f(-3)+2$
46. $g(3)-3$
47. $f(2)+g(2)$
48. $f(-5)-g(-5)$
49. $(f \circ g)(0)$
50. $(g \circ f)(0)$
51. Find $x$ if $f(x)=1$.
52. Find $x$ if $g(x)=-2$.

## Applying the Concepts

53. Profit, Revenue, and Cost A company manufactures and sells prerecorded videotapes. Here are the equations they use in connection with their business.
Number of tapes sold each day: $n(x)=x$
Selling price for each tape: $p(x)=11.5-0.05 x$

Daily fixed costs: $f(x)=200$
Daily variable costs: $v(x)=2 x$
Find the following functions.
a. Revenue $=R(x)=$ the product of the number of tapes sold each day and the selling price of each tape.
b. Cost $=C(x)=$ the sum of the fixed costs and the variable costs.
c. Profit $=P(X)=$ the difference between revenue and cost.
d. Average cost $=\bar{C}(x)=$ the quotient of cost and the number of tapes sold each day.
54. Profit, Revenue, and Cost A company manufactures and sells diskettes for home computers. Here are the equations they use in connection with their business.

Number of diskettes sold each day: $n(x)=x$
Selling price for each diskette: $p(x)=3-\frac{1}{300} x$
Daily fixed costs: $f(x)=200$
Daily variable costs: $v(x)=2 x$
Find the following functions.
a. Revenue $=R(X)=$ the product of the number of diskettes sold each day and the selling price of each diskette.
b. Cost $=C(x)=$ the sum of the fixed costs and the variable costs.
c. Profit $=P(x)=$ the difference between revenue and cost.
d. Average cost $=\bar{C}(x)=$ the quotient of cost and the number of diskettes sold each day.
55. Training Heart Rate Find the training heart rate function, $T(M)$, for a person with a resting heart rate of 62 beats per minute, then find the following.
a. Find the maximum heart rate function, $M(x)$, for a person $x$ years of age.
b. What is the maximum heart rate for a 24-yearold person?
c. What is the training heart rate for a 24 -year-old person with a resting heart rate of 62 beats per minute?
d. What is the training heart rate for a 36-year-old person with a resting heart rate of 62 beats per minute?
e. What is the training heart rate for a 48-year-old person with a resting heart rate of 62 beats per minute?
56. Training Heart Rate Find the training heart rate function, $T(M)$, for a person with a resting heart rate of 72 beats per minute, then finding the following to the nearest whole number.
a. Find the maximum heart rate function, $M(X)$, for a person $x$ years of age.
b. What is the maximum heart rate for a 20-yearold person?
c. What is the training heart rate for a 20 -year-old person with a resting heart rate of 72 beats per minute?
d. What is the training heart rate for a 30-year-old person with a resting heart rate of 72 beats per minute?
e. What is the training heart rate for a 40-year-old person with a resting heart rate of 72 beats per minute?

## Maintaining Your Skills

Solve each inequality.
57. $|x-3|<1$
58. $|x-3|>1$
59. $|6-x|>2$
60. $\left|1-\frac{1}{2} x\right|>2$
61. $|7 x-1| \leq 6$
62. $|7 x-1| \geq 6$

## Getting Ready for the Next Section

Simplify.
63. $16(3.5)^{2}$
64. $\frac{2,400}{100}$
65. $\frac{180}{45}$
66. $4(2)(4)^{2}$
67. $\frac{0.0005(200)}{(0.25)^{2}}$
68. $\frac{0.2(0.5)^{2}}{100}$
69. If $y=K x$, find $K$ if $x=5$ and $y=15$.
70. If $d=K t^{2}$, find $K$ if $t=2$ and $d=64$.
71. If $V=\frac{K}{P}$, find $K$ if $P=48$ and $V=50$.
72. If $y=K x z^{2}$, find $K$ if $x=5, z=3$, and $y=180$.

## 8.6

## OBJECTIVES

A Set up and solve problems with direct, inverse, or joint variation.


## Variation

If you are a runner and you average $t$ minutes for every mile you run during one of your workouts, then your speed $s$ in miles per hour is given by the equation and graph shown here. Figure 1 is shown in the first quadrant only because both $t$ and $s$ are positive.

$$
s=\frac{60}{t}
$$

| Input <br> $\boldsymbol{t}$ | Output <br> $\boldsymbol{s}$ |
| :---: | :---: |
| 4 | 15.0 |
| 6 | 10.0 |
| 8 | 7.5 |
| 10 | 6.0 |
| 12 | 5.0 |
| 14 | 4.3 |

You know intuitively that as your average time per mile ( $t$ ) increases, your speed (s) decreases. Likewise, lowering your time per mile will increase your speed. The equation and Figure 1 also show this to be true: increasing $t$ decreases $s$, and decreasing $t$ increases $s$. Quantities that are connected in this way are said to vary inversely with each other. Inverse variation is one of the topics we will study in this section.

There are two main types of variation: direct variation and inverse variation. Variation problems are most common in the sciences, particularly in chemistry and physics.

## Direct Variation

When we say the variable $y$ varies directly with the variable $x$, we mean that the relationship can be written in symbols as $y=K x$, where $K$ is a nonzero constant called the constant of variation (or proportionality constant). Another way of saying $y$ varies directly with $x$ is to say $y$ is directly proportional to $x$.

Study the following list. It gives the mathematical equivalent of some direct variation statements.

| English Phrase | Algebraic Equation |
| :--- | :---: |
| $y$ varies directly with $x$ | $y=K X$ |
| $S$ varies directly with the square of $t$ | $S=K t^{2}$ |
| $y$ is directly proportional to the cube of $z$ | $y=K Z^{3}$ |
| $u$ is directly proportional to the square root of $v$ | $u=K \sqrt{v}$ |

## EXAMPLE 1

$y$ varies directly with $x$. If $y$ is 15 when $x$ is 5 , find $y$ when
$x$ is 7 .
SOLUTION The first sentence gives us the general relationship between $x$ and $y$. The equation equivalent to the statement " $y$ varies directly with $x$ " is

$$
y=K x
$$

The first part of the second sentence in our example gives us the information necessary to evaluate the constant $K$ :

|  | When | $y$ | $=15$ |
| :--- | ---: | :--- | ---: | :--- |
|  | and | $x$ | $=5$ |
|  | the equation | $y$ | $=K x$ |
|  | becomes | 15 | $=K \cdot 5$ |
|  | or | $K$ | $=3$ |

The equation now can be written specifically as

$$
y=3 x
$$

Letting $x=7$, we have

$$
\begin{aligned}
& y=3 \cdot 7 \\
& y=21
\end{aligned}
$$

EXAMPLE 2 A skydiver jumps from a plane. As with any object that falls toward Earth, the distance the skydiver falls is directly proportional to the square of the time he has been falling until he reaches his terminal velocity. If the skydiver falls 64 feet in the first 2 seconds of the jump, then
a. How far will he have fallen after 3.5 seconds?
b. Graph the relationship between distance and time.
c. How long will it take him to fall 256 feet?

SOLUTION We let $t$ represent the time the skydiver has been falling. Then we can let $d(t)$ represent the distance he has fallen.
a. Because $d(t)$ is directly proportional to the square of $t$, we have the general function that describes this situation:

$$
d(t)=K t^{2}
$$

Next, we use the fact that $d(2)=64$ to find $K$.

$$
\begin{aligned}
64 & =K\left(2^{2}\right) \\
K & =16
\end{aligned}
$$

The specific equation that describes this situation is

$$
d(t)=16 t^{2}
$$

To find how far a skydiver will have fallen after 3.5 seconds, we find $d$ (3.5):

$$
\begin{aligned}
& d(3.5)=16\left(3.5^{2}\right) \\
& d(3.5)=196
\end{aligned}
$$

A skydiver will have fallen 196 feet after 3.5 seconds.
b. To graph this equation, we use a table:

| Input <br> $\boldsymbol{t}$ | Output <br> $\boldsymbol{d}(\boldsymbol{t})$ |
| :---: | :---: |
| 0 | 0 |
| 1 | 16 |
| 2 | 64 |
| 3 | 144 |
| 4 | 256 |
| 5 | 400 |
|  |  |



FIGURE 2
c. From the table or the graph (Figure 2), we see that it will take 4 seconds for the skydiver to fall 256 feet.

## Inverse Variation

Running From the introduction to this section, we know that the relationship between the number of minutes $(t)$ it takes a person to run mile and his or her average speed in miles per hour ( $s$ ) can be described with the following equation, table, and Figure 3.

$$
\left.S=\begin{array}{c}
\frac{60}{t} \\
\end{array} \begin{array}{cc}
\text { Input } \\
\boldsymbol{t}
\end{array} \quad \begin{array}{c}
\text { Output } \\
\boldsymbol{s}
\end{array}\right]
$$



FIGURE 3

If $t$ decreases, then $s$ will increase, and if $t$ increases, then $s$ will decrease. The variable $s$ is inversely proportional to the variable $t$. In this case, the constant of proportionality is 60 .

Photography If you are familiar with the terminology and mechanics associated with photography, you know that the $f$-stop for a particular lens will increase as the aperture (the maximum diameter of the opening of the lens) decreases. In mathematics we say that $f$-stop and aperture vary inversely with each other. The diagram illustrates this relationship.


If $f$ is the $f$-stop and $d$ is the aperture, then their relationship can be written

$$
f=\frac{K}{d}
$$

In this case, $K$ is the constant of proportionality. (Those of you familiar with photography know that $K$ is also the focal length of the camera lens.)

In General We generalize this discussion of inverse variation as follows: If $y$ varies inversely with $x$, then

$$
y=K \frac{1}{x} \quad \text { or } \quad y=\frac{K}{x}
$$

We can also say $y$ is inversely proportional to $x$. The constant $K$ is again called the constant of variation or proportionality constant.

| English Phrase | Algebraic Equation |
| :--- | ---: |
| $y$ is inversely proportional to $x$ | $y=\frac{K}{X}$ |
| $S$ varies inversely with the square of $t$ | $S=\frac{K}{t^{2}}$ |
| $y$ is inversely proportional to $x^{4}$ | $y=\frac{K}{x^{4}}$ |
| $z$ varies inversely with the cube root of $t$ | $z=\frac{K}{\sqrt[3]{t}}$ |

EXAMPLE 3 The volume of a gas is inversely proportional to the pressure of the gas on its container. If a pressure of 48 pounds per square inch corresponds to a volume of 50 cubic feet, what pressure is needed to produce a volume of 100 cubic feet?

## Note

The relationship between pressure and volume as given in this example is known as Boyle's law and applies to situations such as those encountered in a piston-cylinder arrangement. It was Robert Boyle (1627-1691) who, in 1662, published the results of some of his experiments that showed, among other things, that the volume of a gas decreases as the pressure increases. This is an example of inverse variation.

SOLUTION We can represent volume with $V$ and pressure with $P$ :

$$
V=\frac{K}{P}
$$

Using $P=48$ and $V=50$, we have

$$
\begin{aligned}
& 50=\frac{K}{48} \\
& K=50(48) \\
& K=2,400
\end{aligned}
$$

The equation that describes the relationship between $P$ and $V$ is

$$
V=\frac{2,400}{P}
$$

Here is a graph of this relationship.


Substituting $V=100$ into our last equation, we get

$$
\begin{aligned}
100 & =\frac{2,400}{P} \\
100 P & =2,400 \\
P & =\frac{2,400}{100} \\
P & =24
\end{aligned}
$$

A volume of 100 cubic feet is produced by a pressure of 24 pounds per square inch.

## Joint Variation and Other Variation Combinations

Many times relationships among different quantities are described in terms of more than two variables. If the variable $y$ varies directly with two other variables, say $x$ and $z$, then we say $y$ varies jointly with $x$ and $z$. In addition to joint variation, there are many other combinations of direct and inverse variation involving more than two variables. The following table is a list of some variation statements and their equivalent mathematical forms:

| English Phrase | Algebraic Equation |
| :--- | :---: |
| $y$ varies jointly with $x$ and $z$ | $y=K x z$ |
| $Z$ varies jointly with $r$ and the square of $s$ | $Z=K r s^{2}$ |
| $V$ is directly proportional to $T$ and inversely proportional to $P$ | $V=\frac{K T}{P}$ |
| $F$ varies jointly with $m_{1}$ and $m_{2}$ and inversely with the square of $r$ | $F=\frac{K m_{1} m_{2}}{r^{2}}$ |

EXAMPLE $4 \quad y$ varies jointly with $x$ and the square of $z$. When $x$ is 5 and $z$ is $3, y$ is 180 . Find $y$ when $x$ is 2 and $z$ is 4 .
SOLUTION The general equation is given by

$$
y=K x z^{2}
$$

Substituting $x=5, z=3$, and $y=180$, we have

$$
\begin{aligned}
180 & =K(5)(3)^{2} \\
180 & =45 K \\
K & =4
\end{aligned}
$$

The specific equation is

$$
y=4 x z^{2}
$$

When $x=2$ and $z=4$, the last equation becomes

$$
\begin{aligned}
& y=4(2)(4)^{2} \\
& y=128
\end{aligned}
$$

EXAMPLE 5 In electricity, the resistance of a cable is directly proportional to its length and inversely proportional to the square of the diameter. If a 100 -foot cable 0.5 inch in diameter has a resistance of 0.2 ohm , what will be the resistance of a cable made from the same material if it is 200 feet long with a diameter of 0.25 inch?
SOLUTION Let $R=$ resistance, $l=$ length, and $d=$ diameter. The equation is

$$
R=\frac{K l}{d^{2}}
$$

When $R=0.2, l=100$, and $d=0.5$, the equation becomes

$$
0.2=\frac{K(100)}{(0.5)^{2}}
$$

or

$$
K=0.0005
$$

Using this value of $K$ in our original equation, the result is

$$
R=\frac{0.00051}{d^{2}}
$$

When $l=200$ and $d=0.25$, the equation becomes

$$
\begin{aligned}
& R=\frac{0.0005(200)}{(0.25)^{2}} \\
& R=1.6 \mathrm{ohms}
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $1-5$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Give an example of a direct variation statement, and then translate it into symbols.
2. Translate the equation $y=\frac{k}{x}$ into words.
3. For the inverse variation equation $y=\frac{3}{x}$, what happens to the values of $y$ as $\times$ gets larger?
4. How are direct variation statements and linear equations in two variables related?

## Problem Set 8.6

Online support materials can be found at www.thomsonedu.com/login

Describe each relationship below as direct or inverse.

1. The number of homework problems assigned each night and the time it takes to complete the assignment
2. The number of years of school and starting salary
3. The length of the line for an amusement park ride and the number of people in line
4. The age of a home computer and its relative speed
5. The age of a bottle of wine and its price
6. The speed of an Internet connection and the time to download a song
7. The length of a song and the time it takes to download
8. The speed of a rollercoaster and your willingness to ride it
9. The diameter of a Ferris wheel and circumference
10. The measure angle $\alpha$ to the measure of angle $\beta$ if $\alpha+\beta=90^{\circ}(\alpha=0)$
11. The altitude above sea level and the air temperature
12. The uncommonness of a search term and the number of search results

For the following problems, $y$ varies directly with $x$.
13. If $y$ is 10 when $x$ is 2 , find $y$ when $x$ is 6 .
14. If $y$ is 20 when $x$ is 5 , find $y$ when $x$ is 3 .
15. If $y$ is -32 when $x$ is 4 , find $x$ when $y$ is -40 .
16. If $y$ is -50 when $x$ is 5 , find $x$ when $y$ is -70 .

For the following problems, $r$ is inversely proportional to $s$.
17. If $r$ is -3 when $s$ is 4 , find $r$ when $s$ is 2 .
18. If $r$ is -10 when $s$ is 6 , find $r$ when $s$ is -5 .
19. If $r$ is 8 when $s$ is 3 , find $s$ when $r$ is 48 .
20. If $r$ is 12 when $s$ is 5 , find $s$ when $r$ is 30 .

For the following problems, $d$ varies directly with the square of $r$.
21. If $d=10$ when $r=5$, find $d$ when $r=10$.
22. If $d=12$ when $r=6$, find $d$ when $r=9$.
23. If $d=100$ when $r=2$, find $d$ when $r=3$.
24. If $d=50$ when $r=5$, find $d$ when $r=7$.

For the following problems, $y$ varies inversely with the absolute value of $x$.
25. If $y=6$ when $x=3$, find $y$ when $x=9$.
26. If $y=6$ when $x=-3$, find $y$ when $x=-9$.
27. If $y=20$ when $x=-5$, find $y$ when $x=10$.
28. If $y=20$ when $x=5$, find $y$ when $x=10$.

For the following problems, $y$ varies inversely with the square of $x$.
29. If $y=45$ when $x=3$, find $y$ when $x$ is 5 .
30. If $y=12$ when $x=2$, find $y$ when $x$ is 6 .
31. If $y=18$ when $x=3$, find $y$ when $x$ is 2 .
32. If $y=45$ when $x=4$, find $y$ when $x$ is 5 .

For the following problems, $z$ varies jointly with $x$ and the square of $y$.
33. If $z$ is 54 when $x$ and $y$ are 3 , find $z$ when $x=2$ and $y=4$.
34. If $z$ is 80 when $x$ is 5 and $y$ is 2 , find $z$ when $x=2$ and $y=5$.
35. If $z$ is 64 when $x=1$ and $y=4$, find $x$ when $z=32$ and $y=1$.
36. If $z$ is 27 when $x=6$ and $y=3$, find $x$ when $z=50$ and $y=4$.

## Applying the Concepts

37. Length of a Spring The length a spring stretches is directly proportional to the force applied. If a force of 5 pounds stretches a spring 3 inches, how much force is necessary to stretch the same spring 10 inches?

38. Weight and Surface Area The weight of a certain material varies directly with the surface area of that material. If 8 square feet weighs half a pound, how much will 10 square feet weigh?
39. Pressure and Temperature The temperature of a gas varies directly with its pressure. A temperature of 200 K produces a pressure of 50 pounds per square inch.
a. Find the equation that relates pressure and temperature.
b. Graph the equation from part (a) in the first quadrant only.
c. What pressure will the gas have at $280^{\circ} \mathrm{K}$ ?
40. Circumference and Diameter The circumference of a wheel is directly proportional to its diameter. A wheel has a circumference of 8.5 feet and a diameter of 2.7 feet.
a. Find the equation that relates circumference and diameter.
b. Graph the equation from part (a) in the first quadrant only.
c. What is the circumference of a wheel that has a diameter of 11.3 feet?
41. Volume and Pressure The volume of a gas is inversely proportional to the pressure. If a pressure of 36 pounds per square inch corresponds to a volume of 25 cubic feet, what pressure is needed to produce a volume of 75 cubic feet?
42. Wave Frequency The frequency of an electromagnetic wave varies inversely with the wavelength. If a wavelength of 200 meters has a frequency of 800 kilocycles per second, what frequency will be associated with a wavelength of 500 meters?
43. f-Stop and Aperture Diameter The relative aperture or $f$-stop for a camera lens is inversely proportional to the diameter of the aperture. An $f$-stop of 2 corresponds to an aperture diameter of 40 millimeters for the lens on an automatic camera.
a. Find the equation that relates $f$-stop and diameter.
b. Graph the equation from part (a) in the first quadrant only.
c. What is the $f$-stop of this camera when the aperture diameter is 10 millimeters?
44. f-Stop and Aperture Diameter The relative aperture or $f$-stop for a camera lens is inversely proportional to the diameter of the aperture. An $f$-stop of 2.8
corresponds to an aperture diameter of 75 millimeters for a certain telephoto lens.
a. Find the equation that relates $f$-stop and diameter.
b. Graph the equation from part (a) in the first quadrant only.
c. What aperture diameter corresponds to an $f$-stop of 5.6 ?
45. Surface Area of a Cylinder The surface area of a hollow cylinder varies jointly with the height and radius of the cylinder. If a cylinder with radius 3 inches and height 5 inches has a surface area of 94 square inches, what is the surface area of a cylinder with radius 2 inches and height 8 inches?
46. Capacity of a Cylinder The capacity of a cylinder varies jointly with its height and the square of its radius. If a cylinder with a radius of 3 centimeters and a height of 6 centimeters has a capacity of 169.56 cubic centimeters, what will be the capacity of a cylinder with radius 4 centimeters and height 9 centimeters?
47. Electrical Resistance The resistance of a wire varies directly with its length and inversely with the square of its diameter. If 100 feet of wire with diameter 0.01 inch has a resistance of 10 ohms, what is the resistance of 60 feet of the same type of wire if its diameter is 0.02 inch?
48. Volume and Temperature The volume of a gas varies directly with its temperature and inversely with the pressure. If the volume of a certain gas is 30 cubic feet at a temperature of 300 K and a pressure of 20 pounds per square inch, what is the volume of the same gas at 340 K when the pressure is 30 pounds per square inch?
49. Music A musical tone's pitch varies inversely with its wavelength. If one tone has a pitch of 420 vibrations each second and a wavelength of 2.2 meters, find the wavelength of a tone that has a pitch of 720 vibrations each second.
50. Hooke's Law Hooke's law states that the stress (force per unit area) placed on a solid object varies directly with the strain (deformation) produced.
a. Using the variables $S_{1}$ for stress and $S_{2}$ for strain, state this law in algebraic form.
b. Find the constant, $K$, if for one type of material $S_{1}=24$ and $S_{2}=72$.
51. Gravity In Book Three of his Principia, Isaac Newton (depicted on the postage stamp) states that there is a single force in the universe that holds everything together, called the force of universal gravity.
Newton stated that the force of universal gravity, $F$, is directly proportional
 with the product of two masses, $m_{1}$ and $m_{2}$, and inversely proportional with the square of the distance $d$ between them. Write the equation for Newton's force of universal gravity, using the symbol $G$ as the constant of proportionality.
52. Boyle's Law and Charles's Law Boyle's law states that for low pressures, the pressure of an ideal gas kept at a constant temperature varies inversely with the volume of the gas. Charles's law states that for low pressures, the density of an ideal gas kept at a constant pressure varies inversely with the absolute temperature of the gas.
a. State Boyle's law as an equation using the symbols $P, K$, and $V$.
b. State Charles's law as an equation using the symbols $D, K$, and $T$.

## Maintaining Your Skills

Solve the following equations.
53. $x-5=7$
54. $3 y=-4$
55. $5-\frac{4}{7} a=-11$
56. $\frac{1}{5} x-\frac{1}{2}-\frac{1}{10} x+\frac{2}{5}=\frac{3}{10} x+\frac{1}{2}$
57. $5(x-1)-2(2 x+3)=5 x-4$
58. $0.07-0.02(3 x+1)=-0.04 x+0.01$

Solve for the indicated variable.
59. $P=2 l+2 w$ for $w$
60. $A=\frac{1}{2} h(b+B)$ for $B$

Solve the following inequalities. Write the solution set using interval notation, then graph the solution set.
61. $-5 t \leq 30$
62. $5-\frac{3}{2} x>-1$
63. $1.6 x-2<0.8 x+2.8$
64. $3(2 y+4) \geq 5(y-8)$

Solve the following equations.
65. $\left|\frac{1}{4} x-1\right|=\frac{1}{2}$
66. $\left|\frac{2}{3} a+4\right|=6$
67. $|3-2 x|+5=2$
68. $5=|3 y+6|-4$

Solve each inequality and graph the solution set.
69. $\left|\frac{x}{5}+1\right| \geq \frac{4}{5}$
70. $|2-6 t|<-5$
71. $|3-4 t|>-5$
72. $|6 y-1|-4 \leq 2$

## Extending the Concepts

73. Human Cannonball A circus company is deciding where to position the net for the human cannonball so that he will land safely during the act. They do this by firing a 100-pound sack of potatoes out of the cannon at different speeds and then measuring how far from the cannon the sack lands. The results are shown in the table.

| Speed in <br> Miles/Hour | Distance in Feet |
| :---: | :---: |
| 40 | 108 |
| 50 | 169 |
| 60 | 243 |
| 70 | 331 |

a. Does distance vary directly with the speed, or directly with the square of the speed?
b. Write the equation that describes the relationship between speed and distance.
c. If the cannon will fire a human safely at 55 miles/hour, where should they position the net so the cannonball has a safe landing?
d. How much farther will he land if his speed out of the cannon is 56 miles/hour?

## Chapter 8 SUMMARYIX <br> Linear Equations in Two Variables [8.1, 8.3] <br> \section*{EXAMPLES}

1. The equation $3 x+2 y=6$ is an example of a linear equation in two variables.

A linear equation in two variables is any equation that can be put in standard form $a x+b y=c$. The graph of every linear equation is a straight line.

## Intercepts [8.1]

The $x$-intercept of an equation is the $x$-coordinate of the point where the graph crosses the $x$-axis. The $y$-intercept is the $y$-coordinate of the point where the graph crosses the $y$-axis. We find the $y$-intercept by substituting $x=0$ into the equation and solving for $y$. The $x$-intercept is found by letting $y=0$ and solving for $x$.

## The Slope of a Line [8.1]

The slope of the line containing points $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ is given by

$$
\text { Slope }=m=\frac{\text { Rise }}{\text { Run }}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
$$

Horizontal lines have 0 slope, and vertical lines have no slope.
Parallel lines have equal slopes, and perpendicular lines have slopes that are negative reciprocals.

## The Slope Intercept Form of a Line [8.2]

The equation of a line with slope $m$ and $y$-intercept $b$ is given by

$$
y=m x+b
$$

## The Point Slope Form of a Line [8.2]

The equation of the line through $\left(x_{1}, y_{1}\right)$ that has slope $m$ can be written as

$$
y-y_{1}=m\left(x-x_{1}\right)
$$

## Relations and Functions [8.3]

6. The relation

$$
\{(8,1),(6,1),(-3,0)\}
$$

is also a function because no ordered pairs have the same first coordinates. The domain is $\{8,6,-3\}$ and the range is $\{1,0\}$.

A function is a rule that pairs each element in one set, called the domain, with exactly one element from a second set, called the range.

A relation is any set of ordered pairs. The set of all first coordinates is called the domain of the relation, and the set of all second coordinates is the range of the relation. A function is a relation in which no two different ordered pairs have the same first coordinates.

## Vertical Line Test [8.4]

If a vertical line crosses the graph of a relation in more than one place, the relation cannot be a function. If no vertical line can be found that crosses the graph in more than one place, the relation must be a function.

## Function Notation [8.4]

The alternative notation for $y$ is $f(x)$. It is read " $f$ of $x$ " and can be used instead of the variable $y$ when working with functions. The notation $y$ and the notation $f(x)$ are equivalent; that is, $y=f(x)$.

## Algebra with Functions [8.5]

If $f$ and $g$ are any two functions with a common domain, then:
$(f+g)(x)=f(x)+g(x) \quad$ The function $f+g$ is the sum of the functions $f$ and $g$.
$(f-g)(x)=f(x)-g(x) \quad$ The function $f-g$ is the difference of the functions $f$ and $g$.
$(f g)(x)=f(x) g(x) \quad$ The function $f g$ is the product of the functions $f$ and $g$.
$\frac{f}{g}(x)=\frac{f(x)}{g(x)}$ The function $\frac{f}{g}$ is the quotient of the functions $f$ and $g$, where $g(x) \neq 0$

## Composition of Functions [8.5]

10. If $f(x)=4 x$ and $g(x)=x^{2}-3$, then

$$
\begin{aligned}
(f \circ g)(x) & =f[g(x)]=f\left(x^{2}-3\right) \\
& =4 x^{2}-12 \\
(g \circ f)(x) & =g[f(x)]=g(4 x) \\
& =16 x^{2}-3
\end{aligned}
$$

11. If $y$ varies directly with $x$, then

$$
y=K x
$$

Then if $y$ is 18 when $x$ is 6 ,

$$
18=K \cdot 6
$$

or

$$
K=3
$$

So the equation can be writ ten more specifically as

$$
y=3 x
$$

If we want to know what $y$ is when $x$ is 4 , we simply substitute:

$$
\begin{aligned}
& y=3 \cdot 4 \\
& y=12
\end{aligned}
$$

If $f$ and $g$ are two functions for which the range of each has numbers in common with the domain of the other, then we have the following definitions:

The composition of $f$ with $g:(f \circ g)(x)=f[g(x)]$
The composition of $g$ with $f:(g \circ f)(x)=g[f(x)]$

## Variation [8.6]

If $y$ varies directly with $x$ ( $y$ is directly proportional to $x$ ), then

$$
y=K x
$$

If $y$ varies inversely with $x$ ( $y$ is inversely proportional to $x$ ), then

$$
y=\frac{K}{x}
$$

If $z$ varies jointly with $x$ and $y$ ( $z$ is directly proportional to both $x$ and $y$ ), then

$$
z=K x y
$$

In each case, $K$ is called the constant of variation.

## COMMON MISTAKES

1. When graphing ordered pairs, the most common mistake is to associate the first coordinate with the $y$-axis and the second with the $x$-axis. If you make this mistake you would graph $(3,1)$ by going up 3 and to the right 1 , which is just the reverse of what you should do. Remember, the first coordinate is always associated with the horizontal axis, and the second coordinate is always associated with the vertical axis.
2. The two most common mistakes students make when first working with the formula for the slope of a line are the following:
a. Putting the difference of the $x$-coordinates over the difference of the $y$ coordinates.
b. Subtracting in one order in the numerator and then subtracting in the opposite order in the denominator.

## Chapter 8 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

Graph each line. [8.1]

1. $3 x+2 y=6$
2. $y=-\frac{3}{2} x+1$
3. $x=3$

Find the slope of the line through the following pairs of points. [8.1]
4. $(5,2),(3,6)$
5. $(-4,2),(3,2)$

Find $x$ if the line through the two given points has the given slope. [8.1]
6. $(4, x),(1,-3) ; m=2$
7. $(-4,7),(2, x) ; m=-\frac{1}{3}$
8. Find the slope of any line parallel to the line through $(3,8)$ and $(5,-2)$. [8.1]
9. The line through $(5,3 y)$ and $(2, y)$ is parallel to a line with slope 4 . What is the value of $y$ ? [8.1]

Give the equation of the line with the following slope and $y$-intercept. [8.2]
10. $m=3, b=5$
11. $m=-2, b=0$

Give the slope and $y$-intercept of each equation. [8.2]
12. $3 x-y=6$
13. $2 x-3 y=9$

Find the equation of the line that contains the given point and has the given slope. [8.2]
14. $(2,4), m=2$
15. $(-3,1), m=-\frac{1}{3}$

Find the equation of the line that contains the given pair of points. [8.2]
16. $(2,5),(-3,-5)$
17. $(-3,7),(4,7)$
18. $(-5,-1),(-3,-4)$
19. Find the equation of the line that is parallel to $2 x-y=4$ and contains the point $(2,-3)$. [8.2]
20. Find the equation of the line perpendicular to $y=-3 x+1$ that has an $x$-intercept of 2. [8.2]

State the domain and range of each relation, and then indicate which relations are also functions. [8.3]
21. $\{(2,4),(3,3),(4,2)\}$
22. $\{(6,3),(-4,3),(-2,0)\}$

If $f=\left\{(2,-1),(-3,0),\left(4, \frac{1}{2}\right),(\pi, 2)\right\}$ and $g=\{(2,2)$, $(-1,4),(0,0)\}$, find the following. [8.4]
23. $f(\pi)$
24. $f(-3)+g(-1)$
25. $f(-3)$
26. $f(2)+g(2)$

Let $f(x)=2 x^{2}-4 x+1$ and $g(x)=3 x+2$, and evaluate each of the following. [8.4]
27. $f(0)$
28. $g(a)$
29. $f[g(0)]$
30. $f[g(1)]$

For the following problems, $y$ varies directly with $x$. [8.6]
31. If $y$ is 6 when $x$ is 2 , find $y$ when $x$ is 8 .
32. If $y$ is -3 when $x$ is 5 , find $y$ when $x$ is -10 .

For the following problems, $y$ varies inversely with the square of $x$. [8.6]
33. If $y$ is 9 when $x$ is 2 , find $y$ when $x$ is 3 .
34. If $y$ is 4 when $x$ is 5 , find $y$ when $x$ is 2 .

Solve each application problem. [8.6]
35. Tension in a Spring The tension $t$ in a spring varies directly with the distance $d$ the spring is stretched. If the tension is 42 pounds when the spring is stretched 2 inches, find the tension when the spring is stretched twice as far.
36. Light Intensity The intensity of a light source varies inversely with the square of the distance from the source. Four feet from the source the intensity is 9 foot-candles. What is the intensity 3 feet from the source?

## Chapter 8 Projects

## Equaiotns and Inequalities in Two Variables

## GROUP PROJECT Light Intensity

## Number of People 2-3

Time Needed 15 minutes
Equipment Paper and pencil
Background I found the following diagram while shopping for some track lighting for my home. I was impressed by the diagram because it displays a lot of
useful information in a very efficient manner. As the diagram indicates, the amount of light that falls on a surface depends on how far above the surface the light is placed and how much the light spreads out on the surface. Assume that this light illuminates a circle on a flat surface, and work the following problems.

b. Construct line graphs from the data in the tables.
c. Which of the relationships is direct variation, and which is inverse variation?
d. Let $F$ represent the number of foot-candles that fall on the surface, $h$ the distance the light source is above the surface, and $A$ the area of the illuminated region. Write an equation that shows the relationship between $A$ and $h$, then write another equation that gives the relationship between $F$ and $h$.

## RESEARCH PROJECT

## Descartes and Pascal



In this chapter, we mentioned that René Descartes, the inventor of René Descartes, 1596-1650


Blaise Pascal, 1623-1662 the rectangular coordinate system, is the person who made the statement, "I think, therefore, I am." Blaise Pascal, another French philosopher, is responsible for the statement, "The heart has its reasons which reason does not know." Although Pascal and Descartes were contemporaries, the philosophies of the two men differed greatly. Research the philosophy of both Descartes and Pascal, and then write an essay that gives the main points of each man's philosophy. In the essay, show how the quotations given here fit in with the philosophy of the man responsible for the quotation.

## Rational Exponents and Roots

## 5



E cology and conservation are topics that interest most college students. If our rivers and oceans are to be preserved for future generations, we need to work to eliminate pollution from our waters. If a river is flowing at 1 meter per second and a pollutant is entering the river at a constant rate, the shape of the pollution plume can often be modeled by the simple equation

$$
y=\sqrt{x}
$$

The following table and graph were produced from the equation.

| Width of a Pollutant Plume |  |
| :---: | :---: |
| Distance from <br> Source (meters) <br> $x$ | Width of <br> Plume (meters) <br> $y$ |
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |
| 9 | 3 |
| 16 | 4 |



FIGURE 1

To visualize how Figure 1 models the pollutant plume, imagine that the river is flowing from left to right, parallel to the $x$-axis, with the $x$-axis as one of its banks. The pollutant is entering the river from the bank at $(0,0)$.

By modeling pollution with mathematics, we can use our knowledge of mathematics to help control and eliminate pollution.

## CHAPTER OUTLINE

9.1 Rational Exponents
9.2 More Expressions Involving Rational Exponents
9.3 Simplified Form for Radicals
9.4 Addition and Subtraction of Radical Expressions
9.5 Multiplication and Division of Radical Expressions
9.6 Equations with Radicals
9.7 Complex Numbers

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## Rational Exponents

## OBJECTIVES

A
Simplify radical expressions using the definition for roots.

B Simplify expressions with rational exponents.

## Note

It is a common mistake to assume that an expression like $\sqrt{25}$ indicates both square roots, 5 and -5 . The expression $\sqrt{25}$ indicates only the positive square root of 25 , which is 5 . If we want the negative square root, we must use a negative sign: $-\sqrt{25}=-5$.

Figure 1 shows a square in which each of the four sides is 1 inch long. To find the square of the length of the diagonal $c$, we apply the Pythagorean theorem:


Because we know that $c$ is positive and that its square is 2 , we call $c$ the positive square root of 2 , and we write $c=\sqrt{2}$. Associating numbers, such as $\sqrt{2}$, with the diagonal of a square or rectangle allows us to analyze some interesting items from geometry. One particularly interesting geometric object that we will study in this section is shown in Figure 2. It is constructed from a right triangle, and the length of the diagonal is found from the Pythagorean theorem. We will come back to this figure at the end of this section.


The Golden Rectangle
FIGURE 2
Previously, we developed notation (exponents) to give us the square, cube, or any other power of a number. For instance, if we wanted the square of 3 , we wrote $3^{2}=9$. If we wanted the cube of 3 , we wrote $3^{3}=27$. In this section, we will develop notation that will take us in the reverse direction-that is, from the square of a number, say 25 , back to the original number, 5 .

DEFINITION If $x$ is a nonnegative real number, then the expression $\sqrt{x}$ is called the positive square root of $x$ and is the nonnegative number such that

$$
(\sqrt{x})^{2}=x
$$

In words: $\sqrt{x}$ is the nonnegative number we square to get $x$.

The negative square root of $x,-\sqrt{x}$, is defined in a similar manner.

EXAMPLE 1 The positive square root of 64 is 8 because 8 is the positive number with the property $8^{2}=64$. The negative square root of 64 is -8 since -8 is the negative number whose square is 64 . We can summarize both of these facts by saying

$$
\sqrt{64}=8 \quad \text { and } \quad-\sqrt{64}=-8
$$

## Note

We have restricted the even roots in this definition to nonnegative numbers. Even roots of negative numbers exist but are not represented by real numbers; that is, $\sqrt{-4}$ is not a real number since there is no real number whose square is -4 .

The higher roots, cube roots, fourth roots, and so on are defined by definitions similar to that of square roots.

DEFINITION If $x$ is a real number and $n$ is a positive integer, then
Positive square root of $x, \sqrt{x}$, is such that $(\sqrt{x})^{2}=x \quad x \geq 0$
Cube root of $x, \sqrt[3]{x}$, is such that $(\sqrt[3]{x})^{3}=x$
Positive fourth root of $x, \sqrt[4]{x}$, is such that $(\sqrt[4]{x})^{4}=x \quad x \geq 0$
Fifth root of $x, \sqrt[5]{x}$, is such that $(\sqrt[5]{x})^{5}=x$

The $n$th root of $x, \sqrt[n]{x}$, is such that $(\sqrt[n]{x})^{n}=x \quad x \geq 0$ if $n$ is even

The following is a table of the most common roots used in this book. Any of the roots that are unfamiliar should be memorized.

| Square Roots |  | Cube Roots | Fourth Roots |
| ---: | ---: | ---: | ---: |
| $\sqrt{0}=0$ | $\sqrt{49}=7$ | $\sqrt[3]{0}=0$ | $\sqrt[4]{0}=0$ |
| $\sqrt{1}=1$ | $\sqrt{64}=8$ | $\sqrt[3]{1}=1$ | $\sqrt[4]{1}=1$ |
| $\sqrt{4}=2$ | $\sqrt{81}=9$ | $\sqrt[3]{8}=2$ | $\sqrt[4]{16}=2$ |
| $\sqrt{9}=3$ | $\sqrt{100}=10$ | $\sqrt[3]{27}=3$ |  |
| $\sqrt{16}=4$ | $\sqrt{121}=11$ | $\sqrt[3]{64}=4$ |  |
| $\sqrt{25}=5$ | $\sqrt{144}=12$ | $\sqrt[3]{125}=5$ |  |
| $\sqrt{36}=6$ | $\sqrt{169}=13$ |  |  |

Notation An expression like $\sqrt[3]{8}$ that involves a root is called a radical expression. In the expression $\sqrt[3]{8}$, the ${ }^{3}$ is called the index, the $\sqrt{ }$ is the radical sign, and 8 is called the radicand. The index of a radical must be a positive integer greater than 1. If no index is written, it is assumed to be 2 .

## Roots and Negative Numbers

When dealing with negative numbers and radicals, the only restriction concerns negative numbers under even roots. We can have negative signs in front of radicals and negative numbers under odd roots and still obtain real numbers. Here are some examples to help clarify this. In the last section of this chapter we will see how to deal with even roots of negative numbers.

## EXAMPLES Simplify each expression, if possible.

2. $\sqrt[3]{-8}=-2$ because $(-2)^{3}=-8$.
3. $\sqrt{-4}$ is not a real number since there is no real number whose square is -4 .
4. $-\sqrt{25}=-5$ is the negative square root of 25 .
5. $\sqrt[5]{-32}=-2$ because $(-2)^{5}=-32$.
6. $\sqrt[4]{-81}$ is not a real number since there is no real number we can raise to the fourth power and obtain -81 .

## Variables Under a Radical

From the preceding examples it is clear that we must be careful that we do not try to take an even root of a negative number. For this reason, we will assume that all variables appearing under a radical sign represent nonnegative numbers.

EXAMPLES Assume all variables represent nonnegative numbers and simplify each expression as much as possible.
7. $\sqrt{25 a^{4} b^{6}}=5 a^{2} b^{3}$ because $\left(5 a^{2} b^{3}\right)^{2}=25 a^{4} b^{6}$.
8. $\sqrt[3]{x^{6} y^{12}}=x^{2} y^{4}$ because $\left(x^{2} y^{4}\right)^{3}=x^{6} y^{12}$.
9. $\sqrt[4]{81 r^{8} S^{20}}=3 r^{2} S^{5}$ because $\left(3 r^{2} S^{5}\right)^{4}=81 r^{8} S^{20}$.

## Rational Numbers as Exponents

Next we develop a second kind of notation involving exponents that will allow us to designate square roots, cube roots, and so on in another way.

Consider the equation $x=8^{1 / 3}$. Although we have not encountered fractional exponents before, let's assume that all the properties of exponents hold in this case. Cubing both sides of the equation, we have

$$
\begin{aligned}
& x^{3}=\left(8^{1 / 3}\right)^{3} \\
& x^{3}=8^{(1 / 3)(3)} \\
& x^{3}=8^{1} \\
& x^{3}=8
\end{aligned}
$$

The last line tells us that $x$ is the number whose cube is 8 . It must be true, then, that $x$ is the cube root of $8, x=\sqrt[3]{8}$. Since we started with $x=8^{1 / 3}$, it follows that

$$
8^{1 / 3}=\sqrt[3]{8}
$$

It seems reasonable, then, to define fractional exponents as indicating roots. Here is the formal definition.

DEFINITION If $x$ is a real number and $n$ is a positive integer greater than 1, then

$$
x^{1 / n}=\sqrt[n]{x} \quad(x \geq 0 \text { when } n \text { is even })
$$

In words: The quantity $x^{1 / n}$ is the $n$th root of $x$.

With this definition we have a way of representing roots with exponents. Here are some examples.

EXAMPLES Write each expression as a root and then simplify, if possible.
10. $8^{1 / 3}=\sqrt[3]{8}=2$
11. $36^{1 / 2}=\sqrt{36}=6$
12. $-25^{1 / 2}=-\sqrt{25}=-5$

## Note

Unless we state otherwise, assume all variables throughout this section and problem set represent nonnegative numbers.
13. $(-25)^{1 / 2}=\sqrt{-25}$, which is not a real number
14. $\left(\frac{4}{9}\right)^{1 / 2}=\sqrt{\frac{4}{9}}=\frac{2}{3}$

The properties of exponents developed in a previous chapter were applied to integer exponents only. We will now extend these properties to include rational exponents also. We do so without proof.

Properties of Exponents If $a$ and $b$ are real numbers and $r$ and $s$ are rational numbers, and $a$ and $b$ are nonnegative whenever $r$ and $s$ indicate even roots, then

1. $a^{r} \cdot a^{s}=a^{r+s}$
2. $\left(a^{r}\right)^{s}=a^{r s}$
3. $(a b)^{r}=a^{r} b^{r}$
4. $a^{-r}=\frac{1}{a^{r}} \quad(a \neq 0)$
5. $\left(\frac{a}{b}\right)^{r}=\frac{a^{r}}{b^{r}} \quad(b \neq 0)$
6. $\frac{a^{r}}{a^{s}}=a^{r-s} \quad(a \neq 0)$

There are times when rational exponents can simplify our work with radicals. Here are Examples 8 and 9 again, but this time we will work them using rational exponents.

## EXAMPLES

Write each radical with a rational exponent and then simplify.
15. $\sqrt[3]{x^{6} y^{12}}=\left(x^{6} y^{12}\right)^{1 / 3}$

$$
\begin{aligned}
& =\left(x^{6}\right)^{1 / 3}\left(y^{12}\right)^{1 / 3} \\
& =x^{2} y^{4}
\end{aligned}
$$

16. $\sqrt[4]{81 r^{8} S^{20}}=\left(81 r^{8} S^{20}\right)^{1 / 4}$

$$
\begin{aligned}
& =81^{1 / 4}\left(r^{8}\right)^{1 / 4}\left(S^{20}\right)^{1 / 4} \\
& =3 r^{2} S^{5}
\end{aligned}
$$

So far, the numerators of all the rational exponents we have encountered have been 1. The next theorem extends the work we can do with rational exponents to rational exponents with numerators other than 1.

We can extend our properties of exponents with the following theorem.

The Rational Exponent Theorem If $a$ is a nonnegative real number, $m$ is an integer, and $n$ is a positive integer, then

$$
a^{m / n}=\left(a^{1 / n}\right)^{m}=\left(a^{m}\right)^{1 / n}
$$

Proof We can prove this theorem using the properties of exponents. Since $\frac{m}{n}=m\left(\frac{1}{n}\right)$ we have

$$
\begin{aligned}
a^{m / n} & =a^{m(1 / n)} & a^{m / n} & =a^{(1 / n)(m)} \\
& =\left(a^{m}\right)^{1 / n} & & =\left(a^{1 / n}\right)^{m}
\end{aligned}
$$

Here are some examples that illustrate how we use this theorem.


## Note

In the introduction to this section we mentioned the golden rectangle. Its origins can be traced back more than 2,000 years to the Greek civilization that produced Pythagoras, Socrates, Plato, Aristotle, and Euclid. The most important mathematical work to come from that Greek civilization was Euclid's Elements, an elegantly written summary of all that was known about geometry at that time in history. Euclid's Elements, according to Howard Eves, an authority on the history of mathematics, exercised a greater influence on scientific thinking than any other work. Here is how we construct a golden rectangle from a square of side 2 , using the same method that Euclid used in his Elements.

## FACTS FROM GEOMETRY

## The Pythagorean Theorem (Again) and the Golden Rectangle

Now that we have had some experience working with square roots, we can rewrite the Pythagorean theorem using a square root. If triangle $A B C$ is a right triangle with $C=90^{\circ}$, then the length of the longest side is the positive square root of the sum of the squares of the other two sides (see Figure 3).


FIGURE 3

Constructing a Golden Rectangle from a Square of Side 2
Step 1: Draw a square with a side of length two. Connect the midpoint of side $C D$ to corner $B$. (Note that we have labeled the midpoint of segment $C D$ with the letter $O$.)


Step 2: Drop the diagonal from step 1 down so it aligns with side $C D$.


Step 3: Form rectangle $A C E F$. This is a golden rectangle.


All golden rectangles are constructed from squares. Every golden rectangle, no matter how large or small it is, will have the same shape. To associate a number with the shape of the golden rectangle, we use the ratio of its length to its width. This ratio is called the golden ratio. To calculate the golden ratio, we must first find the length of the diagonal we used to construct the golden rec-
tangle. Figure 4 shows the golden rectangle that we constructed from a square of side 2. The length of the diagonal $O B$ is found by applying the Pythagorean theorem to triangle $O B D$.


FIGURE 4
The length of segment $O E$ is equal to the length of diagonal $O B$; both are $\sqrt{5}$. Since the distance from $C$ to $O$ is 1 , the length $C E$ of the golden rectangle is $1+\sqrt{5}$. Now we can find the golden ratio:

$$
\text { Golden ratio }=\frac{\text { length }}{\text { width }}=\frac{C E}{E F}=\frac{1+\sqrt{5}}{2}
$$

## USING TECHNOLOGY

Graphing Calculators-A Word of Caution
Some graphing calculators give surprising results when evaluating expressions such as $(-8)^{2 / 3}$. As you know from reading this section, the expression $(-8)^{2 / 3}$ simplifies to 4 , either by taking the cube root first and then squaring the result, or by squaring the base first and then taking the cube root of the result. Here are three different ways to evaluate this expression on your calculator:

1. $(-8)^{\wedge}(2 / 3) \quad$ To evaluate $(-8)^{2 / 3}$
2. $\left((-8)^{\wedge} 2\right)^{\wedge}(1 / 3) \quad$ To evaluate $\left((-8)^{2}\right)^{1 / 3}$
3. $\left((-8)^{\wedge}(1 / 3)\right)^{\wedge} 2 \quad$ To evaluate $\left((-8)^{1 / 3}\right)^{2}$

Note any difference in the results.
Next, graph each of the following functions, one at a time.

1. $Y_{1}=X^{2 / 3}$
2. $Y_{2}=\left(X^{2}\right)^{1 / 3}$
3. $Y_{3}=\left(\mathrm{X}^{1 / 3}\right)^{2}$

The correct graph is shown in Figure 5. Note which of your graphs match the correct graph.


FIGURE 5
Different calculators evaluate exponential expressions in different ways. You should use the method (or methods) that gave you the correct graph.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-9
B $\quad 10-24$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Every real number has two square roots. Explain the notation we use to tell them apart. Use the square roots of 3 for examples.
2. Explain why a square root of -4 is not a real number.
3. We use the notation $\sqrt{2}$ to represent the positive square root of 2 . Explain why there isn't a simpler way to express the positive square root of 2 .
4. For the expression $a^{m / n}$, explain the significance of the numerator $m$ and the significance of the denominator $n$ in the exponent.

## Problem Set 9.1

Find each of the following roots, if possible.

1. $\sqrt{144}$
2. $\sqrt{-144}$
3. $-\sqrt{49}$
4. $\sqrt[3]{-27}$
5. $\sqrt[4]{16}$
6. $\sqrt[4]{-16}$
7. $\sqrt{0.04}$
8. $\sqrt[3]{0.008}$
9. $\sqrt[3]{125}$
10. $-\sqrt[3]{216}$
11. $\sqrt{\frac{1}{36}}$
12. $\sqrt[3]{\frac{8}{125}}$

Simplify each expression. Assume all variables represent nonnegative numbers.
25. $\sqrt{36 a^{8}}$
26. $\sqrt{49 a^{10}}$
27. $\sqrt[3]{27 a^{12}}$
28. $\sqrt[3]{8 a^{15}}$
29. $\sqrt[5]{32 x^{10} y^{5}}$
30. $\sqrt[5]{32 x^{5} y^{10}}$
31. $\sqrt[4]{16 a^{12} b^{20}}$
32. $\sqrt[4]{81 a^{24} b^{8}}$

Use the definition of rational exponents to write each of the following with the appropriate root. Then simplify.
33. $36^{1 / 2}$
34. $49^{1 / 2}$
35. $-9^{1 / 2}$
36. $-16^{1 / 2}$
37. $8^{1 / 3}$
38. $-8^{1 / 3}$
39. $(-8)^{1 / 3}$
40. $-27^{1 / 3}$
41. $32^{1 / 5}$
42. $81^{1 / 4}$
43. $\left(\frac{81}{25}\right)^{1 / 2}$
44. $\left(\frac{64}{125}\right)^{1 / 3}$

Use the rational exponent theorem to simplify each of the following as much as possible.
45. $27^{2 / 3}$
46. $8^{4 / 3}$
47. $25^{3 / 2}$
48. $81^{3 / 4}$

Simplify each expression. Remember, negative exponents give reciprocals.
49. $27^{-1 / 3}$
50. $9^{-1 / 2}$
51. $81^{-3 / 4}$
53. $\left(\frac{25}{36}\right)^{-1 / 2}$
55. $\left(\frac{81}{16}\right)^{-3 / 4}$
57. $16^{1 / 2}+27^{1 / 3}$
59. $8^{-2 / 3}+4^{-1 / 2}$
52. $4^{-3 / 2}$
54. $\left(\frac{16}{49}\right)^{-1 / 2}$
56. $\left(\frac{27}{8}\right)^{-2 / 3}$
58. $25^{1 / 2}+100^{1 / 2}$
60. $49^{-1 / 2}+25^{-1 / 2}$
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Use the properties of exponents to simplify each of the following as much as possible. Assume all bases are positive.
61. $X^{3 / 5} \cdot X^{1 / 5}$
62. $x^{3 / 4} \cdot x^{5 / 4}$
63. $\left(a^{3 / 4}\right)^{4 / 3}$
64. $\left(a^{2 / 3}\right)^{3 / 4}$
65. $\frac{X^{1 / 5}}{X^{3 / 5}}$
66. $\frac{x^{2 / 7}}{x^{5 / 7}}$
67. $\frac{x^{5 / 6}}{x^{2 / 3}}$
68. $\frac{x^{7 / 8}}{x^{8 / 7}}$
69. $\left(x^{3 / 5} y^{5 / 6} Z^{1 / 3}\right)^{3 / 5}$
70. $\left(x^{3 / 4} y^{1 / 8} Z^{5 / 6}\right)^{4 / 5}$
71. $\frac{a^{3 / 4} b^{2}}{a^{7 / 8} b^{1 / 4}}$
72. $\frac{a^{1 / 3} b^{4}}{a^{3 / 5} b^{1 / 3}}$
73. $\frac{\left(y^{2 / 3}\right)^{3 / 4}}{\left(y^{1 / 3}\right)^{3 / 5}}$
74. $\frac{\left(y^{5 / 4}\right)^{2 / 5}}{\left(y^{1 / 4}\right)^{4 / 3}}$
75. $\left(\frac{a^{-1 / 4}}{b^{1 / 2}}\right)^{8}$
76. $\left(\frac{a^{-1 / 5}}{b^{1 / 3}}\right)^{15}$
77. $\frac{\left(r^{-2} S^{1 / 3}\right)^{6}}{r^{8} S^{3 / 2}}$
78. $\frac{\left(r^{-5} S^{1 / 2}\right)^{4}}{r^{12} S^{5 / 2}}$
79. $\frac{\left(25 a^{6} b^{4}\right)^{1 / 2}}{\left(8 a^{-9} b^{3}\right)^{-1 / 3}}$
80. $\frac{\left(27 a^{3} b^{6}\right)^{1 / 3}}{\left(81 a^{8} b^{-4}\right)^{1 / 4}}$

## Applying the Concepts

81. Maximum Speed The maximum speed ( $v$ ) that an automobile can travel around a curve of radius $r$ without skidding is given by the equation

$$
V=\left(\frac{5 r}{2}\right)^{1 / 2}
$$

where $v$ is in miles per hour and $r$ is measured in feet. What is the maximum speed a car can travel around a curve with a radius of 250 feet without skidding?
82. Golden Ratio The golden ratio is the ratio of the length to the width in any golden rectangle. The exact value of this number is $\frac{1+\sqrt{5}}{2}$. Use a calculator to find a decimal approximation to this number and round it to the nearest thousandth.
83. Chemistry Figure 6 shows part of a model of a magnesium oxide (MgO) crystal. Each corner of the square is at the center of one oxygen ion $\left(\mathrm{O}^{2-}\right)$, and the center of the middle ion is at the center of the square. The radius for each oxygen ion is 126 picometers (pm), and the
radius for each magnesium ion $\left(\mathrm{Mg}^{2+}\right)$ is 86 picometers.
a. Find the length of the side of the square. Write your answer in picometers.
b. Find the length of the diagonal of the square to the nearest picometer.
c. If 1 meter is $10^{12}$ picometers, give the length of the diagonal of the square in meters.


FIGURE 6 (Susan M. Young)
84. Geometry The length of each side of the cube shown in Figure 7 is 1 inch.
a. Find the length of the diagonal CH .
b. Find the length of the diagonal $C F$.


FIGURE 7
85. Comparing Graphs Identify the graph with the correct equation.
a. $y=x$
b. $y=x^{2}$
c. $y=x^{2 / 3}$
d. What are the two points of intersection of all three graphs?

86. Falling Objects The time $t$ in seconds it takes an object to fall $d$ feet is given by the equation

$$
t=\frac{1}{4} \sqrt{d}
$$

a. The Sears Tower in Chicago is 1,450 feet tall. How long would it take a penny to fall to the ground from the top of the Sears Tower? Round to the nearest hundredth.
b. An object took 30 seconds to fall to the ground. From what distance must it have been dropped?

Factor.
103. $x^{2}-3 x-10$
104. $x^{2}+x-12$
105. $6 x^{2}+11 x-10$
106. $10 x^{2}-x-3$

Use rules of exponents to simplify.
107. $x^{2 / 3} \cdot x^{4 / 3}$
108. $x^{1 / 4} \cdot x^{3 / 4}$
109. $\left(t^{1 / 2}\right)^{2}$
110. $\left(x^{3 / 2}\right)^{2}$
111. $\frac{X^{2 / 3}}{X^{1 / 3}}$
112. $\frac{X^{1 / 2}}{X^{1 / 2}}$

## Maintaining Your Skills

Multiply.
87. $x^{2}\left(x^{4}-x\right)$
88. $5 x^{2}\left(2 x^{3}-x\right)$
89. $(x-3)(x+5)$
90. $(x-2)(x+2)$
91. $\left(x^{2}-5\right)^{2}$
92. $\left(x^{2}+5\right)^{2}$
93. $(x-3)\left(x^{2}+3 x+9\right)$
94. $(x+3)\left(x^{2}-3 x+9\right)$

## Getting Ready for the Next Section

Simplify.
95. $X^{2}\left(X^{4}-x^{3}\right)$
96. $\left(x^{2}-3\right)\left(x^{2}+5\right)$
97. $(3 a-2 b)(4 a-b)$
98. $\left(x^{2}+3\right)^{2}$
99. $\left(x^{3}-2\right)\left(x^{3}+2\right)$
100. $(a-b)\left(a^{2}+a b+b^{2}\right)$
101. $\frac{15 x^{2} y-20 x^{4} y^{2}}{5 x y}$
102. $\frac{12 a^{3} b^{2}-24 a^{2} b^{4}}{3 a b}$

## Extending the Concepts

113. Show that the expression $\left(a^{1 / 2}+b^{1 / 2}\right)^{2}$ is not equal to $a+b$ by replacing $a$ with 9 and $b$ with 4 in both expressions and then simplifying each.
114. Show that the statement $\left(a^{2}+b^{2}\right)^{1 / 2}=a+b$ is not, in general, true by replacing $a$ with 3 and $b$ with 4 and then simplifying both sides.
115. You may have noticed, if you have been using a calculator to find roots, that you can find the fourth root of a number by pressing the square root button twice. Written in symbols, this fact looks like this:

$$
\sqrt{\sqrt{a}}=\sqrt[4]{a} \quad(a \geq 0)
$$

Show that this statement is true by rewriting each side with exponents instead of radical notation and then simplifying the left side.
116. Show that the following statement is true by rewriting each side with exponents instead of radical notation and then simplifying the left side.

$$
\sqrt[3]{\sqrt{a}}=\sqrt[6]{a} \quad(a \geq 0)
$$

## More Expressions Involving Rational Exponents

## OBJECTIVES

A Multiply expressions with rational exponents.

B Divide expressions with rational exponents.

C Factor expressions with rational exponents.

D Add and subtract expressions with rational exponents.

Suppose you purchased 10 silver proof coin sets in 1997 for $\$ 21$ each, for a total investment of $\$ 210$. Three years later, in 2000, you find that each set is worth $\$ 30$, which means that your 10 sets have a total value of $\$ 300$.


You can calculate the annual rate of return on this investment using a formula that involves rational exponents. The annual rate of return will tell you at what interest rate you would have to invest your original \$210 for it to be worth \$300 three years later. As you will see at the end of this section, the annual rate of return on this investment is $12.6 \%$, which is a good return on your money.

In this section we will look at multiplication, division, factoring, and simplification of some expressions that resemble polynomials but contain rational exponents. The problems in this section will be of particular interest to you if you are planning to take either an engineering calculus class or a business calculus class. As was the case in the previous section, we will assume all variables represent nonnegative real numbers. That way, we will not have to worry about the possibility of introducing undefined terms-even roots of negative numbersinto any of our examples. Let's begin this section with a look at multiplication of expressions containing rational exponents.

## EXAMPLE 1 Multiply $x^{2 / 3}\left(x^{4 / 3}-x^{1 / 3}\right)$.

SOLUTION Applying the distributive property and then simplifying the resulting terms, we have:

$$
\begin{aligned}
X^{2 / 3}\left(X^{4 / 3}-X^{1 / 3}\right) & =x^{2 / 3} X^{4 / 3}-x^{2 / 3} X^{1 / 3} & & \text { Distributive property } \\
& =x^{6 / 3}-x^{3 / 3} & & \text { Add exponents } \\
& =x^{2}-x & & \text { Simplify }
\end{aligned}
$$

## EXAMPLE 2 Multiply $\left(x^{2 / 3}-3\right)\left(x^{2 / 3}+5\right)$.

SOLUTION Applying the FOIL method, we multiply as if we were multiplying two binomials:

$$
\begin{aligned}
\left(x^{2 / 3}-3\right)\left(x^{2 / 3}+5\right) & =x^{2 / 3} x^{2 / 3}+5 x^{2 / 3}-3 x^{2 / 3}-15 \\
& =x^{4 / 3}+2 x^{2 / 3}-15
\end{aligned}
$$

## EXAMPLE 3 Multiply $\left(3 a^{1 / 3}-2 b^{1 / 3}\right)\left(4 a^{1 / 3}-b^{1 / 3}\right)$.

SOLUTION Again, we use the FOIL method to multiply:

$$
\begin{aligned}
\left(3 a^{1 / 3}-2 b^{1 / 3}\right)\left(4 a^{1 / 3}\right. & \left.-b^{1 / 3}\right) \\
& =3 a^{1 / 3} 4 a^{1 / 3}-3 a^{1 / 3} b^{1 / 3}-2 b^{1 / 3} 4 a^{1 / 3}+2 b^{1 / 3} b^{1 / 3} \\
& =12 a^{2 / 3}-11 a^{1 / 3} b^{1 / 3}+2 b^{2 / 3}
\end{aligned}
$$

## EXAMPLE $4 \quad$ Expand $\left(t^{1 / 2}-5\right)^{2}$.

SOLUTION We can use the definition of exponents and the FOIL method:

$$
\begin{aligned}
\left(t^{1 / 2}-5\right)^{2} & =\left(t^{1 / 2}-5\right)\left(t^{1 / 2}-5\right) \\
& =t^{1 / 2} t^{1 / 2}-5 t^{1 / 2}-5 t^{1 / 2}+25 \\
& =t-10 t^{1 / 2}+25
\end{aligned}
$$

We can obtain the same result by using the formula for the square of a binomial, $(a-b)^{2}=a^{2}-2 a b+b^{2}$.

$$
\begin{aligned}
\left(t^{1 / 2}-5\right)^{2} & =\left(t^{1 / 2}\right)^{2}-2 t^{1 / 2} \cdot 5+5^{2} \\
& =t-10 t^{1 / 2}+25
\end{aligned}
$$

## EXAMPLE $5 \quad$ Multiply $\left(x^{3 / 2}-2^{3 / 2}\right)\left(x^{3 / 2}+2^{3 / 2}\right)$.

SOLUTION This product has the form $(a-b)(a+b)$, which will result in the difference of two squares, $a^{2}-b^{2}$ :

$$
\begin{aligned}
\left(x^{3 / 2}-2^{3 / 2}\right)\left(x^{3 / 2}+2^{3 / 2}\right) & =\left(x^{3 / 2}\right)^{2}-\left(2^{3 / 2}\right)^{2} \\
& =x^{3}-2^{3} \\
& =x^{3}-8
\end{aligned}
$$

## EXAMPLE $6 \quad$ Multiply $\left(a^{1 / 3}-b^{1 / 3}\right)\left(a^{2 / 3}+a^{1 / 3} b^{1 / 3}+b^{2 / 3}\right)$.

SOLUTION We can find this product by multiplying in columns:

$$
\begin{aligned}
& \begin{array}{l}
a^{2 / 3}+a^{1 / 3} b^{1 / 3}+b^{2 / 3} \\
\\
\hline a \quad+a^{2 / 3} b^{1 / 3}+a^{1 / 3} b^{2 / 3} \\
\\
\hline a
\end{array} \quad-a^{2 / 3} b^{1 / 3}-a^{1 / 3} b^{2 / 3}-b \\
& \hline a
\end{aligned}
$$

The product is $a-b$.
Our next example involves division with expressions that contain rational exponents. As you will see, this kind of division is very similar to division of a polynomial by a monomial.

## Note

For Example 7, assume $x$ and $y$ are positive numbers.

EXAMPLE $7 \quad$ Divide $\frac{15 x^{2 / 3} y^{1 / 3}-20 x^{4 / 3} y^{2 / 3}}{5 x^{1 / 3} y^{1 / 3}}$.
SOLUTION We can approach this problem in the same way we approached division by a monomial. We simply divide each term in the numerator by the term in the denominator:

$$
\begin{aligned}
\frac{15 x^{2 / 3} y^{1 / 3}-20 x^{4 / 3} y^{2 / 3}}{5 x^{1 / 3} y^{1 / 3}} & =\frac{15 x^{2 / 3} y^{1 / 3}}{5 x^{1 / 3} y^{1 / 3}}-\frac{20 x^{4 / 3} y^{2 / 3}}{5 x^{1 / 3} y^{1 / 3}} \\
& =3 x^{1 / 3}-4 x y^{1 / 3}
\end{aligned}
$$

The next three examples involve factoring. In the first example, we are told what to factor from each term of an expression.

EXAMPLE $8 \quad$ Factor $3(x-2)^{1 / 3}$ from $12(x-2)^{4 / 3}-9(x-2)^{1 / 3}$, and then simplify, if possible.

SOLUTION This solution is similar to factoring out the greatest common factor:

$$
\begin{aligned}
12(x-2)^{4 / 3}-9(x-2)^{1 / 3} & =3(x-2)^{1 / 3}[4(x-2)-3] \\
& =3(x-2)^{1 / 3}(4 x-11)
\end{aligned}
$$

Although an expression containing rational exponents is not a polynomialremember, a polynomial must have exponents that are whole numbers-we are going to treat the expressions that follow as if they were polynomials.

## EXAMPLE 9 Factor $x^{2 / 3}-3 x^{1 / 3}-10$ as if it were a trinomial.

SOLUTION We can think of $x^{2 / 3}-3 x^{1 / 3}-10$ as if it is a trinomial in which the variable is $x^{1 / 3}$. To see this, replace $x^{1 / 3}$ with $y$ to get

$$
y^{2}-3 y-10
$$

Since this trinomial in $y$ factors as $(y-5)(y+2)$, we can factor our original expression similarly:

$$
x^{2 / 3}-3 x^{1 / 3}-10=\left(x^{1 / 3}-5\right)\left(x^{1 / 3}+2\right)
$$

Remember, with factoring, we can always multiply our factors to check that we have factored correctly.

EXAMPLE 10 Factor $6 x^{2 / 5}+11 x^{1 / 5}-10$ as if it were a trinomial.
SOLUTION We can think of the expression in question as a trinomial in $x^{1 / 5}$.

$$
6 x^{2 / 5}+11 x^{1 / 5}-10=\left(3 x^{1 / 5}-2\right)\left(2 x^{1 / 5}+5\right)
$$

In our next example, we combine two expressions by applying the methods we used to add and subtract fractions or rational expressions.

## EXAMPLE 11 Subtract $\left(x^{2}+4\right)^{1 / 2}-\frac{x^{2}}{\left(x^{2}+4\right)^{1 / 2}}$.

SOLUTION To combine these two expressions, we need to find a least common denominator, change to equivalent fractions, and subtract numerators. The least common denominator is $\left(x^{2}+4\right)^{1 / 2}$.

$$
\begin{aligned}
\left(x^{2}+4\right)^{1 / 2}-\frac{x^{2}}{\left(x^{2}+4\right)^{1 / 2}} & =\frac{\left(x^{2}+4\right)^{1 / 2}}{1} \cdot \frac{\left(x^{2}+4\right)^{1 / 2}}{\left(x^{2}+\mathbf{4}\right)^{1 / 2}}-\frac{x^{2}}{\left(x^{2}+4\right)^{1 / 2}} \\
& =\frac{x^{2}+4-x^{2}}{\left(x^{2}+4\right)^{1 / 2}} \\
& =\frac{4}{\left(x^{2}+4\right)^{1 / 2}}
\end{aligned}
$$

EXAMPLE 12 If you purchase an investment for $P$ dollars and $t$ years later it is worth $A$ dollars, then the annual rate of return $r$ on that investment is given by the formula

$$
r=\left(\frac{A}{P}\right)^{1 / t}-1
$$

Find the annual rate of return on a coin collection that was purchased for $\$ 210$ and sold 3 years later for $\$ 300$.

SOLUTION Using $A=300, P=210$, and $t=3$ in the formula, we have

$$
r=\left(\frac{300}{210}\right)^{1 / 3}-1
$$

The easiest way to simplify this expression is with a calculator.


Allowing three decimal places, the result is 0.126 . The annual return on the coin collection is approximately $12.6 \%$. To do as well with a savings account, we would have to invest the original $\$ 210$ in an account that paid $12.6 \%$, compounded annually.

IINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $1-6$
B 7
C $\quad 8-10$
D 11

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. When multiplying expressions with fractional exponents, when do we add the fractional exponents?
2. Is it possible to multiply two expressions with fractional exponents and end up with an expression containing only integer exponents? Support your answer with examples.
3. Write an application modeled by the equation $r=\left(\frac{1,000}{600}\right)^{1 / 8}-1$.
4. When can you use the FOIL method with expressions that contain rational exponents?

Multiply. (Assume all variables in this problem set represent nonnegative real numbers.)

1. $x^{2 / 3}\left(X^{1 / 3}+x^{4 / 3}\right)$
2. $x^{2 / 5}\left(x^{3 / 5}-x^{8 / 5}\right)$
3. $a^{1 / 2}\left(a^{3 / 2}-a^{1 / 2}\right)$
4. $a^{1 / 4}\left(a^{3 / 4}+a^{7 / 4}\right)$
5. $2 x^{1 / 3}\left(3 x^{8 / 3}-4 x^{5 / 3}+5 x^{2 / 3}\right)$
6. $5 x^{1 / 2}\left(4 x^{5 / 2}+3 x^{3 / 2}+2 x^{1 / 2}\right)$
7. $4 x^{1 / 2} y^{3 / 5}\left(3 x^{3 / 2} y^{-3 / 5}-9 x^{-1 / 2} y^{7 / 5}\right)$
8. $3 x^{4 / 5} y^{1 / 3}\left(4 x^{6 / 5} y^{-1 / 3}-12 x^{-4 / 5} y^{5 / 3}\right)$
9. $\left(x^{2 / 3}-4\right)\left(x^{2 / 3}+2\right)$
10. $\left(x^{2 / 3}-5\right)\left(x^{2 / 3}+2\right)$
11. $\left(a^{1 / 2}-3\right)\left(a^{1 / 2}-7\right)$
12. $\left(a^{1 / 2}-6\right)\left(a^{1 / 2}-2\right)$
13. $\left(4 y^{1 / 3}-3\right)\left(5 y^{1 / 3}+2\right)$
14. $\left(5 y^{1 / 3}-2\right)\left(4 y^{1 / 3}+3\right)$
15. $\left(5 x^{2 / 3}+3 y^{1 / 2}\right)\left(2 x^{2 / 3}+3 y^{1 / 2}\right)$
16. $\left(4 x^{2 / 3}-2 y^{1 / 2}\right)\left(5 x^{2 / 3}-3 y^{1 / 2}\right)$
17. $\left(t^{1 / 2}+5\right)^{2}$
18. $\left(t^{1 / 2}-3\right)^{2}$
19. $\left(x^{3 / 2}+4\right)^{2}$
20. $\left(x^{3 / 2}-6\right)^{2}$
21. $\left(a^{1 / 2}-b^{1 / 2}\right)^{2}$
22. $\left(a^{1 / 2}+b^{1 / 2}\right)^{2}$
23. $\left(2 x^{1 / 2}-3 y^{1 / 2}\right)^{2}$
24. $\left(5 x^{1 / 2}+4 y^{1 / 2}\right)^{2}$
25. $\left(a^{1 / 2}-3^{1 / 2}\right)\left(a^{1 / 2}+3^{1 / 2}\right)$
26. $\left(a^{1 / 2}-5^{1 / 2}\right)\left(a^{1 / 2}+5^{1 / 2}\right)$
27. $\left(x^{3 / 2}+y^{3 / 2}\right)\left(x^{3 / 2}-y^{3 / 2}\right)$
28. $\left(x^{5 / 2}+y^{5 / 2}\right)\left(x^{5 / 2}-y^{5 / 2}\right)$
29. $\left(t^{1 / 2}-2^{3 / 2}\right)\left(t^{1 / 2}+2^{3 / 2}\right)$
30. $\left(t^{1 / 2}-5^{3 / 2}\right)\left(t^{1 / 2}+5^{3 / 2}\right)$
31. $\left(2 x^{3 / 2}+3^{1 / 2}\right)\left(2 x^{3 / 2}-3^{1 / 2}\right)$
32. $\left(3 x^{1 / 2}+2^{3 / 2}\right)\left(3 x^{1 / 2}-2^{3 / 2}\right)$
33. $\left(x^{1 / 3}+y^{1 / 3}\right)\left(x^{2 / 3}-x^{1 / 3} y^{1 / 3}+y^{2 / 3}\right)$
34. $\left(x^{1 / 3}-y^{1 / 3}\right)\left(x^{2 / 3}+x^{1 / 3} y^{1 / 3}+y^{2 / 3}\right)$
35. $\left(a^{1 / 3}-2\right)\left(a^{2 / 3}+2 a^{1 / 3}+4\right)$
36. $\left(a^{1 / 3}+3\right)\left(a^{2 / 3}-3 a^{1 / 3}+9\right)$
37. $\left(2 x^{1 / 3}+1\right)\left(4 x^{2 / 3}-2 x^{1 / 3}+1\right)$
38. $\left(3 x^{1 / 3}-1\right)\left(9 x^{2 / 3}+3 x^{1 / 3}+1\right)$
39. $\left(t^{1 / 4}-1\right)\left(t^{1 / 4}+1\right)\left(t^{1 / 2}+1\right)$
40. $\left(t^{1 / 4}-2\right)\left(t^{1 / 4}+2\right)\left(t^{1 / 2}+4\right)$

Divide. (Assume all variables represent positive numbers.)
41. $\frac{18 x^{3 / 4}+27 x^{1 / 4}}{9 x^{1 / 4}}$
42. $\frac{25 x^{1 / 4}+30 x^{3 / 4}}{5 x^{1 / 4}}$
43. $\frac{12 x^{2 / 3} y^{1 / 3}-16 x^{1 / 3} y^{2 / 3}}{4 x^{1 / 3} y^{1 / 3}}$
44. $\frac{12 x^{4 / 3} y^{1 / 3}-18 x^{1 / 3} y^{4 / 3}}{6 x^{1 / 3} y^{1 / 3}}$
45. $\frac{21 a^{7 / 5} b^{3 / 5}-14 a^{2 / 5} b^{8 / 5}}{7 a^{2 / 5} b^{3 / 5}}$
46. $\frac{24 a^{9 / 5} b^{3 / 5}-16 a^{4 / 5} b^{8 / 5}}{8 a^{4 / 5} b^{3 / 5}}$
47. Factor $3(x-2)^{1 / 2}$ from $12(x-2)^{3 / 2}-9(x-2)^{1 / 2}$.
48. Factor $4(x+1)^{1 / 3}$ from $4(x+1)^{4 / 3}+8(x+1)^{1 / 3}$.
49. Factor $5(x-3)^{7 / 5}$ from $5(x-3)^{12 / 5}-15(x-3)^{7 / 5}$.
50. Factor $6(x+3)^{8 / 7}$ from $6(x+3)^{15 / 7}-12(x+3)^{8 / 7}$.
51. Factor $3(x+1)^{1 / 2}$ from $9 x(x+1)^{3 / 2}+6(x+1)^{1 / 2}$.
52. Factor $4 x(x+1)^{1 / 2}$ from $4 x^{2}(x+1)^{1 / 2}+8 x(x+1)^{3 / 2}$.

Factor each of the following as if it were a trinomial.
53. $x^{2 / 3}-5 x^{1 / 3}+6$
54. $x^{2 / 3}-x^{1 / 3}-6$
55. $a^{2 / 5}-2 a^{1 / 5}-8$
56. $a^{2 / 5}+2 a^{1 / 5}-8$
57. $2 y^{2 / 3}-5 y^{1 / 3}-3$
58. $3 y^{2 / 3}+5 y^{1 / 3}-2$
59. $9 t^{2 / 5}-25$
60. $16 t^{2 / 5}-49$
61. $4 x^{2 / 7}+20 x^{1 / 7}+25$
62. $25 x^{2 / 7}-20 x^{1 / 7}+4$

Evaluate the following functions for the given value.
63. If $f(x)=x-2 \sqrt{x}-8$, find $f(4)$.
64. If $g(x)=x-2 \sqrt{x}-3$, find $g(9)$.
65. If $f(x)=2 x+9 \sqrt{x}-5$, find $f(25)$.
66. If $f(t)=t-2 \sqrt{t}-15$, find $f(9)$.
67. If $g(x)=2 x-\sqrt{x}-6$, find $g\left(\frac{9}{4}\right)$.
68. If $f(x)=2 x+\sqrt{x}-15$, find $f(9)$.
69. If $f(x)=x^{2 / 3}-2 x^{1 / 3}-8$, find $f(-8)$.
70. If $g(x)=x^{2 / 3}+4 x^{1 / 3}-12$, find $g(8)$.

Simplify each of the following to a single fraction. (Assume all variables represent positive numbers.)
71. $\frac{3}{x^{1 / 2}}+x^{1 / 2}$
72. $\frac{2}{x^{1 / 2}}-x^{1 / 2}$
73. $x^{2 / 3}+\frac{5}{x^{1 / 3}}$
74. $x^{3 / 4}-\frac{7}{X^{1 / 4}}$
75. $\frac{3 x^{2}}{\left(x^{3}+1\right)^{1 / 2}}+\left(x^{3}+1\right)^{1 / 2}$
76. $\frac{x^{3}}{\left(x^{2}-1\right)^{1 / 2}}+2 x\left(x^{2}-1\right)^{1 / 2}$
77. $\frac{x^{2}}{\left(x^{2}+4\right)^{1 / 2}}-\left(x^{2}+4\right)^{1 / 2}$
78. $\frac{x^{5}}{\left(X^{2}-2\right)^{1 / 2}}+4 x^{3}\left(X^{2}-2\right)^{1 / 2}$

## Applying the Concepts

79. Investing A coin collection is purchased as an investment for $\$ 500$ and sold 4 years later for $\$ 900$. Find the annual rate of return on the investment.
80. Investing An investor buys stock in a company for $\$ 800$. Five years later, the same stock is worth $\$ 1,600$. Find the annual rate of return on the stocks.

## Maintaining Your Skills

81. Find the slope of the line that contains $(-4,-1)$ and $(-2,5)$.
82. A line has a slope of $\frac{2}{3}$. Find the slope of any line a. Parallel to it.
b. Perpendicular to it.
83. Give the slope and $y$-intercept of the line $2 x-3 y=6$.
84. Give the equation of the line with slope -3 and $y$-intercept 5 .
85. Find the equation of the line with slope $\frac{2}{3}$ that contains the point $(-6,2)$.
86. Find the equation of the line through $(1,3)$ and $(-1,-5)$.
87. Find the equation of the line with $x$-intercept 3 and $y$-intercept -2 .
88. Find the equation of the line through $(-1,4)$ whose graph is perpendicular to the graph of $y=2 x+3$.

## Getting Ready for the Next Section

Simplify. Assume all variable are positive real numbers.
89. $\sqrt{25}$
90. $\sqrt{4}$
91. $\sqrt{6^{2}}$
92. $\sqrt{3^{2}}$
93. $\sqrt{16 x^{4} y^{2}}$
94. $\sqrt{4 x^{6} y^{8}}$
95. $\sqrt{(5 y)^{2}}$
96. $\sqrt{\left(8 x^{3}\right)^{2}}$
97. $\sqrt[3]{27}$
98. $\sqrt[3]{-8}$
99. $\sqrt[3]{2^{3}}$
100. $\sqrt[3]{(-5)^{3}}$
101. $\sqrt[3]{8 a^{3} b^{3}}$
102. $\sqrt[3]{64 a^{6} b^{3}}$

Fill in the blank.
103. $50=$ $\qquad$ - 2
104. $12=$ $\qquad$ - 3
105. $48 x^{4} y^{3}=$ $\qquad$ - y
106. $40 a^{5} b^{4}=$ $\qquad$ - $5 a^{2} b$
107. $12 x^{7} y^{6}=$ $\qquad$ -3x
108. $54 a^{6} b^{2} c^{4}=$ $\qquad$ - $2 b^{2} c$

## Simplified Form for Radicals

## OBJECTIVES

A Write radical expressions in simplified form.

B Rationalize a denominator that contains only one term.

## Note

There is no property for radicals that says the $n$th root of a sum is the sum of the $n$th roots; that is, in general

$$
\sqrt[n]{a+b} \neq \sqrt[n]{a}+\sqrt[n]{b}
$$

Earlier in this chapter, we showed how the Pythagorean theorem can be used to construct a golden rectangle. In a similar manner, the Pythagorean theorem can be used to construct the attractive spiral shown here.

## The Spiral of Roots



This spiral is called the Spiral of Roots because each of the diagonals is the positive square root of one of the positive integers. At the end of this section, we will use the Pythagorean theorem and some of the material in this section to construct this spiral.

In this section we will use radical notation instead of rational exponents. We will begin by stating two properties of radicals. Following this, we will give a definition for simplified form for radical expressions. The examples in this section show how we use the properties of radicals to write radical expressions in simplified form.

There are two properties of radicals. For these two properties, we will assume $a$ and $b$ are nonnegative real numbers whenever $n$ is an even number.

## Property 1 for Radicals

$$
\sqrt[n]{a b}=\sqrt[n]{a} \sqrt[n]{b}
$$

In words: The $n$th root of a product is the product of the $n$th roots.

## Proof of Property 1

$$
\begin{aligned}
\sqrt[n]{a b} & =(a b)^{1 / n} & & \text { Definition of fractional exponents } \\
& =a^{1 / n} b^{1 / n} & & \text { Exponents distribute over products } \\
& =\sqrt[n]{a} \sqrt[n]{b} & & \text { Definition of fractional exponents }
\end{aligned}
$$

Property 2 for Radicals

$$
\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad(b \neq 0)
$$

In words: The $n$th root of a quotient is the quotient of the $n$th roots.

## Note

Writing a radical expression in simplified form does not always result in a simpler-looking expression. Simplified form for radicals is a way of writing radicals so they are easiest to work with.

The proof of property 2 is similar to the proof of property 1.
The two properties of radicals allow us to change the form of and simplify radical expressions without changing their value.

## Simplified Form for Radical Expressions A radical expression is in simplified form if

1. None of the factors of the radicand (the quantity under the radical sign) can be written as powers greater than or equal to the index-that is, no perfect squares can be factors of the quantity under a square root sign, no perfect cubes can be factors of what is under a cube root sign, and so forth;
2. There are no fractions under the radical sign; and
3. There are no radicals in the denominator.

Satisfying the first condition for simplified form actually amounts to taking as much out from under the radical sign as possible. The following examples illustrate the first condition for simplified form.

## EXAMPLE 1 Write $\sqrt{50}$ in simplified form.

SOLUTION The largest perfect square that divides 50 is 25 . We write 50 as $25 \cdot 2$ and apply property 1 for radicals:

$$
\begin{aligned}
\sqrt{50} & =\sqrt{25 \cdot 2} & & 50=\mathbf{2 5} \cdot \mathbf{2} \\
& =\sqrt{25} \sqrt{2} & & \text { Property } \mathbf{1} \\
& =5 \sqrt{2} & & \sqrt{\mathbf{2 5}}=\mathbf{5}
\end{aligned}
$$

We have taken as much as possible out from under the radical sign—in this case, factoring 25 from 50 and then writing $\sqrt{25}$ as 5 .

## Note

Unless we state otherwise, assume all variables throughout this section represent nonnegative numbers. Further, if a variable appears in a denominator, assume the variable cannot be 0 .

EXAMPLE 2 Write in simplified form: $\sqrt{48 x^{4} y^{3}}$, where $x, y \geq 0$.
SOLUTION The largest perfect square that is a factor of the radicand is $16 x^{4} y^{2}$.
Applying property 1 again, we have

$$
\begin{aligned}
\sqrt{48 x^{4} y^{3}} & =\sqrt{16 x^{4} y^{2} \cdot 3 y} \\
& =\sqrt{16 x^{4} y^{2}} \sqrt{3 y} \\
& =4 x^{2} y \sqrt{3 y}
\end{aligned}
$$

EXAMPLE 3 Write $\sqrt[3]{40 a^{5} b^{4}}$ in simplified form.
SOLUTION We now want to factor the largest perfect cube from the radicand. We write $40 a^{5} b^{4}$ as $8 a^{3} b^{3} \cdot 5 a^{2} b$ and proceed as we did in Examples 1 and 2 .

$$
\begin{aligned}
\sqrt[3]{40 a^{5} b^{4}} & =\sqrt[3]{8 a^{3} b^{3} \cdot 5 a^{2} b} \\
& =\sqrt[3]{8 a^{3} b^{3}} \sqrt[3]{5 a^{2} b} \\
& =2 a b \sqrt[3]{5 a^{2} b}
\end{aligned}
$$

## Note

The idea behind rationalizing the denominator is to produce a perfect square under the square root sign in the denominator. This is accomplished by multiplying both the numerator and denominator by the appropriate radical.

Here are some further examples concerning the first condition for simplified form.

## EXAMPLES Write each expression in simplified form.

4. $\sqrt{12 x^{7} y^{6}}=\sqrt{4 x^{6} y^{6} \cdot 3 x}$

$$
\begin{aligned}
& =\sqrt{4 x^{6} y^{6}} \sqrt{3 x} \\
& =2 x^{3} y^{3} \sqrt{3 x}
\end{aligned}
$$

5. $\sqrt[3]{54 a^{6} b^{2} c^{4}}=\sqrt[3]{27 a^{6} c^{3} \cdot 2 b^{2} c}$

$$
\begin{aligned}
& =\sqrt[3]{27 a^{6} c^{3}} \sqrt[3]{2 b^{2} c} \\
& =3 a^{2} c \sqrt[3]{2 b^{2} c}
\end{aligned}
$$

The second property of radicals is used to simplify a radical that contains a fraction.

## EXAMPLE 6 Simplify $\sqrt{\frac{3}{4}}$.

SOLUTION Applying property 2 for radicals, we have

$$
\begin{array}{rlrl}
\sqrt{\frac{3}{4}} & =\frac{\sqrt{3}}{\sqrt{4}} & & \text { Property } 2 \\
& =\frac{\sqrt{3}}{2} & \sqrt{4}=2
\end{array}
$$

The last expression is in simplified form because it satisfies all three conditions for simplified form.

EXAMPLE 7 Write $\sqrt{\frac{5}{6}}$ in simplified form.
SOLUTION Proceeding as in Example 6, we have

$$
\sqrt{\frac{5}{6}}=\frac{\sqrt{5}}{\sqrt{6}}
$$

The resulting expression satisfies the second condition for simplified form since neither radical contains a fraction. It does, however, violate condition 3 since it has a radical in the denominator. Getting rid of the radical in the denominator is called rationalizing the denominator and is accomplished, in this case, by multiplying the numerator and denominator by $\sqrt{6}$ :

$$
\begin{aligned}
\frac{\sqrt{5}}{\sqrt{6}} & =\frac{\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} \\
& =\frac{\sqrt{30}}{\sqrt{6^{2}}} \\
& =\frac{\sqrt{30}}{6}
\end{aligned}
$$

## EXAMPLES Rationalize the denominator.

8. $\frac{4}{\sqrt{3}}=\frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$$
=\frac{4 \sqrt{3}}{\sqrt{3^{2}}}
$$

$$
=\frac{4 \sqrt{3}}{3}
$$

9. $\frac{2 \sqrt{3 x}}{\sqrt{5 y}}=\frac{2 \sqrt{3 x}}{\sqrt{5 y}} \cdot \frac{\sqrt{5 y}}{\sqrt{5 y}}$

$$
\begin{aligned}
& =\frac{2 \sqrt{15 x y}}{\sqrt{(5 y)^{2}}} \\
& =\frac{2 \sqrt{15 x y}}{5 y}
\end{aligned}
$$

When the denominator involves a cube root, we must multiply by a radical that will produce a perfect cube under the cube root sign in the denominator, as our next example illustrates.

EXAMPLE $10 \quad$ Rationalize the denominator in $\frac{7}{\sqrt[3]{4}}$.
SOLUTION Since $4=2^{2}$, we can multiply both numerator and denominator by $\sqrt[3]{2}$ and obtain $\sqrt[3]{2^{3}}$ in the denominator.

$$
\begin{aligned}
\frac{7}{\sqrt[3]{4}} & =\frac{7}{\sqrt[3]{2^{2}}} \\
& =\frac{7}{\sqrt[3]{2^{2}}} \cdot \frac{\sqrt[3]{\mathbf{2}}}{\sqrt[3]{\mathbf{2}}} \\
& =\frac{7 \sqrt[3]{2}}{\sqrt[3]{2^{3}}} \\
& =\frac{7 \sqrt[3]{2}}{2}
\end{aligned}
$$

EXAMPLE 11 Simplify $\sqrt{\frac{12 x^{5} y^{3}}{5 z}}$.
SOLUTION We use property 2 to write the numerator and denominator as two separate radicals:

$$
\sqrt{\frac{12 x^{5} y^{3}}{5 z}}=\frac{\sqrt{12 x^{5} y^{3}}}{\sqrt{5 z}}
$$

Simplifying the numerator, we have

$$
\begin{aligned}
\frac{\sqrt{12 x^{5} y^{3}}}{\sqrt{5 z}} & =\frac{\sqrt{4 x^{4} y^{2}} \sqrt{3 x y}}{\sqrt{5 z}} \\
& =\frac{2 x^{2} y \sqrt{3 x y}}{\sqrt{5 z}}
\end{aligned}
$$

To rationalize the denominator, we multiply the numerator and denominator by $\sqrt{5 z}$ :

$$
\begin{aligned}
\frac{2 x^{2} y \sqrt{3 x y}}{\sqrt{5 z}} \cdot \frac{\sqrt{5 z}}{\sqrt{5 z}} & =\frac{2 x^{2} y \sqrt{15 x y z}}{\sqrt{(5 z)^{2}}} \\
& =\frac{2 x^{2} y \sqrt{15 x y z}}{5 z}
\end{aligned}
$$

## The Square Root of a Perfect Square

So far in this chapter we have assumed that all our variables are nonnegative when they appear under a square root symbol. There are times, however, when this is not the case.

Consider the following two statements:

$$
\sqrt{3^{2}}=\sqrt{9}=3 \quad \text { and } \quad \sqrt{(-3)^{2}}=\sqrt{9}=3
$$

Whether we operate on 3 or -3 , the result is the same: Both expressions simplify to 3 . The other operation we have worked with in the past that produces the same result is absolute value; that is,

$$
|3|=3 \quad \text { and } \quad|-3|=3
$$

This leads us to the next property of radicals.

Property 3 for Radicals If $a$ is a real number, then $\sqrt{a^{2}}=|a|$.

The result of this discussion and property 3 is simply this:
If we know $a$ is positive, then $\sqrt{a^{2}}=a$.
If we know $a$ is negative, then $\sqrt{a^{2}}=|a|$.
If we don't know if $a$ is positive or negative, then $\sqrt{a^{2}}=|a|$.

EXAMPLES Simplify each expression. Do not assume the variables represent positive numbers.
12. $\sqrt{9 x^{2}}=3|x|$
13. $\sqrt{x^{3}}=|x| \sqrt{x}$
14. $\sqrt{x^{2}-6 x+9}=\sqrt{(x-3)^{2}}=|x-3|$
15. $\sqrt{x^{3}-5 x^{2}}=\sqrt{x^{2}(x-5)}=|x| \sqrt{x-5}$

As you can see, we must use absolute value symbols when we take a square root of a perfect square, unless we know the base of the perfect square is a positive number. The same idea holds for higher even roots, but not for odd roots. With odd roots, no absolute value symbols are necessary.

## EXAMPLES Simplify each expression.

16. $\sqrt[3]{(-2)^{3}}=\sqrt[3]{-8}=-2$
17. $\sqrt[3]{(-5)^{3}}=\sqrt[3]{-125}=-5$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-6, 12-17
B 7-11

We can extend this discussion to all roots as follows:

Extending Property 3 for Radicals If $a$ is a real number, then

$$
\begin{array}{lll}
\sqrt[n]{a^{n}}=|a| & \text { if } & n \text { is even } \\
\sqrt[n]{a^{n}}=a & \text { if } & n \text { is odd }
\end{array}
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Explain why this statement is false: "The square root of a sum is the sum of the square roots."
2. What is simplified form for an expression that contains a square root?
3. Why is it not necessarily true that $\sqrt{a^{2}}=a$ ?
4. What does it mean to rationalize the denominator in an expression?

Use property 1 for radicals to write each of the following expressions in simplified form. (Assume all variables are nonnegative through Problem 84.)

1. $\sqrt{8}$
2. $\sqrt{32}$
3. $\sqrt{98}$
4. $\sqrt{75}$
5. $\sqrt{288}$
6. $\sqrt{128}$
7. $\sqrt{80}$
8. $\sqrt{200}$
9. $\sqrt{48}$
10. $\sqrt{27}$
11. $\sqrt{675}$
12. $\sqrt{972}$
13. $\sqrt[3]{54}$
14. $\sqrt[3]{24}$
15. $\sqrt[3]{128}$
16. $\sqrt[3]{162}$
17. $\sqrt[3]{432}$
18. $\sqrt[3]{1,536}$
19. $\sqrt[5]{64}$
20. $\sqrt[4]{48}$
21. $\sqrt{18 x^{3}}$
22. $\sqrt{27 x^{5}}$
23. $\sqrt[4]{32 y^{7}}$
24. $\sqrt[5]{32 y^{7}}$
25. $\sqrt[3]{40 x^{4} y^{7}}$
26. $\sqrt[3]{128 x^{6} y^{2}}$
27. $\sqrt{48 a^{2} b^{3} c^{4}}$
28. $\sqrt{72 a^{4} b^{3} c^{2}}$
29. $\sqrt[3]{48 a^{2} b^{3} c^{4}}$
30. $\sqrt[3]{72 a^{4} b^{3} c^{2}}$
31. $\sqrt[5]{64 x^{8} y^{12}}$
32. $\sqrt[4]{32 x^{9} y^{10}}$
33. $\sqrt[5]{243 x^{7} y^{10} z^{5}}$
34. $\sqrt[5]{64 x^{8} y^{4} z^{11}}$

Substitute the given number into the expression $\sqrt{b^{2}-4 a c}$, and then simplify.
35. $a=2, b=-6, c=3$
36. $a=6, b=7, c=-5$
37. $a=1, b=2, c=6$
38. $a=2, b=5, c=3$
39. $a=\frac{1}{2}, b=-\frac{1}{2}, c=-\frac{5}{4}$
40. $a=\frac{7}{4}, b=-\frac{3}{4}, c=-2$

Rationalize the denominator in each of the following expressions.
41. $\frac{2}{\sqrt{3}}$
42. $\frac{3}{\sqrt{2}}$
43. $\frac{5}{\sqrt{6}}$
44. $\frac{7}{\sqrt{5}}$
45. $\sqrt{\frac{1}{2}}$
46. $\sqrt{\frac{1}{3}}$
47. $\sqrt{\frac{1}{5}}$
48. $\sqrt{\frac{1}{6}}$
49. $\frac{4}{\sqrt[3]{2}}$
50. $\frac{5}{\sqrt[3]{3}}$
51. $\frac{2}{\sqrt[3]{9}}$
52. $\frac{3}{\sqrt[3]{4}}$
53. $\sqrt[4]{\frac{3}{2 x^{2}}}$
54. $\sqrt[4]{\frac{5}{3 x^{2}}}$
55. $\sqrt[4]{\frac{8}{y}}$
56. $\sqrt[4]{\frac{27}{y}}$
57. $\sqrt[3]{\frac{4 x}{3 y}}$
58. $\sqrt[3]{\frac{7 x}{6 y}}$
59. $\sqrt[3]{\frac{2 x}{9 y}}$
60. $\sqrt[3]{\frac{5 x}{4 y}}$
61. $\sqrt[4]{\frac{1}{8 x^{3}}}$
62. $\sqrt[4]{\frac{8}{9 x^{3}}}$

Write each of the following in simplified form.
63. $\sqrt{\frac{27 x^{3}}{5 y}}$
64. $\sqrt{\frac{12 x^{5}}{7 y}}$
65. $\sqrt{\frac{75 x^{3} y^{2}}{2 z}}$
66. $\sqrt{\frac{50 x^{2} y^{3}}{3 z}}$
67. $\sqrt[3]{\frac{16 a^{4} b^{3}}{9 c}}$
68. $\sqrt[3]{\frac{54 a^{5} b^{4}}{25 c^{2}}}$
69. $\sqrt[3]{\frac{8 x^{3} y^{6}}{9 z}}$
70. $\sqrt[3]{\frac{27 x^{6} y^{3}}{2 z^{2}}}$
71. $\sqrt{\sqrt{x^{2}}}$
72. $\sqrt{\sqrt{2 x^{3}}}$
73. $\sqrt[3]{\sqrt{x y}}$
74. $\sqrt{\sqrt{4 x}}$
75. $\sqrt[3]{\sqrt[4]{a}}$
76. $\sqrt[6]{\sqrt[4]{x}}$
77. $\sqrt[3]{\sqrt[3]{6 x^{10}}}$
78. $\sqrt[5]{\sqrt{x^{14} y^{11} Z}}$
79. $\sqrt[4]{\sqrt[3]{a^{12} b^{24} c^{14}}}$
80. $\sqrt{\sqrt{4 a^{17}}}$
81. $\sqrt[3]{\sqrt[5]{3 a^{17} b^{16} c^{30}}}$
82. $\left(\sqrt{\sqrt[4]{x^{4} y^{8} Z^{9}}}\right)^{2}$
83. $\left(\sqrt{\sqrt[3]{8 a b^{6}}}\right)^{2}$
84. $\left(\sqrt[4]{\sqrt[3]{16 x^{8} y^{12} z^{3}}}\right)^{3}$

Simplify each expression. Do not assume the variables represent positive numbers.
85. $\sqrt{25 x^{2}}$
86. $\sqrt{49 x^{2}}$
87. $\sqrt{27 x^{3} y^{2}}$
88. $\sqrt{40 x^{3} y^{2}}$
89. $\sqrt{x^{2}-10 x+25}$
90. $\sqrt{x^{2}-16 x+64}$
91. $\sqrt{4 x^{2}+12 x+9}$
92. $\sqrt{16 x^{2}+40 x+25}$
93. $\sqrt{4 a^{4}+16 a^{3}+16 a^{2}}$
94. $\sqrt{9 a^{4}+18 a^{3}+9 a^{2}}$
95. $\sqrt{4 x^{3}-8 x^{2}}$
96. $\sqrt{18 x^{3}-9 x^{2}}$
97. Show that the statement $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ is not true by replacing $a$ with 9 and $b$ with 16 and simplifying both sides.
98. Find a pair of values for $a$ and $b$ that will make the statement $\sqrt{a+b}=\sqrt{a}+\sqrt{b}$ true.

## Applying the Concepts

99. Diagonal Distance The distance $d$ between opposite corners of a rectangular room with length $l$ and width $w$ is given by

$$
d=\sqrt{l^{2}+w^{2}}
$$

How far is it between opposite corners of a living room that measures 10 by 15 feet?

100. Radius of a Sphere The radius $r$ of a sphere with volume $V$ can be found by using the formula

$$
r=\sqrt[3]{\frac{3 V}{4 \pi}}
$$

Find the radius of a sphere with volume 9 cubic feet. Write your answer in simplified form. (Use $\frac{22}{7}$ for $\pi$.)

101. Diagonal of a Box The length of the diagonal of a rectangular box with length 1 , width $w$, and height $h$ is given by $d=\sqrt{l^{2}+w^{2}+h^{2}}$.

a. Find the length of the diagonal of a rectangular box that is 3 feet wide, 4 feet long, and 12 feet high.
b. Find the length of the diagonal of a rectangular box that is 2 feet wide, 4 feet high, and 6 feet long.
102. Distance to the Horizon If you are at a point $k$ miles above the surface of the Earth, the distance you can see, in miles, is approximated by the equation $d=\sqrt{8000 k+k^{2}}$.
a. How far can you see from a point that is 1 mile above the surface of the Earth?
b. How far can you see from a point that is 2 miles above the surface of the Earth?
c. How far can you see from a point that is 3 miles above the surface of the Earth?
103. Spiral of Roots Construct your own spiral of roots by using a ruler. Draw the first triangle by using two 1-inch lines. The first diagonal will have a length of $\sqrt{2}$ inches. Each new triangle will be formed by drawing a 1 -inch line segment at the end of the previous diagonal so that the angle formed is $90^{\circ}$.
104. Spiral of Roots Construct a spiral of roots by using line segments of length 2 inches. The length of the first diagonal will be $2 \sqrt{2}$ inches. The length of the second diagonal will be $2 \sqrt{3}$ inches.

## Maintaining Your Skills

Let $f(x)=\frac{1}{2} x+3$ and $g(x)=x^{2}-4$, and find
105. $f(0)$
106. $g(0)$
107. $g(2)$
108. $f(2)$
109. $f(-4)$
110. $g(-6)$
111. $f[g(2)]$
112. $g[f(2)]$

## Getting Ready for the Next Section

Simplify the following.
113. $35 x y^{2}-8 x y^{2}$
114. $20 a^{2} b+33 a^{2} b$
115. $\frac{1}{2} x+\frac{1}{3} x$
116. $\frac{2}{3} x+\frac{5}{8} x$

Write in simplified form for radicals.
117. $\sqrt{18}$
118. $\sqrt{8}$
119. $\sqrt{75 x y^{3}}$
120. $\sqrt{12 x y}$
121. $\sqrt[3]{8 a^{4} b^{2}}$
122. $\sqrt[3]{27 a b^{2}}$

## Extending the Concepts

Factor each radicand into the product of prime factors. Then simplify each radical.
123. $\sqrt[3]{8,640}$
124. $\sqrt{8,640}$
125. $\sqrt[3]{10,584}$
126. $\sqrt{10,584}$

Assume $a$ is a positive number, and rationalize each denominator.
127. $\frac{1}{\sqrt[10]{a^{3}}}$
128. $\frac{1}{\sqrt[12]{a^{7}}}$
129. $\frac{1}{\sqrt[20]{a^{11}}}$
130. $\frac{1}{\sqrt[15]{a^{13}}}$
131. Show that the two expressions $\sqrt{x^{2}+1}$ and $x+1$ are not, in general, equal to each other by graphing $y=\sqrt{x^{2}+1}$ and $y=x+1$ in the same viewing window.
132. Show that the two expressions $\sqrt{x^{2}+9}$ and $x+3$ are not, in general, equal to each other by graphing $y=\sqrt{x^{2}+9}$ and $y=x+3$ in the same viewing window.
133. Approximately how far apart are the graphs in Problem 131 when $x=2$ ?
134. Approximately how far apart are the graphs in Problem 132 when $x=2$ ?
135. For what value of $x$ are the expressions $\sqrt{x^{2}+1}$ and $x+1$ in Problem 131 equal?
136. For what value of $x$ are the expressions $\sqrt{x^{2}+9}$ and $x+3$ in Problem 132 equal?

## Q. 4 Addition and Subtraction of Radical Expressions

## OBJECTIVES

A Add and subtract radicals.
B Construct golden rectangles from squares.

We have been able to add and subtract polynomials by combining similar terms. The same idea applies to addition and subtraction of radical expressions.

DEFINITION Two radicals are said to be similar radicals if they have the same index and the same radicand.

The expressions $5 \sqrt[3]{7}$ and $-8 \sqrt[3]{7}$ are similar since the index is ${ }^{3}$ in both cases and the radicands are 7 . The expressions $3 \sqrt[4]{5}$ and $7 \sqrt[3]{5}$ are not similar since they have different indices, and the expressions $2 \sqrt[5]{8}$ and $3 \sqrt[5]{9}$ are not similar because the radicands are not the same.

## EXAMPLE 1 Combine $5 \sqrt{3}-4 \sqrt{3}+6 \sqrt{3}$.

SOLUTION All three radicals are similar. We apply the distributive property to get

$$
\begin{aligned}
5 \sqrt{3}-4 \sqrt{3}+6 \sqrt{3} & =(5-4+6) \sqrt{3} \\
& =7 \sqrt{3}
\end{aligned}
$$

## EXAMPLE 2 Combine $3 \sqrt{8}+5 \sqrt{18}$.

SOLUTION The two radicals do not seem to be similar. We must write each in simplified form before applying the distributive property.

$$
\begin{aligned}
3 \sqrt{8}+5 \sqrt{18} & =3 \sqrt{4 \cdot 2}+5 \sqrt{9 \cdot 2} \\
& =3 \sqrt{4} \sqrt{2}+5 \sqrt{9} \sqrt{2} \\
& =3 \cdot 2 \sqrt{2}+5 \cdot 3 \sqrt{2} \\
& =6 \sqrt{2}+15 \sqrt{2} \\
& =(6+15) \sqrt{2} \\
& =21 \sqrt{2}
\end{aligned}
$$

The result of Example 2 can be generalized to the following rule for sums and differences of radical expressions.

Rule To add or subtract radical expressions, put each in simplified form and apply the distributive property if possible. We can add only similar radicals. We must write each expression in simplified form for radicals before we can tell if the radicals are similar.

## EXAMPLE 3 Combine $7 \sqrt{75 x y^{3}}-4 y \sqrt{12 x y}$, where $x, y \geq 0$.

SOLUTION We write each expression in simplified form and combine similar radicals:

$$
\begin{aligned}
7 \sqrt{75 x y^{3}}-4 y \sqrt{12 x y} & =7 \sqrt{25 y^{2}} \sqrt{3 x y}-4 y \sqrt{4} \sqrt{3 x y} \\
& =35 y \sqrt{3 x y}-8 y \sqrt{3 x y} \\
& =(35 y-8 y) \sqrt{3 x y} \\
& =27 y \sqrt{3 x y}
\end{aligned}
$$

## EXAMPLE 4 Combine $10 \sqrt[3]{8 a^{4} b^{2}}+11 a \sqrt[3]{27 a b^{2}}$.

SOLUTION Writing each radical in simplified form and combining similar terms, we have

$$
\begin{aligned}
10 \sqrt[3]{8 a^{4} b^{2}}+11 a \sqrt[3]{27 a b^{2}} & =10 \sqrt[3]{8 a^{3}} \sqrt[3]{a b^{2}}+11 a \sqrt[3]{27} \sqrt[3]{a b^{2}} \\
& =20 a \sqrt[3]{a b^{2}}+33 a \sqrt[3]{a b^{2}} \\
& =53 a \sqrt[3]{a b^{2}}
\end{aligned}
$$

## EXAMPLE 5 Combine $\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}$.

SOLUTION We begin by writing the second term in simplified form.

$$
\begin{aligned}
\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}} & =\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} \\
& =\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{3} \\
& =\frac{1}{2} \sqrt{3}+\frac{1}{3} \sqrt{3} \\
& =\left(\frac{1}{2}+\frac{1}{3}\right) \sqrt{3} \\
& =\frac{5}{6} \sqrt{3}=\frac{5 \sqrt{3}}{6}
\end{aligned}
$$

EXAMPLE 6 Construct a golden rectangle from a square of side 4. Then show that the ratio of the length to the width is the golden ratio $\frac{1+\sqrt{5}}{2}$.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-5

B 6

The length of the diagonal $O B$ is found from the Pythagorean theorem.

$$
O B=\sqrt{2^{2}+4^{2}}=\sqrt{4+16}=\sqrt{20}=2 \sqrt{5}
$$

The ratio of the length to the width for the rectangle is the golden ratio.

$$
\text { Golden ratio }=\frac{C E}{E F}=\frac{2+2 \sqrt{5}}{4}=\frac{2(1+\sqrt{5})}{2 \cdot 2}=\frac{1+\sqrt{5}}{2}
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What are similar radicals?
2. When can we add two radical expressions?
3. What is the first step when adding or subtracting expressions containing radicals?
4. What is the golden ratio, and where does it come from?

## Problem Set 9.4

Combine the following expressions. (Assume any variables under an even root are nonnegative.)

1. $3 \sqrt{5}+4 \sqrt{5}$
2. $6 \sqrt{3}-5 \sqrt{3}$
3. $3 x \sqrt{7}-4 x \sqrt{7}$
4. $5 \sqrt[3]{10}-4 \sqrt[3]{10}$
5. $6 y \sqrt{a}+7 y \sqrt{a}$
6. $8 \sqrt[5]{6}-2 \sqrt[5]{6}+3 \sqrt[5]{6}$
7. $7 \sqrt[6]{7}-\sqrt[6]{7}+4 \sqrt[6]{7}$
8. $3 x \sqrt{2}-4 x \sqrt{2}+x \sqrt{2}$
9. $5 x \sqrt{6}-3 x \sqrt{6}-2 x \sqrt{6}$
10. $\sqrt{20}-\sqrt{80}+\sqrt{45}$
11. $\sqrt{8}-\sqrt{32}-\sqrt{18}$
12. $4 \sqrt{8}-2 \sqrt{50}-5 \sqrt{72}$
13. $\sqrt{48}-3 \sqrt{27}+2 \sqrt{75}$
14. $5 x \sqrt{8}+3 \sqrt{32 x^{2}}-5 \sqrt{50 x^{2}}$
15. $2 \sqrt{50 x^{2}}-8 x \sqrt{18}-3 \sqrt{72 x^{2}}$
16. $5 \sqrt[3]{16}-4 \sqrt[3]{54}$
17. $\sqrt[3]{81}+3 \sqrt[3]{24}$
18. $\sqrt[3]{x^{4} y^{2}}+7 x \sqrt[3]{x y^{2}}$
19. $2 \sqrt[3]{x^{8} y^{6}}-3 y^{2} \sqrt[3]{8 x^{8}}$
20. $5 a^{2} \sqrt{27 a b^{3}}-6 b \sqrt{12 a^{5} b}$
21. $9 a \sqrt{20 a^{3} b^{2}}+7 b \sqrt{45 a^{5}}$
22. $b \sqrt[3]{24 a^{5} b}+3 a \sqrt[3]{81 a^{2} b^{4}}$
23. $7 \sqrt[3]{a^{4} b^{3} c^{2}}-6 a b \sqrt[3]{a c^{2}}$
24. $5 x \sqrt[4]{3 y^{5}}+y \sqrt[4]{243 x^{4} y}+\sqrt[4]{48 x^{4} y^{5}}$
25. $x \sqrt[4]{5 x y^{8}}+y \sqrt[4]{405 x^{5} y^{4}}+y^{2} \sqrt[4]{80 x^{5}}$
26. $\frac{\sqrt{2}}{2}+\frac{1}{\sqrt{2}}$
27. $\frac{\sqrt{3}}{3}+\frac{1}{\sqrt{3}}$
28. $\frac{\sqrt{5}}{3}+\frac{1}{\sqrt{5}}$
29. $\frac{\sqrt{6}}{2}+\frac{1}{\sqrt{6}}$
30. $\sqrt{x}-\frac{1}{\sqrt{x}}$
31. $\sqrt{x}+\frac{1}{\sqrt{x}}$
32. $\frac{\sqrt{18}}{6}+\sqrt{\frac{1}{2}}+\frac{\sqrt{2}}{2}$
33. $\frac{\sqrt{12}}{6}+\sqrt{\frac{1}{3}}+\frac{\sqrt{3}}{3}$
34. $\sqrt{6}-\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{6}}$
35. $\sqrt{15}-\sqrt{\frac{3}{5}}+\sqrt{\frac{5}{3}}$
36. $\sqrt[3]{25}+\frac{3}{\sqrt[3]{5}}$
37. $\sqrt[4]{8}+\frac{1}{\sqrt[4]{2}}$

The following problems apply to what you have learned with the algebra of functions.
139. Let $f(x)=\sqrt{8 x}$ and $g(x)=\sqrt{72 x}$, then find
a. $f(x)+g(x)$
b. $f(x)-g(x)$
40. Let $f(x)=x+\sqrt{3}$ and $g(x)=x+2 \sqrt{3}$, then find
a. $f(x)+g(x)$
b. $f(x)-g(x)$
41. Let $f(x)=3 \sqrt{2 x}$ and $g(x)=\sqrt{2 x}$, then find
a. $f(x)+g(x)$
b. $f(x)-g(x)$
42. Let $f(x)=x \sqrt[3]{64}$ and $g(x)=x \sqrt{81}$, then find
a. $f(x)+g(x)$
b. $f(x)-g(x)$
43. Let $f(x)=x \sqrt{2}$ and $g(x)=2 x \sqrt{2}$, then find
a. $f(x)+g(x)$
b. $f(x)-g(x)$
44. Let $f(x)=5+2 \sqrt{5 x}$ and $g(x)=3 \sqrt{5 x}$, then find
a. $f(x)+g(x)$
b. $f(x)-g(x)$
45. Let $f(x)=\sqrt{2 x}-2$ and $g(x)=2 \sqrt{2 x}+5$, then find
a. $f(x)+g(x)$
b. $f(x)-g(x)$
46. Let $f(x)=1-2 \sqrt[3]{3 x}$ and $g(x)=2+3 \sqrt[3]{3 x}$, then find
a. $f(x)+g(x)$
b. $f(x)-g(x)$
47. Use a calculator to find a decimal approximation for $\sqrt{12}$ and for $2 \sqrt{3}$.
48. Use a calculator to find decimal approximations for $\sqrt{50}$ and $5 \sqrt{2}$.
49. Use a calculator to find a decimal approximation for $\sqrt{8}+\sqrt{18}$. Is it equal to the decimal approximation for $\sqrt{26}$ or $\sqrt{50}$ ?
50. Use a calculator to find a decimal approximation for $\sqrt{3}+\sqrt{12}$. Is it equal to the decimal approximation for $\sqrt{15}$ or $\sqrt{27}$ ?

## Applying the Concepts

51. Golden Rectangle Construct a golden rectangle from a square of side 8 . Then show that the ratio of the length to the width is the golden ratio $\frac{1+\sqrt{5}}{2}$.
52. Golden Rectangle Construct a golden rectangle from a square of side 10 . Then show that the ratio of the length to the width is the golden ratio $\frac{1+\sqrt{5}}{2}$.
53. Golden Rectangle To show that all golden rectangles have the same ratio of length to width, construct a golden rectangle from a square of side $2 x$. Then show that the ratio of the length to the width is the golden ratio.
54. Golden Rectangle To show that all golden rectangles have the same ratio of length to width, construct a golden rectangle from a square of side $x$. Then show that the ratio of the length to the width is the golden ratio.
55. Equilateral Triangles A triangle is equilateral if it has three equal sides. The triangle in the figure is equilateral with each side of length $2 x$. Find the ratio of the height to a side.

56. Pyramids Refer to the diagram of a square pyramid below. Find the ratio of the height $h$ of the pyramid to the altitude $a$.


## Maintaining Your Skills

Graph each inequality.
57. $2 x+3 y<6$
58. $2 x+y<-5$
59. $y \geq-3 x-4$
60. $y \geq 2 x-1$
61. $x \geq 3$
62. $y>-5$

Solve each inequality.
63. $-2 \leq 5 x-1 \leq 2$
64. $2 x-3<7$ or $2 x+1>15$

## Getting Ready for the Next Section

Simplify the following.
65. $3 \cdot 2$
66. $5 \cdot 7$
67. $(x+y)(4 x-y)$
68. $(2 x+y)(x-y)$
69. $(x+3)^{2}$
70. $(3 x-2 y)^{2}$
71. $(x-2)(x+2)$
72. $(2 x+5)(2 x-5)$

Simplify the following expressions.
73. $2 \sqrt{18}$
74. $5 \sqrt{36}$
75. $(\sqrt{6})^{2}$
76. $(\sqrt{2})^{2}$
77. $(3 \sqrt{x})^{2}$
78. $(2 \sqrt{y})^{2}$

Rationalize the denominator.
79. $\frac{\sqrt{3}}{\sqrt{2}}$
80. $\frac{\sqrt{5}}{\sqrt{6}}$

## Extending the Concepts

Assume all variables represent positive numbers. Simplify.
81. $\sqrt[5]{32 x^{5} y^{5}}-y \sqrt[3]{27 x^{3}}$
82. $\sqrt[6]{x^{4}}+4 \sqrt[3]{8 x^{2}}$
83. $3 \sqrt[9]{x^{9} y^{18} Z^{27}}-4 \sqrt[6]{x^{6} y^{12} Z^{18}}$
84. $4 a \sqrt{b^{4} c^{6}}+3 b \sqrt[3]{a^{3} b^{3} c^{9}}$
85. $3 c \sqrt[8]{4 a^{6} b^{18}}+b \sqrt[4]{32 a^{3} b^{5} c^{4}}$
86. $4 x \sqrt[6]{16 y^{6} z^{8}}-y \sqrt[3]{32 x^{3} z^{4}}$
87. $3 \sqrt[9]{8 a^{12} b^{9}}+b \sqrt[3]{16 a^{4}}-8 \sqrt[6]{4 a^{8} b^{6}}$
88. $-a c \sqrt{108 b c^{3}}-4 \sqrt[6]{27 a^{6} b^{3} c^{15}}+3 \sqrt[4]{9 a^{4} b^{2} c^{10}}$

## 9.5

## OBJECTIVES

A Multiply expressions containing radicals.

B Rationalize a denominator containing two terms.

## Multiplication and Division of Radical Expressions

We have worked with the golden rectangle more than once in this chapter. The following is one such golden rectangle.

By now you know that in any golden rectangle constructed from a square (of any size) the ratio of the length to the width will be

$$
\frac{1+\sqrt{5}}{2}
$$


which we call the golden ratio. What is interesting is that the smaller rectangle on the right, $B F E D$, is also a golden rectangle. We will use the mathematics developed in this section to confirm this fact.
In this section we will look at multiplication and division of expressions that contain radicals. As you will see, multiplication of expressions that contain radicals is very similar to multiplication of polynomials. The division problems in
this section are just an extension of the work we did previously when we rationalized denominators.

## EXAMPLE 1 Multiply $(3 \sqrt{5})(2 \sqrt{7})$.

SOLUTION We can rearrange the order and grouping of the numbers in this product by applying the commutative and associative properties. Following this, we apply property 1 for radicals and multiply:

$$
\begin{aligned}
(3 \sqrt{5})(2 \sqrt{7}) & =(3 \cdot 2)(\sqrt{5} \sqrt{7}) & & \text { Commutative and associative properties } \\
& =(3 \cdot 2)(\sqrt{5 \cdot 7}) & & \text { Property } \mathbf{1} \text { for radicals } \\
& =6 \sqrt{35} & & \text { Multiplication }
\end{aligned}
$$

In practice, it is not necessary to show the first two steps.
EXAMPLE 2 Multiply $\sqrt{3}(2 \sqrt{6}-5 \sqrt{12})$.
SOLUTION Applying the distributive property, we have

$$
\begin{aligned}
\sqrt{3}(2 \sqrt{6}-5 \sqrt{12}) & =\sqrt{3} \cdot 2 \sqrt{6}-\sqrt{3} \cdot 5 \sqrt{12} \\
& =2 \sqrt{18}-5 \sqrt{36}
\end{aligned}
$$

Writing each radical in simplified form gives

$$
\begin{aligned}
2 \sqrt{18}-5 \sqrt{36} & =2 \sqrt{9} \sqrt{2}-5 \sqrt{36} \\
& =6 \sqrt{2}-30
\end{aligned}
$$

## EXAMPLE 3 Multiply $(\sqrt{3}+\sqrt{5})(4 \sqrt{3}-\sqrt{5})$.

SOLUTION The same principle that applies when multiplying two binomials applies to this product. We must multiply each term in the first expression by each term in the second one. Any convenient method can be used. Let's use the FOIL method.

$$
\begin{aligned}
(\sqrt{3}+\sqrt{5})(4 \sqrt{3}-\sqrt{5}) & =\sqrt{3} \cdot 4 \sqrt{3}-\sqrt{3} \sqrt{5}+\sqrt{5} \cdot 4 \sqrt{3}-\sqrt{5} \sqrt{5} \\
& =4 \cdot 3-\sqrt{15}+4 \sqrt{15}-5 \\
& =12+3 \sqrt{15}-5 \\
& =7+3 \sqrt{15}
\end{aligned}
$$

EXAMPLE $4 \quad$ Expand and simplify $(\sqrt{x}+3)^{2}$.
SOLUTION 1 We can write this problem as a multiplication problem and proceed as we did in Example 3:

$$
\begin{aligned}
(\sqrt{x}+3)^{2} & =(\sqrt{x}+3)(\sqrt{x}+3) \\
& \mathrm{F} \quad \mathrm{O} \quad \mathrm{I} \quad \mathrm{~L} \\
& =\sqrt{x} \cdot \sqrt{x}+3 \sqrt{x}+3 \sqrt{x}+3 \cdot 3 \\
& =x+3 \sqrt{x}+3 \sqrt{x}+9 \\
& =x+6 \sqrt{x}+9
\end{aligned}
$$

SOLUTION 2 We can obtain the same result by applying the formula for the square of a sum: $(a+b)^{2}=a^{2}+2 a b+b^{2}$.

$$
\begin{aligned}
(\sqrt{x}+3)^{2} & =(\sqrt{x})^{2}+2(\sqrt{x})(3)+3^{2} \\
& =x+6 \sqrt{x}+9
\end{aligned}
$$

EXAMPLE 5 Expand $(3 \sqrt{x}-2 \sqrt{y})^{2}$ and simplify the result.
SOLUTION Let's apply the formula for the square of a difference, $(a-b)^{2}=a^{2}-2 a b+b^{2}$.

$$
\begin{aligned}
(3 \sqrt{x}-2 \sqrt{y})^{2} & =(3 \sqrt{x})^{2}-2(3 \sqrt{x})(2 \sqrt{y})+(2 \sqrt{y})^{2} \\
& =9 x-12 \sqrt{x y}+4 y
\end{aligned}
$$

EXAMPLE $6 \quad$ Expand and simplify $(\sqrt{x+2}-1)^{2}$.
SOLUTION Applying the formula $(a-b)^{2}=a^{2}-2 a b+b^{2}$, we have

$$
\begin{align*}
(\sqrt{x+2}-1)^{2} & =(\sqrt{x+2})^{2}-2 \sqrt{x+2}(1)+1^{2} \\
& =x+2-2 \sqrt{x+2}+1 \\
& =x+3-2 \sqrt{x+2} \tag{111112}
\end{align*}
$$

EXAMPLE $7 \quad$ Multiply $(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})$.
SOLUTION We notice the product is of the form $(a+b)(a-b)$, which always gives the difference of two squares, $a^{2}-b^{2}$ :

$$
\begin{aligned}
(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2}) & =(\sqrt{6})^{2}-(\sqrt{2})^{2} \\
& =6-2 \\
& =4
\end{aligned}
$$

The two expressions $(\sqrt{6}+\sqrt{2})$ and $(\sqrt{6}-\sqrt{2})$ are called conjugates. In genal, the conjugate of $\sqrt{a}+\sqrt{b}$ is $\sqrt{a}-\sqrt{b}$. If $a$ and $b$ are integers, multiplying njugates of this form always produces a rational number.
Division with radical expressions is the same as rationalizing the denomina$r$. In a previous section we were able to divide $\sqrt{3}$ by $\sqrt{2}$ by rationalizing the nominator:

$$
\frac{\sqrt{3}}{\sqrt{2}}=\frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{\mathbf{2}}}{\sqrt{2}}=\frac{\sqrt{6}}{2}
$$

We can accomplish the same result with expressions such as

$$
\frac{6}{\sqrt{5}-\sqrt{3}}
$$

by multiplying the numerator and denominator by the conjugate of the denominator.

EXAMPLE 8 Divide $\frac{6}{\sqrt{5}-\sqrt{3}}$. (Rationalize the denominator.)
SOLUTION Since the product of two conjugates is a rational number, we multiply the numerator and denominator by the conjugate of the denominator.

$$
\begin{aligned}
\frac{6}{\sqrt{5}-\sqrt{3}} & =\frac{6}{\sqrt{5}-\sqrt{3}} \cdot \frac{(\sqrt{5}+\sqrt{3})}{(\sqrt{5}+\sqrt{3})} \\
& =\frac{6 \sqrt{5}+6 \sqrt{3}}{(\sqrt{5})^{2}-(\sqrt{3})^{2}} \\
& =\frac{6 \sqrt{5}+6 \sqrt{3}}{5-3} \\
& =\frac{6 \sqrt{5}+6 \sqrt{3}}{2}
\end{aligned}
$$

The numerator and denominator of this last expression have a factor of 2 in common. We can reduce to lowest terms by factoring 2 from the numerator and then dividing both the numerator and denominator by 2 :

$$
\begin{aligned}
& =\frac{2(3 \sqrt{5}+3 \sqrt{3})}{2} \\
& =3 \sqrt{5}+3 \sqrt{3}
\end{aligned}
$$

EXAMPLE 9 Rationalize the denominator $\frac{\sqrt{5}-2}{\sqrt{5}+2}$.
SOLUTION To rationalize the denominator, we multiply the numerator and denominator by the conjugate of the denominator:

$$
\begin{aligned}
\frac{\sqrt{5}-2}{\sqrt{5}+2} & =\frac{\sqrt{5}-2}{\sqrt{5}+2} \cdot \frac{(\sqrt{5}-2)}{(\sqrt{5}-2)} \\
& =\frac{5-2 \sqrt{5}-2 \sqrt{5}+4}{(\sqrt{5})^{2}-2^{2}} \\
& =\frac{9-4 \sqrt{5}}{5-4} \\
& =\frac{9-4 \sqrt{5}}{1} \\
& =9-4 \sqrt{5}
\end{aligned}
$$

EXAMPLE 10 A golden rectangle constructed from a square of side 2 is shown in Figure 1. Show that the smaller rectangle $B D E F$ is also a golden rectangle by finding the ratio of its length to its width.


FIGURE 1

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

$$
\begin{array}{lc}
\text { A } & 1-7 \\
\text { B } & 8-10
\end{array}
$$

SOLUTION First we find expressions for the length and width of the smaller rectangle.

$$
\begin{aligned}
\text { Length } & =E F=2 \\
\text { Width } & =D E=\sqrt{5}-1
\end{aligned}
$$

Next, we find the ratio of length to width.

$$
\text { Ratio of length to width }=\frac{E F}{D E}=\frac{2}{\sqrt{5}-1}
$$

To show that the small rectangle is a golden rectangle, we must show that the ratio of length to width is the golden ratio. We do so by rationalizing the denominator.

$$
\begin{aligned}
\frac{2}{\sqrt{5}-1} & =\frac{2}{\sqrt{5}-1} \cdot \frac{\sqrt{5}+\mathbf{1}}{\sqrt{5}+\mathbf{1}} \\
& =\frac{2(\sqrt{5}+1)}{5-1} \\
& =\frac{2(\sqrt{5}+1)}{4} \\
& =\frac{\sqrt{5}+1}{2} \quad \text { Divide out common factor } \mathbf{2}
\end{aligned}
$$

Since addition is commutative, this last expression is the golden ratio. Therefore, the small rectangle in Figure 1 is a golden rectangle.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Explain why $(\sqrt{5}+\sqrt{2})^{2} \neq 5+2$.
2. Explain in words how you would rationalize the denominator in the expression $\frac{\sqrt{3}}{\sqrt{5}-\sqrt{2}}$
3. What are conjugates?
4. What result is guaranteed when multiplying radical expressions that are conjugates?

## Problem Set 9.5

Multiply. (Assume all expressions appearing under a square root symbol represent nonnegative numbers throughout this problem set.)

1. $\sqrt{6} \sqrt{3}$
2. $\sqrt{6} \sqrt{2}$
3. $(2 \sqrt{3})(5 \sqrt{7})$
4. $(4 \sqrt{6})(2 \sqrt{15})(3 \sqrt{10})$
5. $(4 \sqrt{35})(2 \sqrt{21})(5 \sqrt{15})$
6. $(3 \sqrt[3]{3})(6 \sqrt[3]{9})$
7. $(2 \sqrt[3]{2})(6 \sqrt[3]{4})$
8. $\sqrt{3}(\sqrt{2}-3 \sqrt{3})$
9. $\sqrt{2}(5 \sqrt{3}+4 \sqrt{2})$
10. $6 \sqrt[3]{4}(2 \sqrt[3]{2}+1)$
11. $7 \sqrt[3]{5}(3 \sqrt[3]{25}-2)$
12. $\sqrt[3]{4}(\sqrt[3]{2}+\sqrt[3]{6})$
13. $\sqrt[3]{5}(\sqrt[3]{8}-\sqrt[3]{25})$
14. $\sqrt[3]{x}\left(\sqrt[3]{\mathrm{x}^{2} \mathrm{y}^{4}}+\sqrt[3]{\mathrm{x}^{5} \mathrm{y}}\right)$
15. $\sqrt[3]{x^{2} y}\left(2 \sqrt[3]{x^{4} y^{2}}-\sqrt[3]{x^{2} y^{4}}\right)$
16. $\sqrt[4]{2 x^{3}}\left(\sqrt[4]{8 x^{6}}+\sqrt[4]{16 x^{9}}\right)$
17. $\sqrt[4]{8 y^{2}}\left(\sqrt[4]{6 y^{6}}-\sqrt[4]{8 y^{9}}\right)$
18. $(\sqrt{3}+\sqrt{2})(3 \sqrt{3}-\sqrt{2})$
19. $(\sqrt{5}-\sqrt{2})(3 \sqrt{5}+2 \sqrt{2})$
20. $(\sqrt{x}+5)(\sqrt{x}-3)$
21. $(\sqrt{x}+4)(\sqrt{x}+2)$
22. $(3 \sqrt{6}+4 \sqrt{2})(\sqrt{6}+2 \sqrt{2})$
23. $(\sqrt{7}-3 \sqrt{3})(2 \sqrt{7}-4 \sqrt{3})$
24. $(\sqrt{3}+4)^{2}$
25. $(\sqrt{5}-2)^{2}$
26. $(\sqrt{x}-3)^{2}$
27. $(\sqrt{x}+4)^{2}$
28. $(2 \sqrt{a}-3 \sqrt{b})^{2}$
29. $(5 \sqrt{a}-2 \sqrt{b})^{2}$
30. $(\sqrt{x-4}+2)^{2}$
31. $(\sqrt{x-3}+2)^{2}$
32. $(\sqrt{x-5}-3)^{2}$
33. $(\sqrt{x-3}-4)^{2}$
34. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$
35. $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$
36. $(\sqrt{a}+7)(\sqrt{a}-7)$
37. $(\sqrt{a}+5)(\sqrt{a}-5)$
38. $(5-\sqrt{x})(5+\sqrt{x})$
39. $(3-\sqrt{x})(3+\sqrt{x})$
40. $(\sqrt{x+3}+5)(\sqrt{x+3}-5)$
41. $(\sqrt{3}+1)^{3}$
42. $(\sqrt{5}-2)^{3}$
43. $(\sqrt[3]{3}+\sqrt[3]{2})(\sqrt[3]{9}+\sqrt[3]{4})$
44. $(\sqrt[3]{5}+\sqrt[3]{3})(\sqrt[3]{9}+\sqrt[3]{25})$
45. $\left(\sqrt[3]{x^{5}}+\sqrt[3]{y}\right)\left(\sqrt[3]{x}+\sqrt[3]{y^{2}}\right)$
46. $\left(\sqrt[3]{x^{7}}-\sqrt[3]{y^{4}}\right)\left(\sqrt[3]{x^{2}}-\sqrt[3]{y^{2}}\right)$

Rationalize the denominator in each of the following.
49. $\frac{1}{\sqrt{2}}$
50. $\frac{x}{\sqrt{2}}$
51. $\frac{1}{\sqrt{x}}$
52. $\frac{3}{\sqrt{x}}$
53. $\frac{4}{\sqrt{3}}$
54. $\frac{4}{\sqrt{2 x}}$
55. $\frac{2 x}{\sqrt{6}}$
56. $\frac{6 x}{\sqrt{3}}$
57. $\frac{4}{\sqrt{10 x}}$
58. $\sqrt{\frac{4 x^{2}}{3 x}}$
59. $\frac{2}{\sqrt{8}}$
60. $\sqrt{\frac{16 x y}{4 x^{2} y}}$
61. $\sqrt{\frac{32 x^{3} y}{4 x y^{2}}}$
62. $\frac{4}{\sqrt{6 x^{2} y^{5}}}$
63. $\sqrt{\frac{12 a^{3} b^{3} c}{8 a b^{5} c^{2}}}$
65. $\sqrt[3]{\frac{16 a^{4} b^{2} c^{4}}{2 a b^{3} c}}$
67. $\frac{\sqrt{2}}{\sqrt{6}-\sqrt{2}}$
69. $\frac{\sqrt{5}}{\sqrt{5}+1}$
71. $\frac{\sqrt{x}}{\sqrt{x}-3}$
72. $\frac{\sqrt{x}}{\sqrt{x}+2}$
73. $\frac{\sqrt{5}}{2 \sqrt{5}-3}$
74. $\frac{\sqrt{7}}{3 \sqrt{7}-2}$
75. $\frac{3}{\sqrt{x}-\sqrt{y}}$
76. $\frac{2}{\sqrt{x}+\sqrt{y}}$
77. $\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$
78. $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$
79. $\frac{\sqrt{7}-2}{\sqrt{7}+2}$
80. $\frac{\sqrt{11}+3}{\sqrt{11}-3}$
81. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$
82. $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$
83. $\frac{\sqrt{x}+2}{\sqrt{x}-2}$
84. $\frac{\sqrt{x}-3}{\sqrt{x}+3}$
85. $\frac{2 \sqrt{3}-\sqrt{7}}{3 \sqrt{3}+\sqrt{7}}$
86. $\frac{5 \sqrt{6}+2 \sqrt{2}}{\sqrt{6}-\sqrt{2}}$
87. $\frac{3 \sqrt{x}+2}{1+\sqrt{x}}$
88. $\frac{5 \sqrt{x}-1}{2+\sqrt{x}}$
89. $\frac{2}{\sqrt{3}}+\sqrt{12}$
90. $\sqrt{2}+\frac{5}{\sqrt{8}}$
91. $\frac{1}{\sqrt{5}}+\sqrt{20}$
92. $\frac{4}{\sqrt{2}}+\sqrt{72}$
93. $\frac{6}{\sqrt{12}}-\sqrt{75}$
94. $\frac{5}{\sqrt{7}}-\sqrt{63}+\sqrt{28}$
95. $\frac{1}{\sqrt{3}}+\sqrt{48}+\frac{4}{\sqrt{12}}$
96. $\frac{4}{\sqrt{2}}+\sqrt{8}+\frac{10}{\sqrt{50}}$
97. Show that the product $(\sqrt[3]{2}+\sqrt[3]{3})(\sqrt[3]{4}-\sqrt[3]{6}+\sqrt[3]{9})$ is 5 .
98. Show that the product $(\sqrt[3]{x}+2)\left(\sqrt[3]{x^{2}}-2 \sqrt[3]{x}+4\right)$ is $x+8$.

Each statement below is false. Correct the right side of each one.
99. $5(2 \sqrt{3})=10 \sqrt{15}$
100. $3(2 \sqrt{x})=6 \sqrt{3 x}$
101. $(\sqrt{x}+3)^{2}=x+9$
102. $(\sqrt{x}-7)^{2}=x-49$
103. $(5 \sqrt{3})^{2}=15$
104. $(3 \sqrt{5})^{2}=15$

## Applying the Concepts

105. Gravity If an object is dropped from the top of a 100 -foot building, the amount of time $t$ (in seconds) that it takes for the object to be $h$ feet from the ground is given by the formula

$$
t=\frac{\sqrt{100-h}}{4}
$$

How long does it take before the object is 50 feet from the ground? How long does it take to reach the ground? (When it is on the ground, $h$ is 0 .)
106. Gravity Use the formula given in Problem 105 to determine $h$ if $t$ is 1.25 seconds.
107. Golden Rectangle Rectangle $A C E F$ in Figure 2 is a golden rectangle. If side $A C$ is 6 inches, show that the smaller rectangle $B D E F$ is also a golden rectangle.


FIGURE 2
108. Golden Rectangle Rectangle $A C E F$ in Figure 2 is a golden rectangle. If side $A C$ is 1 inch, show that the smaller rectangle $B D E F$ is also a golden rectangle.
109. Golden Rectangle If side $A C$ in Figure 2 is $2 x$, show the rectangle $B D E F$ is a golden rectangle.
110. Golden Rectangle If side $A C$ in Figure 2 is $x$, show that rectangle $B D E F$ is a golden rectangle.

## Maintaining Your Skills

111. If $y$ varies directly with square of $x$, and $y$ is 75 when $x$ is 5 , find $y$ when $x$ is 7 .
112. Suppose $y$ varies directly with the cube of $x$. If $y$ is 16 when $x$ is 2 , find $y$ when $x$ is 3 .
113. Suppose $y$ varies inversely with $x$. If $y$ is 10 when $x$ is 25 , find $x$ when $y$ is 5 .
114. If $y$ varies inversely with the cube of $x$, and $y$ is 2 when $x$ is 2 , find $y$ when $x$ is 4 .
115. Suppose $z$ varies jointly with $x$ and the square of $y$. If $z$ is 40 when $x$ is 5 and $y$ is 2 , find $z$ when $x$ is 2 and $y$ is 5 .
116. Suppose $z$ varies jointly with $x$ and the cube of $y$. If $z$ is 48 when $x$ is 3 and $y$ is 2 , find $z$ when $x$ is 4 and $y$ is $\frac{1}{2}$.

## Getting Ready for the Next Section

Simplify.
117. $(t+5)^{2}$
119. $\sqrt{x} \cdot \sqrt{X}$

Solve.
121. $3 x+4=5^{2}$
122. $4 x-7=3^{2}$
123. $t^{2}+7 t+12=0$
124. $x^{2}-3 x-10=0$
125. $t^{2}+10 t+25=t+7$
126. $x^{2}-4 x+4=x-2$
127. $(x+4)^{2}=x+6$
128. $(x-6)^{2}=x-4$

## Extending the Concepts

Rationalize the denominator in each of the following.
129. $\frac{x}{\sqrt{x-2}+4}$
130. $\frac{3}{\sqrt{x-4}-7}$
131. $\frac{x}{\sqrt{x+5}-5}$
132. $\frac{2}{\sqrt{2 x-1}+3}$
133. $\frac{3 x}{\sqrt{5 x}+x}$
134. $\frac{4 x}{\sqrt{5 x^{3}}+2 x}$


## OBJECTIVES

A Solve equations containing radicals.

B Graph simple square root and cube root equations in two variables.

## Equations with Radicals

This section is concerned with solving equations that involve one or more radicals. The first step in solving an equation that contains a radical is to eliminate the radical from the equation. To do so, we need an additional property.

Squaring Property of Equality If both sides of an equation are squared, the solutions to the original equation are solutions to the resulting equation.

We will never lose solutions to our equations by squaring both sides. We may, however, introduce extraneous solutions. Extraneous solutions satisfy the equation obtained by squaring both sides of the original equation, but they do not satisfy the original equation.

We know that if two real numbers $a$ and $b$ are equal, then so are their squares:

$$
\begin{array}{rlrl}
\text { If } & & a & =b \\
\text { then } & a^{2} & =b^{2}
\end{array}
$$

However, extraneous solutions are introduced when we square opposites; that is, even though opposites are not equal, their squares are. For example,

$$
\begin{aligned}
5 & =-5 & & \text { A false statement } \\
(5)^{2} & =(-5)^{2} & & \text { Square both sides } \\
25 & =25 & & \text { A true statement }
\end{aligned}
$$

## Note

The fact that there is no solution to the equation in Example 2 was obvious to begin with. Notice that the left side of the equation is the positive square root of $4 x-7$, which must be a nonnegative number. The right side of the equation is -3 . Since we cannot have a number that is nonnegative equal to a negative number, there is no solution to the equation.

We are free to square both sides of an equation any time it is convenient. We must be aware, however, that doing so may introduce extraneous solutions. We must, therefore, check all our solutions in the original equation if at any time we square both sides of the original equation.

## EXAMPLE 1 Solve for $x: \sqrt{3 x+4}=5$.

SOLUTION We square both sides and proceed as usual:

$$
\begin{aligned}
\sqrt{3 x+4} & =5 \\
(\sqrt{3 x+4})^{2} & =5^{2} \\
3 x+4 & =25 \\
3 x & =21 \\
x & =7
\end{aligned}
$$

Checking $x=7$ in the original equation, we have

$$
\begin{array}{r}
\sqrt{3(7)+4} \stackrel{?}{=} 5 \\
\sqrt{21+4} \stackrel{?}{=} 5 \\
\sqrt{25} \stackrel{?}{=} 5 \\
5=5
\end{array}
$$

The solution $x=7$ satisfies the original equation.

EXAMPLE 2 Solve $\sqrt{4 x-7}=-3$.
SOLUTION Squaring both sides, we have

$$
\begin{aligned}
\sqrt{4 x-7} & =-3 \\
(\sqrt{4 x-7})^{2} & =(-3)^{2} \\
4 x-7 & =9 \\
4 x & =16 \\
x & =4
\end{aligned}
$$

Checking $x=4$ in the original equation gives

$$
\begin{array}{r}
\sqrt{4(4)-7} \stackrel{?}{=}-3 \\
\sqrt{16-7} \stackrel{?}{=}-3 \\
\sqrt{9} \stackrel{?}{\underline{?}}-3 \\
3
\end{array}=-3
$$

The solution $x=4$ produces a false statement when checked in the original equation. Since $x=4$ was the only possible solution, there is no solution to the original equation. The possible solution $x=4$ is an extraneous solution. It satisfies the equation obtained by squaring both sides of the original equation, but it does not satisfy the original equation.

## EXAMPLE 3 Solve $\sqrt{5 x-1}+3=7$.

SOLUTION We must isolate the radical on the left side of the equation. If we attempt to square both sides without doing so, the resulting equation will also contain a radical. Adding -3 to both sides, we have

$$
\begin{array}{r}
\sqrt{5 x-1}+3=7 \\
\sqrt{5 x-1}=4
\end{array}
$$

We can now square both sides and proceed as usual:

$$
\begin{aligned}
(\sqrt{5 x-1})^{2} & =4^{2} \\
5 x-1 & =16 \\
5 x & =17 \\
x & =\frac{17}{5}
\end{aligned}
$$

Checking $x=\frac{17}{5}$, we have

$$
\begin{array}{r}
\sqrt{5\left(\frac{17}{5}\right)-1}+3 \stackrel{?}{=} 7 \\
\sqrt{17-1}+3 \stackrel{?}{=} 7 \\
\sqrt{16}+3 \stackrel{?}{=} 7 \\
4+3 \stackrel{?}{=} 7 \\
7=7
\end{array}
$$

## EXAMPLE 4 Solve $t+5=\sqrt{t+7}$.

SOLUTION This time, squaring both sides of the equation results in a quadratic equation:

| $(t+5)^{2}$ | $=(\sqrt{t+7})^{2}$ |  | Square both sides |
| ---: | :--- | ---: | :--- |
| $t^{2}+10 t+25$ | $=t+7$ |  |  |
| $t^{2}+9 t+18$ | $=0$ |  | Standard form |
| $(t+3)(t+6)$ | $=0$ |  | Factor the left side |
| $t+3=0$ |  | or $t+6$ | $=0$ |
| $t$ | $=-3$ |  | or |$\quad$| Set factors equal to 0 |  |
| :--- | :--- |
|  | $=-6$ |

We must check each solution in the original equation:

$$
\begin{array}{rlrl}
\text { Check } t & =-3 & \text { Check } t & =-6 \\
-3+5 & \stackrel{?}{=} \sqrt{-3+7} & -6+5 \stackrel{?}{=} \sqrt{-6+7} \\
2 \stackrel{?}{=} \sqrt{4} & -1 \stackrel{?}{=} \sqrt{1} \\
2 & =2 & -1 & =1 \\
\text { A true statement } & \text { A false statement }
\end{array}
$$

Since $t=-6$ does not check, our only solution is $t=-3$.

## Note

It is very important that you realize that the square of $(\sqrt{x}-3)$ is not $x+9$. Remember, when we square a difference with two terms, we use the formula

$$
(a-b)^{2}=a^{2}-2 a b+b^{2}
$$

Applying this formula to $(\sqrt{x}-3)^{2}$, we have

$$
\begin{aligned}
(\sqrt{x}-3)^{2}= & (\sqrt{x})^{2}-2(\sqrt{x})(3)+3^{2} \\
& =x-6 \sqrt{x}+9
\end{aligned}
$$

## EXAMPLE 5 Solve $\sqrt{x-3}=\sqrt{x}-3$.

SOLUTION We begin by squaring both sides. Note what happens when we square the right side of the equation, and compare the square of the right side with the square of the left side. You must convince yourself that these results are correct. (The note in the margin will help if you are having trouble convincing yourself that what is written below is true.)

$$
\begin{aligned}
(\sqrt{x-3})^{2} & =(\sqrt{x}-3)^{2} \\
x-3 & =x-6 \sqrt{x}+9
\end{aligned}
$$

Now we still have a radical in our equation, so we will have to square both sides again. Before we do, though, let's isolate the remaining radical.

$$
\begin{aligned}
x-3 & =x-6 \sqrt{x}+9 & & \\
-3 & =-6 \sqrt{x}+9 & & \text { Add }-x \text { to each side } \\
-12 & =-6 \sqrt{x} & & \text { Add }-9 \text { to each side } \\
2 & =\sqrt{x} & & \text { Divide each side by }-6 \\
4 & =x & & \text { Square each side }
\end{aligned}
$$

Our only possible solution is $x=4$, which we check in our original equation as follows:

$$
\begin{aligned}
\sqrt{4-3} & \stackrel{?}{=} \sqrt{4}-3 \\
\sqrt{1} & \stackrel{?}{=} 2-3 \\
1 & =-1 \quad \text { A false statement }
\end{aligned}
$$

Substituting 4 for $x$ in the original equation yields a false statement. Since 4 was our only possible solution, there is no solution to our equation.

Here is another example of an equation for which we must apply our squaring property twice before all radicals are eliminated.

## EXAMPLE $6 \quad$ Solve $\sqrt{x+1}=1-\sqrt{2 x}$.

SOLUTION This equation has two separate terms involving radical signs. Squaring both sides gives

$$
\begin{aligned}
x+1 & =1-2 \sqrt{2 x}+2 x & & \\
-x & =-2 \sqrt{2 x} & & \text { Add }-\mathbf{2 x} \text { and }-\mathbf{1} \text { to both sides } \\
x^{2} & =4(2 x) & & \text { Square both sides } \\
x^{2}-8 x & =0 & & \text { Standard form }
\end{aligned}
$$

Our equation is a quadratic equation in standard form. To solve for $x$, we factor the left side and set each factor equal to 0 .

$$
\begin{aligned}
x(x-8) & =0 & & \text { Factor left side } \\
x=0 \quad \text { or } \quad x-8 & =0 & & \text { Set factors equal to } 0 \\
x & =8 & &
\end{aligned}
$$

Since we squared both sides of our equation, we have the possibility that one or both of the solutions are extraneous. We must check each one in the original equation:

$$
\begin{gathered}
\text { Check } x=8 \\
\begin{array}{c}
\sqrt{8+1} \stackrel{?}{=} 1-\sqrt{2 \cdot 8} \\
\sqrt{9} \stackrel{?}{=} 1-\sqrt{16} \\
3 \stackrel{?}{=} 1-4 \\
3=-3 \\
\text { A false statement }
\end{array}
\end{gathered}
$$

$$
\begin{gathered}
\text { Check } x=0 \\
\begin{aligned}
& \sqrt{0+1} \stackrel{?}{=} 1-\sqrt{2 \cdot 0} \\
& \sqrt{1} \stackrel{?}{=} 1-\sqrt{0} \\
& 1 \stackrel{?}{=} 1-0 \\
& 1=1 \\
& \text { A true statement }
\end{aligned}
\end{gathered}
$$

Since $x=8$ does not check, it is an extraneous solution. Our only solution is $x=0$.

## EXAMPLE 7 Solve $\sqrt{x+1}=\sqrt{x+2}-1$.

SOLUTION Squaring both sides we have

$$
\begin{aligned}
(\sqrt{x+1})^{2} & =(\sqrt{x+2}-1)^{2} \\
x+1 & =x+2-2 \sqrt{x+2}+1
\end{aligned}
$$

Once again we are left with a radical in our equation. Before we square each side again, we must isolate the radical on the right side of the equation.

$$
\begin{aligned}
x+1 & =x+3-2 \sqrt{x+2} & & \text { Simplify the right side } \\
1 & =3-2 \sqrt{x+2} & & \text { Add }-x \text { to each side } \\
-2 & =-2 \sqrt{x+2} & & \text { Add }-\mathbf{3} \text { to each side } \\
1 & =\sqrt{x+2} & & \text { Divide each side by } \mathbf{- 2} \\
1 & =x+2 & & \text { Square both sides } \\
-1 & =x & & \text { Add }-\mathbf{2} \text { to each side }
\end{aligned}
$$

Checking our only possible solution, $x=-1$, in our original equation, we have

$$
\begin{aligned}
\sqrt{-1+1} & \stackrel{?}{=} \sqrt{-1+2}-1 \\
\sqrt{0} & \stackrel{?}{=} \sqrt{1}-1 \\
0 & \stackrel{?}{=} 1-1
\end{aligned}
$$

$$
0=0 \quad \text { A true statement }
$$

Our solution checks.

It is also possible to raise both sides of an equation to powers greater than 2. We only need to check for extraneous solutions when we raise both sides of an equation to an even power. Raising both sides of an equation to an odd power will not produce extraneous solutions.

## EXAMPLE 8 Solve $\sqrt[3]{4 x+5}=3$.

SOLUTION Cubing both sides we have

$$
\begin{aligned}
(\sqrt[3]{4 x+5})^{3} & =3^{3} \\
4 x+5 & =27 \\
4 x & =22 \\
x & =\frac{22}{4} \\
x & =\frac{11}{2}
\end{aligned}
$$

We do not need to check $x=\frac{11}{2}$ since we raised both sides to an odd power.

We end this section by looking at graphs of some equations that contain radicals.

## EXAMPLE $9 \quad$ Graph $y=\sqrt{x}$ and $y=\sqrt[3]{x}$.

SOLUTION The graphs are shown in Figures 1 and 2. Notice that the graph of $y=\sqrt{x}$ appears in the first quadrant only because in the equation $y=\sqrt{x}, x$ and $y$ cannot be negative.

The graph of $y=\sqrt[3]{x}$ appears in quadrants 1 and 3 since the cube root of a positive number is also a positive number and the cube root of a negative number is a negative number; that is, when $x$ is positive, $y$ will be positive and when $x$ is negative, $y$ will be negative.
The graphs of both equations will contain the origin since $y=0$ when $x=0$ in both equations.

| $x$ | $y$ |
| :---: | :---: |
| -4 | undefined |
| -1 | undefined |
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |
| 9 | 3 |
| 16 | 4 |



FIGURE 1


FIGURE 2

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A $1-8$
B 9

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the squaring property of equality?
2. Under what conditions do we obtain extraneous solutions to equations that contain radical expressions?
3. If we have raised both sides of an equation to a power, when is it not necessary to check for extraneous solutions?
4. When will you need to apply the squaring property of equality twice in the process of solving an equation containing radicals?

## Problem Set 9.6

## Online support materials can be found at www.thomsonedu.com/login

Solve each of the following equations.

1. $\sqrt{2 x+1}=3$
2. $\sqrt{3 x+1}=4$
3. $\sqrt{4 x+1}=-5$
4. $\sqrt{6 x+1}=-5$
5. $\sqrt{2 y-1}=3$
6. $\sqrt{3 y-1}=2$
7. $\sqrt{5 x-7}=-1$
8. $\sqrt{8 x+3}=-6$
9. $\sqrt{2 x-3}-2=4$
10. $\sqrt{3 x+1}-4=1$
11. $\sqrt{4 x+1}+3=2$
12. $\sqrt{5 a-3}+6=2$
13. $\sqrt[4]{3 x+1}=2$
14. $\sqrt[4]{4 x+1}=3$
15. $\sqrt[3]{2 x-5}=1$
16. $\sqrt[3]{5 x+7}=2$
17. $\sqrt[3]{3 a+5}=-3$
18. $\sqrt{y-3}=y-3$
19. $\sqrt{y+3}=y-3$
20. $\sqrt{a+2}=a+2$
21. $\sqrt{a+10}=a-2$
22. $\sqrt{2 x+3}=\frac{2 x-7}{3}$
23. $\sqrt{3 x-2}=\frac{2 x-3}{3}$
24. $\sqrt{4 x-3}=\frac{x+3}{2}$
25. $\sqrt{4 x+5}=\frac{x+3}{2}$
26. $\sqrt{7 x+2}=\frac{2 x+2}{3}$
27. $\sqrt{7 x-3}=\frac{4 x-1}{3}$
28. $\sqrt{2 x+4}=\sqrt{1-x}$
29. $\sqrt{3 x+4}=-\sqrt{2 x+3}$
= Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
30. $\sqrt{4 a+7}=-\sqrt{a+2}$
31. $\sqrt{7 a-1}=\sqrt{2 a+4}$
32. $\sqrt[4]{5 x-8}=\sqrt[4]{4 x-1}$
33. $\sqrt[4]{6 x+7}=\sqrt[4]{x+2}$
34. $x+1=\sqrt{5 x+1}$
35. $x-1=\sqrt{6 x+1}$
36. $t+5=\sqrt{2 t+9}$
37. $t+7=\sqrt{2 t+13}$
38. $\sqrt{y-8}=\sqrt{8-y}$
39. $\sqrt[3]{3 x+5}=\sqrt[3]{5-2 x}$
40. $\sqrt{2 y+5}=\sqrt{5 y+2}$
41. $\sqrt[3]{4 x+9}=\sqrt[3]{3-2 x}$

The following equations will require that you square both sides twice before all the radicals are eliminated. Solve each equation using the methods shown in Examples 5, 6, and 7 .
43. $\sqrt{x-8}=\sqrt{x}-2$
44. $\sqrt{x+3}=\sqrt{x}-3$
45. $\sqrt{x+1}=\sqrt{x}+1$
46. $\sqrt{x-1}=\sqrt{x}-1$
47. $\sqrt{x+8}=\sqrt{x-4}+2$
48. $\sqrt{x+5}=\sqrt{x-3}+2$
49. $\sqrt{x-5}-3=\sqrt{x-8}$
50. $\sqrt{x-3}-4=\sqrt{x-3}$
51. $\sqrt{x+4}=2-\sqrt{2 x}$
52. $\sqrt{5 x+1}=1+\sqrt{5 x}$
53. $\sqrt{2 x+4}=\sqrt{x+3}+1$
54. $\sqrt{2 x-1}=\sqrt{x-4}+2$

Let $f(x)=\sqrt{2 x-1}$. Find all values for the variable $x$ that produce the following values of $f(x)$.
55. $f(x)=0$
56. $f(x)=-10$
57. $f(x)=2 x-1$
58. $f(x)=\sqrt{5 x-10}$
59. $f(x)=\sqrt{x-4}+2$
60. $f(x)=9$

Let $g(x)=\sqrt{2 x+3}$. Find all values for the variable $x$ that produce the following values of $g(x)$.
61. $g(x)=0$
62. $g(x)=x$
63. $g(x)=-\sqrt{5 x}$
64. $g(x)=\sqrt{x^{2}-5}$

Let $f(x)=\sqrt{2 x}-1$. Find all values for the variable $x$ for which $f(x)=g(x)$.
65. $g(x)=0$
66. $g(x)=5$
67. $g(x)=\sqrt{2 x+5}$
68. $g(x)=x-1$

Let $h(x)=\sqrt[3]{3 x+5}$. Find all values for the variable $x$ for which $h(x)=f(x)$.
69. $f(x)=2$
70. $f(x)=-1$

Let $h(x)=\sqrt[3]{5-2 x}$. Find all values for the variable $x$ for which $h(x)=f(x)$.
71. $f(x)=3$
72. $f(x)=-1$

Graph each equation.
73. $y=2 \sqrt{x}$
74. $y=-2 \sqrt{x}$
75. $y=\sqrt{x}-2$
76. $y=\sqrt{x}+2$
77. $y=\sqrt{x-2}$
78. $y=\sqrt{x+2}$
79. $y=3 \sqrt[3]{x}$
80. $y=-3 \sqrt[3]{x}$
81. $y=\sqrt[3]{x}+3$
82. $y=\sqrt[3]{x}-3$
83. $y=\sqrt[3]{x+3}$
84. $y=\sqrt[3]{x-3}$

## Applying the Concepts

85. Solving a Formula Solve the following formula for $h$ :

$$
t=\frac{\sqrt{100-h}}{4}
$$

86. Solving a Formula Solve the following formula for $h$ :

$$
t=\sqrt{\frac{2 h-40 t}{g}}
$$

87. Pendulum Clock The length of time $(T)$ in seconds it takes the pendulum of a grandfather clock to swing through one complete cycle is given by the formula

$$
T=2 \pi \sqrt{\frac{L}{32}}
$$

where $L$ is the length, in feet, of the pendulum, and $\pi$ is approximately $\frac{22}{7}$. How long must the pendulum be if one complete cycle takes 2 seconds?
88. Pendulum Clock Solve the formula in Problem 87 for $L$.


Pollution A long straight river, 100 meters wide, is flowing at 1 meter per second. A pollutant is entering the river at a constant rate from one of its banks. As the pollutant disperses in the water, it forms a plume that is modeled by the equation $y=\sqrt{x}$. Use this information to answer the following questions.

89. How wide is the plume 25 meters down river from the source of the pollution?
90. How wide is the plume 100 meters down river from the source of the pollution?
91. How far down river from the source of the pollution does the plume reach halfway across the river?
92. How far down river from the source of the pollution does the plume reach the other side of the river?

## Maintaining Your Skills

Multiply.
93. $\sqrt{2}(\sqrt{3}-\sqrt{2})$
94. $(\sqrt{x}-4)(\sqrt{x}+5)$
95. $(\sqrt{x}+5)^{2}$
96. $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$

Rationalize the denominator.
97. $\frac{\sqrt{x}}{\sqrt{x}+3}$
98. $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

## Getting Ready for the Next Section

Simplify.
99. $\sqrt{25}$
100. $\sqrt{49}$
101. $\sqrt{12}$
102. $\sqrt{50}$
103. $(-1)^{15}$
104. $(-1)^{20}$
105. $(-1)^{50}$
106. $(-1)^{5}$

Solve.
107. $3 x=12$
108. $4=8 y$
109. $4 x-3=5$
110. $7=2 y-1$

Perform the indicated operation.
111. $(3+4 x)+(7-6 x)$
112. $(2-5 x)+(-1+7 x)$
113. $(7+3 x)-(5+6 x)$
114. $(5-2 x)-(9-4 x)$
115. $(3-4 x)(2+5 x)$
116. $(8+x)(7-3 x)$
117. $2 x(4-6 x)$
118. $3 x(7+2 x)$
119. $(2+3 x)^{2}$
120. $(3+5 x)^{2}$
121. $(2-3 x)(2+3 x)$
122. $(4-5 x)(4+5 x)$

## Complex Numbers

## OBJECTIVES

A Simplify square roots of negative numbers.

B Simplify powers of $i$.
C Solve for unknown variables by equating real parts and equating imaginary parts of two complex numbers.

D Add and subtract complex numbers.

E Multiply complex numbers.

F Divide complex numbers.


## Note

In Examples 3 and 4 we wrote $i$ before the radical simply to avoid confusion. If we were to write the answer to 3 as $2 \sqrt{3} i$, some people would think the $i$ was under the radical sign and it is not.

The stamp shown in the margin was issued by Germany in 1977 to commemorate the 200th anniversary of the birth of mathematician Carl Gauss. The number $-5+6 i$ shown on the stamp is a complex number, as are the other numbers on the stamp. Working with complex numbers gives us a way to solve a wider variety of equations. For example, the equation $x^{2}=-9$ has no real number solutions since the square of a real number is always positive. We have been unable to work with square roots of negative numbers like $\sqrt{-25}$ and $\sqrt{-16}$ for the same reason. Complex numbers allow us to expand our work with radicals to include square roots of negative numbers and solve equations like $x^{2}=-9$ and $x^{2}=-64$. Our work with complex numbers is based on the following definition.

DEFINITION The number $i$ is such that $i=\sqrt{-1}$ (which is the same as saying $i^{2}=-1$ ).

The number $i$, as we have defined it here, is not a real number. Because of the way we have defined $i$, we can use it to simplify square roots of negative numbers.

Square Roots of Negative Numbers If $a$ is a positive number, then $\sqrt{-a}$ can always be written as $i \sqrt{a}$; that is,

$$
\sqrt{-a}=i \sqrt{a} \text { if } a \text { is a positive number }
$$

To justify our rule, we simply square the quantity $i \sqrt{a}$ to obtain $-a$. Here is what it looks like when we do so:

$$
\begin{aligned}
(i \sqrt{a})^{2} & =i^{2} \cdot(\sqrt{a})^{2} \\
& =-1 \cdot a \\
& =-a
\end{aligned}
$$

Here are some examples that illustrate the use of our new rule.

## EXAMPLES Write each square root in terms of the number $i$.

1. $\sqrt{-25}=i \sqrt{25}=i \cdot 5=5 i$
2. $\sqrt{-49}=i \sqrt{49}=i \cdot 7=7 i$
3. $\sqrt{-12}=i \sqrt{12}=i \cdot 2 \sqrt{3}=2 i \sqrt{3}$
4. $\sqrt{-17}=i \sqrt{17}$

If we assume all the properties of exponents hold when the base is $i$, we can write any power of $i$ as $i,-1,-i$, or 1 . Using the fact that $i^{2}=-1$, we have

$$
\begin{aligned}
& i^{1}=i \\
& i^{2}=-1 \\
& i^{3}=i^{2} \cdot i=-1(i)=-i \\
& i^{4}=i^{2} \cdot i^{2}=-1(-1)=1
\end{aligned}
$$

Since $i^{4}=1, i^{5}$ will simplify to $i$, and we will begin repeating the sequence $i$, $-1,-i, 1$ as we simplify higher powers of $i$ : Any power of $i$ simplifies to $i,-1,-i$,
or 1 . The easiest way to simplify higher powers of $i$ is to write them in terms of $i^{2}$. For instance, to simplify $i^{21}$, we would write it as

$$
\left(i^{2}\right)^{10} \cdot i \quad \text { because } 2 \cdot 10+1=21
$$

Then, since $i^{2}=-1$, we have

$$
(-1)^{10} \cdot i=1 \cdot i=i
$$

## EXAMPLES Simplify as much as possible.

5. $i^{30}=\left(i^{2}\right)^{15}=(-1)^{15}=-1$
6. $i^{11}=\left(i^{2}\right)^{5} \cdot i=(-1)^{5} \cdot i=(-1) i=-i$
7. $i^{40}=\left(i^{2}\right)^{20}=(-1)^{20}=1$

DEFINITION A complex number is any number that can be put in the form

$$
a+b i
$$

where $a$ and $b$ are real numbers and $i=\sqrt{-1}$. The form $a+b i$ is called standard form for complex numbers. The number $a$ is called the real part of the complex number. The number $b$ is called the imaginary part of the complex number.

Every real number is a complex number. For example, 8 can be written as $8+0 i$. Likewise, $-\frac{1}{2}, \pi, \sqrt{3}$, and -9 are complex numbers because they can all be written in the form $a+b i$ :

$$
\begin{aligned}
-\frac{1}{2} & =-\frac{1}{2}+0 i & \pi & =\pi+0 i \\
\sqrt{3} & =\sqrt{3}+0 i & -9 & =-9+0 i
\end{aligned}
$$

The real numbers occur when $b=0$. When $b \neq 0$, we have complex numbers that contain $i$, such as $2+5 i, 6-i, 4 i$, and $\frac{1}{2} i$. These numbers are called imaginary numbers. The diagram explains this further.


## Equality for Complex Numbers

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal; that is, for real numbers $a, b, c$, and $d$,

$$
a+b i=c+d i \quad \text { if and only if } a=c \text { and } b=d
$$

EXAMPLE $8 \quad$ Find $x$ and $y$ if $3 x+4 i=12-8 y i$
SOLUTION Since the two complex numbers are equal, their real parts are equal and their imaginary parts are equal:

$$
\left.\begin{array}{rlrl}
3 x & =12 & \text { and } & 4
\end{array}=-8 y\right\}
$$

EXAMPLE $9 \quad$ Find $x$ and $y$ if $(4 x-3)+7 i=5+(2 y-1) i$.
SOLUTION The real parts are $4 x-3$ and 5. The imaginary parts are 7 and $2 y-1$ :

$$
\begin{aligned}
4 x-3 & =5 & \text { and } & 7
\end{aligned}=2 y-1
$$

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## Addition and Subtraction of Complex Numbers

To add two complex numbers, add their real parts and add their imaginary parts; that is, if $a, b, c$, and $d$ are real numbers, then

$$
(a+b i)+(c+d i)=(a+c)+(b+d) i
$$

If we assume that the commutative, associative, and distributive properties hold for the number $i$, then the definition of addition is simply an extension of these properties.

We define subtraction in a similar manner. If $a, b, c$, and $d$ are real numbers, then

$$
(a+b i)-(c+d i)=(a-c)+(b-d) i
$$

## EXAMPLES Add or subtract as indicated.

10. $(3+4 i)+(7-6 i)=(3+7)+(4-6) i=10-2 i$
11. $(7+3 i)-(5+6 i)=(7-5)+(3-6) i=2-3 i$
12. $(5-2 i)-(9-4 i)=(5-9)+(-2+4) i=-4+2 i$

## Multiplication of Complex Numbers

Since complex numbers have the same form as binomials, we find the product of two complex numbers the same way we find the product of two binomials.

## EXAMPLE 13 Multiply $(3-4 i)(2+5 i)$.

SOLUTION Multiplying each term in the second complex number by each term in the first, we have

$$
\begin{aligned}
(3-4 i)(2+5 i) & =3 \cdot 2+3 \cdot 5 i-2 \cdot 4 i-5 i(4 i) \\
& =6+15 i-8 i-20 i^{2}
\end{aligned}
$$

Combining similar terms and using the fact that $i^{2}=-1$, we can simplify as follows:

$$
\begin{aligned}
6+15 i-8 i-20 i^{2} & =6+7 i-20(-1) \\
& =6+7 i+20 \\
& =26+7 i
\end{aligned}
$$

The product of the complex numbers $3-4 i$ and $2+5 i$ is the complex number $26+7 i$.

EXAMPLE 14 Multiply $2 i(4-6 i)$.
SOLUTION Applying the distributive property gives us

$$
\begin{aligned}
2 i(4-6 i) & =2 i \cdot 4-2 i \cdot 6 i \\
& =8 i-12 i^{2} \\
& =12+8 i
\end{aligned}
$$

## EXAMPLE 15 Expand $(3+5 i)^{2}$.

SOLUTION We treat this like the square of a binomial. Remember, $(a+b)^{2}=a^{2}+2 a b+b^{2}$.

$$
\begin{aligned}
(3+5 i)^{2} & =3^{2}+2(3)(5 i)+(5 i)^{2} \\
& =9+30 i+25 i^{2} \\
& =9+30 i-25 \\
& =-16+30 i
\end{aligned}
$$

EXAMPLE 16 Multiply $(2-3 i)(2+3 i)$.
SOLUTION This product has the form $(a-b)(a+b)$, which we know results in the difference of two squares, $a^{2}-b^{2}$ :

$$
\begin{aligned}
(2-3 i)(2+3 i) & =2^{2}-(3 i)^{2} \\
& =4-9 i^{2} \\
& =4+9 \\
& =13
\end{aligned}
$$

The product of the two complex numbers $2-3 i$ and $2+3 i$ is the real number 13. The two complex numbers $2-3 i$ and $2+3 i$ are called complex conjugates. The fact that their product is a real number is very useful.

DEFINITION The complex numbers $a+b i$ and $a-b i$ are called complex conjugates. One important property they have is that their product is the real number $a^{2}+b^{2}$. Here's why:

$$
\begin{aligned}
(a+b i)(a-b i) & =a^{2}-(b i)^{2} \\
& =a^{2}-b^{2} i^{2} \\
& =a^{2}-b^{2}(-1) \\
& =a^{2}+b^{2}
\end{aligned}
$$

## Division with Complex Numbers

The fact that the product of two complex conjugates is a real number is the key to division with complex numbers.

## EXAMPLE 17 Divide $\frac{2+i}{3-2 i}$.

SOLUTION We want a complex number in standard form that is equivalent to the quotient $\frac{2+i}{3-2 i}$. We need to eliminate $i$ from the denominator. Multiplying the numerator and denominator by $3+2 i$ will give us what we want:

$$
\begin{aligned}
\frac{2+i}{3-2 i} & =\frac{2+i}{3-2 i} \cdot \frac{(\mathbf{3}+\mathbf{2 i} \mathbf{i}}{(\mathbf{3}+\mathbf{2 i} \mathbf{)}} \\
& =\frac{6+4 i+3 i+2 i^{2}}{9-4 i^{2}} \\
& =\frac{6+7 i-2}{9+4} \\
& =\frac{4+7 i}{13} \\
& =\frac{4}{13}+\frac{7}{13} i
\end{aligned}
$$

Dividing the complex number $2+i$ by $3-2 i$ gives the complex number $\frac{4}{13}+\frac{7}{13} i$.

EXAMPLE 18 Divide $\frac{7-4 i}{i}$.
SOLUTION The conjugate of the denominator is $-i$. Multiplying numerator and denominator by this number, we have

$$
\begin{aligned}
\frac{7-4 i}{i} & =\frac{7-4 i}{i} \cdot \frac{-\mathbf{i}}{-\mathbf{i}} \\
& =\frac{-7 i+4 i^{2}}{-i^{2}} \\
& =\frac{-7 i+4(-1)}{-(-1)} \\
& =-4-7 i
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-4

B ${ }^{5-7}$
C 8,9
D 10-12
E $\quad 13-16$
F $\quad 17,18$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the number i?
2. What is a complex number?
3. What kind of number results when we multiply complex conjugates?
4. Explain how to divide complex numbers.

## Problem Set 9.7

Write the following in terms of $i$, and simplify as much as possible.

1. $\sqrt{-36}$
2. $\sqrt{-49}$
3. $-\sqrt{-25}$
4. $-\sqrt{-81}$
5. $\sqrt{-72}$
6. $\sqrt{-48}$
7. $-\sqrt{-12}$
8. $-\sqrt{-75}$

Write each of the following as $i,-1,-i$, or 1 .
9. $i^{28}$
10. $i^{31}$
11. $i^{26}$
12. $i^{37}$
13. $i^{75}$
14. $i^{42}$
21. $(2 x-4)-3 i=10-6 y i$
22. $(4 x-3)-2 i=8+y i$
23. $(7 x-1)+4 i=2+(5 y+2) i$
24. $(5 x+2)-7 i=4+(2 y+1) i$

Combine the following complex numbers.
25. $(2+3 i)+(3+6 i)$
26. $(4+i)+(3+2 i)$
27. $(3-5 i)+(2+4 i)$
28. $(7+2 i)+(3-4 i)$
29. $(5+2 i)-(3+6 i)$
30. $(6+7 i)-(4+i)$
31. $(3-5 i)-(2+i)$
32. $(7-3 i)-(4+10 i)$
33. $[(3+2 i)-(6+i)]+(5+i)$
34. $[(4-5 i)-(2+i)]+(2+5 i)$
35. $[(7-i)-(2+4 i)]-(6+2 i)$
36. $[(3-i)-(4+7 i)]-(3-4 i)$
37. $(3+2 i)-[(3-4 i)-(6+2 i)]$
38. $(7-4 i)-[(-2+i)-(3+7 i)]$
39. $(4-9 i)+[(2-7 i)-(4+8 i)]$
40. $(10-2 i)-[(2+i)-(3-i)]$

Find the following products.
41. $3 i(4+5 i)$
42. $2 i(3+4 i)$
43. $6 i(4-3 i)$
45. $(3+2 i)(4+i)$
47. $(4+9 i)(3-i)$
49. $(1+i)^{3}$
51. $(2-i)^{3}$
53. $(2+5 i)^{2}$
55. $(1-i)^{2}$
57. $(3-4 i)^{2}$
59. $(2+i)(2-i)$
61. $(6-2 i)(6+2 i)$
63. $(2+3 i)(2-3 i)$
65. $(10+8 i)(10-8 i)$
44. $11 i(2-i)$
46. $(2-4 i)(3+i)$
48. $(5-2 i)(1+i)$
50. $(1-i)^{3}$
52. $(2+i)^{3}$
54. $(3+2 i)^{2}$
56. $(1+i)^{2}$
58. $(6-5 i)^{2}$
60. $(3+i)(3-i)$
62. $(5+4 i)(5-4 i)$
64. $(2-7 i)(2+7 i)$
66. $(11-7 i)(11+7 i)$

Find the following quotients. Write all answers in standard form for complex numbers.
67. $\frac{2-3 i}{i}$
68. $\frac{3+4 i}{i}$
69. $\frac{5+2 i}{-i}$
70. $\frac{4-3 i}{-i}$
71. $\frac{4}{2-3 i}$
72. $\frac{3}{4-5 i}$
73. $\frac{6}{-3+2 i}$
74. $\frac{-1}{-2-5 i}$
75. $\frac{2+3 i}{2-3 i}$
76. $\frac{4-7 i}{4+7 i}$
77. $\frac{5+4 i}{3+6 i}$
78. $\frac{2+i}{5-6 i}$

## Applying the Concepts

79. Electric Circuits Complex numbers may be applied to electrical circuits. Electrical engineers use the fact
that resistance $R$ to electrical flow of the electrical current $I$ and the voltage $V$ are related by the formula $V=R I$. (Voltage is measured in volts, resistance in ohms, and current in amperes.) Find the resistance to electrical flow in a circuit that has a voltage $V=(80+20 i)$ volts and current $I=(-6+2 i)$ amps.
80. Electric Circuits Refer to the information about electrical circuits in Problem 79, and find the current in a circuit that has a resistance of $(4+10 i)$ ohms and a voltage of $(5-7 i)$ volts.

## Maintaining Your Skills

For each relation that follows, state the domain and the range, and indicate which are also functions.
81. $\{(1,2),(3,4),(4,2)\}$
82. $\{(0,0),(1,1),(0,1)\}$
83. $\{(3,1),(2,3),(1,2)\}$
84. $\{(-1,1),(2,-2),(-3,-3)\}$

State whether each of the following graphs is the graph of a function
85.

86.

87.

88.


## Chapter 9 SUMMARY $\mathbb{X}$ <br> Square Roots [9.1]

Every positive real number $x$ has two square roots. The positive square root of $x$ is written $\sqrt{x}$, and the negative square root of $x$ is written $-\sqrt{x}$. Both the positive and the negative square roots of $x$ are numbers we square to get $x$; that is,

$$
\text { and } \left.\quad \begin{array}{r}
(\sqrt{x})^{2}=x \\
(-\sqrt{x})^{2}=x
\end{array}\right\} \quad \text { for } x \geq 0
$$

## Higher Roots [9.1]

2. $\sqrt[3]{8}=2$
$\sqrt[3]{-27}=-3$

In the expression $\sqrt[n]{a}, n$ is the index, $a$ is the radicand, and $\sqrt{ }$ is the radical sign. The expression $\sqrt[n]{a}$ is such that

$$
(\sqrt[n]{a})^{n}=a \quad a \geq 0 \text { when } n \text { is even }
$$

## Rational Exponents [9.1, 9.2]

3. $25^{1 / 2}=\sqrt{25}=5$
$8^{2 / 3}=(\sqrt[3]{8})^{2}=2^{2}=4$
$9^{3 / 2}=(\sqrt{9})^{3}=3^{3}=27$

Rational exponents are used to indicate roots. The relationship between rational exponents and roots is as follows:

$$
\begin{gathered}
a^{1 / n}=\sqrt[n]{a} \quad \text { and } \quad a^{m / n}=\left(a^{1 / n}\right)^{m}=\left(a^{m}\right)^{1 / n} \\
a \geq 0 \text { when } n \text { is even }
\end{gathered}
$$

## Properties of Radicals [9.3]

4. $\sqrt{4 \cdot 5}=\sqrt{4} \sqrt{5}=2 \sqrt{5}$

$$
\sqrt{\frac{7}{9}}=\frac{\sqrt{7}}{\sqrt{9}}=\frac{\sqrt{7}}{3}
$$

5. $\sqrt{\frac{4}{5}}=\frac{\sqrt{4}}{\sqrt{5}}$

$$
=\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}
$$

$$
=\frac{2 \sqrt{5}}{5}
$$

If $a$ and $b$ are nonnegative real numbers whenever $n$ is even, then

1. $\sqrt[n]{a b}=\sqrt[n]{a} \sqrt[n]{b}$
2. $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad(b \neq 0)$

## Simplified Form for Radicals [9.3]

## A radical expression is said to be in simplified form

1. If there is no factor of the radicand that can be written as a power greater than or equal to the index;
2. If there are no fractions under the radical sign; and
3. If there are no radicals in the denominator.

## Addition and Subtraction of Radical Expressions [9.4]

6. $5 \sqrt{3}-7 \sqrt{3}=(5-7) \sqrt{3}$
$=-2 \sqrt{3}$
$\sqrt{20}+\sqrt{45}=2 \sqrt{5}+3 \sqrt{5}$
$=(2+3) \sqrt{5}$ $=5 \sqrt{5}$

We add and subtract radical expressions by using the distributive property to combine similar radicals. Similar radicals are radicals with the same index and the same radicand.

## Multiplication of Radical Expressions [9.5]

We multiply radical expressions in the same way that we multiply polynomials. We can use the distributive property and the FOIL method.
$=\sqrt{x} \sqrt{x}+3 \sqrt{x}+2 \sqrt{x}+2 \cdot 3$
$=x+5 \sqrt{x}+6$

## Rationalizing the Denominator [9.3, 9.5]

When a fraction contains a square root in the denominator, we rationalize the denominator by multiplying numerator and denominator by

1. The square root itself if there is only one term in the denominator, or
2. The conjugate of the denominator if there are two terms in the denominator. Rationalizing the denominator is also called division of radical expressions.

## Squaring Property of Equality [9.6]

We may square both sides of an equation any time it is convenient to do so, as long as we check all resulting solutions in the original equation.

## Complex Numbers [9.7]

A complex number is any number that can be put in the form

$$
a+b i
$$

where $a$ and $b$ are real numbers and $i=\sqrt{-1}$. The real part of the complex number is $a$, and $b$ is the imaginary part.
If $a, b, c$, and $d$ are real numbers, then we have the following definitions associated with complex numbers:

1. Equality

$$
a+b i=c+d i \quad \text { if and only if } a=c \text { and } b=d
$$

$$
\begin{aligned}
& \text { Division } \\
& \begin{aligned}
\frac{2}{3+4 i} & =\frac{2}{3+4 i} \cdot \frac{3-4 i}{3-4 i} \\
& =\frac{6-8 i}{9+16} \\
& =\frac{6}{25}-\frac{8}{25} i
\end{aligned}
\end{aligned}
$$

2. Addition and subtraction

$$
\begin{aligned}
& (a+b i)+(c+d i)=(a+c)+(b+d) i \\
& (a+b i)-(c+d i)=(a-c)+(b-d) i
\end{aligned}
$$

3. Multiplication

$$
(a+b i)(c+d i)=(a c-b d)+(a d+b c) i
$$

4. Division is similar to rationalizing the denominator.

## Chapter 9 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

Simplify each expression as much as possible. [9.1]

1. $\sqrt{49}$
2. $(-27)^{1 / 3}$
3. $16^{1 / 4}$
4. $9^{3 / 2}$
5. $\sqrt[5]{32 x^{15} y^{10}}$
6. $8^{-4 / 3}$

Use the properties of exponents to simplify each expression. Assume all bases represent positive numbers. [9.1]
7. $x^{2 / 3} \cdot x^{4 / 3}$
8. $\left(a^{2 / 3} b^{4 / 3}\right)^{3}$
9. $\frac{a^{3 / 5}}{a^{1 / 4}}$
10. $\frac{a^{2 / 3} b^{3}}{a^{1 / 4} b^{1 / 3}}$

Multiply. [9.2]
11. $\left(3 x^{1 / 2}+5 y^{1 / 2}\right)\left(4 x^{1 / 2}-3 y^{1 / 2}\right)$
12. $\left(a^{1 / 3}-5\right)^{2}$
13. Divide: $\frac{28 x^{5 / 6}+14 x^{7 / 6}}{7 x^{1 / 3}}$. (Assume $x>0$.) [9.2]
14. Factor $2(x-3)^{1 / 4}$ from $8(x-3)^{5 / 4}-2(x-3)^{1 / 4}$. [9.2]
15. Simplify $x^{3 / 4}+\frac{5}{x^{1 / 4}}$ into a single fraction.
(Assume $x>0$. .) [9.2]

Write each expression in simplified form for radicals. (Assume all variables represent nonnegative numbers.) [9.3]
16. $\sqrt{12}$
17. $\sqrt{50}$
18. $\sqrt[3]{16}$
19. $\sqrt{18 x^{2}}$
20. $\sqrt{80 a^{3} b^{4} c^{2}}$
21. $\sqrt[4]{32 a^{4} b^{5} c^{6}}$

Rationalize the denominator in each expression. [9.3]
22. $\frac{3}{\sqrt{2}}$
23. $\frac{6}{\sqrt[3]{2}}$

Write each expression in simplified form. (Assume all variables represent positive numbers.) [9.3]
24. $\sqrt{\frac{48 x^{3}}{7 y}}$
25. $\sqrt[3]{\frac{40 x^{2} y^{3}}{3 z}}$

Combine the following expressions. (Assume all variables represent positive numbers.) [9.4]
26. $5 x \sqrt{6}+2 x \sqrt{6}-9 x \sqrt{6}$
27. $\sqrt{12}+\sqrt{3}$
28. $\frac{3}{\sqrt{5}}+\sqrt{5}$
29. $3 \sqrt{8}-4 \sqrt{72}+5 \sqrt{50}$
30. $3 b \sqrt{27 a^{5} b}+2 a \sqrt{3 a^{3} b^{3}}$
31. $2 x \sqrt[3]{x y^{3} Z^{2}}-6 y \sqrt[3]{x^{4} Z^{2}}$

Multiply. [9.5]
32. $\sqrt{2}(\sqrt{3}-2 \sqrt{2})$
33. $(\sqrt{x}-2)(\sqrt{x}-3)$

Rationalize the denominator. (Assume $x, y>0$.) [9.5]
34. $\frac{3}{\sqrt{5}-2}$
35. $\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$
36. $\frac{3 \sqrt{7}}{3 \sqrt{7}-4}$

Solve each equation. [9.6]
37. $\sqrt{4 a+1}=1$
38. $\sqrt[3]{3 x-8}=1$
39. $\sqrt{3 x+1}-3=1$
40. $\sqrt{x+4}=\sqrt{x}-2$

Graph each equation. [9.6]
41. $y=3 \sqrt{x}$
42. $y=\sqrt[3]{x}+2$

Write each of the following as $i,-1,-i$, or 1. [9.7]
43. $i^{24}$
44. $i^{27}$

Find $x$ and $y$ so that each of the following equations is true. [9.7]
45. $3-4 i=-2 x+8 y i$
46. $(3 x+2)-8 i=-4+2 y i$

Combine the following complex numbers. [9.7]
47. $(3+5 i)+(6-2 i)$
48. $(2+5 i)-[(3+2 i)+(6-i)]$

Multiply. [9.7]
49. $3 i(4+2 i)$
50. $(2+3 i)(4+i)$
51. $(4+2 i)^{2}$
52. $(4+3 i)(4-3 i)$

Divide. Write all answers in standard form for complex numbers. [9.7]
53. $\frac{3+i}{i}$
54. $\frac{-3}{2+i}$
55. Construction The roof of the house shown in Figure 1 is to extend up 13.5 feet above the ceiling, which is 36 feet across. Find the length of one side of the roof.


FIGURE 1
56. Surveying A surveyor is attempting to find the distance across a pond. From a point on one side of the pond he walks 25 yards to the end of the pond and then makes a 90-degree turn and walks another 60 yards before coming to a point directly across the pond from the point at which he started. What is the distance across the pond? (See Figure 2.)


FIGURE 2

## Chapter 9 Projects

## Rational Exponents and Roots

## GROUP PROJECT Constructing the Spiral of Roots

## Number of People 3

Time Needed 20 minutes
Equipment Two sheets of graph paper (4 or 5 squares per inch) and pencils.

Background The spiral of roots gives us a way to visualize the positive square roots of the counting numbers, and in so doing, we see many line segments whose lengths are irrational numbers.


Procedure You are to construct a spiral of roots from a line segment 1 inch long. The graph paper you have contains either 4 or 5 squares per inch, allowing you to accurately draw 1 -inch line segments. Because the lines on the graph paper are perpendicular to one another, if you are careful, you can use the graph paper to connect one line segment to another so that they form a right angle.

1. Fold one of the pieces of graph paper so it can be used as a ruler.
2. Use the folded paper to draw a line segment 1 -inch long, just to the right of the middle of the unfolded paper. On the end of this segment, attach another
segment of 1 -inch length at a right angle to the first one. Connect the end points of the segments to form a right triangle. Label each side of this triangle. When you are finished, your work should resemble Figure 1.


FIGURE 1
3. On the end of the hypotenuse of the triangle, attach a 1 -inch line segment so that the two segments form a right angle. (Use the folded paper to do this.) Draw the hypotenuse of this triangle. Label all the sides of this second triangle. Your work should resemble Figure 2.


FIGURE 2
4. Continue to draw a new right triangle by attaching 1 -inch line segments at right angles to the previous hypotenuse. Label all the sides of each triangle.
5. Stop when you have drawn a hypotenuse $\sqrt{8}$ inches long.

## RESEARCH PROJECT

## Connections

A
lthough it may not look like it, the three items shown here are related very closely to one another. Your job is to find the connection.

A Continued Fraction The Fibonacci Sequence

$$
1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}}
$$

The Golden Rectangle


Step 1: The dots in the continued fraction indicate that the pattern shown continues indefinitely. This means that there is no way for us to simplify this expression, as we have simplified the expressions in this chapter. However, we can begin to understand the continued fraction, and what it simplifies to, by working with the following sequence of expressions. Simplify each expression. Write each answer as a fraction, in lowest terms.

$$
1+\frac{1}{1+1} \quad 1+\frac{1}{1+\frac{1}{1+1}} \quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}} \quad 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}
$$

Step 2: Compare the fractional answers to step 1 with the numbers in the Fibonacci sequence. Based on your observation, give the answer to the following problem, without actually doing any arithmetic.

$$
1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}}}}
$$

Step 3: Continue the sequence of simplified fractions you have written in steps 1 and 2, until you have nine numbers in the sequence. Convert each of these numbers to a decimal, accurate to four places past the decimal point.
Step 4: Find a decimal approximation to the golden ratio $\frac{1+\sqrt{5}}{2}$, accurate to four places past the decimal point.

Step 5: Compare the results in steps 3 and 4, and then make a conjecture about what number the continued fraction would simplify to, if it was actually possible to simplify it.

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## Quadratic Functions

## 10


f you have been to the circus or the county fair, you may have witnessed one of the more spectacular acts, the human cannonball. The human cannonball shown in the photograph will reach a height of 70 feet and travel a distance of 160 feet before landing in a safety net. In this chapter, we use this information to derive the equation

$$
f(x)=-\frac{7}{640}(x-80)^{2}+70 \text { for } 0 \leq x \leq 160
$$

which describes the path flown by this particular cannonball. The table and graph below were constructed from this equation.

| Path of a Human Cannonball <br> $\boldsymbol{x}$ <br> (feet) | $\boldsymbol{f}(\boldsymbol{x})$ <br> (nearest foot) |
| :---: | :---: |
| 0 | 0 |
| 40 | 53 |
| 80 | 70 |
| 120 | 53 |
| 160 | 0 |



FIGURE 1

All objects that are projected into the air, whether they are basketballs, bullets, arrows, or coins, follow parabolic paths like the one shown in Figure 1. Studying the material in this chapter will give you a more mathematical hold on the world around you.

## CHAPTER OUTLINE

10.1 Completing the Square
10.2 The Quadratic Formula
10.3 Additional Items Involving Solutions to Equations
10.4 Equations Quadratic in Form
10.5 Graphing Parabolas
10.6 Quadratic Inequalities

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## Completing the Square

## OBJECTIVES

A
Solve quadratic equations by taking the square root of both sides.

B Solve quadratic equations by completing the square.

C Use quadratic equations to solve for missing parts of right triangles.

Table 1 is taken from the trail map given to skiers at the Northstar at Tahoe Ski Resort in Lake Tahoe, California. The table gives the length of each chair lift at Northstar, along with the change in elevation from the beginning of the lift to the end of the lift.

Right triangles are good mathematical models for chair lifts. In this section, we will use our knowledge of right triangles, along with the new material developed in the section, to solve problems involving chair lifts and a variety of other examples.


TABLE 1
From the Trail Map for the Northstar at Tahoe Ski Resort

| Lift Information |  |  |
| :--- | ---: | ---: |
| Lift | Vertical <br> Rise <br> (feet) | Length <br> (feet) |
| Big Springs Gondola | 480 | 4,100 |
| Bear Paw Double | 120 | 790 |
| Echo Triple | 710 | 4,890 |
| Aspen Express Quad | 900 | 5,100 |
| Forest Double | 1,170 | 5,750 |
| Lookout Double | 960 | 4,330 |
| Comstock Express Quad | 1,250 | 5,900 |
| Rendezvous Triple | 650 | 2,900 |
| Schaffer Camp Triple | 1,860 | 6,150 |
| Chipmunk Tow Lift | 28 | 280 |
| Bear Cub Tow Lift | 120 | 750 |
|  |  |  |

In this section, we will develop the first of our new methods of solving quadratic equations. The new method is called completing the square. Completing the square on a quadratic equation allows us to obtain solutions, regardless of whether the equation can be factored. Before we solve equations by completing the square, we need to learn how to solve equations by taking square roots of both sides.

Consider the equation

$$
x^{2}=16
$$

We could solve it by writing it in standard form, factoring the left side, and proceeding as we have done previously. We can shorten our work considerably, however, if we simply notice that $x$ must be either the positive square root of 16 or the negative square root of 16 ; that is,

$$
\left.\begin{array}{llrlrl}
\text { If } & & x^{2} & =16 & & \\
\text { then } & x & =\sqrt{16} & & \text { or } & x
\end{array}\right)=-\sqrt{16}
$$

We can generalize this result into a theorem as follows.

Theorem 1 If $a^{2}=b$ where $b$ is a real number, then $a=\sqrt{b}$ or $a=-\sqrt{b}$.

Notation The expression $a=\sqrt{b}$ or $a=-\sqrt{b}$ can be written in shorthand form as $a= \pm \sqrt{b}$. The symbol $\pm$ is read "plus or minus."

We can apply Theorem 1 to some fairly complicated quadratic equations.

## EXAMPLE 1 Solve $(2 x-3)^{2}=25$.

## SOLUTION

$$
\begin{aligned}
(2 x-3)^{2} & =25 & & \\
2 x-3 & = \pm \sqrt{25} & & \text { Theorem } \mathbf{1} \\
2 x-3 & = \pm 5 & & \sqrt{\mathbf{2 5}}=\mathbf{5} \\
2 x & =3 \pm 5 & & \text { Add } 3 \text { to both sides } \\
x & =\frac{3 \pm 5}{2} & & \text { Divide both sides by } \mathbf{2}
\end{aligned}
$$

The last equation can be written as two separate statements:

$$
\begin{array}{rlrlrl}
x & =\frac{3+5}{2} & & \text { or } & x & =\frac{3-5}{2} \\
& =\frac{8}{2} & & & =\frac{-2}{2} \\
& =4 & & \text { or } & & =-1
\end{array}
$$

The solution set is $\{4,-1\}$.

Notice that we could have solved the equation in Example 1 by expanding the left side, writing the resulting equation in standard form, and then factoring. The problem would look like this:

## Note

We cannot solve the equation in Example 2 by factoring. If we expand the left side and write the resulting equation in standard form, we are left with a quadratic equation that does not factor:
$(3 x-1)^{2}=-12$
$9 x^{2}-6 x+1=-12$ $9 x^{2}-6 x+13=0$

$$
\begin{array}{rlrlrl}
(2 x-3)^{2} & =25 & & \text { Original equation } \\
4 x^{2}-12 x+9 & =25 & & \text { Expand the left side } \\
4 x^{2}-12 x-16 & =0 & & \text { Add }-\mathbf{2 5} \text { to each side } \\
4\left(x^{2}-3 x-4\right) & =0 & & \text { Begin factoring } \\
4(x-4)(x+1) & =0 & & \text { Factor completely } \\
x-4=0 & & \text { or } x+1 & =0 & & \text { Set variable factors equal to } \mathbf{0} \\
x=4 & \text { or } & & x & =-1 &
\end{array}
$$

As you can see, solving the equation by factoring leads to the same two solutions.

EXAMPLE 2 Solve for $x:(3 x-1)^{2}=-12$.

## SOLUTION

$$
\begin{aligned}
(3 x-1)^{2} & =-12 & & \\
3 x-1 & = \pm \sqrt{-12} & & \text { Theorem } \mathbf{1} \\
3 x-1 & = \pm 2 i \sqrt{3} & & \sqrt{-\mathbf{1 2}}=\mathbf{2 i} \sqrt{\mathbf{3}} \\
3 x & =1 \pm 2 i \sqrt{3} & & \text { Add 1 to both sides } \\
X & =\frac{1 \pm 2 i \sqrt{3}}{3} & & \text { Divide both sides by } \mathbf{3}
\end{aligned}
$$

The solution set is $\left\{\frac{1+2 i \sqrt{3}}{3}, \frac{1-2 i \sqrt{3}}{3}\right\}$

Both solutions are complex. Here is a check of the first solution:
When $\quad x=\frac{1+2 i \sqrt{3}}{3}$
the equation
becomes
or

$$
\begin{aligned}
(3 x-1)^{2} & =-12 \\
\left(3 \cdot \frac{1+2 i \sqrt{3}}{3}-1\right)^{2} & \stackrel{?}{=}-12 \\
(1+2 i \sqrt{3}-1)^{2} & \stackrel{?}{=}-12 \\
(2 i \sqrt{3})^{2} & \stackrel{?}{=}-12 \\
4 \cdot i^{2} \cdot 3 & \stackrel{?}{=}-12 \\
12(-1) & \stackrel{?}{=}-12 \\
-12 & =-12
\end{aligned}
$$

111415

## EXAMPLE 3 Solve $x^{2}+6 x+9=12$.

SOLUTION We can solve this equation as we have the equations in Examples 1 and 2 if we first write the left side as $(x+3)^{2}$.

$$
\begin{aligned}
x^{2}+6 x+9 & =12 & & \text { Original equation } \\
(x+3)^{2} & =12 & & \text { Write } x^{2}+6 x+9 \text { as }(x+3)^{2} \\
x+3 & = \pm 2 \sqrt{3} & & \text { Theorem } \mathbf{1} \\
x & =-3 \pm 2 \sqrt{3} & & \text { Add }-\mathbf{3} \text { to each side }
\end{aligned}
$$

We have two irrational solutions: $-3+2 \sqrt{3}$ and $-3-2 \sqrt{3}$. What is important about this problem, however, is the fact that the equation was easy to solve because the left side was a perfect square trinomial.

## Completing the Square

The method of completing the square is simply a way of transforming any quadratic equation into an equation of the form found in the preceding three examples.
The key to understanding the method of completing the square lies in recognizing the relationship between the last two terms of any perfect square trinomial whose leading coefficient is 1 .

Consider the following list of perfect square trinomials and their corresponding binomial squares:

$$
\begin{aligned}
x^{2}-6 x+9 & =(x-3)^{3} \\
x^{2}+8 x+16 & =(x+4)^{2} \\
x^{2}-10 x+25 & =(x-5)^{2} \\
x^{2}+12 x+36 & =(x+6)^{2}
\end{aligned}
$$

In each case the leading coefficient is 1. A more important observation comes from noticing the relationship between the linear and constant terms (middle and last terms) in each trinomial. Observe that the constant term in each case is the square of half the coefficient of $x$ in the middle term. For example, in the last expression, the constant term 36 is the square of half of 12 , where 12 is the coefficient of $x$ in the middle term. (Notice also that the second terms in all the binomials on the right side are half the coefficients of the middle terms of the tri-

## Note

This is the step in which we actually complete the square.

## Note

We can use a calculator to get decimal approximations to these solutions. If $\sqrt{33} \approx 5.74$, then

$$
\begin{aligned}
& \frac{-5+5.74}{2}=0.37 \\
& \frac{-5-5.74}{2}=-5.37
\end{aligned}
$$

nomials on the left side.) We can use these observations to build our own perfect square trinomials and, in doing so, solve some quadratic equations.

Consider the following equation:

$$
x^{2}+6 x=3
$$

We can think of the left side as having the first two terms of a perfect square trinomial. We need only add the correct constant term. If we take half the coefficient of $x$, we get 3 . If we then square this quantity, we have 9 . Adding the 9 to both sides, the equation becomes

$$
x^{2}+6 x+\mathbf{9}=3+\mathbf{9}
$$

The left side is the perfect square $(x+3)^{2}$; the right side is 12 :

$$
(x+3)^{2}=12
$$

The equation is now in the correct form. We can apply Theorem 1 and finish the solution:

$$
\begin{aligned}
(x+3)^{2} & =12 \\
x+3 & = \pm \sqrt{12} \quad \text { Theorem } 1 \\
x+3 & = \pm 2 \sqrt{3} \\
x & =-3 \pm 2 \sqrt{3}
\end{aligned}
$$

The solution set is $\{-3+2 \sqrt{3},-3-2 \sqrt{3}\}$. The method just used is called completing the square since we complete the square on the left side of the original equation by adding the appropriate constant term.

## EXAMPLE 4

Solve by completing the square: $x^{2}+5 x-2=0$.
SOLUTION We must begin by adding 2 to both sides. (The left side of the equation, as it is, is not a perfect square because it does not have the correct constant term. We will simply "move" that term to the other side and use our own constant term.)

$$
x^{2}+5 x=2 \quad \text { Add } 2 \text { to each side }
$$

We complete the square by adding the square of half the coefficient of the linear term to both sides:

$$
\begin{array}{rlrl}
x^{2}+5 x+\frac{\mathbf{2 5}}{\mathbf{4}} & =2+\frac{\mathbf{2 5}}{\mathbf{4}} & & \text { Half of } 5 \text { is } \frac{5}{2}, \text { the square of which is } \frac{25}{4} \\
\left(x+\frac{5}{2}\right)^{2} & =\frac{33}{4} & & 2+\frac{25}{4}=\frac{8}{4}+\frac{25}{4}=\frac{33}{4} \\
x+\frac{5}{2} & = \pm \sqrt{\frac{33}{4}} & & \text { Theorem } 1 \\
x+\frac{5}{2} & = \pm \frac{\sqrt{33}}{2} & & \text { Simplify the radical } \\
x & =-\frac{5}{2} \pm \frac{\sqrt{33}}{2} & \text { Add }-\frac{5}{2} \text { to both sides } \\
& =\frac{-5 \pm \sqrt{33}}{2} & &
\end{array}
$$

The solution set is $\left\{\frac{-5+\sqrt{33}}{2}, \frac{-5-\sqrt{33}}{2}\right\}$.

## EXAMPLE 5 Solve for $x: 3 x^{2}-8 x+7=0$.

## Note

A quadratic function $f$ has the form

$$
f(x)=a x^{2}+b x+c
$$

where $a, b$, and $c$ are real numbers and $a$ is not 0 . Using function notation, the problem shown in Example 5 could be stated this way

If $f(x)=3 x^{2}-8 x+7$, find all
values of $x$ for which $f(x)=0$

## SOLUTION

$$
\begin{aligned}
3 x^{2}-8 x+7 & =0 \\
3 x^{2}-8 x & =-7 \quad \text { Add }-7 \text { to both sides }
\end{aligned}
$$

We cannot complete the square on the left side because the leading coefficient is not 1 . We take an extra step and divide both sides by 3 :

$$
\begin{aligned}
\frac{3 x^{2}}{3}-\frac{8 x}{3} & =-\frac{7}{3} \\
x^{2}-\frac{8}{3} x & =-\frac{7}{3}
\end{aligned}
$$

Half of $\frac{8}{3}$ is $\frac{4}{3}$, the square of which is $\frac{16}{9}$.

$$
\begin{aligned}
x^{2}-\frac{8}{3} x+\frac{16}{9} & =-\frac{7}{3}+\frac{16}{9} & & \text { Add } \frac{16}{9} \text { to both sides } \\
\left(x-\frac{4}{3}\right)^{2} & =-\frac{5}{9} & & \text { Simplify right side } \\
x-\frac{4}{3} & = \pm \sqrt{-\frac{5}{9}} & & \text { Theorem } 1 \\
x-\frac{4}{3} & = \pm \frac{i \sqrt{5}}{3} & & \sqrt{-\frac{5}{9}}=\frac{\sqrt{-5}}{3}=\frac{i \sqrt{5}}{3} \\
X & =\frac{4}{3} \pm \frac{i \sqrt{5}}{3} & & \text { Add } \frac{4}{3} \text { to both sides } \\
x & =\frac{4 \pm i \sqrt{5}}{3} & &
\end{aligned}
$$

The solution set is $\left\{\frac{4+i \sqrt{5}}{3}, \frac{4-i \sqrt{5}}{3}\right\}$.

To Solve a Quadratic Equation by Completing the Square To summarize the method used in the preceding two examples, we list the following steps:
Step 1: Write the equation in the form $a x^{2}+b x=c$.
Step 2: If the leading coefficient is not 1, divide both sides by the coefficient so that the resulting equation has a leading coefficient of 1 ; that is, if $a \neq 1$, then divide both sides by $a$.

Step 3: Add the square of half the coefficient of the linear term to both sides of the equation.

Step 4: Write the left side of the equation as the square of a binomial, and simplify the right side if possible.

Step 5: Apply Theorem 1, and solve as usual.

## FACTS FROM GEOMETRY

## More Special Triangles

The triangles shown in Figures 1 and 2 occur frequently in mathematics.


FIGURE 1


FIGURE 2

Note that both of the triangles are right triangles. We refer to the triangle in Figure 1 as a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle and the triangle in Figure 2 as a $45^{\circ}-45^{\circ}-90^{\circ}$ triangle.

EXAMPLE 6 If the shortest side in a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle is 1 inch, find the lengths of the other two sides.

SOLUTION In Figure 3 triangle $A B C$ is a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle in which the shortest side $A C$ is 1 inch long. Triangle $D B C$ is also a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle in which the shortest side $D C$ is 1 inch long.

Notice that the large triangle $A B D$ is an equilateral triangle because each of its interior angles is $60^{\circ}$. Each side of triangle $A B D$ is 2 inches long. Side $A B$ in triangle


FIGURE 3 $A B C$ is therefore 2 inches. To find the length of side $B C$, we use the Pythagorean theorem.

$$
\begin{aligned}
B C^{2}+A C^{2} & =A B^{2} \\
x^{2}+1^{2} & =2^{2} \\
x^{2}+1 & =4 \\
x^{2} & =3 \\
x & =\sqrt{3} \text { inches }
\end{aligned}
$$

Note that we write only the positive square root because $x$ is the length of a side in a triangle and is therefore a positive number.


EXAMPLE 7 Table 1 in the introduction to this section gives the vertical rise of the Forest Double chair lift as 1,170 feet and the length of the chair lift as 5,750 feet. To the nearest foot, find the horizontal distance covered by a person riding this lift.

SOLUTION Figure 4 is a model of the Forest Double chair lift. A rider gets on the lift at point $A$ and exits at point $B$. The length of the lift is $A B$.


FIGURE 4
To find the horizontal distance covered by a person riding the chair lift, we use the Pythagorean theorem.

$$
\begin{array}{rlrl}
5,750^{2} & =x^{2}+1,170^{2} & & \text { Pythagorean theorem } \\
33,062,500 & =x^{2}+1,368,900 & & \text { Simplify squares } \\
x^{2} & =33,062,500-1,368,900 & & \text { Solve for } x^{2} \\
x^{2} & =31,693,600 & & \text { Simplify the right side } \\
x & =\sqrt{31,693,600} & & \text { Theorem } 1 \\
& =5,630 \text { feet (to the nearest foot) }
\end{array}
$$

A rider getting on the lift at point $A$ and riding to point $B$ will cover a horizontal distance of approximately 5,630 feet.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A ${ }^{1-3}$

B 4,5
C 6,7

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What kind of equation do we solve using the method of completing the square?
2. Explain in words how you would complete the square on $x^{2}-16 x=4$
3. What is the relationship between the shortest side and the longest side in a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle?
4. What two expressions together are equivalent to $x= \pm 4$ ?

## Problem Set 10.1

Solve the following equations.

1. $x^{2}=25$
2. $x^{2}=16$
3. $y^{2}=\frac{3}{4}$
4. $y^{2}=\frac{5}{9}$
5. $x^{2}+12=0$
6. $x^{2}+8=0$
7. $4 a^{2}-45=0$
8. $9 a^{2}-20=0$
9. $(2 y-1)^{2}=25$
10. $(3 y+7)^{2}=1$
11. $(2 a+3)^{2}=-9$
12. $(3 a-5)^{2}=-49$
13. $x^{2}+8 x+16=-27$
14. $x^{2}-12 x+36=-8$
15. $4 a^{2}-12 a+9=-4$
16. $9 a^{2}-12 a+4=-9$

Copy each of the following, and fill in the blanks so that the left side of each is a perfect square trinomial; that is, complete the square.
17. $x^{2}+12 x+$ $\qquad$ $=(x+$ $\qquad$ ) ${ }^{2}$
18. $x^{2}+6 x+$ $\qquad$ $=(x+$ $\qquad$ $)^{2}$
19. $x^{2}-4 x+$ $\qquad$ $=(x-$ $\qquad$ $)^{2}$
20. $x^{2}-2 x+$ $\qquad$ $=(x-$ $\qquad$ $)^{2}$
21. $a^{2}-10 a+$ $\qquad$ $=(a-$ $\qquad$ $)^{2}$
22. $a^{2}-8 a+$ $\qquad$ $=(a-$ $\qquad$ $)^{2}$
23. $x^{2}+5 x+$ $\qquad$ $=(x+$ $\qquad$ $)^{2}$
24. $x^{2}+3 x+$ $\qquad$ $=(x+$ $\qquad$ ) ${ }^{2}$
25. $y^{2}-7 y+$ $\qquad$ $=(y-$ $\qquad$ $)^{2}$
26. $y^{2}-y+$ $\qquad$ $=(y-$ $\qquad$ $)^{2}$
27. $x^{2}+\frac{1}{2} x+$ $\qquad$ $=(x+$ $\qquad$ $)^{2}$
28. $x^{2}-\frac{3}{4} x+$ $\qquad$ $=(x-$ $\qquad$ $)^{2}$
29. $x^{2}+\frac{2}{3} x+$ $\qquad$ $=(x+$ $\qquad$ $)^{2}$
30. $x^{2}-\frac{4}{5} x+$ $\qquad$ $=(x-$ $\qquad$ $)^{2}$

Solve each of the following quadratic equations by completing the square.
31. $x^{2}+12 x=-27$
32. $x^{2}-6 x=16$
33. $a^{2}-2 a+5=0$
34. $a^{2}+10 a+22=0$
35. $y^{2}-8 y+1=0$
36. $y^{2}+6 y-1=0$
37. $x^{2}-5 x-3=0$
38. $x^{2}-5 x-2=0$
39. $2 x^{2}-4 x-8=0$
40. $3 x^{2}-9 x-12=0$
41. $3 t^{2}-8 t+1=0$
42. $5 t^{2}+12 t-1=0$
43. $4 x^{2}-3 x+5=0$
44. $7 x^{2}-5 x+2=0$
45. For the equation $x^{2}=-9$
a. Can it be solved by factoring?
b. Solve it.
46. For the equation $x^{2}-10 x+18=0$
a. Can it be solved by factoring?
b. Solve it.
47. Solve each equation below by the indicated method.
a. $x^{2}-6 x=0$ Factoring
b. $x^{2}-6 x=0$ Completing the square
48. Solve each equation below by the indicated method.
a. $x^{2}+a x=0$ Factoring
b. $x^{2}+a x=0$ Completing the square
49. Solve the equation $x^{2}+2 x=35$
a. By factoring.
b. By completing the square.
50. Solve the equation $8 x^{2}-10 x-25=0$
a. By factoring.
b. By completing the square.
51. Is $x=-3+\sqrt{2}$ a solution to $x^{2}-6 x=7$ ?
52. Is $x=2-\sqrt{5}$ a solution to $x^{2}-4 x=1$ ?
53. Solve each equation
a. $5 x-7=0$
b. $5 x-7=8$
c. $(5 x-7)^{2}=8$
d. $\sqrt{5 x-7}=8$
e. $\frac{5}{2}-\frac{7}{2 x}=\frac{4}{x}$
54. Solve each equation
a. $5 x+11=0$
b. $5 x+11=9$
c. $(5 x+11)^{2}=9$
d. $\sqrt{5 x+11}=9$
e. $\frac{5}{3}-\frac{11}{3 x}=\frac{3}{x}$

Simplify the left side of each equation, and then solve for $x$.
55. $(x+5)^{2}+(x-5)^{2}=52$
56. $(2 x+1)^{2}+(2 x-1)^{2}=10$
57. $(2 x+3)^{2}+(2 x-3)^{2}=26$
58. $(3 x+2)^{2}+(3 x-2)^{2}=26$
59. $(3 x+4)(3 x-4)-(x+2)(x-2)=-4$
60. $(5 x+2)(5 x-2)-(x+3)(x-3)=29$
61. Fill in the table below given the following functions. $f(x)=(2 x-3)^{2}, g(x)=4 x^{2}-12 x+9, h(x)=4 x^{2}+9$

|  | $f(x)$ | $g(x)$ | $h(x)$ |
| ---: | ---: | ---: | ---: |
| -2 |  |  |  |
| -1 |  |  |  |
| 0 |  |  |  |
| 1 |  |  |  |

62. Fill in the table below given the following functions. $f(x)=\left(x+\frac{1}{2}\right)^{2}, g(x)=x^{2}+x+\frac{1}{4}, h(x)=x^{2}+\frac{1}{4}$

| $\boldsymbol{x}$ | $f(x)$ | $g(x)$ | $\boldsymbol{h}(x)$ |
| ---: | :--- | :--- | :--- |
| -2 |  |  |  |
| -1 |  |  |  |
| 0 |  |  |  |
| 1 |  |  |  |
| 2 |  |  |  |

63. If $f(x)=(x-3)^{2}$, find $x$ if $f(x)=0$.
64. If $f(x)=(3 x-5)^{2}$, find $x$ if $f(x)=0$.
65. If $f(x)=x^{2}-5 x-6$ find
a. The $x$-intercepts
b. The value of $x$ for which $f(x)=0$
c. $f(0)$
d. $f(1)$
66. If $f(x)=9 x^{2}-12 x+4$ find
a. The $x$-intercepts
b. The value of $x$ for which $f(x)=0$
c. $f(0)$
d. $f\left(\frac{2}{3}\right)$

## Applying the Concepts

67. Geometry If the shortest side in a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle is $\frac{1}{2}$ inch long, find the lengths of the other two sides.

68. Geometry If the length of the longest side of a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle is $x$, find the length of the other two sides in terms of $x$.
69. Geometry If the length of the shorter sides of a $45^{\circ}-45^{\circ}-90^{\circ}$ triangle is 1 inch, find the length of the hypotenuse.

70. Geometry If the length of the shorter sides of a $45^{\circ}-45^{\circ}-90^{\circ}$ triangle is $x$, find the length of the hypotenuse, in terms of $x$.

71. Chair Lift Use Table 1 from the introduction to this chapter to find the horizontal distance covered by a person riding the Bear Paw Double chair lift. Round your answer to the nearest foot.

72. Fermat's Last Theorem As we mentioned in a previous chapter, the postage stamp shows Fermat's last theorem, which states that if $n$ is an integer greater than 2 , then there are no positive integers $x, y$, and $z$ that will make the formula $x^{n}+y^{n}=z^{n}$ true.

Use the formula $x^{n}+y^{n}=z^{n}$ to
a. find $z$ if $n=2, x=6$, and $y=8$.
b. find $y$ if $n=2, x=5$, and $z=13$.

73. Interest Rate Suppose a deposit of \$3,000 in a savings account that paid an annual interest rate $r$ (compounded yearly) is worth $\$ 3,456$ after 2 years. Using the formula $A=P(1+r)^{t}$, we have

$$
3,456=3,000(1+r)^{2}
$$

Solve for $r$ to find the annual interest rate.
74. Special Triangles In Figure 5, triangle $A B C$ has angles $45^{\circ}$ and $30^{\circ}$ and height $x$. Find the lengths of sides $A B, B C$, and $A C$, in terms of $x$.


FIGURE 5
75. Length of an Escalator An escalator in a department store is to carry people a vertical distance of 20 feet between floors. How long is the escalator if it makes an angle of $45^{\circ}$ with the ground? (See Figure 6.)


FIGURE 6
76. Dimensions of a Tent A two-person tent is to be made so the height at the center is 4 feet. If the sides of the tent are to meet the ground at an angle of $60^{\circ}$ and the tent is to be 6 feet in length, how many square feet of material will be needed to make the tent? (Figure 7, assume that the tent has a floor and is closed at both ends.) Give your answer to the nearest tenth of a square foot.


FIGURE 7

## Maintaining Your Skills

Write each of the following in simplified form for radicals.
77. $\sqrt{45}$
78. $\sqrt{24}$
79. $\sqrt{27 y^{5}}$
80. $\sqrt{8 y^{3}}$
81. $\sqrt[3]{54 x^{6} y^{5}}$
82. $\sqrt[3]{16 x^{9} y^{7}}$

Rationalize the denominator.
83. $\frac{3}{\sqrt{2}}$
84. $\frac{5}{\sqrt{3}}$
85. $\frac{2}{\sqrt[3]{4}}$
86. $\frac{3}{\sqrt[3]{2}}$

## Getting Ready for the Next Section

Simplify.
87. $\sqrt{49-4(6)(-5)}$
88. $\sqrt{49-4(6) 2}$
89. $\sqrt{(-27)^{2}-4(0.1)(1,700)}$
90. $\sqrt{25-4(4)(-10)}$
91. $\frac{-7+\sqrt{169}}{12}$
92. $\frac{-7-\sqrt{169}}{12}$
93. Simplify $\sqrt{b^{2}-4 a c}$ when $a=6, b=7$, and $c=-5$.
94. Simplify $\sqrt{b^{2}-4 a c}$ when $a=2, b=-6$, and $c=3$.

Factor.
95. $27 t^{3}-8$
96. $125 t^{3}+1$

## Extending the Concepts

Solve for $x$.
97. $(x+a)^{2}+(x-a)^{2}=10 a^{2}$
98. $(a x+1)^{2}+(a x-1)^{2}=10$

Assume $p$ and $q$ are positive numbers and solve for $x$ by completing the square on $x$.
99. $x^{2}+p x+q=0$
100. $x^{2}-p x+q=0$
101. $3 x^{2}+p x+q=0$
102. $3 x^{2}+2 p x+q=0$

Complete the square on $x$ and also on $y$ so that each equation below is written in the form

$$
(x-a)^{2}+(y-b)^{2}=r^{2}
$$

which you will see later in the book as the equation of a circle with center $(a, b)$ and radius $r$.
103. $x^{2}-10 x+y^{2}-6 y=-30$
104. $x^{2}-2 x+y^{2}-4 y=20$

## 10.2

## OBJECTIVES

A Solve quadratic equations by the quadratic formula.

B Solve application problems using quadratic equations.

## The Quadratic Formula

In this section we will use the method of completing the square from the preceding section to derive the quadratic formula. The quadratic formula is a very useful tool in mathematics. It allows us to solve all types of quadratic equations.

The Quadratic Theorem For any quadratic equation in the form $a x^{2}+b x+c=0$, where $a \neq 0$, the two solutions are

$$
x=\frac{-b+\sqrt{b^{2}-4 a c}}{2 a} \quad \text { and } \quad x=\frac{-b-\sqrt{b^{2}-4 a c}}{2 a}
$$

Proof We will prove the quadratic theorem by completing the square on $a x^{2}+b x+c=0 ;$

$$
\begin{array}{rlrl}
a x^{2}+b x+c & =0 & \\
a x^{2}+b x & =-c & \text { Add }-c \text { to both sides } \\
x^{2}+\frac{b}{a} x & =-\frac{c}{a} & & \text { Divide both sides by } a
\end{array}
$$

To complete the square on the left side, we add the square of $\frac{1}{2}$ of $\frac{b}{a}$ to both sides. $\left(\frac{1}{2}\right.$ of $\frac{b}{a}$ is $\frac{b}{2 a}$.)

$$
x^{2}+\frac{b}{a} x+\left(\frac{b}{2 a}\right)^{2}=-\frac{c}{a}+\left(\frac{b}{2 a}\right)^{2}
$$

We now simplify the right side as a separate step. We square the second term and combine the two terms by writing each with the least common denominator $4 a^{2}$.

$$
-\frac{c}{a}+\left(\frac{b}{2 a}\right)^{2}=-\frac{c}{a}+\frac{b^{2}}{4 a^{2}}=\frac{\mathbf{4} \boldsymbol{a}}{\mathbf{4 a}}\left(\frac{-c}{a}\right)+\frac{b^{2}}{4 a^{2}}=\frac{-4 a c+b^{2}}{4 a^{2}}
$$

It is convenient to write this last expression as

$$
\frac{b^{2}-4 a c}{4 a^{2}}
$$

Continuing with the proof, we have

$$
\begin{array}{rlrl}
x^{2}+\frac{b}{a} x+\left(\frac{b}{2 a}\right)^{2} & =\frac{b^{2}-4 a c}{4 a^{2}} & \\
\left(x+\frac{b}{2 a}\right)^{2} & =\frac{b^{2}-4 a c}{4 a^{2}} & & \begin{array}{l}
\text { Write left side as a } \\
\text { binomial square }
\end{array} \\
x+\frac{b}{2 a} & = \pm \frac{\sqrt{b^{2}-4 a c}}{2 a} & & \text { Theorem } 1 \\
x & =-\frac{b}{2 a} \pm \frac{\sqrt{b^{2}-4 a c}}{2 a} & & \text { Add }-\frac{b}{2 a} \text { to both sides } \\
& =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} & &
\end{array}
$$

## Note

Using function notation, the problem shown in Example 1 could be stated this way:

If $f(x)=6 x^{2}+7 x-5$, find all values of $x$ for which $f(x)=0$.

EXAMPLE 1 Use the quadratic formula to solve $6 x^{2}+7 x-5=0$

SOLUTION Using $a=6, b=7$, and $c=-5$ in the formula

$$
\text { we have } \begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
\text { or } \quad & x
\end{aligned} \quad \begin{aligned}
x & =\frac{-7 \pm \sqrt{49-4(6)(-5)}}{2(6)} \\
& =\frac{-7 \pm \sqrt{49+120}}{12} \\
& =\frac{-7 \pm 13}{12}
\end{aligned}
$$

We separate the last equation into the two statements

$$
\begin{array}{lll}
x=\frac{-7+13}{12} & \text { or } & x=\frac{-7-13}{12} \\
x=\frac{1}{2} & \text { or } & x=-\frac{5}{3}
\end{array}
$$

The solution set is $\left\{\frac{1}{2},-\frac{5}{3}\right\}$.

Whenever the solutions to a quadratic equation are rational numbers, as they are in Example 1, it means that the original equation was solvable by factoring. To illustrate, let's solve the equation from Example 1 again but this time by factoring:

$$
\begin{array}{rlrl}
6 x^{2}+7 x-5 & =0 & & \text { Equation in standard form } \\
(3 x+5)(2 x-1) & =0 & & \text { Factor the left side } \\
3 x+5=0 & \text { or } 2 x-1 & =0 & \\
\text { Set factors equal to 0 } \\
x=-\frac{5}{3} & \text { or } & x & =\frac{1}{2}
\end{array}
$$

When an equation can be solved by factoring, then factoring is usually the faster method of solution. It is best to try to factor first, and then if you have trouble factoring, go to the quadratic formula. It always works.

EXAMPLE 2 Solve $\frac{x^{2}}{3}-x=-\frac{1}{2}$.
SOLUTION Multiplying through by 6 and writing the result in standard form, we have

$$
2 x^{2}-6 x+3=0
$$

the left side of which is not factorable. Therefore, we use the quadratic formula with $a=2, b=-6$, and $c=3$. The two solutions are given by

$$
\begin{aligned}
x & =\frac{-(-6) \pm \sqrt{36-4(2)(3)}}{2(2)} \\
& =\frac{6 \pm \sqrt{12}}{4} \\
& =\frac{6 \pm 2 \sqrt{3}}{4} \quad \sqrt{\mathbf{1 2}}=\sqrt{\mathbf{4} \cdot \mathbf{3}}=\sqrt{4} \sqrt{\mathbf{3}}=\mathbf{2} \sqrt{\mathbf{3}}
\end{aligned}
$$

We can reduce this last expression to lowest terms by factoring 2 from the numerator and denominator and then dividing the numerator and denominator by 2 :

$$
x=\frac{2(3 \pm \sqrt{3})}{2 \cdot 2}=\frac{3 \pm \sqrt{3}}{2}
$$

## EXAMPLE 3 Solve $\frac{1}{x+2}-\frac{1}{x}=\frac{1}{3}$.

SOLUTION To solve this equation, we must first put it in standard form. To do so, we must clear the equation of fractions by multiplying each side by the LCD for all the denominators, which is $3 x(x+2)$. Multiplying both sides by the LCD, we have

$$
\begin{array}{rlrl}
\mathbf{3 x}(\boldsymbol{x}+\mathbf{2})\left(\frac{1}{x+2}-\frac{1}{x}\right) & =\frac{1}{3} \cdot \mathbf{3 x}(\boldsymbol{x}+\mathbf{2}) & \begin{array}{l}
\text { Multiply each } \\
\text { by the LCD }
\end{array} \\
3 x(x+2) \cdot \frac{1}{x+2}-3 x(x+2) \cdot \frac{1}{x} & =\frac{1}{\nexists} \cdot \not 3 x(x+2) & & \\
3 x-3(x+2) & =x(x+2) & & \\
3 x-3 x-6 & =x^{2}+2 x & & \text { Multiplication } \\
-6 & =x^{2}+2 x & & \text { Simplify left side } \\
0 & =x^{2}+2 x+6 & & \text { Add 6 to each side }
\end{array}
$$

Since the right side of our last equation is not factorable, we use the quadratic formula. From our last equation, we have $a=1, b=2$, and $c=6$. Using these numbers for $a, b$, and $c$ in the quadratic formula gives us

$$
\begin{array}{rlr}
x & =\frac{-2 \pm \sqrt{4-4(1)(6)}}{2(1)} & \\
& =\frac{-2 \pm \sqrt{4-24}}{2} & \text { Simplify inside the radical } \\
& =\frac{-2 \pm \sqrt{-20}}{2} & \mathbf{4 - 2 4 = - \mathbf { 2 0 }} \\
& =\frac{-2 \pm 2 i \sqrt{5}}{2} & \\
& =\frac{20=i \sqrt{\mathbf{2 0}}=i \sqrt{\mathbf{4}} \sqrt{\mathbf{5}}=\mathbf{2} i \sqrt{5}}{2(-1 \pm i \sqrt{5})} \\
& =-1 \pm i \sqrt{5} & \\
\text { Factor } \mathbf{2} \text { from the numerator } \\
\text { Divide numerator and denominator by } \mathbf{2}
\end{array}
$$

Since neither of the two solutions, $-1+i \sqrt{5}$ nor $-1-i \sqrt{5}$, will make any of the denominators in our original equation 0 , they are both solutions.

Although the equation in our next example is not a quadratic equation, we solve it by using both factoring and the quadratic formula.

## EXAMPLE $4 \quad$ Solve $27 t^{3}-8=0$.

SOLUTION It would be a mistake to add 8 to each side of this equation and then take the cube root of each side because we would lose two of our solutions. Instead, we factor the left side, and then set the factors equal to 0 :

$$
\begin{aligned}
27 t^{3}-8 & =0 & & \text { Equation in standard form } \\
(3 t-2)\left(9 t^{2}+6 t+4\right) & =0 & & \text { Factor as the difference of } \\
3 t-2=0 \quad \text { or } \quad 9 t^{2}+6 t+4 & =0 & & \text { Set each factor equal to } 0
\end{aligned}
$$

The first equation leads to a solution of $t=\frac{2}{3}$. The second equation does not factor, so we use the quadratic formula with $a=9, b=6$, and $c=4$ :

$$
\begin{aligned}
t & =\frac{-6 \pm \sqrt{36-4(9)(4)}}{2(9)} \\
& =\frac{-6 \pm \sqrt{36-144}}{18} \\
& =\frac{-6 \pm \sqrt{-108}}{18} \\
& =\frac{-6 \pm 6 i \sqrt{3}}{18} \quad \sqrt{-108}=i \sqrt{36 \cdot \mathbf{3}}=6 i \sqrt{3} \\
& =\frac{6(-1 \pm i \sqrt{3})}{6 \cdot 3} \quad \text { Factor } 6 \text { from the numerator and denominator } \\
& =\frac{-1 \pm i \sqrt{3}}{3} \quad \text { Divide out common factor } 6
\end{aligned}
$$

The three solutions to our original equation are

$$
\frac{2}{3}, \quad \frac{-1+i \sqrt{3}}{3}, \quad \text { and } \quad \frac{-1-i \sqrt{3}}{3}
$$

## EXAMPLE 5 If an object is thrown downward with an initial velocity

 of 20 feet per second, the distance $s(t)$, in feet, it travels in $t$ seconds is given by the function $s(t)=20 t+16 t^{2}$. How long does it take the object to fall 40 feet?SOLUTION We let $s(t)=40$, and solve for $t$ :

$$
\begin{aligned}
& \text { When } \quad \begin{aligned}
s(t) & =40 \\
\text { the function } \quad s(t) & =20 t+16 t^{2} \\
\text { becomes } \quad 40 & =20 t+16 t^{2} \\
\text { or } \quad 16 t^{2}+20 t-40 & =0 \\
4 t^{2}+5 t-10 & =0 \quad \text { Divide by } 4
\end{aligned} \\
&
\end{aligned}
$$

Using the quadratic formula, we have

$$
\begin{aligned}
t & =\frac{-5 \pm \sqrt{25-4(4)(-10)}}{2(4)} \\
& =\frac{-5 \pm \sqrt{185}}{8} \\
& =\frac{-5+\sqrt{185}}{8} \text { or } \frac{-5-\sqrt{185}}{8}
\end{aligned}
$$



The second solution is impossible since it is a negative number and time $t$ must be positive. It takes

$$
t=\frac{-5+\sqrt{185}}{8} \text { or approximately } \frac{-5+13.60}{8} \approx 1.08 \text { seconds }
$$

for the object to fall 40 feet.

Recall that the relationship between profit, revenue, and cost is given by the formula

$$
P(x)=R(x)-C(x)
$$

where $P(x)$ is the profit, $R(x)$ is the total revenue, and $C(x)$ is the total cost of producing and selling $x$ items.

EXAMPLE 6 A company produces and sells copies of an accounting program for home computers. The total weekly cost (in dollars) to produce $x$ copies of the program is $C(x)=8 x+500$, and the weekly revenue for selling all $x$ copies of the program is $R(x)=35 x-0.1 x^{2}$. How many programs must be sold each week for the weekly profit to be $\$ 1,200$ ?

SOLUTION Substituting the given expressions for $R(x)$ and $C(x)$ in the equation $P(x)=R(x)-C(x)$, we have a polynomial in $x$ that represents the weekly profit $P(x)$ :

$$
\begin{aligned}
P(x) & =R(x)-C(x) \\
& =35 x-0.1 x^{2}-(8 x+500) \\
& =35 x-0.1 x^{2}-8 x-500 \\
& =-500+27 x-0.1 x^{2}
\end{aligned}
$$

Setting this expression equal to 1,200 , we have a quadratic equation to solve that gives us the number of programs $x$ that need to be sold each week to bring in a profit of $\$ 1,200$ :

$$
1,200=-500+27 x-0.1 x^{2}
$$

We can write this equation in standard form by adding the opposite of each term on the right side of the equation to both sides of the equation. Doing so produces the following equation:

$$
0.1 x^{2}-27 x+1,700=0
$$

Applying the quadratic formula to this equation with $a=0.1, b=-27$, and $c=1,700$, we have

$$
\begin{aligned}
x & =\frac{27 \pm \sqrt{(-27)^{2}-4(0.1)(1,700)}}{2(0.1)} \\
& =\frac{27 \pm \sqrt{729-680}}{0.2} \\
& =\frac{27 \pm \sqrt{49}}{0.2} \\
& =\frac{27 \pm 7}{0.2}
\end{aligned}
$$

Writing this last expression as two separate expressions, we have our two solutions:

$$
\left.\begin{array}{rlrl}
x & =\frac{27+7}{0.2} & \text { or } & x
\end{array}=\frac{27-7}{0.2}\right)
$$

The weekly profit will be $\$ 1,200$ if the company produces and sells 100 programs or 170 programs.

What is interesting about the equation we solved in Example 6 is that it has rational solutions, meaning it could have been solved by factoring. But looking back at the equation, factoring does not seem like a reasonable method of solution because the coefficients are either very large or very small. So, there are times when using the quadratic formula is a faster method of solution, even though the equation you are solving is factorable.

## USING TECHNOLOGY

## Graphing Calculators

## More About Example 5

We can solve the problem discussed in Example 5 by graphing the function $\mathrm{Y}_{1}=20 \mathrm{X}+$ $16 \mathrm{X}^{2}$ in a window with X from 0 to 2 (because X is taking the place of $t$ and we know $t$ is a positive quantity) and $Y$ from 0 to 50 (because we are looking for X when $\mathrm{Y}_{1}$ is 40). Graphing
 $Y_{1}$ gives a graph similar to the graph in Figure 1. Using the Zoom and Trace features at $Y_{1}=40$ gives us $X=1.08$ to the nearest hundredth, matching the results we obtained by solving the original equation algebraically.

## More About Example 6

To visualize the functions in Example 6, we set up our calculator this way:

| $\mathrm{Y}_{1}=35 \mathrm{X}-.1 \mathrm{X}^{2}$ | Revenue function |
| :--- | :--- |
| $\mathrm{Y}_{2}=8 \mathrm{X}+500$ | Cost function |
| $\mathrm{Y}_{3}=\mathrm{Y}_{1}-\mathrm{Y}_{2}$ | Profit function |

Window: X from 0 to 350, Y from 0 to 3500
Graphing these functions produces graphs similar to the ones shown in Figure 2. The lower graph is the graph of the profit function. Using the Zoom and Trace features on the lower graph at $Y_{3}=$ 1,200 produces two corresponding values of $\mathrm{X}, 170$ and 100, which match the
 results in Example 6.

We will continue this discussion of the relationship between graphs of functions and solutions to equations in the Using Technology material in the next section.

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-4
B 5,6

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the quadratic formula?
2. Under what circumstances should the quadratic formula be applied?
3. When would the quadratic formula result in complex solutions?
4. When will the quadratic formula result in only one solution?

## Problem Set 10.2

Online support materials can be found at www.thomsonedu.com/login

Solve each equation in each problem using the quadratic formula.

1. a. $3 x^{2}+4 x-2=0$
b. $3 x^{2}-4 x-2=0$
c. $3 x^{2}+4 x+2=0$
d. $2 x^{2}+4 x-3=0$
e. $2 x^{2}-4 x+3=0$
2. a. $3 x^{2}+6 x-2=0$
b. $3 x^{2}-6 x-2=0$
c. $3 x^{2}+6 x+2=0$
d. $2 x^{2}+6 x+3=0$
e. $2 x^{2}+6 x-3=0$
3. a. $x^{2}-2 x+2=0$
b. $x^{2}-2 x+5=0$
c. $x^{2}+2 x+2=0$
4. a. $x^{2}-4 x+5=0$
b. $x^{2}+4 x+5=0$
c. $a^{2}+4 a+1=0$

Solve each equation. Use factoring or the quadratic formula, whichever is appropriate. (Try factoring first. If you have any difficulty factoring, then go right to the quadratic formula.)
5. $\frac{1}{6} x^{2}-\frac{1}{2} x+\frac{1}{3}=0$
6. $\frac{1}{4} x^{2}+\frac{1}{4} x-\frac{1}{2}=0$
7. $\frac{x^{2}}{2}+1=\frac{2 x}{3}$
8. $\frac{x^{2}}{2}+\frac{2}{3}=-\frac{2 x}{3}$
9. $y^{2}-5 y=0$
10. $2 y^{2}+10 y=0$
11. $30 x^{2}+40 x=0$
12. $50 x^{2}-20 x=0$
13. $\frac{2 t^{2}}{3}-t=-\frac{1}{6}$
14. $\frac{t^{2}}{3}-\frac{t}{2}=-\frac{3}{2}$
15. $0.01 x^{2}+0.06 x-0.08=0$
16. $0.02 x^{2}-0.03 x+0.05=0$
17. $2 x+3=-2 x^{2}$
18. $2 x-3=3 x^{2}$
19. $100 x^{2}-200 x+100=0$
20. $100 x^{2}-600 x+900=0$
21. $\frac{1}{2} r^{2}=\frac{1}{6} r-\frac{2}{3}$
22. $\frac{1}{4} r^{2}=\frac{2}{5} r+\frac{1}{10}$
23. $(x-3)(x-5)=1$
24. $(x-3)(x+1)=-6$
25. $(x+3)^{2}+(x-8)(x-1)=16$
26. $(x-4)^{2}+(x+2)(x+1)=9$
27. $\frac{x^{2}}{3}-\frac{5 x}{6}=\frac{1}{2}$
28. $\frac{x^{2}}{6}+\frac{5}{6}=-\frac{x}{3}$

Multiply both sides of each equation by its LCD. Then solve the resulting equation.
29. $\frac{1}{x+1}-\frac{1}{x}=\frac{1}{2}$
30. $\frac{1}{x+1}+\frac{1}{x}=\frac{1}{3}$
31. $\frac{1}{y-1}+\frac{1}{y+1}=1$
32. $\frac{2}{y+2}+\frac{3}{y-2}=1$
33. $\frac{1}{x+2}+\frac{1}{x+3}=1$
34. $\frac{1}{x+3}+\frac{1}{x+4}=1$
35. $\frac{6}{r^{2}-1}-\frac{1}{2}=\frac{1}{r+1}$
36. $2+\frac{5}{r-1}=\frac{12}{(r-1)^{2}}$

Solve each equation. In each case you will have three solutions.
37. $x^{3}-8=0$
38. $x^{3}-27=0$
39. $8 a^{3}+27=0$
40. $27 a^{3}+8=0$
41. $125 t^{3}-1=0$
42. $64 t^{3}+1=0$

Each of the following equations has three solutions. Look for the greatest common factor, then use the quadratic formula to find all solutions.
43. $2 x^{3}+2 x^{2}+3 x=0$
44. $6 x^{3}-4 x^{2}+6 x=0$
45. $3 y^{4}=6 y^{3}-6 y^{2}$
46. $4 y^{4}=16 y^{3}-20 y^{2}$
47. $6 t^{5}+4 t^{4}=-2 t^{3}$
48. $8 t^{5}+2 t^{4}=-10 t^{3}$
49. Which two of the expressions below are equivalent?
a. $\frac{6+2 \sqrt{3}}{4}$
b. $\frac{3+\sqrt{3}}{2}$
c. $6+\frac{\sqrt{3}}{2}$
50. Which two of the expressions below are equivalent?
a. $\frac{8-4 \sqrt{2}}{4}$
b. $2-4 \sqrt{3}$
c. $2-\sqrt{2}$
51. Solve $3 x^{2}-5 x=0$
a. By factoring
b. By the quadratic formula
52. Solve $3 x^{2}+23 x-70=0$
a. By factoring
b. By the quadratic formula
53. Can the equation $x^{2}-4 x+7=0$ be solved by factoring? Solve the equation.
54. Can the equation $x^{2}=5$ be solved by factoring? Solve the equation.
55. Is $x=-1+i$ a solution to $x^{2}+2 x=-2$ ?
56. Is $x=2+2 i$ a solution to $(x-2)^{2}=-4$ ?
57. Let $f(x)=x^{2}-2 x-3$. Find all values for the variable $x$, which produce the following values of $f(x)$.
a. $f(x)=0$
b. $f(x)=-11$
c. $f(x)=-2 x+1$
d. $f(x)=2 x+1$
58. Let $g(x)=x^{2}+16$. Find all values for the variable $x$, which produce the following values of $g(x)$.
a. $g(x)=0$
b. $g(x)=20$
c. $g(x)=8 x$
d. $g(x)=-8 x$
59. Let $f(x)=\frac{10}{x^{2}}$. Find all values for the variable $x$, for which $f(x)=g(x)$.
a. $g(x)=3+\frac{1}{x}$
b. $g(x)=8 x-\frac{17}{x^{2}}$
c. $g(x)=0$
d. $g(x)=10$
60. Let $h(x)=\frac{x+2}{x}$. Find all values for the variable $x$, for which $h(x)=f(x)$.
a. $f(x)=2$
b. $f(x)=x+2$
c. $f(x)=x-2$
d. $f(x)=\frac{x-2}{4}$

## Applying the Concepts

61. Falling Object An object is thrown downward with an initial velocity of 5 feet per second. The relationship between the distance $s$ it travels and time $t$ is given by $s=5 t+16 t^{2}$. How long does it take the object to fall 74 feet?
62. Coin Toss A coin is tossed upward with an initial velocity of 32 feet per second from a height of 16 feet above the ground. The equation giving the object's height $h$ at any time $t$ is $h=16+32 t-16 t^{2}$. Does the object ever reach a height of 32 feet?

63. Profit The total cost (in dollars) for a company to manufacture and sell $x$ items per week is $C=60 x+300$, whereas the revenue brought in by selling all $x$ items is $R=100 x-0.5 x^{2}$. How many items must be sold to obtain a weekly profit of $\$ 300$ ?
64. Profit Suppose a company manufactures and sells $x$ picture frames each month with a total cost of $C=1,200+3.5 x$ dollars. If the revenue
 obtained by selling $x$ frames is $R=9 x-0.002 x^{2}$, find the number of frames it must sell each month if its monthly profit is to be \$2,300.
65. Photograph Cropping The following figure shows a photographic image on a 10.5-centimeter by 8.2-centimeter background. The overall area of the background is
 to be reduced to $80 \%$ of its original area by cutting off (cropping) equal strips on all four sides. What is the width of the strip that is cut from each side?
66. Area of a Garden A garden measures 20.3 meters by 16.4 meters. To double the area of the garden, strips of equal width are added to all four sides.
a. Draw a diagram that illustrates these conditions.
b. What are the new overall dimensions of the garden?
67. Area and Perimeter A rectangle has a perimeter of 20 yards and an area of 15 square yards.
a. Write two equations that state these facts in terms of the rectangle's length, $l$, and its width, $w$.
b. Solve the two equations from part (a) to determine the actual length and width of the rectangle.
c. Explain why two answers are possible to part (b).
68. Population Size Writing in 1829, former President James Madison made some predictions about the
growth of the population of the United States. The populations he predicted fit the equation

$$
y=0.029 x^{2}-1.39 x+42
$$

where $y$ is the population in millions of people $x$ years from 1829.
a. Use the equation to determine the approximate year President Madison would have predicted that the U.S. population would reach 100,000,000.
b. If the U.S. population in 2006 was approximately 300 million, were President Madison's predictions accurate in the long term? Explain why or why not.
79. $\frac{\left(49 x^{8} y^{-4}\right)^{1 / 2}}{\left(27 x^{-3} y^{9}\right)^{-1 / 3}}$
80. $\frac{\left(x^{-2} y^{1 / 3}\right)^{6}}{x^{-10} y^{3 / 2}}$

## Getting Ready for the Next Section

Find the value of $b^{2}-4 a c$ when
81. $a=1, b=-3, c=-40$
82. $a=2, b=3, c=4$
83. $a=4, b=12, c=9$
84. $a=-3, b=8, c=-1$

Solve.
85. $k^{2}-144=0$
86. $36-20 k=0$

Multiply.
87. $(x-3)(x+2)$
88. $(t-5)(t+5)$
89. $(x-3)(x-3)(x+2)$
90. $(t-5)(t+5)(t-3)$

## Extending the Concepts

So far, all the equations we have solved have had coefficients that were rational numbers. Here are some equations that have irrational coefficients and some that have complex coefficients. Solve each equation.
(Remember, $i^{2}=-1$.)
91. $x^{2}+\sqrt{3} x-6=0$
92. $x^{2}-\sqrt{5} x-5=0$
93. $\sqrt{2} x^{2}+2 x-\sqrt{2}=0$
94. $\sqrt{7} x^{2}+2 \sqrt{2} x-\sqrt{7}=0$
95. $x^{2}+i x+2=0$
96. $x^{2}+3 i x-2=0$

## 10.3

## Additional Items Involving Solutions to Equations

## OBJECTIVES

A Find the number and kind of solutions to a quadratic equation by using the discriminant.

B Find an unknown constant in quadratic equation so that there is exactly one solution.

C Find an equation from its solutions.

In this section we will do two things. First, we will define the discriminant and use it to find the kind of solutions a quadratic equation has without solving the equation. Second, we will use the zero-factor property to build equations from their solutions.

## The Discriminant

The quadratic formula

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

gives the solutions to any quadratic equation in standard form. When working with quadratic equations, there are times when it is important only to know what kind of solutions the equation has.

DEFINITION The expression under the radical in the quadratic formula is called the discriminant:

$$
\text { Discriminant }=D=b^{2}-4 a c
$$

The discriminant indicates the number and type of solutions to a quadratic equation, when the original equation has integer coefficients. For example, if we were to use the quadratic formula to solve the equation $2 x^{2}+2 x+3=0$, we would find the discriminant to be

$$
b^{2}-4 a c=2^{2}-4(2)(3)=-20
$$

Since the discriminant appears under a square root symbol, we have the square root of a negative number in the quadratic formula. Our solutions therefore would be complex numbers. Similarly, if the discriminant were 0 , the quadratic formula would yield

$$
x=\frac{-b \pm \sqrt{0}}{2 a}=\frac{-b \pm 0}{2 a}=\frac{-b}{2 a}
$$

and the equation would have one rational solution, the number $\frac{-b}{2 a}$.
The following table gives the relationship between the discriminant and the type of solutions to the equation.
For the equation $a x^{2}+b x+c=0$ where $a, b$, and $c$ are integers and $a \neq 0$ :

| If the Discriminant $\boldsymbol{b}^{\mathbf{2}}-\mathbf{4} \boldsymbol{a} \boldsymbol{c}$ is | Then the Equation Will Have |
| :--- | :--- |
| Negative | Two complex solutions containing $i$ |
| Zero | One rational solution |
| A positive number that is also a perfect square | Two rational solutions |
| A positive number that is not a perfect square | Two irrational solutions |

In the second and third cases, when the discriminant is 0 or a positive perfect square, the solutions are rational numbers. The quadratic equations in these two cases are the ones that can be factored.

## EXAMPLES <br> For each equation, give the number and kind of solutions.

1. $x^{2}-3 x-40=0$

SOLUTION Using $a=1, b=-3$, and $c=-40$ in $b^{2}-4 a c$, we have $(-3)^{2}-4(1)(-40)=9+160=169$.

The discriminant is a perfect square. The equation therefore has two rational solutions.
2. $2 x^{2}-3 x+4=0$

SOLUTION Using $a=2, b=-3$, and $c=4$, we have

$$
b^{2}-4 a c=(-3)^{2}-4(2)(4)=9-32=-23
$$

The discriminant is negative, implying the equation has two complex solutions that contain $i$.
3. $4 x^{2}-12 x+9=0$

SOLUTION Using $a=4, b=-12$, and $c=9$, the discriminant is $b^{2}-4 a c=(-12)^{2}-4(4)(9)=144-144=0$

Since the discriminant is 0 , the equation will have one rational solution.
4. $x^{2}+6 x=8$

SOLUTION We first must put the equation in standard form by adding -8 to each side. If we do so, the resulting equation is

$$
x^{2}+6 x-8=0
$$

Now we identify $a, b$, and $c$ as 1,6 , and -8 , respectively:

$$
b^{2}-4 a c=6^{2}-4(1)(-8)=36+32=68
$$

The discriminant is a positive number but not a perfect square. Therefore, the equation will have two irrational solutions.

EXAMPLE $5 \quad$ Find an appropriate $k$ so that the equation $4 x^{2}-k x=-9$ has exactly one rational solution.

SOLUTION We begin by writing the equation in standard form:

$$
4 x^{2}-k x+9=0
$$

Using $a=4, b=-k$, and $c=9$, we have

$$
\begin{aligned}
b^{2}-4 a c & =(-k)^{2}-4(4)(9) \\
& =k^{2}-144
\end{aligned}
$$

An equation has exactly one rational solution when the discriminant is 0 . We set the discriminant equal to 0 and solve:

$$
\begin{aligned}
k^{2}-144 & =0 \\
k^{2} & =144 \\
k & = \pm 12
\end{aligned}
$$

Choosing $k$ to be 12 or -12 will result in an equation with one rational solution.

## Building Equations from Their Solutions

Suppose we know that the solutions to an equation are $x=3$ and $x=-2$. We can find equations with these solutions by using the zero-factor property. First, let's write our solutions as equations with 0 on the right side:

| If | $x$ | $=3$ |  | First solution |
| :--- | ---: | :--- | ---: | :--- |
| then | $x-3$ | $=0$ |  | Add -3 to each side |
| and if | $x$ | $=-2$ |  | Second solution |
| then | $x+2$ | $=0$ |  | Add 2 to each side |

Now, since both $x-3$ and $x+2$ are 0 , their product must be 0 also. Therefore, we can write

$$
\begin{aligned}
(x-3)(x+2)=0 & \text { Zero-factor property } \\
x^{2}-x-6=0 & \text { Multiply out the left side }
\end{aligned}
$$

Many other equations have 3 and -2 as solutions. For example, any constant multiple of $x^{2}-x-6=0$, such as $5 x^{2}-5 x-30=0$, also has 3 and -2 as solutions. Similarly, any equation built from positive integer powers of the factors $x-3$ and $x+2$ will also have 3 and -2 as solutions. One such equation is

$$
\begin{aligned}
(x-3)^{2}(x+2) & =0 \\
\left(x^{2}-6 x+9\right)(x+2) & =0 \\
x^{3}-4 x^{2}-3 x+18 & =0
\end{aligned}
$$

In mathematics we distinguish between the solutions to this last equation and those to the equation $x^{2}-x-6=0$ by saying $x=3$ is a solution of multiplicity 2 in the equation $x^{3}-4 x^{2}-3 x+18=0$ and a solution of multiplicity 1 in the equation $x^{2}-x-6=0$.

## EXAMPLE 6 Find an equation that has solutions $t=5, t=-5$, and $t=3$.

SOLUTION First, we use the given solutions to write equations that have 0 on their right sides:

$$
\begin{aligned}
& \text { If } \quad t=5 \quad t=-5 \quad t=3 \\
& \text { then } t-5=0 \quad t+5=0 \quad t-3=0
\end{aligned}
$$

Since $t-5, t+5$, and $t-3$ are all 0 , their product is also 0 by the zero-factor property. An equation with solutions of $5,-5$, and 3 is

$$
\begin{aligned}
(t-5)(t+5)(t-3) & =0
\end{aligned} \text { Zero-factor property } \begin{aligned}
\left(t^{2}-25\right)(t-3) & =0
\end{aligned} \text { Multiply first two binomials } \quad \text { Complete the multiplication }
$$

The last line gives us an equation with solutions of $5,-5$, and 3 . Remember, many other equations have these same solutions.

EXAMPLE 7 Find an equation with solutions $x=-\frac{2}{3}$ and $x=\frac{4}{5}$.
SOLUTION The solution $x=-\frac{2}{3}$ can be rewritten as $3 x+2=0$ as follows:

| $x$ | $=-\frac{2}{3}$ |  | The first solution |
| ---: | :--- | ---: | :--- |
| $3 x$ | $=-2$ |  | Multiply each side by 3 |
| $3 x+2$ | $=0$ |  | Add 2 to each side |

Similarly, the solution $x=\frac{4}{5}$ can be rewritten as $5 x-4=0$ :

$$
\begin{aligned}
x & =\frac{4}{5} & & \text { The second solution } \\
5 x & =4 & & \text { Multiply each side by } 5 \\
5 x-4 & =0 & & \text { Add }-4 \text { to each side }
\end{aligned}
$$

Since both $3 x+2$ and $5 x-4$ are 0 , their product is 0 also, giving us the equation we are looking for:

$$
\begin{array}{rll}
(3 x+2)(5 x-4) & =0 & \text { Zero-factor property } \\
15 x^{2}-2 x-8 & =0 & \text { Multiplication }
\end{array}
$$

## USING TECHNOLOGY

Graphing Calculators

## Solving Equations

Now that we have explored the relationship between equations and their solutions, we can look at how a graphing calculator can be used in the solution process. To begin, let's solve the equation $x^{2}=x+2$ using techniques from algebra: writing it in standard form, factoring, and then setting each factor equal to 0 .

$$
\begin{array}{rlrlrl}
x^{2}-x-2 & =0 & & \text { Standard form } \\
& (x-2)(x+1) & =0 & & \text { Factor } \\
x-2=0 & & \text { or } & x+1 & =0 & \\
\text { Set each factor equal to } 0 \\
x=2 & & \text { or } & x & =-1 & \\
\text { Solve }
\end{array}
$$

Our original equation, $x^{2}=x+2$, has two solutions: $x=2$ and $x=-1$. To solve the equation using a graphing calculator, we need to associate it with an equation (or equations) in two variables. One way to do this is to associate the left side with the equation $y=x^{2}$ and the right side of the equation with $y=x+2$. To do so, we set up the functions list in our calculator this way:

$$
\begin{aligned}
& Y_{1}=X^{2} \\
& Y_{2}=X+2
\end{aligned}
$$

Window: $X$ from -5 to $5, Y$ from -5 to 5

Graphing these functions in this window will produce a graph similar to the one shown in Figure 1.


Figure 1
If we use the Trace feature to find the coordinates of the points of intersection, we find that the two curves intersect at $(-1,1)$ and $(2,4)$. We note that the $x$-coordinates of these two points match the solutions to the equation $x^{2}=x+2$, which we found using algebraic techniques. This makes sense because if two graphs intersect at a point $(x, y)$, then the coordinates of that point satisfy both equations. If a point $(x, y)$ satisfies both $y=x^{2}$ and $y=x+2$, then, for that particular point, $x^{2}=x+2$. From this we conclude that the $x$-coordinates of the points of intersection are solutions to our original equation. Here is a summary of what we have discovered:

Conclusion 1 If the graph of two functions $y=f(x)$ and $y=g(x)$ intersect in the coordinate plane, then the $x$-coordinates of the points of intersection are solutions to the equation $f(x)=g(x)$.

A second method of solving our original equation $x^{2}=x+2$ graphically requires the use of one function instead of two. To begin, we write the equation in standard form as $x^{2}-x-2=0$. Next, we graph the function $y=x^{2}-x-2$. The $x$-intercepts of the graph are the points with $y$-coordinates of 0 . They therefore satisfy the equation $0=x^{2}-x-2$, which is equivalent to our original equation. The graph in Figure 2 shows $Y_{1}=X^{2}-X-2$ in a window with $X$ from -5 to 5 and $Y$ from -5 to 5.


Figure 2

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-4
B 5
C 6,7

Using the Trace feature, we find that the $x$-intercepts of the graph are $x=-1$ and $x=2$, which match the solutions to our original equation $x^{2}=x+2$. We can summarize the relationship between solutions to an equation and the intercepts of its associated graph this way:

Conclusion 2 If $y=f(x)$ is a function, then any $x$-intercept on the graph of $y=f(x)$ is a solution to the equation $f(x)=0$.

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the discriminant?
2. What kind of solutions do we get to a quadratic equation when the discriminant is negative?
3. What does it mean for a solution to have multiplicity 3 ?
4. When will a quadratic equation have two rational solutions?

## Problem Set 10.3

Use the discriminant to find the number and kind of solutions for each of the following equations.

1. $x^{2}-6 x+5=0$
2. $x^{2}-x-12=0$
3. $4 x^{2}-4 x=-1$
4. $9 x^{2}+12 x=-4$
5. $x^{2}+x-1=0$
6. $x^{2}-2 x+3=0$
7. $2 y^{2}=3 y+1$
8. $3 y^{2}=4 y-2$
9. $x^{2}-9=0$
10. $4 x^{2}-81=0$
11. $5 a^{2}-4 a=5$
12. $3 a=4 a^{2}-5$

Determine $k$ so that each of the following has exactly one real solution.
13. $x^{2}-k x+25=0$
14. $x^{2}+k x+25=0$
15. $x^{2}=k x-36$
16. $x^{2}=k x-49$
17. $4 x^{2}-12 x+k=0$
18. $9 x^{2}+30 x+k=0$
19. $k x^{2}-40 x=25$
20. $k x^{2}-2 x=-1$
21. $3 x^{2}-k x+2=0$
22. $5 x^{2}+k x+1=0$

For each of the following problems, find an equation that has the given solutions.
23. $x=5, x=2$
24. $x=-5, x=-2$
25. $t=-3, t=6$
26. $t=-4, t=2$
27. $y=2, y=-2, y=4$
28. $y=1, y=-1, y=3$
29. $x=\frac{1}{2}, x=3$
30. $x=\frac{1}{3}, x=5$
31. $t=-\frac{3}{4}, t=3$
$=$ Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
32. $t=-\frac{4}{5}, t=2$
33. $x=3, x=-3, x=\frac{5}{6}$
34. $x=5, x=-5, x=\frac{2}{3}$
35. $a=-\frac{1}{2}, a=\frac{3}{5}$
36. $a=-\frac{1}{3}, a=\frac{4}{7}$
37. $x=-\frac{2}{3}, x=\frac{2}{3}, x=1$
38. $x=-\frac{4}{5}, x=\frac{4}{5}, x=-1$
39. $x=2, x=-2, x=3, x=-3$
40. $x=1, x=-1, x=5, x=-5$
41. $x=\sqrt{5}, x=-\sqrt{5}$
42. $x=\sqrt{13}, x=-\sqrt{13}$
43. $x=\sqrt{3}, x=-\sqrt{7}$
44. $x=\sqrt{2}, x=-\sqrt{5}$
45. $x=4 i, x=-4 i$
46. $x=9 i, x=-9 i$
47. $x=2 i, x=3 i$
48. $x=-2 i, x=-3 i$
49. $x=2+i, x=2-i$
50. $x=3+2 i, x=3-2 i$
51. $x=\frac{1+\sqrt{5}}{2}, x=\frac{1-\sqrt{5}}{2}$
52. $x=\frac{-1+\sqrt{5}}{2}, x=\frac{-1-\sqrt{5}}{2}$
53. The graphs of three quadratic functions are shown below. Match each function with the number of solutions to equation $y=0$, that it contains.
a. One real solution
b. No real solutions
c. Two real solutions


## Maintaining Your Skills

Multiply. (Assume all variables represent positive numbers for the rest of the problems in this section.)
54. $a^{4}\left(a^{3 / 2}-a^{1 / 2}\right)$
55. $\left(a^{1 / 2}-5\right)\left(a^{1 / 2}+3\right)$
56. $\left(x^{3 / 2}-3\right)^{2}$
57. $\left(x^{1 / 2}-8\right)\left(x^{1 / 2}+8\right)$

Divide.
58. $\frac{30 x^{3 / 4}-25 x^{5 / 4}}{5 x^{1 / 4}}$
59. $\frac{45 x^{5 / 3} y^{7 / 3}-36 x^{8 / 3} y^{4 / 3}}{9 x^{2 / 3} y^{1 / 3}}$
60. Factor $5(x-3)^{1 / 2}$ from $10(x-3)^{3 / 2}-15(x-3)^{1 / 2}$.
61. Factor $2(x+1)^{1 / 3}$ from $8(x+1)^{4 / 3}-2(x+1)^{1 / 3}$.

Factor each of the following.
62. $2 x^{2 / 3}-11 x^{1 / 3}+12$
63. $9 x^{2 / 3}+12 x^{1 / 3}+4$

## Getting Ready for the Next Section

Simplify
64. $(x+3)^{2}-2(x+3)-8$
65. $(x-2)^{2}-3(x-2)-10$
66. $(2 a-3)^{2}-9(2 a-3)+20$
67. $(3 a-2)^{2}+2(3 a-2)-3$
68. $2(4 a+2)^{2}-3(4 a+2)-20$
69. $6(2 a+4)^{2}-(2 a+4)-2$

Solve.
70. $x^{2}=\frac{1}{4}$
72. $\sqrt{x}=-3$
74. $x+3=4$
76. $y^{2}-2 y-8=0$
71. $x^{2}=-2$
73. $\sqrt{x}=2$
75. $x+3=-2$
77. $y^{2}+y-6=0$
78. $4 y^{2}+7 y-2=0$
79. $6 x^{2}-13 x-5=0$

## Extending the Concepts

Find all solutions to the following equations. Solve using algebra and by graphing. If rounding is necessary, round to the nearest hundredth. A calculator can be used in these problems.
80. $x^{2}=4 x+5$
81. $4 x^{2}=8 x+5$
82. $x^{2}-1=2 x$
83. $4 x^{2}-1=4 x$

Find all solutions to each equation. If rounding is necessary, round to the nearest hundredth.
84. $2 x^{3}-x^{2}-2 x+1=0$
85. $3 x^{3}-2 x^{2}-3 x+2=0$
86. $2 x^{3}+2=x^{2}+4 x$
87. $3 x^{3}-9 x=2 x^{2}-6$

## Equations Quadratic in Form

## OBJECTIVES

A Solve equations that are reducible to a quadratic equation.
B Solve application problems using equations quadratic in form.

We are now in a position to put our knowledge of quadratic equations to work to solve a variety of equations.

## EXAMPLE $1 \quad$ Solve $(x+3)^{2}-2(x+3)-8=0$.

- SOLUTION We can see that this equation is quadratic in form by replacing $x+3$ with another variable, say $y$. Replacing $x+3$ with $y$ we have

$$
y^{2}-2 y-8=0
$$

We can solve this equation by factoring the left side and then setting each factor equal to 0 .

$$
\begin{aligned}
& y^{2}-2 y-8=0 \\
& (y-4)(y+2)=0 \quad \text { Factor } \\
& y-4=0 \quad \text { or } \quad y+2=0 \quad \text { Set factors to } 0 \\
& y=4 \quad \text { or } \quad y=-2
\end{aligned}
$$

Since our original equation was written in terms of the variable $x$, we would like our solutions in terms of $x$ also. Replacing $y$ with $x+3$ and then solving for $x$ we have

$$
\begin{array}{rlrlrl}
x+3 & =4 & \text { or } & & x+3 & =-2 \\
x & =1 & \text { or } & x & =-5
\end{array}
$$

The solutions to our original equation are 1 and -5 .
The method we have just shown lends itself well to other types of equations that are quadratic in form, as we will see. In this example, however, there is an-
other method that works just as well. Let's solve our original equation again, but this time, let's begin by expanding $(x+3)^{2}$ and $2(x+3)$.

$$
\begin{array}{rlrl}
(x+3)^{2}-2(x+3)-8 & =0 & & \\
x^{2}+6 x+9-2 x-6-8 & =0 & & \text { Multiply } \\
x^{2}+4 x-5 & =0 & & \text { Combine similar terms } \\
(x-1)(x+5) & =0 & & \text { Factor } \\
x-1=0 & \text { or } \quad x+5 & =0 & \\
\text { Set factors to } 0 \\
x=1 \quad \text { or } & x & =-5 &
\end{array}
$$

As you can see, either method produces the same result.

## EXAMPLE 2 Solve $4 x^{4}+7 x^{2}=2$.

SOLUTION This equation is quadratic in $x^{2}$. We can make it easier to look at by using the substitution $y=x^{2}$. (The choice of the letter $y$ is arbitrary. We could just as easily use the substitution $m=x^{2}$.) Making the substitution $y=x^{2}$ and then solving the resulting equation we have

$$
\begin{array}{rlrl}
4 y^{2}+7 y & =2 & & \\
4 y^{2}+7 y-2 & =0 & & \text { Standard form } \\
(4 y-1)(y+2) & =0 & & \text { Factor } \\
4 y-1=0 & \text { or } y+2 & =0 & \\
\text { Set factors to 0 } \\
y=\frac{1}{4} \quad \text { or } & y & =-2 &
\end{array}
$$

Now we replace $y$ with $x^{2}$ to solve for $x$ :

$$
\begin{array}{rlrlrl}
x^{2} & =\frac{1}{4} & \text { or } & x^{2} & =-2 & \\
x & = \pm \sqrt{\frac{1}{4}} & & \text { or } & x & = \pm \sqrt{-2} \\
& \text { Theorem 1 } \\
x & = \pm \frac{1}{2} & & \text { or } & x & = \pm i \sqrt{2}
\end{array}
$$

The solution set is $\left\{\frac{1}{2},-\frac{1}{2}, i \sqrt{2},-i \sqrt{2}\right\}$.

## EXAMPLE 3 Solve for $x: x+\sqrt{x}-6=0$.

SOLUTION To see that this equation is quadratic in form, we have to notice that $(\sqrt{x})^{2}=x$; that is, the equation can be rewritten as

$$
(\sqrt{x})^{2}+\sqrt{x}-6=0
$$

Replacing $\sqrt{x}$ with $y$ and solving as usual, we have

$$
\begin{array}{rlrl}
y^{2}+y-6 & =0 \\
& & (y+3)(y-2) & =0 \\
y+3 & =0 & & \text { or } \\
y & y-2 & =0 \\
y & =-3 & \text { or } & y
\end{array}=2
$$

Again, to find $x$, we replace with $\sqrt{x}$ and solve:

$$
\begin{aligned}
\sqrt{x} & =-3 & \text { or } & \sqrt{x} & =2 \\
x & =9 & & x & =4
\end{aligned}
$$

Since we squared both sides of each equation, we have the possibility of obtaining extraneous solutions. We have to check both solutions in our original equation.

|  | $x$ | $=9$ |  |
| ---: | :--- | ---: | :--- |
| When | When | $x=4$ |  |
| the equation | $x+\sqrt{x}-6$ | $=0$ |  |
| becomes | the equation | $x+\sqrt{x}-6=0$ |  |
| $9+\sqrt{9}-6 \stackrel{?}{=} 0$ |  | becomes | $4+\sqrt{4}-6 \stackrel{?}{=} 0$ |
| $9+3-6 \stackrel{?}{=} 0$ |  | $4+2-6 \stackrel{?}{=} 0$ |  |
| 6 | $\neq 0$ |  | 0 |

This means 9 is extraneous.

This means 4 is a solution.

The only solution to the equation $x+\sqrt{x}-6=0$ is $x=4$.

We should note here that the two possible solutions, 9 and 4 , to the equation in Example 3 can be obtained by another method. Instead of substituting for $\sqrt{x}$, we can isolate it on one side of the equation and then square both sides to clear the equation of radicals.

$$
\begin{aligned}
x+\sqrt{x}-6 & =0 & & \\
\sqrt{x} & =-x+6 & & \text { Isolate } x \\
x & =x^{2}-12 x+36 & & \text { Square both sides } \\
0 & =x^{2}-13 x+36 & & \text { Add }-x \text { to both sides } \\
0 & =(x-4)(x-9) & & \text { Factor } \\
x-4=0 & & \text { or } \quad x-9=0 &
\end{aligned}
$$

We obtain the same two possible solutions. Since we squared both sides of the equation to find them, we would have to check each one in the original equation. As was the case in Example 3, only $x=4$ is a solution; $x=9$ is extraneous.

## EXAMPLE 4 If an object is tossed into the air with an upward velocity

 of 12 feet per second from the top of a building $h$ feet high, the time it takes for the object to hit the ground below is given by the formula

$$
16 t^{2}-12 t-h=0
$$

Solve this formula for $t$.
SOLUTION The formula is in standard form and is quadratic in $t$. The coefficients $a, b$, and $c$ that we need to apply to the quadratic formula are $a=16$, $b=-12$, and $c=-h$. Substituting these quantities into the quadratic formula, we have

$$
\begin{aligned}
t & =\frac{12 \pm \sqrt{144-4(16)(-h)}}{2(16)} \\
& =\frac{12 \pm \sqrt{144+64 h}}{32}
\end{aligned}
$$

We can factor the perfect square 16 from the two terms under the radical and simplify our radical somewhat:

$$
\begin{aligned}
t & =\frac{12 \pm \sqrt{16(9+4 h)}}{32} \\
& =\frac{12 \pm 4 \sqrt{9+4 h}}{32}
\end{aligned}
$$

Now we can reduce to lowest terms by factoring a 4 from the numerator and denominator.

$$
\begin{aligned}
t & =\frac{4(3 \pm \sqrt{9+4 h})}{4 \cdot 8} \\
& =\frac{3 \pm \sqrt{9+4 h}}{8}
\end{aligned}
$$

If we were given a value of $h$, we would find that one of the solutions to this last formula would be a negative number. Since time is always measured in positive units, we wouldn't use that solution.

## More About the Golden Ratio

Previously, we derived the golden ratio $\frac{1+\sqrt{5}}{2}$ by finding the ratio of length to width for a golden rectangle. The golden ratio was actually discovered before the golden rectangle by the Greeks who lived before Euclid. The early Greeks found the golden ratio by dividing a line segment into two parts so that the ratio of the shorter part to the longer part was the same as the ratio of the longer part to the whole segment. When they divided a line segment in this manner, they said it was divided in "extreme and mean ratio." Figure 1 illustrates a line segment divided this way.


FIGURE 1
If point $B$ divides segment $A C$ in "extreme and mean ratio," then

$$
\begin{aligned}
\frac{\text { Length of shorter segment }}{\text { Length of longer segment }} & =\frac{\text { Length of longer segment }}{\text { Length of whole segment }} \\
\qquad \frac{A B}{B C} & =\frac{B C}{A C}
\end{aligned}
$$

EXAMPLE 5 If the length of segment $A B$ in Figure 1 is 1 inch, find the length of $B C$ so that the whole segment $A C$ is divided in "extreme and mean ratio."

SOLUTION Using Figure 1 as a guide, if we let $x=$ the length of segment $B C$, then the length of $A C$ is $x+1$. If $B$ divides $A C$ into "extreme and mean ratio,"
then the ratio of $A B$ to $B C$ must equal the ratio of $B C$ to $A C$. Writing this relationship using the variable $x$, we have

$$
\frac{1}{x}=\frac{x}{x+1}
$$

If we multiply both sides of this equation by the $\operatorname{LCD} x(x+1)$ we have

$$
\begin{aligned}
x+1 & =x^{2} \\
0 & =x^{2}-x-1 \quad \text { Write equation in standard form }
\end{aligned}
$$

Since this last equation is not factorable, we apply the quadratic formula.

$$
\begin{aligned}
x & =\frac{1 \pm \sqrt{(-1)^{2}-4(1)(-1)}}{2} \\
& =\frac{1 \pm \sqrt{5}}{2}
\end{aligned}
$$

Our equation has two solutions, which we approximate using decimals:

$$
\frac{1+\sqrt{5}}{2} \approx 1.618 \quad \frac{1-\sqrt{5}}{2} \approx-0.618
$$

Since we originally let $x$ equal the length of segment $B C$, we use only the positive solution to our equation. As you can see, the positive solution is the golden ratio.

## USING TECHNOLOGY

## Graphing Calculators

## More About Example 1

As we mentioned earlier, algebraic expressions entered into a graphing calculator do not have to be simplified to be evaluated. This fact applies to equations as well. We can graph the equation $y=(x+3)^{2}-2(x+3)-8$ to assist us in solving the equation in Example 1. The graph is shown in Figure 2. Using the Zoom and Trace features at the $x$-intercepts gives us $x=1$ and $x=-5$ as the solutions to the equation $0=(x+3)^{2}-2(x+3)-8$.


FIGURE 2

## More About Example 2

Figure 3 shows the graph of $y=4 x^{4}+7 x^{2}-2$. As we expect, the $x$-intercepts give the real number solutions to the equation $0=4 x^{4}+7 x^{2}-2$. The complex solutions do not appear on the graph.


FIGURE 3

## More About Example 3

In solving the equation in Example 3, we found that one of the possible solutions was an extraneous solution. If we solve the equation $x+\sqrt{x}-6=0$ by graphing the function $y=x+\sqrt{x}-6$, we find that the extraneous solution, 9 , is not an $x$-intercept. Figure 4 shows that the only solution to the equation occurs at the $x$-intercept 4.


FIGURE 4

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.
A 1-3
B 4, 5

## GETTING READY FOR CLASS

## After reading through the preceding section, respond in your own words

 and in complete sentences.1. What does it mean for an equation to be quadratic in form?
2. What are all the circumstances in solving equations (that we have studied) in which it is necessary to check for extraneous solutions?
3. How would you start to solve the equation $x+\sqrt{x}-6=0$ ?
4. What does it mean for a line segment to be divided in "extreme and mean ratio"?

## Problem Set 10.4

Solve each equation.

1. $(x-3)^{2}+3(x-3)+2=0$
2. $(x+4)^{2}-(x+4)-6=0$
3. $2(x+4)^{2}+5(x+4)-12=0$
4. $3(x-5)^{2}+14(x-5)-5=0$
5. $x^{4}-10 x^{2}+9=0$
6. $x^{4}-29 x^{2}+100=0$
7. $x^{4}-7 x^{2}+12=0$
8. $x^{4}-14 x^{2}+45=0$
9. $x^{4}-6 x^{2}-27=0$
10. $x^{4}+2 x^{2}-8=0$
11. $x^{4}+9 x^{2}=-20$
12. $x^{4}-11 x^{2}=-30$
13. $(2 a-3)^{2}-9(2 a-3)=-20$
14. $(3 a-2)^{2}+2(3 a-2)=3$
15. $2(4 a+2)^{2}=3(4 a+2)+20$
16. $6(2 a+4)^{2}=(2 a+4)+2$
17. $6 t^{4}=-t^{2}+5$
18. $3 t^{4}=-2 t^{2}+8$
19. $9 x^{4}-49=0$
20. $25 x^{4}-9=0$

Solve each of the following equations. Remember, if you square both sides of an equation in the process of solving it, you have to check all solutions in the original equation.
21. $x-7 \sqrt{x}+10=0$
22. $x-6 \sqrt{x}+8=0$
23. $t-2 \sqrt{t}-15=0$
24. $t-3 \sqrt{t}-10=0$
25. $6 x+11 \sqrt{x}=35$
26. $2 x+\sqrt{x}=15$
27. $x-2 \sqrt{x}-8=0$
28. $x+2 \sqrt{x}-3=0$
29. $x+3 \sqrt{x}-18=0$
30. $x-5 \sqrt{x}-14=0$
31. $2 x+9 \sqrt{x}-5=0$
32. $2 x-\sqrt{x}-6=0$
33. $(a-2)-11 \sqrt{a-2}+30=0$
34. $(a-3)-9 \sqrt{a-3}+20=0$
35. $(2 x+1)-8 \sqrt{2 x+1}+15=0$
36. $(2 x-3)-7 \sqrt{2 x-3}+12=0$
37. $\left(x^{2}+1\right)^{2}-2\left(x^{2}+1\right)-15=0$
38. $\left(x^{2}-3\right)^{2}-4\left(x^{2}-3\right)-12=0$
39. $\left(x^{2}+5\right)^{2}-6\left(x^{2}+5\right)-27=0$
40. $\left(x^{2}-2\right)^{2}-6\left(x^{2}-2\right)-7=0$
41. $x^{-2}-3 x^{-1}+2=0$
42. $x^{-2}+x^{-1}-2=0$
43. $y^{-4}-5 y^{-2}=0$
44. $2 y^{-4}+10 y^{-2}=0$
45. $x^{2 / 3}+4 x^{1 / 3}-12=0$
46. $x^{2 / 3}-2 x^{1 / 3}-8=0$
47. $x^{4 / 3}+6 x^{2 / 3}+9=0$
48. $x^{4 / 3}-10 x^{2 / 3}+25=0$
49. Solve the formula $16 t^{2}-v t-h=0$ for $t$.
50. Solve the formula $16 t^{2}+v t+h=0$ for $t$.
51. Solve the formula $k x^{2}+8 x+4=0$ for $x$.
52. Solve the formula $k^{2} x^{2}+k x+4=0$ for $x$.
53. Solve $x^{2}+2 x y+y^{2}=0$ for $x$ by using the quadratic formula with $a=1, b=2 y$, and $c=y^{2}$.
54. Solve $x^{2}-2 x y+y^{2}=0$ for $x$ by using the quadratic formula, with $a=1, b=-2 y$, and $c=y^{2}$.

## Applying the Concepts

For Problems 55-56, $t$ is in seconds.
55. Falling Object An object is tossed into the air with an upward velocity of 8 feet per second from the top of a building $h$ feet high. The time it takes for the object to hit the ground below is given by the formula $16 t^{2}-8 t-h=0$. Solve this formula for $t$.

56. Falling Object An object is tossed into the air with an upward velocity of 6 feet per second from the top of a building $h$ feet high. The time it takes for the object to hit the ground below is given by the formula $16 t^{2}-6 t-h=0$. Solve this formula for $t$.
57. Saint Louis Arch The shape of the famous "Gateway to the West" arch in Saint Louis can be modeled by a parabola. The equation for one such parabola is: $y=-\frac{1}{150} x^{2}+\frac{21}{5} x$ where $x$ and $y$ are in feet.
a. Sketch the graph of
 the arch's equation on a coordinate axis.
b. Approximately how far do you have to walk to get from one side of the arch to the other?
58. Area and Perimeter A
total of 160 yards of fencing is to be used to enclose part of a lot that borders on a river.
 This situation is shown in the following diagram.
a. Write an equation that gives the relationship between the length and width and the 160 yards of fencing.
b. The formula for the area that is enclosed by the fencing and the river is $A=l w$. Solve the equation in part (a) for $l$, and then use the result to write the area in terms of $w$ only.
c. Make a table that gives at least five possible values of $w$ and associated area $A$.
d. From the pattern in your table shown in part (c), what is the largest area that can be enclosed by the 160 yards of fencing? (Try some other table values if necessary.)

Golden Ratio Use Figure 1 from this section as a guide to working Problems 59-60.
59. If $A B$ in Figure 1 is 4 inches, and $B$ divides $A C$ in "extreme and mean ratio," find $B C$, and then show that $B C$ is 4 times the golden ratio.
60. If $A B$ in Figure 1 is $\frac{1}{2}$ inch, and $B$ divides $A C$ in "extreme and mean ratio," find $B C$, and then show that the ratio of $B C$ to $A B$ is the golden ratio.

## Maintaining Your Skills

Combine, if possible.
61. $5 \sqrt{7}-2 \sqrt{7}$
62. $6 \sqrt{2}-9 \sqrt{2}$
63. $\sqrt{18}-\sqrt{8}+\sqrt{32}$
64. $\sqrt{50}+\sqrt{72}-\sqrt{8}$
65. $9 x \sqrt{20 x^{3} y^{2}}+7 y \sqrt{45 x^{5}}$
66. $5 x^{2} \sqrt{27 x y^{3}}-6 y \sqrt{12 x^{5} y}$

Multiply.
67. $(\sqrt{5}-2)(\sqrt{5}+8)$
68. $(2 \sqrt{3}-7)(2 \sqrt{3}+7)$
69. $(\sqrt{x}+2)^{2}$
70. $(3-\sqrt{x})(3+\sqrt{x})$

Rationalize the denominator.
71. $\frac{\sqrt{7}}{\sqrt{7}-2}$
72. $\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$

## Getting Ready for the Next Section

73. Evaluate $y=3 x^{2}-6 x+1$ for $x=1$.
74. Evaluate $y=-2 x^{2}+6 x-5$ for $x=\frac{3}{2}$.
75. Let $P(x)=-0.1 x^{2}+27 x-500$ and find $P(135)$.
76. Let $P(x)=-0.1 x^{2}+12 x-400$ and find $P(600)$.

Solve.
77. $0=a(80)^{2}+70$
78. $0=a(80)^{2}+90$
79. $x^{2}-6 x+5=0$
80. $x^{2}-3 x-4=0$
81. $-x^{2}-2 x+3=0$
82. $-x^{2}+4 x+12=0$
83. $2 x^{2}-6 x+5=0$
84. $x^{2}-4 x+5=0$

Fill in the blanks to complete the square.
85. $x^{2}-6 x+\square=(x-\square)^{2}$
86. $x^{2}+10 x+\square=(x+\square)^{2}$
87. $y^{2}+2 y+\square=(y+\square)^{2}$
88. $y^{2}-12 y+\square=(y-\square)^{2}$

## Extending the Concepts

Find the $x$ - and $y$-intercepts.
89. $y=x^{3}-4 x$
90. $y=x^{4}-10 x^{2}+9$
91. $y=3 x^{3}+x^{2}-27 x-9$
92. $y=2 x^{3}+x^{2}-8 x-4$
93. The graph of $y=2 x^{3}-7 x^{2}-5 x+4$ crosses the $x$-axis at $x=4$. Where else does it cross the $x$-axis?
94. The graph of $y=6 x^{3}+x^{2}-12 x+5$ crosses the $x$-axis at $x=1$. Where else does it cross the $x$-axis?

## 10.5

## OBJECTIVES

A Graph a parabola.
B Solve application problems using information from a graph.

C Find an equation from its graph.

## Graphing Parabolas

The solution set to the equation

$$
y=x^{2}-3
$$

consists of ordered pairs. One method of graphing the solution set is to find a number of ordered pairs that satisfy the equation and to graph them. We can obtain some ordered pairs that are solutions to $y=x^{2}-3$ by use of a table as follows:

| $\boldsymbol{x}$ | $\boldsymbol{y}=\boldsymbol{x}^{\mathbf{2}}-\mathbf{3}$ | $\boldsymbol{y}$ | Solutions |
| ---: | :--- | ---: | :--- |
| -3 | $y=(-3)^{2}-3=9-3=6$ | 6 | $(-3,6)$ |
| -2 | $y=(-2)^{2}-3=4-3=1$ | 1 | $(-2,1)$ |
| -1 | $y=(-1)^{2}-3=1-3=-2$ | -2 | $(-1,-2)$ |
| 0 | $y=0^{2}$ | $-3=0-3=-3$ | -3 |
| 1 | $y=1^{2}$ | $-3=1-3=-2$ | -2 |
| 2 | $y=2^{2}$ | $-3=4-3=1$ | 1 |
| 3 | $y=3^{2}$ | $-3=9-3=6$ | 6 |$)(1,-2)$

Graphing these solutions and then connecting them with a smooth curve, we have the graph of $y=x^{2}-3$ (Figure 1).


FIGURE 1
This graph is an example of a parabola. All equations of the form $y=a x^{2}+b x+c, a \neq 0$ have parabolas for graphs.

## Note

What we are doing here is attempting to explain why the vertex of a parabola always has an $x$-coordinate of $\frac{-b}{2 a}$. But the explanation may not be easy to understand the first time you see it. It may be helpful to look over the examples in this section and the notes that accompany these examples and then come back and read over this discussion again.

Although it is always possible to graph parabolas by making a table of values of $x$ and $y$ that satisfy the equation, there are other methods that are faster and, in some cases, more accurate.
The important points associated with the graph of a parabola are the highest (or lowest) point on the graph and the $x$-intercepts. The $y$-intercepts can also be useful.

## Intercepts for Parabolas

The graph of the equation $y=a x^{2}+b x+c$ crosses the $y$-axis at $y=c$ since substituting $x=0$ into $y=a x^{2}+b x+c$ yields $y=c$.

Since the graph crosses the $x$-axis when $y=0$, the $x$-intercepts are those values of $x$ that are solutions to the quadratic equation $0=a x^{2}+b x+c$.

## The Vertex of a Parabola

The highest or lowest point on a parabola is called the vertex. The vertex for the graph of $y=a x^{2}+b x+c$ will occur when

$$
x=\frac{-b}{2 a}
$$

To see this, we must transform the right side of $y=a x^{2}+b x+c$ into an expression that contains $x$ in just one of its terms. This is accomplished by completing the square on the first two terms. Here is what it looks like:

$$
\begin{aligned}
& y=a x^{2}+b x+c \\
& y=a\left(x^{2}+\frac{b}{a} x\right)+c \\
& y=a\left[x^{2}+\frac{b}{a} x+\left(\frac{b}{2 a}\right)^{2}\right]+c-a\left(\frac{b}{2 a}\right)^{2} \\
& y=a\left(x+\frac{b}{2 a}\right)^{2}+\frac{4 a c-b^{2}}{4 a}
\end{aligned}
$$

It may not look like it, but this last line indicates that the vertex of the graph of $y=a x^{2}+b x+c$ has an $x$-coordinate of $\frac{-b}{2 a}$. Since $a, b$, and $c$ are constants, the only quantity that is varying in the last expression is the $x$ in $\left(x+\frac{b}{2 a}\right)^{2}$. Since the quantity $\left(x+\frac{b}{2 a}\right)^{2}$ is the square of $x+\frac{b}{2 a}$, the smallest it will ever be is 0 , and that will happen when $x=\frac{-b}{2 a}$.
We can use the vertex point along with the $x$ - and $y$-intercepts to sketch the graph of any equation of the form $y=a x^{2}+b x+c$. Here is a summary of the preceding information.

Graphing Parabolas I The graph of $y=a x^{2}+b x+c, a \neq 0$, will have

1. A $y$-intercept at $y=c$
2. $x$-intercepts (if they exist) at

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

3. A vertex when $x=\frac{-b}{2 a}$

## EXAMPLE 1 Sketch the graph of $y=x^{2}-6 x+5$.

SOLUTION To find the $x$-intercepts, we let $y=0$ and solve for $x$ :

$$
\begin{aligned}
& 0=x^{2}-6 x+5 \\
& 0=(x-5)(x-1) \\
& x=5 \quad \text { or } \quad x=1
\end{aligned}
$$

To find the coordinates of the vertex, we first find

$$
x=\frac{-b}{2 a}=\frac{-(-6)}{2(1)}=3
$$

The $x$-coordinate of the vertex is 3 . To find the $y$-coordinate, we substitute 3 for $x$ in our original equation:

$$
y=3^{2}-6(3)+5=9-18+5=-4
$$

The graph crosses the $x$-axis at 1 and 5 and has its vertex at ( $3,-4$ ). Plotting these points and connecting them with a smooth curve, we have the graph shown in Figure 2. The graph is a parabola that opens up, so we say the graph is concave up. The vertex is the lowest point on the graph. (Note that the graph crosses the $y$-axis at 5 , which is the value of $y$ we obtain when we let $x=0$.)


FIGURE 2

## Finding the Vertex by Completing the Square

Another way to locate the vertex of the parabola in Example 1 is by completing the square on the first two terms on the right side of the equation $y=x^{2}-6 x+5$. In this case, we would do so by adding 9 to and subtracting 9 from the right side of the equation. This amounts to adding 0 to the equation, so we know we haven't changed its solutions. This is what it looks like:

$$
\begin{aligned}
& y=\left(x^{2}-6 x \quad\right)+5 \\
& y=\left(x^{2}-6 x+9\right)+5-\mathbf{9} \\
& y=(x-3)^{2}-4
\end{aligned}
$$

You may have to look at this last equation a while to see this, but when $x=3$, then $y=(x-3)^{2}-4=0^{2}-4=-4$ is the smallest $y$ will ever be. And that is why
the vertex is at $(3,-4)$. As a matter of fact, this is the same kind of reasoning we used when we derived the formula $x=\frac{-b}{2 a}$ for the $x$-coordinate of the vertex.

## EXAMPLE $2 \quad$ Graph $y=-x^{2}-2 x+3$.

SOLUTION To find the $x$-intercepts, we let $y=0$ :

$$
\begin{aligned}
& 0=-x^{2}-2 x+3 \\
& 0=x^{2}+2 x-3 \quad \text { Multiply each side by } \mathbf{- 1} \\
& 0=(x+3)(x-1) \\
& x=-3 \quad \text { or } \quad x=1
\end{aligned}
$$

The $x$-coordinate of the vertex is given by

$$
x=\frac{-b}{2 a}=\frac{-(-2)}{2(-1)}=\frac{2}{-2}=-1
$$

To find the $y$-coordinate of the vertex, we substitute -1 for $x$ in our original equation to get

$$
y=-(-1)^{2}-2(-1)+3=-1+2+3=4
$$

Our parabola has $x$-intercepts at -3 and 1 and a vertex at $(-1,4)$. Figure 3 shows the graph. We say the graph is concave down since it opens downward.


FIGURE 3
Again, we could have obtained the coordinates of the vertex by completing the square on the first two terms on the right side of our equation. To do so, we must first factor -1 from the first two terms. (Remember, the leading coefficients must be 1 to complete the square.) When we complete the square, we add 1 inside the parentheses, which actually decreases the right side of the equation by -1 since everything in the parentheses is multiplied by -1 . To make up for it, we add 1 outside the parentheses.

$$
\begin{aligned}
& y=-1\left(x^{2}+2 x\right)+3 \\
& y=-1\left(x^{2}+2 x+\mathbf{1}\right)+3+\mathbf{1} \\
& y=-1(x+1)^{2}+4
\end{aligned}
$$

The last line tells us that the largest value of $y$ will be 4, and that will occur when $x=-1$.

## EXAMPLE $3 \quad$ Graph $y=3 x^{2}-6 x+1$.

SOLUTION To find the $x$-intercepts, we let $y=0$ and solve for $x$ :

$$
0=3 x^{2}-6 x+1
$$

Since the right side of this equation does not factor, we can look at the discriminant to see what kind of solutions are possible. The discriminant for this equation is

$$
b^{2}-4 a c=36-4(3)(1)=24
$$

Since the discriminant is a positive number but not a perfect square, the equation will have irrational solutions. This means that the $x$-intercepts are irrational numbers and will have to be approximated with decimals using the quadratic formula. Rather than use the quadratic formula, we will find some other points on the graph, but first let's find the vertex.
Here are both methods of finding the vertex:

Using the formula that gives us the $x$-coordinate of the vertex, we have:

$$
x=\frac{-b}{2 a}=\frac{-(-6)}{2(3)}=1
$$

Substituting 1 for $x$ in the equation gives us the $y$-coordinate of the vertex:

$$
y=3 \cdot 1^{2}-6 \cdot 1+1=-2
$$

To complete the square on the right side of the equation, we factor 3 from the first two terms, add 1 inside the parentheses, and add -3 outside the parentheses (this amounts to adding 0 to the right side):

$$
\begin{aligned}
& y=3\left(x^{2}-2 x \quad\right)+1 \\
& y=3\left(x^{2}-2 x+\mathbf{1}\right)+1-\mathbf{3} \\
& y=3(x-1)^{2}-2
\end{aligned}
$$

In either case, the vertex is $(1,-2)$.
If we can find two points, one on each side of the vertex, we can sketch the graph. Let's let $x=0$ and $x=2$ since each of these numbers is the same distance from $x=1$ and $x=0$ will give us the $y$-intercept.

$$
\text { When } \begin{aligned}
x & =0 \\
y & =3(0)^{2}-6(0)+1 \\
& =0-0+1 \\
& =1
\end{aligned}
$$

$$
\text { When } \begin{aligned}
x & =2 \\
y & =3(2)^{2}-6(2)+1 \\
& =12-12+1 \\
& =1
\end{aligned}
$$

The two points just found are $(0,1)$ and $(2,1)$. Plotting these two points along with the vertex $(1,-2)$, we have the graph shown in Figure 4.


FIGURE 4

EXAMPLE $4 \quad$ Graph $y=-2 x^{2}+6 x-5$.
SOLUTION Letting $y=0$, we have

$$
0=-2 x^{2}+6 x-5
$$

Again, the right side of this equation does not factor. The discriminant is $b^{2}-4 a c=36-4(-2)(-5)=-4$, which indicates that the solutions are complex numbers. This means that our original equation does not have $x$-intercepts. The graph does not cross the $x$-axis.
Let's find the vertex.

Using our formula for the $x$-coordinate of the vertex, we have

$$
x=\frac{-b}{2 a}=\frac{-6}{2(-2)}=\frac{6}{4}=\frac{3}{2}
$$

To find the $y$-coordinate, we let $x=\frac{3}{2}$ :

$$
\begin{aligned}
y & =-2\left(\frac{3}{2}\right)^{2}+6\left(\frac{3}{2}\right)-5 \\
& =\frac{-18}{4}+\frac{18}{2}-5 \\
& =\frac{-18+36-20}{4} \\
& =-\frac{1}{2}
\end{aligned}
$$

Finding the vertex by completing the square is a more complicated matter. To make the coefficient of $x^{2}$ a 1 , we must factor -2 from the first two terms. To complete the square inside the parentheses, we add $\frac{9}{4}$. Since each term inside the parentheses is multiplied by -2 , we add $\frac{9}{2}$ outside the parentheses so that the net result is the same as adding 0 to the right side:

$$
\begin{aligned}
& y=-2\left(x^{2}-3 x \quad\right)-5 \\
& y=-2\left(x^{2}-3 x+\frac{\mathbf{9}}{\mathbf{4}}\right)-5+\frac{\mathbf{9}}{\mathbf{2}} \\
& y=-2\left(x-\frac{3}{2}\right)^{2}-\frac{1}{2}
\end{aligned}
$$

The vertex is $\left(\frac{3}{2},-\frac{1}{2}\right)$. Since this is the only point we have so far, we must find two others, Let's let $x=3$ and $x=0$ since each point is the same distance from $x=\frac{3}{2}$ and on either side:

$$
\text { When } \begin{aligned}
x & =3 \\
y & =-2(3)^{2}+6(3)-5 \\
& =-18+18-5 \\
& =-5
\end{aligned}
$$

$$
\text { When } \begin{aligned}
x & =0 \\
y & =-2(0)^{2}+6(0)-5 \\
& =0+0-5 \\
& =-5
\end{aligned}
$$

The two additional points on the graph are $(3,-5)$ and $(0,-5)$. Figure 5 shows the graph.


FIGURE 5
The graph is concave down. The vertex is the highest point on the graph.

By looking at the equations and graphs in Examples 1 through 4, we can conclude that the graph of $y=a x^{2}+b x+c$ will be concave up when $a$ is positive and concave down when $a$ is negative. Taking this even further, if $a>0$, then the vertex is the lowest point on the graph, and if $a<0$, the vertex is the highest point on the graph. Finally, if we complete the square on $x$ in the equation $y=a x^{2}+b x+c, \quad a \neq 0$, we can rewrite the equation of our parabola as $y=a(x-h)^{2}+k$. When the equation is in this form, the vertex is at the point ( $h$, $k)$. Here is a summary:

## Graphing Parabolas II The graph of

$$
y=a(x-h)^{2}+k, a \neq 0
$$

will be a parabola with a vertex at $(h, k)$. The vertex will be the highest point on the graph when $a<0$, and it will be the lowest point on the graph when $a>0$.

EXAMPLE 5 A company selling copies of an accounting program for home computers finds that it will make a weekly profit of $P$ dollars from selling $x$ copies of the program, according to the equation

$$
P(x)=-0.1 x^{2}+27 x-500
$$

How many copies of the program should it sell to make the largest possible profit, and what is the largest possible profit?

SOLUTION Since the coefficient of $x^{2}$ is negative, we know the graph of this parabola will be concave down, meaning that the vertex is the highest point of the curve. We find the vertex by first finding its $x$-coordinate:

$$
x=\frac{-b}{2 a}=\frac{-27}{2(-0.1)}=\frac{27}{0.2}=135
$$

This represents the number of programs the company needs to sell each week to make a maximum profit. To find the maximum profit, we substitute 135 for $x$ in the original equation. (A calculator is helpful for these kinds of calculations.)

$$
\begin{aligned}
P(135) & =-0.1(135)^{2}+27(135)-500 \\
& =-0.1(18,225)+3,645-500 \\
& =-1,822.5+3,645-500 \\
& =1,322.5
\end{aligned}
$$

The maximum weekly profit is $\$ 1,322.50$ and is obtained by selling 135 programs a week.

EXAMPLE 6 An art supply store finds that they can sell $x$ sketch pads each week at $p$ dollars each, according to the equation $x=900-300 p$. Graph the revenue equation $R=x p$. Then use the graph to find the price $p$ that will bring in the maximum revenue. Finally, find the maximum revenue.

SOLUTION As it stands, the revenue equation contains three variables. Since we are asked to find the value of $p$ that gives us the maximum value of $R$, we rewrite the equation using just the variables $R$ and $p$. Since $x=900-300 p$, we have

$$
R=x p=(900-300 p) p
$$

The graph of this equation is shown in Figure 6. The graph appears in the first quadrant only since $R$ and $p$ are both positive quantities.


From the graph we see that the maximum value of $R$ occurs when $p=\$ 1.50$. We can calculate the maximum value of $R$ from the equation:

$$
\begin{array}{ll}
\text { When } & p
\end{array} \quad \begin{aligned}
& p .5 \\
& \text { The equation } \\
& \text { becomes }
\end{aligned} \quad \begin{aligned}
R & =(900-300 p) p \\
& =(900-300 \cdot 1.5) 1.5 \\
& =(900-450) 1.5 \\
& =450 \cdot 1.5 \\
& =675
\end{aligned}
$$

The maximum revenue is $\$ 675$. It is obtained by setting the price of each sketch pad at $p=\$ 1.50$.

## USING TECHNOLOGY

Graphing Calculators
If you have been using a graphing calculator for some of the material in this course, you are well aware that your calculator can draw all the graphs in this section very easily. It is important, however, that you be able to recognize and sketch the graph of any parabola by hand. It is a skill that all successful intermediate algebra students should possess, even if they are proficient in the use of a graphing calculator. My suggestion is that you work the problems in this section and problem set without your calculator. Then use your calculator to check your results.

## Finding the Equation from the Graph

EXAMPLE 7 At the 1997 Washington County Fair in Oregon, David Smith, Jr., The Bullet, was shot from a cannon. As a human cannonball, he reached a height of 70 feet before landing in a net 160 feet from the cannon. Sketch the graph of his path, and then find the equation of the graph.
SOLUTION We assume that the path taken by the human cannonball is a parabola. If the origin of the coordinate system is at the opening of the cannon, then the net that catches him will be at 160 on the $x$-axis. Figure 7 shows the graph:


FIGURE 7
Since the curve is a parabola, we know the equation will have the form

$$
y=a(x-h)^{2}+k
$$

Since the vertex of the parabola is at $(80,70)$, we can fill in two of the three constants in our equation, giving us

$$
y=a(x-80)^{2}+70
$$

To find $a$, we note that the landing point will be $(160,0)$. Substituting the coordinates of this point into the equation, we solve for $a$ :

$$
\begin{aligned}
& 0=a(160-80)^{2}+70 \\
& 0=a(80)^{2}+70 \\
& 0=6400 a+70 \\
& a=-\frac{70}{6400}=-\frac{7}{640}
\end{aligned}
$$

The equation that describes the path of the human cannonball is

$$
y=-\frac{7}{640}(x-80)^{2}+70 \text { for } 0 \leq x \leq 160
$$

## USING TECHNOLOGY

## Graphing Calculators

Graph the equation found in Example 7 by graphing it on a graphing calculator using the window shown below. (We will use this graph later in the book to find the angle between the cannon and the horizontal.)

Window: X from 0 to 180, increment 20
Y from 0 to 80, increment 10
On the TI-83, an increment of 20 for X means $\mathrm{Xscl}=20$.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-4
B 5
C 6

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a parabola?
2. What part of the equation of a parabola determines whether the graph is concave up or concave down?
3. Suppose $f(x)=a x^{2}+b x+c$ is the equation of a parabola. Explain how $f(4)=1$ relates to the graph of the parabola.
4. A line can be graphed with two points. How many points are necessary to get a reasonable sketch of a parabola? Explain.

## Problem Set 10.5

For each of the following equations, give the $x$-intercepts and the coordinates of the vertex, and sketch the graph.

1. $y=x^{2}+2 x-3$
2. $y=x^{2}-2 x-3$
3. $y=-x^{2}-4 x+5$
4. $y=x^{2}+4 x-5$
5. $y=x^{2}-1$
6. $y=x^{2}-4$
7. $y=-x^{2}+9$
8. $y=-x^{2}+1$
9. $y=2 x^{2}-4 x-6$
10. $y=2 x^{2}+4 x-6$
11. $y=x^{2}-2 x-4$
12. $y=x^{2}-2 x-2$

Find the vertex and any two convenient points to sketch the graphs of the following.
13. $y=x^{2}-4 x-4$
14. $y=x^{2}-2 x+3$
15. $y=-x^{2}+2 x-5$
16. $y=-x^{2}+4 x-2$
17. $y=x^{2}+1$
18. $y=x^{2}+4$
19. $y=-x^{2}-3$
20. $y=-x^{2}-2$

For each of the following equations, find the coordinates of the vertex, and indicate whether the vertex is the highest point on the graph or the lowest point on the graph. (Do not graph.)
21. $y=x^{2}-6 x+5$
22. $y=-x^{2}+6 x-5$
23. $y=-x^{2}+2 x+8$
24. $y=x^{2}-2 x-8$
25. $y=12+4 x-x^{2}$
26. $y=-12-4 x+x^{2}$
27. $y=-x^{2}-8 x$
28. $y=x^{2}+8 x$

## Applying the Concepts

29. Maximum Profit A company finds that it can make a profit of $P$ dollars each month by selling $x$ patterns, according to the formula $P(x)=-0.002 x^{2}+3.5 x-800$. How many patterns must it sell each month to have a maximum profit? What is the maximum profit?
$=$ Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
30. Maximum Profit A company selling picture frames finds that it can make a profit of $P$ dollars each month by selling $x$ frames, according to the formula $P(x)=-0.002 x^{2}+5.5 x-1,200$. How many frames must it sell each month to have a maximum profit? What is the maximum profit?

31. Maximum Height Chaudra is tossing a softball into the air with an underhand motion. The distance of the ball above her hand at any time is given by the function

$$
h(t)=32 t-16 t^{2} \quad \text { for } 0 \leq t \leq 2
$$

where $h(t)$ is the height of the ball (in feet) and $t$ is the time (in seconds). Find the times at which the ball is in her hand and the maximum height of the ball.

32. Maximum Area Justin wants to fence 3 sides of a rectangular exercise yard for his dog. The fourth side of the exercise yard will be a side of the house. He has 80 feet of fencing available. Find the dimensions of the exercise yard that will enclose the maximum area.

33. Maximum Revenue A company that manufactures typewriter ribbons knows that the number of ribbons $x$ it can sell each week is related to the price $p$ of each ribbon by the equation $x=1,200-100 p$. Graph the revenue equation $R=x p$. Then use the graph to find the price $p$ that will bring in the maximum revenue. Finally, find the maximum revenue.
34. Maximum Revenue A company that manufactures diskettes for home computers finds that it can sell $x$ diskettes each day at $p$ dollars per diskette, according to the equation $x=800-100 p$. Graph the revenue equation $R=x p$. Then use the graph to find the price $p$ that will bring in the maximum revenue. Finally, find the maximum revenue.
35. Maximum Revenue The relationship between the number of calculators $x$ a company sells each day and the price $p$ of each calculator is given by the equation $x=1,700-100 p$. Graph the revenue equation $R=x p$, and use the graph to find the price $p$ that will bring in the maximum revenue. Then find the maximum revenue.
36. Maximum Revenue The relationship between the number $x$ of pencil sharpeners a company sells each week and the price $p$ of each sharpener is given by the equation $x=1,800-100 p$. Graph the revenue equation $R=x p$, and use the graph to find the price $p$ that will bring in the maximum revenue. Then find the maximum revenue.
37. Human Cannonball A human cannonball is shot from a cannon at the county fair. He reaches a height of 60 feet before landing in a net 180 feet from the cannon. Sketch the graph of his path, and then find the equation of the graph.

38. Improving Your Quantitative Literacy The graph below shows the different paths taken by the human cannonball when his velocity out of the cannon is 50 miles/hour and his cannon is inclined at varying angles.
a. If his landing net is placed 108 feet from the cannon, at what angle should the cannon be inclined so that he lands in the net?
b. Approximately where do you think he would land if the cannon was inclined at $45^{\circ}$ ?
c. If the cannon was inclined at $45^{\circ}$, approximately what height do you think he would attain?
d. Do you think there is another angle for which he would travel the same distance he travels at $80^{\circ}$ ? Give an estimate of that angle.
e. The fact that every landing point can come from two different paths makes us think that the equations that give us the landing points must be what type of equations?


## Maintaining Your Skills

Perform the indicated operations.
39. $(3-5 i)-(2-4 i)$
40. $2 i(5-6 i)$
41. $(3+2 i)(7-3 i)$
42. $(4+5 i)^{2}$
43. $\frac{i}{3+i}$
44. $\frac{2+3 i}{2-3 i}$

## Getting Ready for the Next Section

Solve.
45. $x^{2}-2 x-8=0$
46. $x^{2}-x-12=0$
47. $6 x^{2}-x=2$
48. $3 x^{2}-5 x=2$
49. $x^{2}-6 x+9=0$
50. $x^{2}+8 x+16=0$

## Extending the Concepts

Finding the Equation from the Graph For each problem below, the graph is a parabola. In each case, find an equation in the form $y=a(x-h)^{2}+k$ that describes the graph.
51.

52.


## 10.6

## OBJECTIVES

A
Solve quadratic inequalities and graph the solution set.

## Quadratic Inequalities

Quadratic inequalities in one variable are inequalities of the form

$$
\begin{array}{ll}
a x^{2}+b x+c<0 & a x^{2}+b x+c>0 \\
a x^{2}+b x+c \leq 0 & a x^{2}+b x+c \geq 0
\end{array}
$$

where $a, b$, and $c$ are constants, with $a \neq 0$. The technique we will use to solve inequalities of this type involves graphing. Suppose, for example, we wish to find the solution set for the inequality $x^{2}-x-6>0$. We begin by factoring the left side to obtain

$$
(x-3)(x+2)>0
$$

We have two real numbers $x-3$ and $x+2$ whose product $(x-3)(x+2)$ is greater than zero; that is, their product is positive. The only way the product can be positive is either if both factors, $(x-3)$ and $(x+2)$, are positive or if they are both negative. To help visualize where $x-3$ is positive and where it is negative, we draw a real number line and label it accordingly:


Here is a similar diagram showing where the factor $x+2$ is positive and where it is negative:


Drawing the two number lines together and eliminating the unnecessary numbers, we have


We can see from the preceding diagram that the graph of the solution to $x^{2}-x-6>0$ is


## USING TECHNOLOGY

## Graphical Solutions to Quadratic Inequalities

We can solve the preceding problem by using a graphing calculator to visualize where the product $(x-3)(x+2)$ is positive. First, we graph the function $y=(x-3)(x+2)$ as shown in Figure 1


FIGURE 1
Next, we observe where the graph is above the $x$-axis. As you can see, the graph is above the $x$-axis to the right of 3 and to the left of -2 , as shown in Figure 2.


FIGURE 2
When the graph is above the $x$-axis, we have points whose $y$-coordinates are positive. Since these $y$-coordinates are the same as the expression $(x-3)(x+2)$, the values of $x$ for which the graph of $y=(x-3)(x+2)$ is above the $x$-axis are the values of $x$ for which the inequality $(x-3)(x+2)>0$ is true. Our solution set is therefore


## EXAMPLE 1 Solve for $x: x^{2}-2 x-8 \leq 0$.

ALGEBRAIC SOLUTION We begin by factoring:

$$
\begin{array}{r}
x^{2}-2 x-8 \leq 0 \\
(x-4)(x+2) \leq 0
\end{array}
$$

The product $(x-4)(x+2)$ is negative or zero. The factors must have opposite signs. We draw a diagram showing where each factor is positive and where each factor is negative:


From the diagram we have the graph of the solution set:


GRAPHIC SOLUTION To solve this inequality with a graphing calculator, we graph the function $y=(x-4)(x+2)$ and observe where the graph is below the $x$-axis. These points have negative $y$-coordinates, which means that the product $(x-4)(x+2)$ is negative for these points. Figure 3 shows the graph of $y=$ $(x-4)(x+2)$, along with the region on the $x$-axis where the graph contains points with negative $y$-coordinates.

As you can see, the graph is below the $x$-axis when $x$ is between -2 and 4 . Since our original inequality includes the possibility that $(x-4)(x+2)$ is 0 , we include the endpoints, -2 and 4 , with our solution set.


EXAMPLE 2 Solve for $x$ : $6 x^{2}-x \geq 2$.

## ALGEBRAIC SOLUTION

$$
\begin{aligned}
6 x^{2}-x & \geq 2 \\
6 x^{2}-x-2 & \geq 0 \leftarrow \text { Standard form } \\
(3 x-2)(2 x+1) & \geq 0
\end{aligned}
$$



FIGURE 4

The product is positive, so the factors must agree in sign. Here is the diagram showing where that occurs:


Since the factors agree in sign below $-\frac{1}{2}$ and above $\frac{2}{3}$, the graph of the solution set is


GRAPHICAL SOLUTION To solve this inequality with a graphing calculator, we graph the function $y=(3 x-2)(2 x+1)$ and observe where the graph is above the $x$-axis. These are the points that have positive $y$-coordinates, which means that the product $(3 x-2)(2 x+1)$ is positive for these points. Figure 4 shows the graph of $y=(3 x-2)(2 x+1)$, along with the regions on the $x$-axis where the graph is on or above the $x$-axis.

To find the points where the graph crosses the $x$-axis, we need to use either the Trace and Zoom features to zoom in on each point, or the calculator function that finds the intercepts automatically (on the TI-82/83 this is the root/zero function under the CALC key). Whichever method we use, we will obtain the following result:


111115

EXAMPLE 3 Solve $x^{2}-6 x+9 \geq 0$.

## ALGEBRAIC SOLUTION

$$
\begin{aligned}
& x^{2}-6 x+9 \geq 0 \\
& (x-3)^{2} \geq 0
\end{aligned}
$$

This is a special case in which both factors are the same. Since $(x-3)^{2}$ is always positive or zero, the solution set is all real numbers; that is, any real number that is used in place of $x$ in the original inequality will produce a true statement.
GRAPHICAL SOLUTION The graph of $y=(x-3)^{2}$ is shown in Figure 5. Notice that it touches the $x$-axis at 3 and is above the $x$-axis everywhere else. This means that every point on the graph has a $y$-coordinate greater than or equal to 0 , no matter what the value of $x$. The conclusion that we draw from the graph is that the inequality $(x-3)^{2} \geq 0$ is true for all values of $x$.

EXAMPLE 4 Solve: $\frac{x-4}{x+1} \leq 0$
SOLUTION The inequality indicates that the quotient of $(x-4)$ and $(x+1)$ is negative or 0 (less than or equal to 0 ). We can use the same reasoning we used
to solve the first three examples, because quotients are positive or negative under the same conditions that products are positive or negative. Here is the diagram that shows where each factor is positive and where each factor is negative:


Between -1 and 4 the factors have opposite signs, making the quotient negative. Thus, the region between -1 and 4 is where the solutions lie, because the original inequality indicates the quotient $\frac{x-4}{x+1}$ is negative. The solution set and its graph are shown here:


Notice that the left endpoint is open-that is, it is not included in the solution set-because $x=-1$ would make the denominator in the original inequality 0 . It is important to check all endpoints of solution sets to inequalities that involve rational expressions.

## EXAMPLE 5 Solve: $\frac{3}{x-2}-\frac{2}{x-3}>0$

SOLUTION We begin by adding the two rational expressions on the left side. The common denominator is $(x-2)(x-3)$ :

$$
\begin{aligned}
& \frac{3}{x-2} \cdot \frac{(\mathbf{x}-\mathbf{3})}{(\mathbf{x}-\mathbf{3})}-\frac{2}{x-3} \cdot \frac{(\boldsymbol{x}-\mathbf{2})}{(\boldsymbol{x}-\mathbf{2})}>0 \\
& \frac{3 x-9-2 x+4}{(x-2)(x-3)}>0 \\
& \frac{x-5}{(x-2)(x-3)}>0
\end{aligned}
$$

This time the quotient involves three factors. Here is the diagram that shows the signs of the three factors:


The original inequality indicates that the quotient is positive. For this to happen, either all three factors must be positive, or exactly two factors must be negative.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A $1-3$

Looking back at the diagram, we see the regions that satisfy these conditions are between 2 and 3 or above 5. Here is our solution set:


$$
2<x<3 \quad \text { or } \quad x>5
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the first step in solving a quadratic inequality?
2. How do you show that the endpoint of a line segment is not part of the graph of a quadratic inequality?
3. How would you use the graph of $y=a x^{2}+b x+c$ to help you find the graph of $a x^{2}+b x+c<0$ ?
4. Can a quadratic inequality have exactly one solution? Give an example.

## Problem Set 10.6

Solve each of the following inequalities and graph the solution set.

1. $x^{2}+x-6>0$
2. $x^{2}+x-6<0$
3. $x^{2}-x-12 \leq 0$
4. $x^{2}-x-12 \geq 0$
5. $x^{2}+5 x \geq-6$
6. $x^{2}-5 x>6$
7. $6 x^{2}<5 x-1$
8. $4 x^{2} \geq-5 x+6$
9. $x^{2}-9<0$
10. $x^{2}-16 \geq 0$
11. $4 x^{2}-9 \geq 0$
12. $9 x^{2}-4<0$
13. $2 x^{2}-x-3<0$
14. $3 x^{2}+x-10 \geq 0$
15. $x^{2}-4 x+4 \geq 0$
16. $x^{2}-4 x+4<0$
17. $x^{2}-10 x+25<0$
18. $x^{2}-10 x+25>0$
19. $\frac{x-2}{(x+3)(x-4)}<0$
20. $\frac{x-1}{(x+2)(x-5)}<0$
21. $(x-2)(x-3)(x-4)>0$
22. $(x-2)(x-3)(x-4)<0$
23. $(x+1)(x+2)(x+3) \leq 0$
24. $(x+1)(x+2)(x+3) \geq 0$
25. $\frac{x-1}{x+4} \leq 0$
26. $\frac{x+4}{x-1} \leq 0$
27. $\frac{3 x-8}{x+6}<0$
28. $\frac{5 x-3}{x+1}<0$
29. $\frac{x-2}{x-6}>0$
30. $\frac{x-1}{x-3} \geq 0$
31. $\frac{2}{x-4}-\frac{1}{x-3}>0$
32. $\frac{4}{x+3}-\frac{3}{x+2}>0$
33. Write each statement using inequality notation.
a. $x-1$ is always positive.
b. $x-1$ is never negative.
c. $x-1$ is greater than or equal to 0 .
34. Match each expression on the left with a phrase on the right
a. $(x-1)^{2} \geq 0$
i. Never true
b. $(x-1)^{2}<0$
ii. Sometimes true
c. $(x-1)^{2} \leq 0$
iii. Always true

Solve each inequality by inspection without showing any work.
35. $(x-1)^{2}<0$
36. $(x+2)^{2}<0$
37. $(x-1)^{2} \leq 0$
38. $(x+2)^{2} \leq 0$
39. $(x-1)^{2} \geq 0$
40. $(x+2)^{2} \geq 0$
41. $\frac{1}{(x-1)^{2}} \geq 0$
42. $\frac{1}{(x+2)^{2}}>0$
= Videos available by instructor request
> = Online student support materials available at www.thomsonedu.com/login
43. $x^{2}-6 x+9<0$
44. $x^{2}-6 x+9 \leq 0$
45. $x^{2}-6 x+9>0$
46. $\frac{1}{x^{2}-6 x+9}>0$
47. The graph of $y=x^{2}-4$ is shown in Figure 6. Use the graph to write the solution set for each of the following:
a. $X^{2}-4<0$
b. $X^{2}-4>0$
c. $X^{2}-4=0$


FIGURE 6
48. The graph of $y=4-x^{2}$ is shown in Figure 7. Use the graph to write the solution set for each of the following:
a. $4-x^{2}<0$
b. $4-x^{2}>0$
c. $4-x^{2}=0$


FIGURE 7
49. The graph of $y=x^{2}-3 x-10$ is shown in Figure 8 . Use the graph to write the solution set for each of the following:
a. $x^{2}-3 x-10<0$
b. $x^{2}-3 x-10>0$
c. $x^{2}-3 x-10=0$

50. The graph of $y=x^{2}+x-12$ is shown in Figure 9. Use the graph to write the solution set for each of the following:
a. $x^{2}+x-12<0$
b. $x^{2}-x-12>0$
c. $x^{2}+x-12=0$

51. The graph of $y=x^{3}-3 x^{2}-x+3$ is shown in Figure 10. Use the graph to write the solution set for each of the following:
a. $x^{3}-3 x^{2}-x+3<0$
b. $x^{3}-3 x^{2}-x+3>0$
c. $x^{3}-3 x^{2}-x+3=0$


FIGURE 10
52. The graph of $y=x^{3}+4 x^{2}-4 x-16$ is shown in Figure 11. Use the graph to write the solution set for each of the following:
a. $x^{3}+4 x^{2}-4 x-16<0$
b. $x^{3}+4 x^{2}-4 x-16>0$
c. $x^{3}+4 x^{2}-4 x-16=0$


## Applying the Concepts

53. Dimensions of a Rectangle The length of a rectangle is 3 inches more than twice the width. If the area is to be at least 44 square inches, what are the possibilities for the width?
54. Dimensions of a Rectangle The length of a rectangle is 5 inches less than 3 times the width. If the area is to be less than 12 square inches, what are the possibilities for the width?
55. Revenue A manufacturer of portable radios knows that the weekly revenue produced by selling $x$ radios is given by the equation $R=1,300 p-100 p^{2}$, where $p$ is the price of each radio. What price
should she charge for each radio if she wants her weekly revenue to be at least $\$ 4,000$ ?
56. Revenue A manufacturer of small calculators knows that the weekly revenue produced by selling $x$ calculators is given by the equation $R=1,700 p-100 p^{2}$, where $p$ is the price of each calculator. What price should be charged for each calculator if the revenue is to be at least $\$ 7,000$ each week?

## Maintaining Your Skills

Use a calculator to evaluate. Give answers to 4 decimal places.
57. $\frac{50,000}{32,000}$
58. $\frac{2.4362}{1.9758}-1$
59. $\frac{1}{2}\left(\frac{4.5926}{1.3876}-2\right)$
60. $1+\frac{0.06}{12}$

Solve each equation.
61. $\sqrt{3 t-1}=2$
62. $\sqrt{4 t+5}+7=3$
63. $\sqrt{x+3}=x-3$
64. $\sqrt{x+3}=\sqrt{x}-3$

Graph each equation.
65. $y=\sqrt[3]{x-1}$
66. $y=\sqrt[3]{x}-1$

## Extending the Concepts

Graph the solution set for each inequality.
67. $x^{2}-2 x-1<0$
68. $x^{2}-6 x+7<0$
69. $x^{2}-8 x+13>0$
70. $x^{2}-10 x+18>0$

## Chapter 10 SUMMARY

## Theorem 1 [10.1]

## EXAMPLES

1. If

$$
\begin{aligned}
(x-3)^{2} & =25 \\
x-3 & = \pm 5 \\
x & =3 \pm 5 \\
x=8 \text { or } x & =-2
\end{aligned}
$$

then
2. Solve $x^{2}-6 x-6=0$

$$
x^{2}-6 x=6
$$

$$
x^{2}-6 x+9=6+\mathbf{9}
$$

$$
(x-3)^{2}=15
$$

$$
x-3= \pm \sqrt{15}
$$

$$
x=3 \pm \sqrt{15}
$$

If $a^{2}=b$, where $b$ is a real number, then

$$
a=\sqrt{b} \quad \text { or } \quad a=-\sqrt{b}
$$

which can be written as $a= \pm \sqrt{b}$.

## To Solve a Quadratic Equation by Completing the Square [10.1]

Step 1: Write the equation in the form $a x^{2}+b x=c$.
Step 2: If $a \neq 1$, divide through by the constant $a$ so the coefficient of $x^{2}$ is 1 .
Step 3: Complete the square on the left side by adding the square of $\frac{1}{2}$ the coefficient of $x$ to both sides.

Step 4: Write the left side of the equation as the square of a binomial. Simplify the right side if possible.

Step 5: Apply Theorem 1, and solve as usual.

## The Quadratic Theorem [10.2]

3. If $2 x^{2}+3 x-4=0$, then

$$
\begin{aligned}
x & =\frac{-3 \pm \sqrt{9-4(2)(-4)}}{2(2)} \\
& =\frac{-3 \pm \sqrt{41}}{4}
\end{aligned}
$$

For any quadratic equation in the form $a x^{2}+b x+c=0, a \neq 0$, the two solutions are

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

This last expression is known as the quadratic formula.

## The Discriminant [10.3]

4. The discriminant for

$$
x^{2}+6 x+9=0
$$

is $D=36-4(1)(9)=0$, which means the equation has one rational solution.

The expression $b^{2}-4 a c$ that appears under the radical sign in the quadratic formula is known as the discriminant.
We can classify the solutions to $a x^{2}+b x+c=0$ :

| The Solutions Are | When the Discriminant Is |
| :--- | :--- |
| Two complex numbers containing $i$ | Negative |
| One rational number | Zero |
| Two rational numbers | A positive perfect square |
| Two irrational numbers | A positive number, but not a perfect square |

5. The equation $x^{4}-x^{2}-12=0$ is quadratic in $x^{2}$. Letting $y=x^{2}$ we have

$$
\begin{aligned}
y^{2}-y-12 & =0 \\
(y-4)(y+3) & =0 \\
y=4 \quad \text { or } \quad y & =-3
\end{aligned}
$$

Resubstituting $x^{2}$ for $y$, we have

$$
\begin{aligned}
x^{2} & =4 & \text { or } & & x^{2} & =-3 \\
x & = \pm 2 & & \text { or } & & x= \pm i \sqrt{3}
\end{aligned}
$$

## Equations Quadratic in Form [10.4]

There are a variety of equations whose form is quadratic. We solve most of them by making a substitution so the equation becomes quadratic, and then solving the equation by factoring or the quadratic formula. For example,

$$
\begin{array}{lc}
\text { The equation } & \text { is quadratic } \\
(2 x-3)^{2}+5(2 x-3)-6=0 & 2 x-3 \\
4 x^{4}-7 x^{2}-2=0 & x^{2} \\
2 x-7 \sqrt{x}+3=0 & \sqrt{x}
\end{array}
$$

## Graphing Parabolas [10.5]

The graph of any equation of the form

$$
y=a x^{2}+b x+c \quad a \neq 0
$$

is a parabola. The graph is concave up if $a>0$ and concave down if $a<0$. The highest or lowest point on the graph is called the vertex and always has an $x$-coordinate of $x=\frac{-b}{2 a}$.

## Quadratic Inequalities [10.6]

We solve quadratic inequalities by manipulating the inequality to get 0 on the right side and then factoring the left side. We then make a diagram that indicates where the factors are positive and where they are negative. From this sign diagram and the original inequality we graph the appropriate solution set.

## Chapter 10 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

Solve each equation. [10.1]

1. $(2 t-5)^{2}=25$
2. $(3 t-2)^{2}=4$
3. $(3 y-4)^{2}=-49$
4. $(2 x+6)^{2}=12$

Solve by completing the square. [10.1]
5. $2 x^{2}+6 x-20=0$
6. $3 x^{2}+15 x=-18$
7. $a^{2}+9=6 a$
8. $a^{2}+4=4 a$
9. $2 y^{2}+6 y=-3$
10. $3 y^{2}+3=9 y$

Solve each equation. [10.2]
11. $\frac{1}{6} x^{2}+\frac{1}{2} x-\frac{5}{3}=0$
12. $8 x^{2}-18 x=0$
13. $4 t^{2}-8 t+19=0$
14. $100 x^{2}-200 x=100$
15. $0.06 a^{2}+0.05 a=0.04$
16. $9-6 x=-x^{2}$
17. $(2 x+1)(x-5)-(x+3)(x-2)=-17$
18. $2 y^{3}+2 y=10 y^{2}$
19. $5 x^{2}=-2 x+3$
20. $x^{3}-27=0$
21. $3-\frac{2}{x}+\frac{1}{x^{2}}=0$
22. $\frac{1}{x-3}+\frac{1}{x+2}=1$
23. Profit The total cost (in dollars) for a company to produce $x$ items per week is $C=7 x+400$. The revenue for selling all $x$ items is $R=34 x-0.1 x^{2}$. How many items must it produce and sell each week for its weekly profit to be $\$ 1,300$ ? [10.2]
24. Profit The total cost (in dollars) for a company to produce $x$ items per week is $C=70 x+300$. The revenue for selling all $x$ items is $R=110 x-0.5 x^{2}$. How many items must it produce and sell each week for its weekly profit to be $\$ 300$ ? [10.2]

Use the discriminant to find the number and kind of solutions for each equation. [10.3]
25. $2 x^{2}-8 x=-8$
26. $4 x^{2}-8 x=-4$
27. $2 x^{2}+x-3=0$
28. $5 x^{2}+11 x=12$
29. $x^{2}-x=1$
30. $x^{2}-5 x=-5$
31. $3 x^{2}+5 x=-4$
32. $4 x^{2}-3 x=-6$

Determine $k$ so that each equation has exactly one real solution. [10.3]
33. $25 x^{2}-k x+4=0$
34. $4 x^{2}+k x+25=0$
35. $k x^{2}+12 x+9=0$
36. $k x^{2}-16 x+16=0$
37. $9 x^{2}+30 x+k=0$
38. $4 x^{2}+28 x+k=0$

For each of the following problems, find an equation that has the given solutions. [10.3]
39. $x=3, x=5$
40. $x=-2, x=4$
41. $y=\frac{1}{2}, y=-4$
42. $t=3, t=-3, t=5$

Find all solutions. [10.4]
43. $(x-2)^{2}-4(x-2)-60=0$
44. $6(2 y+1)^{2}-(2 y+1)-2=0$
45. $x^{4}-x^{2}=12$
46. $x-\sqrt{x}-2=0$
47. $2 x-11 \sqrt{x}=-12$
48. $\sqrt{x+5}=\sqrt{x}+1$
49. $\sqrt{y+21}+\sqrt{y}=7$
50. $\sqrt{y+9}-\sqrt{y-6}=3$
51. Projectile Motion An object is tossed into the air with an upward velocity of 10 feet per second from the top of a building $h$ feet high. The time it takes for the object to hit the ground below is given by the formula $16 t^{2}-10 t-h=0$. Solve this formula for $t$. [10.4]
52. Projectile Motion An object is tossed into the air with an upward velocity of $v$ feet per second from the top of a 10 -foot wall. The time it takes for the object to hit the ground below is given by the formula $16 t^{2}-v t-10=0$. Solve this formula for $t$. [10.4]

Solve each inequality and graph the solution set. [10.6]
53. $x^{2}-x-2<0$
54. $3 x^{2}-14 x+8 \leq 0$
55. $2 x^{2}+5 x-12 \geq 0$
56. $(x+2)(x-3)(x+4)>0$

Find the $x$-intercepts, if they exist, and the vertex for each parabola. Then use them to sketch the graph. [10.5]
57. $y=x^{2}-6 x+8$
58. $y=x^{2}-4$

## Chapter 10 Projects

## Quadratic Functions

## GROUP PROJECT Maximum Volume of a Box

## Number of People 5

Time Needed 30 minutes
Equipment Graphing calculator and five pieces of graph paper

Background For many people, having a concrete model to work with allows them to visualize situations that they would have difficulty with if they had only a written description to work with. The purpose of this project is to rework a problem we have worked previously but this time with a concrete model.

Procedure You are going to make boxes of varying dimensions from rectangles that are 11 centimeters wide and 17 centimeters long.

1. Cut a rectangle from your graph paper that is 11 squares for the width and 17 squares for the length. Pretend that each small square is 1 centimeter by 1 centimeter. Do this with five pieces of paper.
2. One person tears off one square from each corner of their paper, then folds up the sides to form a box. Write down the length, width, and height of this box. Then calculate its volume.
3. The next person tears a square that is two units on a side from each corner of their paper, then folds up the sides to form a box. Write down the length, width, and height of this box. Then calculate its volume.

4. The next person follows the same procedure, tearing a still larger square from each corner of their paper. This continues until the squares that are to be torn off are larger than the original piece of paper.
5. Enter the data from each box you have created into the table. Then graph all the points $(x, V)$ from the table.
6. Using what you have learned from filling in the table, write a formula for the volume of the box that is created when a square of side $x$ is cut from each corner of the original piece of paper.
7. Graph the equation you found in step 6 on a graphing calculator. Use the result to connect the points you plotted in the graph.
8. Use the graphing calculator to find the value of $x$ that will give the maximum volume of the box. Your answer should be accurate to the nearest hundredth.

## RESEARCH PROJECT

## A Continued Fraction and the Golden Ratio

You may recall that the continued fraction

$$
1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}}
$$

is equal to the golden ratio.
The same conclusion can be found with the quadratic formula by using the following conditional statement

$$
\text { If } x=1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ddots}}} \text {, then } x=1+\frac{1}{x}
$$

Work with this conditional statement until you see that it is true. Then solve the equation $x=1+\frac{1}{x}$. Write an essay in which you explain in your own words why the conditional statement is true. Then show the details of the solution to the equation $x=1+\frac{1}{x}$. The message that you want to get across in your essay is that the continued fraction shown here is actually the same as the golden ratio.

## Exponential and Logarithmic Functions

## 11


f you have had any problems with or had testing done on your thyroid gland, then you may have come in contact with radioactive iodine-131. Like all radioactive elements, iodine-131 decays naturally. The half-life of iodine-131 is 8 days, which means that every 8 days a sample of iodine- 131 will decrease to half of its original amount. The following table and graph show what happens to a 1,600microgram sample of iodine-131 over time.


The function represented by the information in the table and Figure 1 is

$$
A(t)=1,600 \cdot 2^{-t / 8}
$$

It is one of the types of functions we will study in this chapter.

## CHAPTER OUTLINE

11.1 Exponential Functions
11.2 The Inverse of a Function
11.3 Logarithms Are Exponents
11.4 Properties of Logarithms
11.5 Common Logarithms and Natural Logarithms
11.6 Exponential Equations and Change of Base

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## Exponential Functions

## OBJECTIVES

A
Find function values for exponential functions.

B Graph exponential functions.

Work problems involving exponential growth and decay.

To obtain an intuitive idea of how exponential functions behave, we can consider the heights attained by a bouncing ball. When a ball used in the game of racquetball is dropped from any height, the first bounce will reach a height that is $\frac{2}{3}$ of the original height. The second bounce will reach $\frac{2}{3}$ of the height of the first bounce, and so on, as shown in Figure 1.



If the ball is dropped initially from a height of 1 meter, then during the first bounce it will reach a height of $\frac{2}{3}$ meter. The height of the second bounce will reach $\frac{2}{3}$ of the height reached on the first bounce. The maximum height of any bounce is $\frac{2}{3}$ of the height of the previous bounce.

Initial height: $\quad h=1$
Bounce 1: $\quad h=\frac{2}{3}(1)=\frac{2}{3}$
Bounce 2: $\quad h=\frac{2}{3}\left(\frac{2}{3}\right)=\left(\frac{2}{3}\right)^{2}$
Bounce 3: $\quad h=\frac{2}{3}\left(\frac{2}{3}\right)^{2}=\left(\frac{2}{3}\right)^{3}$
Bounce 4:

$$
h=\frac{2}{3}\left(\frac{2}{3}\right)^{3}=\left(\frac{2}{3}\right)^{4}
$$

Bounce $n$ :

$$
h=\frac{2}{3}\left(\frac{2}{3}\right)^{n-1}=\left(\frac{2}{3}\right)^{n}
$$

This last equation is exponential in form. We classify all exponential functions together with the following definition.

DEFINITION An exponential function is any function that can be written in the form

$$
f(x)=b^{x}
$$

where $b$ is a positive real number other than 1.

Each of the following is an exponential function:

$$
f(x)=2^{x} \quad y=3^{x} \quad f(x)=\left(\frac{1}{4}\right)^{x}
$$

The first step in becoming familiar with exponential functions is to find some values for specific exponential functions.

EXAMPLE 1 If the exponential functions $f$ and $g$ are defined by

$$
f(x)=2^{x} \quad \text { and } \quad g(x)=3^{x}
$$

then

$$
\begin{array}{rlrl}
f(0) & =2^{0}=1 & g(0) & =3^{0}=1 \\
f(1) & =2^{1}=2 & g(1) & =3^{1}=3 \\
f(2) & =2^{2}=4 & g(2) & =3^{2}=9 \\
f(3) & =2^{3}=8 & g(3) & =3^{3}=27 \\
f(-2) & =2^{-2}=\frac{1}{2^{2}}=\frac{1}{4} & g(-2) & =3^{-2}=\frac{1}{3^{2}}=\frac{1}{9} \\
f(-3) & =2^{-3}=\frac{1}{2^{3}}=\frac{1}{8} & g(-3) & =3^{-3}=\frac{1}{3^{3}}=\frac{1}{27}
\end{array}
$$

In the introduction to this chapter we indicated that the half-life of iodine-131 is 8 days, which means that every 8 days a sample of iodine- 131 will decrease to half of its original amount. If we start with $A_{0}$ micrograms of iodine-131, then after $t$ days the sample will contain

$$
A(t)=A_{0} \cdot 2^{-t / 8}
$$

micrograms of iodine-131.

EXAMPLE 2 A patient is administered a 1,200-microgram dose of iodine-131. How much iodine-131 will be in the patient's system after 10 days and after 16 days?
SOLUTION The initial amount of iodine-131 is $A_{0}=1,200$, so the function that gives the amount left in the patient's system after $t$ days is

$$
A(t)=1,200 \cdot 2^{-t / 8}
$$

After 10 days, the amount left in the patient's system is

$$
\begin{aligned}
A(10) & =1,200 \cdot 2^{-10 / 8} \\
& =1,200 \cdot 2^{-1.25} \\
& \approx 504.5 \text { micrograms }
\end{aligned}
$$

After 16 days, the amount left in the patient's system is

$$
\begin{aligned}
A(16) & =1,200 \cdot 2^{-16 / 8} \\
& =1,200 \cdot 2^{-2} \\
& =300 \text { micrograms }
\end{aligned}
$$

We will now turn our attention to the graphs of exponential functions. Since the notation $y$ is easier to use when graphing, and $y=f(x)$, for convenience we will write the exponential functions as

$$
y=b^{x}
$$

## EXAMPLE 3 Sketch the graph of the exponential function $y=2^{x}$.

SOLUTION Using the results of Example 1, we have the following table. Graphing the ordered pairs given in the table and connecting them with a smooth curve, we have the graph of $y=2^{x}$ shown in Figure 2 .

| $x$ | $y$ |
| :---: | :---: |
| -3 | $\frac{1}{8}$ |
| -2 | $\frac{1}{4}$ |
| -1 | $\frac{1}{2}$ |
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |



Notice that the graph does not cross the $x$-axis. It approaches the $x$-axis-in fact, we can get it as close to the $x$-axis as we want without it actually intersecting the $x$-axis. For the graph of $y=2^{x}$ to intersect the $x$-axis, we would have to find a value of $x$ that would make $2^{x}=0$. Because no such value of $x$ exists, the graph of $y=2^{x}$ cannot intersect the $x$-axis.

## EXAMPLE 4 sketch the graph of $y=\left(\frac{1}{3}\right)^{x}$.

SOLUTION The table shown here gives some ordered pairs that satisfy the equation. Using the ordered pairs from the table, we have the graph shown in Figure 3.

| $x$ | $y$ |
| :---: | :---: |
| -3 | 27 |
| -2 | 9 |
| -1 | 3 |
| 0 | 1 |
| 1 | $\frac{1}{3}$ |
| 2 | $\frac{1}{9}$ |
| 3 | $\frac{1}{27}$ |



FIGURE 3

The graphs of all exponential functions have two things in common: (1) each crosses the $y$-axis at $(0,1)$ since $b^{0}=1$; and (2) none can cross the $x$-axis since $b^{x}=0$ is impossible because of the restrictions on $b$.
Figures 4 and 5 show some families of exponential curves to help you become more familiar with them on an intuitive level.


FIGURE 4


FIGURE 5

Among the many applications of exponential functions are the applications having to do with interest-bearing accounts. Here are the details.

Compound Interest If $P$ dollars are deposited in an account with annual interest rate $r$, compounded $n$ times per year, then the amount of money in the account after $t$ years is given by the formula

$$
A(t)=P\left(1+\frac{r}{n}\right)^{n t}
$$

## EXAMPLE 5 Suppose you deposit \$500 in an account with an annual

 interest rate of $8 \%$ compounded quarterly. Find an equation that gives the amount of money in the account after $t$ years. Then finda. The amount of money in the account after 5 years.
b. The number of years it will take for the account to contain $\$ 1,000$.

SOLUTION First we note that $P=500$ and $r=0.08$. Interest that is compounded quarterly is compounded 4 times a year, giving us $n=4$. Substituting these numbers into the preceding formula, we have our function

$$
A(t)=500\left(1+\frac{0.08}{4}\right)^{4 t}=500(1.02)^{4 t}
$$

a. To find the amount after 5 years, we let $t=5$ :

$$
\begin{aligned}
A(5) & =500(1.02)^{4.5} \\
& =500(1.02)^{20} \\
& \approx \$ 742.97
\end{aligned}
$$

Our answer is found on a calculator and then rounded to the nearest cent.
b. To see how long it will take for this account to total $\$ 1,000$, we graph the equation $Y_{1}=500(1.02)^{4 \mathrm{X}}$ on a graphing calculator and then look to see where it intersects the line $Y_{2}=1,000$. The two graphs are shown in Figure 6.


Using Zoom and Trace, or the Intersect function on the graphing calculator, we find that the two curves intersect at $X \approx 8.75$ and $Y=1,000$. This means that our account will contain $\$ 1,000$ after the money has been on deposit for 8.75 years.

## The Natural Exponential Function

A very commonly occurring exponential function is based on a special number we denote with the letter $e$. The number $e$ is a number like $\pi$. It is irrational and occurs in many formulas that describe the world around us. Like $\pi$, it can be approximated with a decimal number. Whereas $\pi$ is approximately $3.1416, e$ is approximately 2.7183 . (If you have a calculator with a key labeled $e^{x}$, you can use it to find $e^{1}$ to find a more accurate approximation to $e$.) We cannot give a more precise definition of the number $e$ without using some of the topics taught in calculus. For the work we are going to do with the number $e$, we only need to know that it is an irrational number that is approximately 2.7183 .

Here are a table and graph for the natural exponential function.

$$
y=f(x)=e^{x}
$$

| $\boldsymbol{x}$ | $f(x)=\boldsymbol{e}^{\boldsymbol{x}}$ |
| ---: | :--- |
| -2 | $f(-2)=e^{-2}=\frac{1}{e^{2}}=0.135$ |
| -1 | $f(-1)=e^{-1}=\frac{1}{e} \approx 0.368$ |
| 0 | $f(0)=e^{0}=1$ |
| 1 | $f(1)=e^{1}=e \approx 2.72$ |
| 2 | $f(2)$ |
| 3 | $f(3)=e^{2} \approx 7.39$ |
| 3 |  |



One common application of natural exponential functions is with interestbearing accounts. In Example 5 we worked with the formula

$$
A=P\left(1+\frac{r}{n}\right)^{n t}
$$

which gives the amount of money in an account if $P$ dollars are deposited for $t$ years at annual interest rate $r$, compounded $n$ times per year. In Example 5 the number of compounding periods was 4 . What would happen if we let the number of compounding periods become larger and larger, so that we compounded the interest every day, then every hour, then every second, and so on? If we take this as far as it can go, we end up compounding the interest every moment. When this happens, we have an account with interest that is compounded continuously, and the amount of money in such an account depends on the number $e$.

Here are the details.

Continuously Compounded Interest If $P$ dollars are deposited in an account with annual interest rate $r$, compounded continuously, then the amount of money in the account after $t$ years is given by the formula

$$
A(t)=P e^{r t}
$$

## EXAMPLE 6 Suppose you deposit \$500 in an account with an annual

 interest rate of $8 \%$ compounded continously. Find an equation that gives the amount of money in the account after $t$ years. Then find the amount of money in the account after 5 years.SOLUTION Since the interest is compounded continuously, we use the formula $A(t)=P e^{r t}$. Substituting $P=500$ and $r=0.08$ into this formula we have

$$
A(t)=500 e^{0.08 t}
$$

After 5 years, this account will contain

$$
\begin{aligned}
A(5) & =500 e^{0.08 \cdot 5} \\
& =500 e^{0.4} \\
& \approx \$ 745.91
\end{aligned}
$$

to the nearest cent. Compare this result with the answer to Example 5a.

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,2
B 3-5
C 5,6

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is an exponential function?
2. In an exponential function, explain why the base b cannot equal 1. (What kind of function would you get if the base was equal to 1?)
3. Explain continuously compounded interest.
4. What characteristics do the graphs of $y=2^{x}$ and $y=\left(\frac{1}{2}\right)^{x}$ have in common?

Let $f(x)=3^{x}$ and $g(x)=\left(\frac{1}{2}\right)^{x}$, and evaluate each of the following.

1. $g(0)$
2. $f(0)$
3. $g(-1)$
4. $g(-4)$
5. $f(-3)$
6. $f(-1)$
7. $f(2)+g(-2)$
8. $f(2)-g(-2)$

Graph each of the following functions.
9. $y=4^{x}$
10. $y=2^{-x}$
11. $y=3^{-x}$
12. $y=\left(\frac{1}{3}\right)^{-x}$
13. $y=2^{x+1}$
14. $y=2^{x-3}$
15. $y=e^{x}$
16. $y=e^{-x}$

Graph each of the following functions on the same coordinate system for positive values of $x$ only.
17. $y=2 x, y=x^{2}, y=2^{x}$
18. $y=3 x, y=x^{3}, y=3^{x}$
19. On a graphing calculator, graph the family of curves $y=b^{x}, b=2,4,6,8$.
20. On a graphing calculator, graph the family of curves $y=b^{x}, b=\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$.

## Applying the Concepts

21. Bouncing Ball Suppose the ball mentioned in the introduction to this section is dropped from a height of 6 feet above the ground. Find an exponential equation that gives the height $h$ the ball will attain during the $n$th bounce. How high will it bounce on the fifth bounce?

22. Bouncing Ball A golf ball is manufactured so that if it is dropped from $A$ feet above the ground onto a hard surface, the maximum height of each bounce will be $\frac{1}{2}$ of the height of the previous bounce. Find an exponential equation that gives the height $h$ the ball will attain during the $n$th bounce. If the ball is dropped from 10 feet above the ground onto a hard
surface, how high will it bounce on the 8th bounce?
23. Exponential Decay The half-life of iodine-131 is 8 days. If a patient is administered a 1,400-microgram dose of iodine-131, how much iodine-131 will be in the patient's system after 8 days and after 11 days? (See Example 2.)
24. Exponential Growth Automobiles built before 1993 use Freon in their air conditioners. The federal government now prohibits the manufacture of Freon. Because the supply of Freon is decreasing, the price per pound is increasing exponentially. Current estimates put the formula for the price per pound of Freon at $p(t)=1.89(1.25)^{t}$, where $t$ is the number of years since 1990. Find the price of Freon in 1995 and 1990. How much will Freon cost in the year 2005?
25. Compound Interest Suppose you deposit $\$ 1,200$ in an account with an annual interest rate of $6 \%$ compounded quarterly. (See Example 5.)
a. Find an equation that gives the amount of money in the account after $t$ years.
b. Find the amount of money in the account after 8 years.
c. How many years will it take for the account to contain $\$ 2,400$ ?
d. If the interest were compounded continuously, how much money would the account contain after 8 years?
26. Compound Interest Suppose you deposit $\$ 500$ in an account with an annual interest rate of $8 \%$ compounded monthly.
a. Find an equation that gives the amount of money in the account after $t$ years.
b. Find the amount of money in the account after 5 years.
c. How many years will it take for the account to contain $\$ 1,000$ ?
d. If the interest were compounded continuously, how much money would the account contain after 5 years?
27. Health Care In 1990, $\$ 699$ billion were spent on health care expenditures. The amount of money, $E$, in billions spent on health care expenditures can be esti-
mated using the function $E(t)=78.16(1.11)^{t}$, where $t$ is time in years since 1970 (U.S. Census Bureau).
a. How close was the estimate determined by the function in estimating the actual amount of money spent on health care expenditures in 1990?
b. What are the expected health care expenditures in 2008, 2009, and 2010? Round to the nearest billion.
28. Exponential Growth The cost of a can of Coca-Cola on January 1, 1960, was 10 cents. The function below gives the cost of a can of Coca-Cola $t$ years after that.

$$
C(t)=0.10 e^{0.0576 t}
$$

a. Use the function to fill in the table below. (Round to the nearest cent.)

b. Use the table to find the cost of a can of CocaCola at the beginning of the year 2000 .
c. In what year will a can of Coca-Cola cost \$17.84?
29. Value of a Painting A painting is purchased as an investment for $\$ 150$. If the painting's value doubles every 3 years, then its value is given by the function

$$
V(t)=150 \cdot 2^{t / 3} \quad \text { for } t \geq 0
$$

where $t$ is the number of years since it was purchased, and $V(t)$ is its value (in dollars) at that time. Graph this function.
30. Value of a Painting A painting is purchased as an investment for $\$ 125$. If the painting's value doubles every 5 years, then its value is given by the function

$$
V(t)=125 \cdot 2^{t / 5} \text { for } t \geq 0
$$

where $t$ is the number of years since it was purchased, and $V(t)$ is its value (in dollars) at that time. Graph this function.

31. Value of a Painting When will the painting mentioned in Problem 29 be worth $\$ 600$ ?
32. Value of a Painting When will the painting mentioned in Problem 30 be worth $\$ 250$ ?
33. Value of a Crane The function

$$
V(t)=450,000(1-0.30)^{t}
$$

where $V$ is value and $t$ is time in years, can be used to find the value of a crane for the first 6 years of use.

a. What is the value of the crane after 3 years and 6 months?
b. State the domain of this function.
c. Sketch the graph of this function.
d. State the range of this function.
e. After how many years will the crane be worth only $\$ 85,000$ ?
34. Value of a Printing Press The function $V(t)=375,000(1-0.25)^{t}$, where $V$ is value and $t$ is time in years, can be used to find the value of a printing press during the first 7 years of use.

a. What is the value of the printing press after 4 years and 9 months?
b. State the domain of this function.
c. Sketch the graph of this function.
d. State the range of this function.
e. After how many years will the printing press be worth only $\$ 65,000$ ?

## Maintaining Your Skills

For each of the following relations, specify the domain and range, then indicate which are also functions.
35. $\{(1,2),(3,4),(4,1)\}$
36. $\{(-2,6),(-2,8),(2,3)\}$

State the domain for each of the following functions.
37. $y=\sqrt{3 x+1}$
38. $y=\frac{-4}{x^{2}+2 x-35}$

If $f(x)=2 x^{2}-18$ and $g(x)=2 x-6$, find
39. $f(0)$
40. $g[f(0)]$
41. $\frac{g(x+h)-g(x)}{h}$
42. $\frac{g}{f}(x)$

## Getting Ready for the Next Section

Solve each equation for $y$.
43. $x=2 y-3$
44. $x=\frac{y+7}{5}$
45. $x=y^{2}-2$
46. $x=(y+4)^{3}$
47. $x=\frac{y-4}{y-2}$
48. $x=\frac{y+5}{y-3}$
49. $x=\sqrt{y-3}$
50. $x=\sqrt{y}+5$

## Extending the Concepts

51. Drag Racing Previously we mentioned the dragster equipped with a computer. The table gives the speed of the dragster every second during one race at the 1993 Winternationals. Figure 8 is a line graph constructed from the data in the table.

## Speed of a Dragster

| Elapsed Time <br> (sec) | Speed <br> (mph) |
| :---: | ---: |
| 0 | 0.0 |
| 1 | 72.7 |
| 2 | 129.9 |
| 3 | 162.8 |
| 4 | 192.2 |
| 5 | 212.4 |
| 6 | 228.1 |



FIGURE 8

The graph of the following function contains the first point and the last point shown in Figure 8; that is, both $(0,0)$ and $(6,228.1)$ satisfy the function. Graph the function to see how close it comes to the other points in Figure 8.

$$
s(t)=250\left(1-1.5^{-t}\right)
$$


52. The graphs of two exponential functions are given in Figures 9 and 10. Use the graph to find the following:
a. $f(0)$
b. $f(-1)$
c. $f(1)$
d. $g(0)$
e. $g(1)$
f. $g(-1)$
g. $f[g(0)]$
h. $g[f(0)]$


FIGURE 9


FIGURE 10

## 11.2

## The Inverse of a Function

## OBJECTIVES

A Find the equation of the inverse of a function.

B Sketch a graph of a function and its inverse.

The following diagram (Figure 1) shows the route Justin takes to school. He leaves his home and drives 3 miles east and then turns left and drives 2 miles north. When he leaves school to drive home, he drives the same two segments but in the reverse order and the opposite direction; that is, he drives 2 miles south, turns right, and drives 3 miles west. When he arrives home from school, he is right where he started. His route home "undoes" his route to school, leaving him where he began.


As you will see, the relationship between a function and its inverse function is similar to the relationship between Justin's route from home to school and his route from school to home.
Suppose the function $f$ is given by

$$
f=\{(1,4),(2,5),(3,6),(4,7)\}
$$

The inverse of $f$ is obtained by reversing the order of the coordinates in each ordered pair in $f$. The inverse of $f$ is the relation given by

$$
g=\{(4,1),(5,2),(6,3),(7,4)\}
$$

It is obvious that the domain of $f$ is now the range of $g$, and the range of $f$ is now the domain of $g$. Every function (or relation) has an inverse that is obtained from the original function by interchanging the components of each ordered pair.

Suppose a function $f$ is defined with an equation instead of a list of ordered pairs. We can obtain the equation of the inverse of $f$ by interchanging the role of $x$ and $y$ in the equation for $f$.

EXAMPLE 1 If the function $f$ is defined by $f(x)=2 x-3$, find the equation that represents the inverse of $f$.

SOLUTION Since the inverse of $f$ is obtained by interchanging the components of all the ordered pairs belonging to $f$, and each ordered pair in $f$ satisfies the equation $y=2 x-3$, we simply exchange $x$ and $y$ in the equation $y=2 x-3$ to get the formula for the inverse of $f$ :

$$
x=2 y-3
$$

We now solve this equation for $y$ in terms of $x$ :

$$
\begin{aligned}
x+3 & =2 y \\
\frac{x+3}{2} & =y \\
y & =\frac{x+3}{2}
\end{aligned}
$$

The last line gives the equation that defines the inverse of $f$. Let's compare the graphs of $f$ and its inverse as given above. (See Figure 2.)


FIGURE 2
The graphs of $f$ and its inverse have symmetry about the line $y=x$. This is a reasonable result since the one function was obtained from the other by interchanging $x$ and $y$ in the equation. The ordered pairs $(a, b)$ and $(b, a)$ always have symmetry about the line $y=x$.

EXAMPLE 2 Graph the function $y=x^{2}-2$ and its inverse. Give the equation for the inverse.
SOLUTION We can obtain the graph of the inverse of $y=x^{2}-2$ by graphing $y=x^{2}-2$ by the usual methods and then reflecting the graph about the line $y=x$.


FIGURE 3
The equation that corresponds to the inverse of $y=x^{2}-2$ is obtained by interchanging $x$ and $y$ to get $x=y^{2}-2$.

We can solve the equation $x=y^{2}-2$ for $y$ in terms of $x$ as follows:

$$
\begin{aligned}
x & =y^{2}-2 \\
x+2 & =y^{2} \\
y & = \pm \sqrt{x+2}
\end{aligned}
$$

## USING TECHNOLOGY

## Graphing an Inverse

One way to graph a function and its inverse is to use parametric equations. To graph the function $y=x^{2}-2$ and its inverse from Example 2, first set your graphing calculator to parametric mode. Then define the following


FIGURE 4 set of parametric equations (Figure 4).

$$
\mathrm{X}_{1}=t ; \mathrm{Y}_{1}=t^{2}-2
$$

Set the window variables so that

$$
-3 \leq t \leq 3, \text { step }=0.05 ;-4 \leq x \leq 4 ;-4 \leq y \leq 4
$$

Graph the function using the zoom-square command. Your graph should look similar to Figure 5.


FIGURE 5


FIGURE 6

To graph the inverse, we need to interchange the roles of $x$ and $y$ for the original function. This is easily done by defining a new set of parametric equations that is just the reverse of the pair given above:

$$
\mathrm{X}_{2}=t^{2}-2, \mathrm{Y}_{2}=t
$$

Press GRAPH again, and you should now see the graphs of the original function and its inverse (Figure 6). If you trace to any point on either graph, you can alternate between the two curves to see how the coordinates of the corresponding ordered pairs compare. As Figure 7 illustrates, the coordinates of a point on one graph are reversed for the other graph.


FIGURE 7

Comparing the graphs from Examples 1 and 2, we observe that the inverse of a function is not always a function. In Example 1, both $f$ and its inverse have graphs that are nonvertical straight lines and therefore both represent functions. In Example 2, the inverse of function $f$ is not a function since a vertical line crosses it in more than one place.

## One-to-One Functions

We can distinguish between those functions with inverses that are also functions and those functions with inverses that are not functions with the following definition.

DEFINITION A function is a one-to-one function if every element in the range comes from exactly one element in the domain.

This definition indicates that a one-to-one function will yield a set of ordered pairs in which no two different ordered pairs have the same second coordinates. For example, the function

$$
f=\{(2,3),(-1,3),(5,8)\}
$$

is not one-to-one because the element 3 in the range comes from both 2 and -1 in the domain. On the other hand, the function

$$
g=\{(5,7),(3,-1),(4,2)\}
$$

is a one-to-one function because every element in the range comes from only one element in the domain.

## Horizontal Line Test

If we have the graph of a function, we can determine if the function is one-toone with the following test. If a horizontal line crosses the graph of a function in more than one place, then the function is not a one-to-one function because the points at which the horizontal line crosses the graph will be points with the same $y$-coordinates but different $x$-coordinates. Therefore, the function will have an element in the range (the $y$-coordinate) that comes from more than one element in the domain (the $x$-coordinates).

Of the functions we have covered previously, all the linear functions and exponential functions are one-to-one functions because no horizontal lines can be found that will cross their graphs in more than one place.

## Functions Whose Inverses Are Also Functions

Because one-to-one functions do not repeat second coordinates, when we reverse the order of the ordered pairs in a one-to-one function, we obtain a relation in which no two ordered pairs have the same first coordinate-by definition, this relation must be a function. In other words, every one-to-one function has an inverse that is itself a function. Because of this, we can use function notation to represent that inverse.

## Note

The notation $f^{-1}$ does not represent the reciprocal of $f$; that is, the -1 in this notation is not an exponent. The notation $f^{-1}$ is defined as representing the inverse function for a one-to-one function.

Inverse Function Notation If $y=f(x)$ is a one-to-one function, then the inverse of $f$ is also a function and can be denoted by $y=f^{-1}(x)$.

To illustrate, in Example 1 we found the inverse of $f(x)=2 x-3$ was the function $y=\frac{x+3}{2}$. We can write this inverse function with inverse function notation as

$$
f^{-1}(x)=\frac{x+3}{2}
$$

However, the inverse of the function in Example 2 is not itself a function, so we do not use the notation $f^{-1}(x)$ to represent it.

## EXAMPLE 3 Find the inverse of $g(x)=\frac{x-4}{x-2}$.

SOLUTION To find the inverse for $g$, we begin by replacing $g(x)$ with $y$ to obtain

$$
y=\frac{x-4}{x-2} \quad \text { The original function }
$$

To find an equation for the inverse, we exchange $x$ and $y$.

$$
x=\frac{y-4}{y-2} \quad \text { The inverse of the original function }
$$

To solve for $y$, we first multiply each side by $y-2$ to obtain

$$
\begin{aligned}
x(y-2) & =y-4 & & \\
x y-2 x & =y-4 & & \text { Distributive property } \\
x y-y & =2 x-4 & & \text { Collect all terms containing } y \text { on the left side } \\
y(x-1) & =2 x-4 & & \text { Factor } y \text { from each term on the left side } \\
y & =\frac{2 x-4}{x-1} & & \text { Divide each side by } x-1
\end{aligned}
$$

Because our original function is one-to-one, as verified by the graph in Figure 8 , its inverse is also a function. Therefore, we can use inverse function notation to write

$$
g^{-1}(x)=\frac{2 x-4}{x-1}
$$



FIGURE 8

## EXAMPLE $4 \quad$ Graph the function $y=2^{x}$ and its inverse $x=2^{y}$.

SOLUTION We graphed $y=2^{x}$ in the preceding section. We simply reflect its graph about the line $y=x$ to obtain the graph of its inverse $x=2^{y}$. (See Figure 9.)


As you can see from the graph, $x=2^{y}$ is a function. We do not have the mathematical tools to solve this equation for $y$, however. Therefore, we are unable to use the inverse function notation to represent this function. In the next section, we will give a definition that solves this problem. For now, we simply leave the equation as $x=2^{y}$.

## Functions, Relations, and Inverses-A Summary

Here is a summary of some of the things we know about functions, relations, and their inverses:

1. Every function is a relation, but not every relation is a function.
2. Every function has an inverse, but only one-to-one functions have inverses that are also functions.
3. The domain of a function is the range of its inverse, and the range of a function is the domain of its inverse.
4. If $y=f(x)$ is a one-to-one function, then we can use the notation $y=f^{-1}(x)$ to represent its inverse function.
5. The graph of a function and its inverse have symmetry about the line $y=x$.
6. If $(a, b)$ belongs to the function $f$, then the point $(b, a)$ belongs to its inverse.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.
A 1-3
B $1-4$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the inverse of a function?
2. What is the relationship between the graph of a function and the graph of its inverse?
3. Explain why only one-to-one functions have inverses that are also functions.
4. Describe the vertical line test, and explain the difference between the vertical line test and the horizontal line test.

## Problem Set 11.2

For each of the following one-to-one functions, find the equation of the inverse. Write the inverse using the notation $f^{-1}(x)$.

1. $f(x)=3 x-1$
2. $f(x)=2 x-5$
3. $f(x)=x^{3}$
4. $f(x)=x^{3}-2$
5. $f(x)=\frac{x-3}{x-1}$
6. $f(x)=\frac{x-2}{x-3}$
7. $f(x)=\frac{x-3}{4}$
8. $f(x)=\frac{x+7}{2}$
9. $f(x)=\frac{1}{2} x-3$
10. $f(x)=\frac{1}{3} x+1$
11. $f(x)=\frac{2 x+1}{3 x+1}$
12. $f(x)=\frac{3 x+2}{5 x+1}$

For each of the following relations, sketch the graph of the relation and its inverse, and write an equation for the inverse.
13. $y=2 x-1$
14. $y=3 x+1$
15. $y=x^{2}-3$
16. $y=x^{2}+1$
17. $y=x^{2}-2 x-3$
18. $y=x^{2}+2 x-3$
19. $y=3^{x}$
20. $y=\left(\frac{1}{2}\right)^{x}$
21. $y=4$
22. $y=-2$
23. $y=\frac{1}{2} x^{3}$
24. $y=x^{3}-2$
25. $y=\frac{1}{2} x+2$
26. $y=\frac{1}{3} x-1$
27. $y=\sqrt{x+2}$
28. $y=\sqrt{x}+2$
29. Determine if the following functions are one-toone.
a.

b.

c.

30. Could the following tables of values represent ordered pairs from one-to-one functions? Explain your answer.
a.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| ---: | ---: |
| -2 | 5 |
| -1 | 4 |
| 0 | 3 |
| 1 | 4 |
| 2 | 5 |

b.

| $x$ | $y$ |
| :---: | :---: |
| 1.5 | 0.1 |
| 2.0 | 0.2 |
| 2.5 | 0.3 |
| 3.0 | 0.4 |
| 3.5 | 0.5 |

31. If $f(x)=3 x-2$, then $f^{-1}(x)=\frac{x+2}{3}$. Use these two functions to find
a. $f(2)$
b. $f^{-1}(2)$
c. $f\left[f^{-1}(2)\right]$
d. $f^{-1}[f(2)]$
32. If $f(x)=\frac{1}{2} x+5$, then $f^{-1}(x)=2 x-10$. Use these two functions to find
a. $f(-4)$
b. $f^{-1}(-4)$
c. $f\left[f^{-1}(-4)\right]$
d. $f^{-1}[f(-4)]$
33. Let $f(x)=\frac{1}{x}$, and find $f^{-1}(x)$.
34. Let $f(x)=\frac{a}{X}$, and find $f^{-1}(x)$. ( $a$ is a real number constant.)

## Applying the Concepts

35. Reading Tables Evaluate each of the following functions using the functions defined by Tables 1 and 2.
a. $f[g(-3)]$
b. $g[f(-6)]$
c. $g[f(2)]$
d. $f[g(3)]$
e. $f[g(-2)]$
f. $g[f(3)]$
g. What can you conclude about the relationship between functions $f$ and $g$ ?

| TABLE $\mathbf{1}$ |  | TABLE $\mathbf{2}$ |  |
| :---: | ---: | ---: | ---: |
| $\boldsymbol{x}$ | $\boldsymbol{f}(\boldsymbol{x})$ |  | $\boldsymbol{x}$ |
| -6 | 3 | $\boldsymbol{g}(\boldsymbol{x})$ |  |
| 2 | -3 | -2 | 3 |
| 3 | -2 | 3 | -6 |
| 6 | 4 | 4 | 6 |

36. Reading Tables Use the functions defined in Tables 1 and 2 in Problem 35 to answer the following questions.
a. What are the domain and range of $f$ ?
b. What are the domain and range of $g$ ?
c. How are the domain and range of $f$ related to the domain and range of $g$ ?
d. Is $f$ a one-to-one function?
e. Is $g$ a one-to-one function?

## Maintaining Your Skills

Solve each equation.
37. $(2 x-1)^{2}=25$
38. $(3 x+5)^{2}=-12$
39. What number would you add to $x^{2}-10 x$ to make it a perfect square trinomial?
40. What number would you add to $x^{2}-5 x$ to make it a perfect square trinomial?

Solve by completing the square.
41. $x^{2}-10 x+8=0$
42. $x^{2}-5 x+4=0$
43. $3 x^{2}-6 x+6=0$
44. $4 x^{2}-16 x-8=0$

## Getting Ready for the Next Section

Simplify
45. $3^{-2}$
46. $2^{3}$

Solve.
47. $2=3 x$
48. $3=5 x$
49. $4=x^{3}$
50. $12=x^{2}$

Fill in the blanks to make each statement true.
51. $8=2^{\square}$
52. $27=3^{\square}$
53. $10,000=10^{\square}$
54. $1,000=10^{\square}$
55. $81=3^{\square}$
56. $81=9 \square$
57. $6=6^{\square}$
58. $1=5^{\square}$

## Extending the Concepts

For each of the following functions, find $f^{-1}(x)$. Then show that $f\left[f^{-1}(x)\right]=x$.
59. $f(x)=3 x+5$
60. $f(x)=6-8 x$
61. $f(x)=x^{3}+1$
62. $f(x)=x^{3}-8$
63. $f(x)=\frac{x-4}{x-2}$
64. $f(x)=\frac{x-3}{x-1}$
65. The graphs of a function and its inverse are shown in the figure. Use the graphs to find the following:
a. $f(0)$
b. $f(1)$
c. $f(2)$
d. $f^{-1}(1)$
e. $f^{-1}(2)$
f. $f^{-1}(5)$
g. $f^{-1}[f(2)]$
h. $f\left[f^{-1}(5)\right]$


## 11.3

## OBJECTIVES

A Convert between logarithmic form and exponential form.

B Use the definition of logarithms to solve simple logarithmic equations.

C Sketch the graph of a logarithmic function.

D Simplify expressions involving logarithms.

## Logarithms Are Exponents

In January 1999, ABC News reported that an earthquake had occurred in Colombia, causing massive destruction. They reported the strength of the quake by indicating that it measured 6.0 on the Richter scale. For comparison, Table 1 gives the Richter magnitude of a number of other earthquakes.

| TABLE 1 <br> Earthquakes |  |  |
| :--- | :--- | :---: |
| Year | Earthquake | Richter <br> Magnitude |
| 1971 | Los Angeles | 6.6 |
| 1985 | Mexico City | 8.1 |
| 1989 | San Francisco | 7.1 |
| 1992 | Kobe, Japan | 7.2 |
| 1994 | Northridge | 6.6 |
| 1999 | Armenia, Colombia | 6.0 |



## Note

How to change from logarithmic form to exponential form is the first important thing to know about logarithms.

Although the sizes of the numbers in the table do not seem to be very different, the intensity of the earthquakes they measure can be very different. For example, the 1989 San Francisco earthquake was more than 10 times stronger than the 1999 earthquake in Colombia. The reason behind this is that the Richter scale is a logarithmic scale. In this section, we start our work with logarithms, which will give you an understanding of the Richter scale. Let's begin.

As you know from your work in the previous sections, equations of the form

$$
y=b^{x} \quad b>0, b \neq 1
$$

are called exponential functions. Because the equation of the inverse of a function can be obtained by exchanging $x$ and $y$ in the equation of the original function, the inverse of an exponential function must have the form

$$
x=b^{y} \quad b>0, b \neq 1
$$

Now, this last equation is actually the equation of a logarithmic function, as the following definition indicates:

DEFINITION The equation $y=\log _{b} x$ is read " $y$ is the logarithm to the base $b$ of $x^{\prime \prime}$ and is equivalent to the equation

$$
x=b^{y} \quad b>0, b \neq 1
$$

In words, we say " $y$ is the number we raise $b$ to in order to get $x$."

Notation When an equation is in the form $x=b^{y}$, it is said to be in exponential form. On the other hand, if an equation is in the form $y=\log _{b} x$, it is said to be in logarithmic form.

Here are some equivalent statements written in both forms.

| Exponential Form |  | Logarithmic Form |
| :---: | :--- | :---: |
| $8=2^{3}$ | $\Leftrightarrow$ | $\log _{2} 8=3$ |
| $25=5^{2}$ | $\Leftrightarrow$ | $\log _{5} 25=2$ |
| $0.1=10^{-1}$ | $\Leftrightarrow$ | $\log _{10} 0.1=-1$ |
| $\frac{1}{8}=2^{-3}$ | $\Leftrightarrow$ | $\log _{2} \frac{1}{8}=-3$ |
| $r=Z^{s}$ | $\Leftrightarrow$ | $\log _{z} r=S$ |

## EXAMPLE 1 Solve for $x: \log _{3} x=-2$

SOLUTION In exponential form the equation looks like this:

$$
\begin{aligned}
x & =3^{-2} \\
\text { or } \quad x & =\frac{1}{9}
\end{aligned}
$$

The solution is $\frac{1}{9}$.

## EXAMPLE 2 Solve $\log _{x} 4=3$.

SOLUTION Again, we use the definition of logarithms to write the equation in exponential form:

$$
4=x^{3}
$$

Taking the cube root of both sides, we have

$$
\begin{aligned}
\sqrt[3]{4} & =\sqrt[3]{x^{3}} \\
x & =\sqrt[3]{4}
\end{aligned}
$$

The solution set is $\{\sqrt[3]{4}\}$.

EXAMPLE 3 Solve $\log _{8} 4=x$.
SOLUTION We write the equation again in exponential form:

$$
4=8^{x}
$$

## Note

The first step in each of these first three examples is the same. In each case the first step in solving the equation is to put the equation in exponential form.

Since both 4 and 8 can be written as powers of 2 , we write them in terms of powers of 2 :

$$
\begin{aligned}
& 2^{2}=\left(2^{3}\right)^{x} \\
& 2^{2}=2^{3 x}
\end{aligned}
$$

The only way the left and right sides of this last line can be equal is if the exponents are equal-that is, if

$$
\begin{aligned}
2 & =3 x \\
\text { or } \quad x & =\frac{2}{3}
\end{aligned}
$$

The solution is $\frac{2}{3}$. We check as follows:

$$
\begin{aligned}
\log _{8} 4=\frac{2}{3} \Leftrightarrow 4 & =8^{2 / 3} \\
4 & =(\sqrt[3]{8})^{2} \\
4 & =2^{2} \\
4 & =4
\end{aligned}
$$

The solution checks when used in the original equation.

## Graphing Logarithmic Functions

Graphing logarithmic functions can be done using the graphs of exponential functions and the fact that the graphs of inverse functions have symmetry about the line $y=x$. Here's an example to illustrate.

EXAMPLE 4 Graph the equation $y=\log _{2} x$.
SOLUTION The equation $y=\log _{2} x$ is, by definition, equivalent to the exponential equation

$$
x=2^{y}
$$

which is the equation of the inverse of the function

$$
y=2^{x}
$$

## Note

From the graph of $y=\log _{2} x$ it is apparent that $x>0$ since the graph does not appear to the left of the $y$-axis. In general, the only variable in the expression $y=\log _{b} x$ that can be negative is $y$.

We simply reflect the graph of $y=2^{x}$ about the line $y=x$ to get the graph of $x=2^{y}$, which is also the graph of $y=\log _{2} x$. (See Figure 1.)


FIGURE 1
It is apparent from the graph that $y=\log _{2} x$ is a function since no vertical line will cross its graph in more than one place. The same is true for all logarithmic equations of the form $y=\log _{b} x$, where $b$ is a positive number other than 1 . Note also that the graph of $y=\log _{b} x$ will always appear to the right of the $y$-axis, meaning that $x$ will always be positive in the equation $y=\log _{b} x$.

## USING TECHNOLOGY

## Graphing Logarithmic Functions

As demonstrated in Example 4, we can graph the logarithmic function $y=\log _{2} x$ as the inverse of the exponential function $y=2 x$. Your graphing calculator most likely has a command to do this. First, define the exponential function as $Y_{1}=2 x$. To see the line of symmetry, define a second function $\mathrm{Y}_{2}=x$. Set the window variables so that

$$
-6 \leq x \leq 6 ;-6 \leq y \leq 6
$$

and use your zoom-square command to graph both functions. Your graph should look similar to the one shown in Figure 2. Now use the appropriate command to graph the inverse of the exponential function defined as $Y_{1}$ (Figure 3).


FIGURE 2


FIGURE 3

## Two Special Identities

If $b$ is a positive real number other than 1 , then each of the following is a consequence of the definition of a logarithm:

$$
\text { (1) } b^{\log _{b} x}=x \quad \text { and } \quad \text { (2) } \log _{b} b^{x}=x
$$

The justifications for these identities are similar. Let's consider only the first one. Consider the equation

$$
y=\log _{b} x
$$

By definition, it is equivalent to

$$
x=b^{y}
$$

Substituting $\log _{b} x$ for $y$ in the last line gives us

$$
x=b^{\log _{b} x}
$$

The next examples in this section show how these two special properties can be used to simplify equations involving logarithms.

## EXAMPLE 5 Simplify $\log _{2} 8$.

SOLUTION Substitute $2^{3}$ for 8 :

$$
\begin{aligned}
\log _{2} 8 & =\log _{2} 2^{3} \\
& =3
\end{aligned}
$$

EXAMPLE 6 Simplify $\log _{10} 10,000$.
SOLUTION 10,000 can be written as $10^{4}$ :

$$
\begin{aligned}
\log _{10} 10,000 & =\log _{10} 10^{4} \\
& =4
\end{aligned}
$$

EXAMPLE $7 \quad$ Simplify $\log _{b} b(b>0, b \neq 1)$.
SOLUTION Since $b^{1}=b$, we have

$$
\begin{aligned}
\log _{b} b & =\log _{b} b^{1} \\
& =1
\end{aligned}
$$

EXAMPLE $8 \quad$ Simplify $\log _{b} 1(b>0, b \neq 1)$.
SOLUTION Since $1=b^{0}$, we have

$$
\begin{aligned}
\log _{b} 1 & =\log _{b} b^{0} \\
& =0
\end{aligned}
$$

EXAMPLE 9 Simplify $\log _{4}\left(\log _{5} 5\right)$.
SOLUTION Since $\log _{5} 5=1$,

$$
\begin{aligned}
\log _{4}\left(\log _{5} 5\right) & =\log _{4} 1 \\
& =0
\end{aligned}
$$

## Application

As we mentioned in the introduction to this section, one application of logarithms is in measuring the magnitude of an earthquake. If an earthquake has a shock wave $T$ times greater than the smallest shock wave that can be measured on a seismograph, then the magnitude $M$ of the earthquake, as measured on the Richter scale, is given by the formula

$$
M=\log _{10} T
$$

(When we talk about the size of a shock wave, we are talking about its amplitude. The amplitude of a wave is half the difference between its highest point and its lowest point.)


FIGURE 4
To illustrate the discussion, an earthquake that produces a shock wave that is 10,000 times greater than the smallest shock wave measurable on a seismograph will have a magnitude $M$ on the Richter scale of

$$
M=\log _{10} 10,000=4
$$

## EXAMPLE 10 If an earthquake has a magnitude of $M=5$ on the Richter

 scale, what can you say about the size of its shock wave?SOLUTION To answer this question, we put $M=5$ into the formula $M=\log _{10} T$ to obtain

$$
5=\log _{10} T
$$

Writing this equation in exponential form, we have

$$
T=10^{5}=100,000
$$

We can say that an earthquake that measures 5 on the Richter scale has a shock wave 100,000 times greater than the smallest shock wave measurable on a seismograph.

From Example 10 and the discussion that preceded it, we find that an earthquake of magnitude 5 has a shock wave that is 10 times greater than an earthquake of magnitude 4 because 100,000 is 10 times 10,000.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-4
B ${ }^{1-3}$
C 4
D 5-10

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a logarithm?
2. What is the relationship between $y=2^{x}$ and $y=\log _{2} x$ ? How are their graphs related?
3. Will the graph of $y=\log _{b} x$ ever appear in the second or third quadrants? Explain why or why not.
4. Explain why $\log _{2} O=x$ has no solution for $x$.

## Problem Set 11.3

Write each of the following equations in logarithmic form.

1. $2^{4}=16$
2. $3^{2}=9$
3. $125=5^{3}$
4. $16=4^{2}$
5. $0.01=10^{-2}$
6. $0.001=10^{-3}$
7. $2^{-5}=\frac{1}{32}$
8. $4^{-2}=\frac{1}{16}$
9. $\left(\frac{1}{2}\right)^{-3}=8$
10. $\left(\frac{1}{3}\right)^{-2}=9$
11. $27=3^{3}$
12. $81=3^{4}$

Write each of the following equations in exponential form.
13. $\log _{10} 100=2$
14. $\log _{2} 8=3$
15. $\log _{2} 64=6$
16. $\log _{2} 32=5$
17. $\log _{8} 1=0$
18. $\log _{9} 9=1$
19. $\log _{10} 0.001=-3$
20. $\log _{10} 0.0001=-4$
21. $\log _{6} 36=2$
22. $\log _{7} 49=2$
23. $\log _{5} \frac{1}{25}=-2$
24. $\log _{3} \frac{1}{81}=-4$

Solve each of the following equations for $x$.
25. $\log _{3} x=2$
26. $\log _{4} x=3$
27. $\log _{5} x=-3$
28. $\log _{2} x=-4$
29. $\log _{2} 16=x$
30. $\log _{3} 27=x$
31. $\log _{8} 2=x$
32. $\log _{25} 5=x$
33. $\log _{x} 4=2$
34. $\log _{x} 16=4$
35. $\log _{x} 5=3$
36. $\log _{x} 8=2$
$=$ Videos available by instructor request
> = Online student support materials available at www.thomsonedu.com/login

Sketch the graph of each of the following logarithmic equations.
37. $y=\log _{3} x$
38. $y=\log _{1 / 2} x$
39. $y=\log _{1 / 3} x$
40. $y=\log _{4} x$
41. $y=\log _{5} x$
42. $y=\log _{1 / 5} x$
43. $y=\log _{10} x$
44. $y=\log _{1 / 4} x$

Simplify each of the following.
45. $\log _{2} 16$
46. $\log _{3} 9$
47. $\log _{25} 125$
48. $\log _{9} 27$
49. $\log _{10} 1,000$
50. $\log _{10} 10,000$
51. $\log _{3} 3$
52. $\log _{4} 4$
53. $\log _{5} 1$
54. $\log _{10} 1$
55. $\log _{3}\left(\log _{6} 6\right)$
56. $\log _{5}\left(\log _{3} 3\right)$
57. $\log _{4}\left[\log _{2}\left(\log _{2} 16\right)\right]$
58. $\log _{4}\left[\log _{3}\left(\log _{2} 8\right)\right]$

## Applying the Concepts

$\mathrm{pH}=-\log _{10}\left[\mathrm{H}^{+}\right]$, where $\left[\mathrm{H}^{+}\right]$is the concentration of the hydrogen ions in solution. An acid solution has a pH below 7, and a basic solution has a pH higher than 7 .
59. In distilled water, the concentration of hydrogen ions is $\left[\mathrm{H}^{+}\right]=10^{-7}$. What is the pH ?
60. Find the pH of a bottle of vinegar if the concentration of hydrogen ions is $\left[\mathrm{H}^{+}\right]=10^{-3}$.
61. A hair conditioner has a pH of 6 . Find the concentration of hydrogen ions, $\left[\mathrm{H}^{+}\right]$, in the conditioner.
62. If a glass of orange juice has a pH of 4 , what is the concentration of hydrogen ions, $\left[\mathrm{H}^{+}\right]$, in the juice?
63. Magnitude of an Earthquake Find the magnitude $M$ of an earthquake with a shock wave that measures $T=100$ on a seismograph.
64. Magnitude of an Earthquake Find the magnitude $M$ of an earthquake with a shock wave that measures $T=100,000$ on a seismograph.
65. Shock Wave If an earthquake has a magnitude of 8 on the Richter scale, how many times greater is its shock wave than the smallest shock wave measurable on a seismograph?
66. Shock Wave If an earthquake has a magnitude of 6 on the Richter scale, how many times greater is its shock wave than the smallest shock wave measurable on a seismograph?

## Maintaining Your Skills

Fill in the blanks to complete the square.
67. $x^{2}+10 x+\square=(x+\square)^{2}$
68. $x^{2}+4 x+\square=(x+\square)^{2}$
69. $y^{2}-2 y+\square=(y-\square)^{2}$
70. $y^{2}+3 y+\square=(y+\square)^{2}$

Solve.
71. $-y^{2}=9$
72. $7+y^{2}=11$
73. $-x^{2}-8=-4$
74. $10 x^{2}=100$
75. $2 x^{2}+4 x-3=0$
76. $3 x^{2}+4 x-2=0$
77. $(2 y-3)(2 y-1)=-4$
78. $(y-1)(3 y-3)=10$
79. $t^{3}-125=0$
80. $8 t^{3}+1=0$
81. $4 x^{5}-16 x^{4}=20 x^{3}$
82. $3 x^{4}+6 x^{2}=6 x^{3}$
83. $\frac{1}{x-3}+\frac{1}{x+2}=1$
84. $\frac{1}{x+3}+\frac{1}{x-2}=1$

## Getting Ready for the Next Section

Simplify.
85. $8^{2 / 3}$
86. $27^{2 / 3}$

Solve.
87. $(x+2)(x)=2^{3}$
88. $(x+3)(x)=2^{2}$
89. $\frac{x-2}{x+1}=9$
90. $\frac{x+1}{x-4}=25$

Write in exponential form.
91. $\log _{2}[(x+2)(x)]=3$
92. $\log _{4}[x(x-6)]=2$
93. $\log _{3}\left(\frac{x-2}{x+1}\right)=4$
94. $\log _{5}\left(\frac{x-1}{x-4}\right)=2$

## Extending the Concepts

95. The graph of the exponential function $y=f(x)=b^{x}$ is shown here. Use the graph to complete parts a through d.

a. Fill in the table.

| $\boldsymbol{x}$ | $f(x)$ |
| ---: | :--- |
| -1 |  |
| 0 |  |
| 1 |  |
| 2 |  |

b. Fill in the table.

c. Find the equation for $f(x)$.
d. Find the equation for $f^{-1}(X)$.
96. The graph of the exponential function $y=f(x)=b^{x}$ is shown here. Use the graph to complete parts a through d .

a. Fill in the table.

| $\boldsymbol{x}$ | $\boldsymbol{f ( x )}$ |
| ---: | ---: |
| -1 |  |
| 0 |  |
| 1 |  |
| 2 |  |
|  |  |

b. Fill in the table.

c. Find the equation for $f(x)$.
d. Find the equation for $f^{-1}(x)$.

## 11.4

## OBJECTIVES

A Use the properties of logarithms to convert between expanded form and single logarithms.

B Use the properties of logarithms to solve equations that contain logarithms.

## Properties of Logarithms

If we search for the word decibel in Microsoft Bookshelf, we find the following definition:

A unit used to express relative difference in power or intensity, usually between two acoustic or electric signals, equal to ten times the common logarithm of the ratio of the two levels.

| Decibels | Comparable to |
| :---: | :--- |
| 10 | A light whisper |
| 20 | Quiet conversation |
| 30 | Normal conversation |
| 40 | Light traffic |
| 50 | Typewriter, loud conversation |
| 60 | Noisy office |
| 70 | Normal traffic, quiet train |
| 80 | Rock music, subway |
| 90 | Heavy traffic, thunder |
| 100 | Jet plane at takeoff |
|  |  |



The precise definition for a decibel is

|  |  |
| :---: | :---: |
|  | ләрипч7 'э甲.п КлеәН |
|  | Kemqns 'oisnu yooy |
|  |  |
|  | әэщо Кs!̣N |
|  |  |
|  |  |
|  | ио!̣esiənuos [ruı. |
|  | ио!̣\| |
|  |  |

$$
D=10 \log _{10}\left(\frac{I}{I_{0}}\right)
$$

where $I$ is the intensity of the sound being measured, and $I_{0}$ is the intensity of the least audible sound. (Sound intensity is related to the amplitude of the sound wave that models the sound and is given in units of watts per meter ${ }^{2}$.) In this section, we will see that the preceding formula can also be written as

$$
D=10\left(\log _{10} I-\log _{10} I_{0}\right)
$$

The rules we use to rewrite expressions containing logarithms are called the properties of logarithms. There are three of them.
For the following three properties, $x, y$, and $b$ are all positive real numbers, $b \neq 1$, and $r$ is any real number.

Property 1

$$
\log _{b}(x y)=\log _{b} x+\log _{b} y
$$

In words: The logarithm of a product is the sum of the logarithms.

Property 2

$$
\log _{b}\left(\frac{x}{y}\right)=\log _{b} x-\log _{b} y
$$

In words: The logarithm of a quotient is the difference of the logarithms.

## Property 3

$$
\log _{b} x^{r}=r \log _{b} x
$$

In words: The logarithm of a number raised to a power is the product of the power and the logarithm of the number.

Proof of Property 1 To prove property 1, we simply apply the first identity for logarithms given in the preceding section:

$$
b^{\log _{b} x y}=x y=\left(b^{\log _{b} x}\right)\left(b^{\log _{b} y}\right)=b^{\log _{b} x+\log _{b} y}
$$

Since the first and last expressions are equal and the bases are the same, the exponents $\log _{b} x y$ and $\log _{b} x+\log _{b} y$ must be equal. Therefore,

$$
\log _{b} x y=\log _{b} x+\log _{b} y
$$

The proofs of properties 2 and 3 proceed in much the same manner, so we will omit them here. The examples that follow show how the three properties can be used.

SOLUTION Applying property 2, we can write the quotient of $3 x y$ and $z$ in terms of a difference:

$$
\log _{5} \frac{3 x y}{z}=\log _{5} 3 x y-\log _{5} z
$$

Applying property 1 to the product $3 x y$, we write it in terms of addition:

$$
\log _{5} \frac{3 x y}{z}=\log _{5} 3+\log _{5} x+\log _{5} y-\log _{5} z
$$

EXAMPLE 2 Expand, using the properties of logarithms:

$$
\log _{2} \frac{x^{4}}{\sqrt{y} \cdot z^{3}}
$$

SOLUTION We write $\sqrt{y}$ as $y^{1 / 2}$ and apply the properties:

$$
\begin{aligned}
\log _{2} \frac{x^{4}}{\sqrt{y} \cdot z^{3}} & =\log _{2} \frac{x^{4}}{y^{1 / 2} z^{3}} & & \sqrt{y}=y^{1 / 2} \\
& =\log _{2} x^{4}-\log _{2}\left(y^{1 / 2} \cdot z^{3}\right) & & \text { Property } 2 \\
& =\log _{2} x^{4}-\left(\log _{2} y^{1 / 2}+\log _{2} z^{3}\right) & & \text { Property 1 } \\
& =\log _{2} x^{4}-\log _{2} y^{1 / 2}-\log _{2} z^{3} & & \text { Remove parentheses } \\
& =4 \log _{2} x-\frac{1}{2} \log _{2} y-3 \log _{2} z & & \text { Property 3 }
\end{aligned}
$$

We can also use the three properties to write an expression in expanded form as just one logarithm.

EXAMPLE 3 Write as a single logarithm:

$$
2 \log _{10} a+3 \log _{10} b-\frac{1}{3} \log _{10} c
$$

SOLUTION We begin by applying property 3 :

$$
\begin{array}{rlrl}
2 \log _{10} a+3 \log _{10} b & -\frac{1}{3} \log _{10} c & \\
& =\log _{10} a^{2}+\log _{10} b^{3}-\log _{10} c^{1 / 3} & & \text { Property } 3 \\
& =\log _{10}\left(a^{2} \cdot b^{3}\right)-\log _{10} c^{1 / 3} & & \text { Property } 1 \\
& =\log _{10} \frac{a^{2} b^{3}}{c^{1 / 3}} & & \text { Property } 2 \\
& =\log _{10} \frac{a^{2} b^{3}}{\sqrt[3]{c}} & & c^{1 / 3}=\sqrt[3]{c}
\end{array}
$$

The properties of logarithms along with the definition of logarithms are useful in solving equations that involve logarithms.

EXAMPLE 4 Solve for $x: \log _{2}(x+2)+\log _{2} x=3$
SOLUTION Applying property 1 to the left side of the equation allows us to write it as a single logarithm:

$$
\begin{aligned}
\log _{2}(x+2)+\log _{2} x & =3 \\
\log _{2}[(x+2)(x)] & =3
\end{aligned}
$$

The last line can be written in exponential form using the definition of logarithms:

$$
(x+2)(x)=2^{3}
$$

Solve as usual:

$$
\begin{array}{rlrl}
x^{2}+2 x & =8 \\
x^{2}+2 x-8 & =0 \\
(x+4)(x-2) & =0 \\
x+4=0 & & \text { or } & x-2
\end{array}=0
$$

In the previous section we noted the fact that $x$ in the expression $y=\log _{b} x$ cannot be a negative number. Since substitution of $x=-4$ into the original equation gives

$$
\log _{2}(-2)+\log _{2}(-4)=3
$$

which contains logarithms of negative numbers, we cannot use -4 as a solution. The solution set is $\{2\}$.

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A ${ }^{1-3}$
B 4

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Explain why the following statement is false: "The logarithm of a product is the product of the logarithms."
2. Explain why the following statement is false: "The logarithm of a quotient is the quotient of the logarithms."
3. Explain the difference between $\log _{b} m+\log _{b} n$ and $\log _{b}(m+n)$. Are they equivalent?
4. Explain the difference between $\log _{b}(m n)$ and $\left(\log _{b} m\right)\left(\log _{b} n\right)$. Are they equivalent?

## Problem Set 11.4

Use the three properties of logarithms given in this section to expand each expression as much as possible.

1. $\log _{3} 4 x$
2. $\log _{2} 5 x$
3. $\log _{6} \frac{5}{x}$
4. $\log _{3} \frac{x}{5}$
5. $\log _{2} y^{5}$
6. $\log _{7} y^{3}$
7. $\log _{9} \sqrt[3]{z}$
8. $\log _{8} \sqrt{z}$
9. $\log _{6} x^{2} y^{4}$
10. $\log _{10} x^{2} y^{4}$
11. $\log _{5} \sqrt{x} \cdot y^{4}$
12. $\log _{8} \sqrt[3]{x y^{6}}$
13. $\log _{b} \frac{x y}{z}$
14. $\log _{b} \frac{3 x}{y}$
15. $\log _{10} \frac{4}{x y}$
16. $\log _{10} \frac{5}{4 y}$
17. $\log _{10} \frac{x^{2} y}{\sqrt{z}}$
18. $\log _{10} \frac{\sqrt{x} \cdot y}{z^{3}}$
19. $\log _{10} \frac{x^{3} \sqrt{y}}{z^{4}}$
20. $\log _{10} \frac{x^{4} \sqrt[3]{y}}{\sqrt{z}}$
21. $\log _{b} \sqrt[3]{\frac{x^{2} y}{z^{4}}}$
22. $\log _{b} \sqrt[4]{\frac{x^{4} y^{3}}{Z^{5}}}$

Write each expression as a single logarithm.
23. $\log _{b} x+\log _{b} Z$
24. $\log _{b} x-\log _{b} Z$
25. $2 \log _{3} x-3 \log _{3} y$
26. $4 \log _{2} x+5 \log _{2} y$
27. $\frac{1}{2} \log _{10} x+\frac{1}{3} \log _{10} y$
28. $\frac{1}{3} \log _{10} x-\frac{1}{4} \log _{10} y$
29. $3 \log _{2} x+\frac{1}{2} \log _{2} y-\log _{2} z$
30. $2 \log _{3} x+3 \log _{3} y-\log _{3} z$
31. $\frac{1}{2} \log _{2} x-3 \log _{2} y-4 \log _{2} z$
$\square=$ Videos available by instructor request

- = Online student support materials available at www.thomsonedu.com/login

32. $3 \log _{10} x-\log _{10} y-\log _{10} z$
33. $\frac{3}{2} \log _{10} x-\frac{3}{4} \log _{10} y-\frac{4}{5} \log _{10} z$
34. $3 \log _{10} x-\frac{4}{3} \log _{10} y-5 \log _{10} z$

Solve each of the following equations.
35. $\log _{2} x+\log _{2} 3=1$
36. $\log _{3} x+\log _{3} 3=1$
37. $\log _{3} x-\log _{3} 2=2$
38. $\log _{3} x+\log _{3} 2=2$
39. $\log _{3} x+\log _{3}(x-2)=1$
40. $\log _{6} x+\log _{6}(x-1)=1$
41. $\log _{3}(x+3)-\log _{3}(x-1)=1$
42. $\log _{4}(x-2)-\log _{4}(x+1)=1$
43. $\log _{2} x+\log _{2}(x-2)=3$
44. $\log _{4} x+\log _{4}(x+6)=2$
45. $\log _{8} x+\log _{8}(x-3)=\frac{2}{3}$
46. $\log _{27} x+\log _{27}(x+8)=\frac{2}{3}$
47. $\log _{5} \sqrt{x}+\log _{5} \sqrt{6 x+5}=1$
48. $\log _{2} \sqrt{x}+\log _{2} \sqrt{6 x+5}=1$

## Applying the Concepts

49. Decibel Formula Use the properties of logarithms to rewrite the decibel formula $D=10 \log _{10}\left(\frac{I}{I_{0}}\right)$ as

$$
D=10\left(\log _{10} I-\log _{10} I_{0}\right)
$$

50. Decibel Formula In the decibel formula $D=10$ $\log _{10}\left(\frac{I}{I_{0}}\right)$, the threshold of hearing, $I_{0}$, is

$$
I_{0}=10^{-12} \text { watts } / \text { meter }^{2}
$$

Substitute $10^{-12}$ for $I_{0}$ in the decibel formula, and then show that it simplifies to

$$
D=10\left(\log _{10} I+12\right)
$$

51. Acoustic Powers The formula $N=\log _{10} \frac{P_{1}}{P_{2}}$ is used in radio electronics to find the ratio of the acoustic powers of two electric circuits in terms of their electric powers. Find $N$ if $P_{1}$ is 100 and $P_{2}$ is 1 . Then use the same two values of $P_{1}$ and $P_{2}$ to find $N$ in the formula $N=\log _{10} P_{1}-\log _{10} P_{2}$.
52. Henderson-Hasselbalch Formula Doctors use the Henderson-Hasselbalch formula to calculate the pH of a person's blood. pH is a measure of the acidity and/or the alkalinity of a solution. This formula is represented as

$$
\mathrm{pH}=6.1+\log _{10}\left(\frac{x}{y}\right)
$$

where $x$ is the base concentration and $y$ is the acidic concentration. Rewrite the HendersonHasselbalch formula so that the logarithm of a quotient is not involved.

## Maintaining Your Skills

Divide.
53. $\frac{12 x^{2}+y^{2}}{36}$
54. $\frac{x^{2}+4 y^{2}}{16}$
55. Divide $25 x^{2}+4 y^{2}$ by 100
56. Divide $4 x^{2}+9 y^{2}$ by 36

Use the discriminant to find the number and kind of solutions to the following equations.
57. $2 x^{2}-5 x+4=0$
58. $4 x^{2}-12 x=-9$

For each of the following problems, find an equation with the given solutions.
59. $x=-3, x=5$
60. $x=2, x=-2, x=1$
61. $y=\frac{2}{3}, y=3$
62. $y=-\frac{3}{5}, y=2$

## Getting Ready for the Next Section

Simplify.
63. $5^{0}$
64. $4^{1}$
65. $\log _{3} 3$
66. $\log _{5} 5$
67. $\log _{b} b^{4}$
68. $\log _{a} a^{k}$

Use a calculator to find each of the following. Write your answer in scientific notation with the first number in each answer rounded to the nearest tenth.
69. $10^{-5.6}$
70. $10^{-4.1}$

Divide and round to the nearest whole number.
71. $\frac{2.00 \times 10^{8}}{3.96 \times 10^{6}}$
72. $\frac{3.25 \times 10^{12}}{1.72 \times 10^{10}}$

## 11.5

## OBJECTIVES

A Use a calculator to find common logarithms.

B Use a calculator to find a number given its common logarithm.
C Simplify expressions containing natural logarithms.

## Common Logarithms and Natural Logarithms

Acid rain was first discovered in the 1960s by Gene Likens and his research team, who studied the damage caused by acid rain to Hubbard Brook in New Hampshire. Acid rain is rain with a pH of 5.6 and below. As you will see as you work your way through this section, pH is defined in terms of common logarithms-one of the topics we present in this section. So, when you are finished with this section, you will have a more detailed knowledge of pH and acid rain.


There are two kinds of logarithms that occur more frequently than other logarithms. They are logarithms with base 10 and natural logarithms, or logarithms with base $e$. Logarithms with a base of 10 are very common because our number system is a base-10 number system. For this reason, we call base-10 logarithms common logarithms.

DEFINITION A common logarithm is a logarithm with a base of 10. Because common logarithms are used so frequently, it is customary, in order to save time, to omit notating the base; that is,

$$
\log _{10} x=\log x
$$

When the base is not shown, it is assumed to be 10 .

## Note

Remember, when the base is not written it is assumed to be 10 .

## Common Logarithms

Common logarithms of powers of 10 are simple to evaluate. We need only recognize that $\log 10=\log _{10} 10=1$ and apply the third property of logarithms: $\log _{b} X^{r}=r \log _{b} x$.

$$
\begin{aligned}
& \log 1,000=\log 10^{3}=3 \log 10=3(1)=3 \\
& \log 100=\log 10^{2}=2 \log 10=2(1)=2 \\
& \log 10=\log 10^{1}=1 \log 10=1(1)=1 \\
& \log 1=\log 10^{0}=0 \log 10=0(1)=0 \\
& \log 0.1=\log 10^{-1}=-1 \log 10=-1(1)=-1 \\
& \log 0.01=\log 10^{-2}=-2 \log 10=-2(1)=-2 \\
& \log 0.001=\log 10^{-3}=-3 \log 10=-3(1)=-3
\end{aligned}
$$

To find common logarithms of numbers that are not powers of 10 , we use a calculator with a $\log$ key.

Check the following logarithms to be sure you know how to use your calculator. (These answers have been rounded to the nearest ten-thousandth.)
$\log 7.02 \approx 0.8463$
$\log 1.39 \approx 0.1430$
$\log 6.00 \approx 0.7782$
$\log 9.99 \approx 0.9996$

EXAMPLE 1 Use a calculator to find $\log 2,760$.
SOLUTION $\log 2,760 \approx 3.4409$
To work this problem on a scientific calculator, we simply enter the number 2,760 and press the key labeled $\log$. On a graphing calculator we press the $\log$ key first, then 2,760.
The 3 in the answer is called the characteristic, and the decimal part of the logarithm is called the mantissa.

EXAMPLE $2 \quad$ Find $\log 0.0391$.
SOLUTION $\log 0.0391 \approx-1.4078$

EXAMPLE $3 \quad$ Find $\log 0.00523$.
SOLUTION $\log 0.00523 \approx-2.2815$

EXAMPLE $4 \quad$ Find $x$ if $\log x=3.8774$.
SOLUTION We are looking for the number whose logarithm is 3.8774. On a scientific calculator, we enter 3.8774 and press the key labeled $10^{x}$. On a
graphing calculator we press $10^{x}$ first, then 3.8774 . The result is 7,540 to four significant digits. Here's why:

$$
\text { If } \quad \begin{aligned}
\log x & =3.8774 \\
x & =10^{3.8774} \\
& \approx 7,540
\end{aligned}
$$

The number 7,540 is called the antilogarithm or just antilog of 3.8774 , that is, 7,540 is the number whose logarithm is 3.8774 .

EXAMPLE $5 \quad$ Find $x$ if $\log x=-2.4179$.
SOLUTION Using the $10^{x}$ key, the result is 0.00382 .

$$
\text { If } \quad \begin{aligned}
\text { then } \quad & \\
\quad & \\
& =10^{-2.4179} \\
& \approx 0.00382
\end{aligned}
$$

The antilog of -2.4179 is 0.00382 ; that is, the logarithm of 0.00382 is -2.4179 .

## Applications

Previously, we found that the magnitude $M$ of an earthquake that produces a shock wave $T$ times larger than the smallest shock wave that can be measured on a seismograph is given by the formula

$$
M=\log _{10} T
$$

We can rewrite this formula using our shorthand notation for common logarithms as

$$
M=\log T
$$

## EXAMPLE 6

The San Francisco earthquake of 1906 is estimated to
 have measured 8.3 on the Richter scale. The San Fernando earthquake of 1971 measured 6.6 on the Richter scale. Find $T$ for each earthquake, and then give some indication of how much stronger the 1906 earthquake was than the 1971 earthquake.

SOLUTION For the 1906 earthquake:
If $\log T=8.3$, then $T=2.00 \times 10^{8}$.
For the 1971 earthquake:
If $\log T=6.6$, then $T=3.98 \times 10^{6}$.
Dividing the two values of $T$ and rounding our answer to the nearest whole number, we have

$$
\frac{2.00 \times 10^{8}}{3.98 \times 10^{6}} \approx 50
$$

The shock wave for the 1906 earthquake was approximately 50 times stronger than the shock wave for the 1971 earthquake.

In chemistry, the pH of a solution is the measure of the acidity of the solution. The definition for pH involves common logarithms. Here it is:

$$
\mathrm{pH}=-\log \left[\mathrm{H}^{+}\right]
$$

where $\left[\mathrm{H}^{+}\right]$is the concentration of the hydrogen ion in moles per liter. The range for pH is from 0 to 14. Pure water, a neutral solution, has a pH of 7 . An acidic solution, such as vinegar, will have a pH less than 7 , and an alkaline solution, such as ammonia, has a pH above 7 .


## EXAMPLE 7

Normal rainwater has a pH of 5.6. What is the concentration of the hydrogen ion in normal rainwater?

SOLUTION Substituting 5.6 for pH in the formula $\mathrm{pH}=-\log \left[\mathrm{H}^{+}\right]$, we have

$$
\begin{aligned}
5.6 & =-\log \left[\mathrm{H}^{+}\right] \\
{\left[\mathrm{H}^{+}\right] } & =-5.6 \\
{\left[\mathrm{H}^{+}\right] } & =10^{-5.6} \\
& \approx 2.5 \times 10^{-6} \text { mole per liter }
\end{aligned}
$$

Substitution

Answer in scientific notation

$$
\log \left[\mathrm{H}^{+}\right]=-5.6 \quad \text { Isolate the logarithm }
$$

$$
\left[\mathrm{H}^{+}\right]=10^{-5.6} \quad \text { Write in exponential form }
$$ acid rain known to kill fish is $3.2 \times 10^{-5}$ mole per liter. Find the pH of this acid rain to the nearest tenth.

SOLUTION Substituting $3.2 \times 10^{-5}$ for $\left[\mathrm{H}^{+}\right]$in the formula $\mathrm{pH}=-\log \left[\mathrm{H}^{+}\right]$, we have

$$
\begin{aligned}
\mathrm{pH} & =-\log \left[3.2 \times 10^{-5}\right] & & \text { Substitution } \\
& \approx-(-4.5) & & \text { Evaluate the logarithm } \\
& =4.5 & & \text { Simplify }
\end{aligned}
$$

## Natural Logarithms

DEFINITION A natural logarithm is a logarithm with a base of $e$. The natural logarithm of $x$ is denoted by $\ln x$; that is,

$$
\ln x=\log _{e} x
$$

The postage stamp shown here contains one of the two special identities we mentioned previously in this chapter, but stated in terms of natural logarithms.


We can assume that all our properties of exponents and logarithms hold for expressions with a base of $e$ since $e$ is a real number. Here are some examples intended to make you more familiar with the number e and natural logarithms.

## EXAMPLE 9

Simplify each of the following expressions.
a. $e^{0}=1$
b. $e^{1}=e$
c. $\ln e=1 \quad$ In exponential form, $\boldsymbol{e}^{\mathbf{1}}=\boldsymbol{e}$
d. $\ln 1=0 \quad$ In exponential form, $e^{0}=1$
e. $\ln e^{3}=3$
f. $\ln e^{-4}=-4$
g. $\ln e^{t}=t$

EXAMPLE 10 Use the properties of logarithms to expand the expres-
sion $\ln A e^{5 t}$.
SOLUTION Since the properties of logarithms hold for natural logarithms, we have

$$
\begin{aligned}
\ln A e^{5 t} & =\ln A+\ln e^{5 t} \\
& =\ln A+5 t \ln e \\
& =\ln A+5 t \quad \text { Because } \ln e=\mathbf{1}
\end{aligned}
$$

## EXAMPLE 11 If $\ln 2=0.6931$ and $\ln 3=1.0986$, find

a. $\ln 6$
b. $\ln 0.5$
c. $\ln 8$

SOLUTION
a. Since $6=2 \cdot 3$, we have

$$
\begin{aligned}
\ln 6 & =\ln (2 \cdot 3) \\
& =\ln 2+\ln 3 \\
& =0.6931+1.0986 \\
& =1.7917
\end{aligned}
$$

b. Writing 0.5 as $\frac{1}{2}$ and applying property 2 for logarithms gives us

$$
\begin{aligned}
\ln 0.5 & =\ln \frac{1}{2} \\
& =\ln 1-\ln 2 \\
& =0-0.6931 \\
& =-0.6931
\end{aligned}
$$

c. Writing 8 as $2^{3}$ and applying property 3 for logarithms, we have

$$
\begin{aligned}
\ln 8 & =\ln 2^{3} \\
& =3 \ln 2 \\
& =3(0.6931) \\
& =2.0793
\end{aligned}
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a common logarithm?

B 4-8

C $9-11$
3. Is e a rational number? Explain.
4. Find In $e$, and explain how you arrived at your answer.

Find the following logarithms. Round to four decimal places.

1. $\log 378$
2. $\log 426$
3. $\log 37.8$
4. $\log 42,600$
5. $\log 3,780$
6. $\log 0.4260$
7. $\log 0.0378$
8. $\log 0.0426$
9. $\log 37,800$
10. $\log 4,900$
11. $\log 600$
12. $\log 900$
13. $\log 2,010$
14. $\log 10,200$
15. $\log 0.00971$
16. $\log 0.0312$
17. $\log 0.0314$
18. $\log 0.00052$
19. $\log 0.399$
20. $\log 0.111$

Find $x$ in the following equations.
21. $\log x=2.8802$
22. $\log x=4.8802$
23. $\log x=-2.1198$
24. $\log x=-3.1198$
25. $\log x=3.1553$
26. $\log x=5.5911$
27. $\log x=-5.3497$
28. $\log x=-1.5670$

Find $x$.
29. $\log x=-7.0372$
30. $\log x=-4.2000$
31. $\log x=10$
32. $\log x=-1$
33. $\log x=-10$
34. $\log x=1$
35. $\log x=20$
36. $\log x=-20$
37. $\log x=-2$
38. $\log x=4$
39. $\log x=\log _{2} 8$
40. $\log x=\log _{3} 9$

Simplify each of the following expressions.
41. $\ln e$
42. $\ln 1$
43. $\ln e^{5}$
44. $\ln e^{-3}$
45. $\ln e^{x}$
46. $\ln e^{y}$

Use the properties of logarithms to expand each of the following expressions.
47. $\ln 10 e^{3 t}$
48. $\ln 10 e^{4 t}$
49. $\ln A e^{-2 t}$
50. $\ln A e^{-3 t}$

If $\ln 2=0.6931, \ln 3=1.0986$, and $\ln 5=1.6094$, find each of the following.
51. $\ln 15$
52. $\ln 10$
53. $\ln \frac{1}{3}$
54. $\ln \frac{1}{5}$
55. $\ln 9$
56. $\ln 25$
57. $\ln 16$
58. $\ln 81$

Use a calculator to evaluate each expression. Round your answers 4 decimal places, if necessary.
59. a. $\frac{\log 25}{\log 15}$
b. $\log \frac{25}{15}$
60. a. $\frac{\log 12}{\log 7}$
b. $\log \frac{12}{7}$
61. a. $\frac{\log 4}{\log 8}$
b. $\log \frac{4}{8}$
62. a. $\frac{\log 3}{\log 9}$
b. $\log \frac{3}{9}$

## Applying the Concepts

Measuring Acidity Previously we indicated that the pH of a solution is defined in terms of logarithms as

$$
\mathrm{pH}=-\log \left[\mathrm{H}^{+}\right]
$$

where $\left[\mathrm{H}^{+}\right]$is the concentration of the hydrogen ion in that solution.

Round to the nearest hundredth.
63. Find the pH of orange juice if the concentration of the hydrogen ion in the juice is $\left[\mathrm{H}^{+}\right]=6.50 \times 10^{-4}$.
64. Find the pH of milk if the concentration of the hydrogen ion in milk is $\left[\mathrm{H}^{+}\right]=1.88 \times 10^{-6}$.
65. Find the concentration of hydrogen ions in a glass of wine if the pH is 4.75 .
66. Find the concentration of hydrogen ions in a bottle of vinegar if the pH is 5.75 .

The Richter Scale Find the relative size $T$ of the shock wave of earthquakes with the following magnitudes, as measured on the Richter scale. Round to the nearest hundredth.
67. 5.5
68. 6.6
69. 8.3
70. 8.7
71. Shock Wave How much larger is the shock wave of an earthquake that measures 6.5 on the Richter scale than one that measures 5.5 on the same scale?
72. Shock Wave How much larger is the shock wave of an earthquake that measures 8.5 on the Richter scale than one that measures 5.5 on the same scale?
73. Earthquake The chart below is a partial listing of earthquakes that were recorded in Canada in 2000. Complete the chart by computing the magnitude on the Richter scale, $M$, or the number of times the associated shock wave is larger than the smallest measurable shock wave, $T$.

| Location | Date | Magnitude, $\boldsymbol{M}$ | Shock Wave, $\boldsymbol{T}$ |
| :--- | :--- | :---: | :---: |
| Moresby Island | January 23 | 4.0 |  |
| Vancouver Island | April 30 |  | $1.99 \times 10^{5}$ |
| Quebec City | June 29 | 3.2 |  |
| Mould Bay | November 13 | 5.2 |  |
| St. Lawrence | December 14 |  | $5.01 \times 10^{3}$ |

Source: National Resources Canada, National Earthquake Hazards Program.
74. Earthquake On January 6, 2001, an earthquake with a magnitude of 7.7 on the Richter scale hit southern India (National Earthquake Information Center). By what factor was this earthquake's shock wave greater than the smallest measurable shock wave?

Depreciation The annual rate of depreciation $r$ on a car that is purchased for $P$ dollars and is worth $W$ dollars $t$ years later can be found from the formula

$$
\log (1-r)=\frac{1}{t} \log \frac{W}{P}
$$

75. Find the annual rate of depreciation on a car that is purchased for \$9,000 and sold 5 years later for \$4,500.
76. Find the annual rate of depreciation on a car that is purchased for \$9,000 and sold 4 years later for \$3,000.

Two cars depreciate in value according to the following depreciation tables. In each case, find the annual rate of depreciation.
77.

| Age in <br> Years | Value in <br> Dollars |
| :---: | :---: |
| new | 7,550 |
| 5 | 5,750 |

78. 

| Age in <br> Years | Value in <br> Dollars |
| :---: | :---: |
| new | 7,550 |
| 3 | 5,750 |

79. Getting Close to $\boldsymbol{e}$ Use a calculator to complete the following table.

| $\boldsymbol{x}$ | $(1+\boldsymbol{x})^{1 / x}$ |
| :--- | :--- |
| 1 |  |
| 0.5 |  |
| 0.1 |  |
| 0.01 |  |
| 0.001 |  |
| 0.0001 |  |
| 0.00001 |  |

What number does the expression $(1+x)^{1 / x}$ seem to approach as $x$ gets closer and closer to zero?
80. Getting Close to $\boldsymbol{e}$ Use a calculator to complete the following table.


What number does the expression $\left(1+\frac{1}{x}\right)^{x}$ seem to approach as $x$ gets larger and larger?

## Maintaining Your Skills

Solve each equation.
81. $(y+3)^{2}+y^{2}=9$
82. $(2 y+4)^{2}+y^{2}=4$
83. $(x+3)^{2}+1^{2}=2$
84. $(x-3)^{2}+(-1)^{2}=10$
85. $x^{4}-2 x^{2}-8=0$
86. $x^{2 / 3}-5 x^{1 / 3}+6=0$
87. $2 x-5 \sqrt{x}+3=0$
88. $(3 x+1)-6 \sqrt{3 x+1}+8=0$

## Getting Ready for the Next Section

Solve.
89. $5(2 x+1)=12$
90. $4(3 x-2)=21$

Use a calculator to evaluate. Give answers to 4 decimal places.
91. $\frac{100,000}{32,000}$
92. $\frac{1.4982}{3.5681}+3$
93. $\frac{1}{2}\left(\frac{-0.6931}{1.4289}+3\right)$
94. $1+\frac{0.04}{52}$

Use the power rule to rewrite the following logarithms.
95. $\log 1.05^{t}$
96. $\log 1.033^{t}$

Use identities to simplify.
97. $\ln e^{0.05 t}$
98. $\ln e^{-0.000121 t}$

## 11.6 <br> Exponential Equations and Change of Base

## OBJECTIVES

A Solve exponential equations.

B Use the change-of-base property.

C Solve application problems whose solutions are found by solving logarithmic or exponential equations.

For items involved in exponential growth, the time it takes for a quantity to double is called the doubling time. For example, if you invest $\$ 5,000$ in an account that pays $5 \%$ annual interest, compounded quarterly, you may want to know how long it will take for your money to double in value. You can find this doubling time if you can solve the equation

$$
10,000=5,000(1.0125)^{4 t}
$$

As you will see as you progress through this section, logarithms are the key to solving equations of this type.
Logarithms are very important in solving equations in which the variable appears as an exponent. The equation

$$
5^{x}=12
$$

is an example of one such equation. Equations of this form are called exponential equations. Since the quantities $5^{x}$ and 12 are equal, so are their common logarithms. We begin our solution by taking the logarithm of both sides:

$$
\log 5^{x}=\log 12
$$

We now apply property 3 for $\log$ arithms, $\log x^{r}=r \log x$, to turn $x$ from an exponent into a coefficient:

$$
x \log 5=\log 12
$$

Dividing both sides by $\log 5$ gives us

$$
x=\frac{\log 12}{\log 5}
$$

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## Note

Your answers to the problems in the problem sets may differ from those in the book slightly, depending on whether you round off intermediate answers or carry out the complete problem on a calculator. The differences will occur only in the right-most digit of your answers.

If we want a decimal approximation to the solution, we can find $\log 12$ and $\log 5$ on a calculator and divide:

$$
\begin{aligned}
x & \approx \frac{1.0792}{0.6990} \\
& \approx 1.5439
\end{aligned}
$$

The complete problem looks like this:

$$
5^{x}=12
$$

$$
\begin{aligned}
\log 5^{x} & =\log 12 \\
x \log 5 & =\log 12 \\
x & =\frac{\log 12}{\log 5} \\
& \approx \frac{1.0792}{0.6990} \\
& \approx 1.5439
\end{aligned}
$$

Here is another example of solving an exponential equation using logarithms.

## EXAMPLE 1 Solve for $x: 25^{2 x+1}=15$.

SOLUTION Taking the logarithm of both sides and then writing the exponent $(2 x+1)$ as a coefficient, we proceed as follows:

$$
\begin{aligned}
25^{2 x+1} & =15 & & \\
\log 25^{2 x+1} & =\log 15 & & \text { Take the log of both sides } \\
(2 x+1) \log 25 & =\log 15 & & \text { Property } 3 \\
2 x+1 & =\frac{\log 15}{\log 25} & & \text { Divide by } \log 25 \\
2 x & =\frac{\log 15}{\log 25}-1 & & \text { Add }-\mathbf{1} \text { to both sides } \\
x & =\frac{1}{2}\left(\frac{\log 15}{\log 25}-1\right) & & \text { Multiply both sides by } \frac{1}{2}
\end{aligned}
$$

Using a calculator, we can write a decimal approximation to the answer:

$$
\begin{aligned}
x & \approx \frac{1}{2}\left(\frac{1.1761}{1.3979}-1\right) \\
& \approx \frac{1}{2}(0.8413-1) \\
& \approx \frac{1}{2}(-0.1587) \\
& \approx-0.0794
\end{aligned}
$$

If you invest $P$ dollars in an account with an annual interest rate $r$ that is compounded $n$ times a year, then $t$ years later the amount of money in that account will be

$$
A=P\left(1+\frac{r}{n}\right)^{n t}
$$

EXAMPLE 2 How long does it take for $\$ 5,000$ to double if it is deposited in an account that yields $5 \%$ interest compounded once a year?
SOLUTION Substituting $P=5,000, r=0.05, n=1$, and $A=10,000$ into our formula, we have

$$
\begin{aligned}
10,000 & =5,000(1+0.05)^{t} \\
10,000 & =5,000(1.05)^{t} \\
2 & =(1.05)^{t} \quad \text { Divide by } \mathbf{5 , 0 0 0}
\end{aligned}
$$

This is an exponential equation. We solve by taking the logarithm of both sides:

$$
\begin{aligned}
\log 2 & =\log (1.05)^{t} \\
& =t \log 1.05
\end{aligned}
$$

Dividing both sides by log 1.05, we have

$$
\begin{aligned}
t & =\frac{\log 2}{\log 1.05} \\
& \approx 14.2
\end{aligned}
$$

It takes a little over 14 years for $\$ 5,000$ to double if it earns $5 \%$ interest per year, compounded once a year.

There is a fourth property of logarithms we have not yet considered. This last property allows us to change from one base to another and is therefore called the change-of-base property.

## Property 4 (Change of Base)

If $a$ and $b$ are both positive numbers other than 1 , and if $x>0$, then

| $\log _{a} x=$ | $\frac{\log _{b} x}{\log _{b} a}$ |
| :---: | :---: |
| $\uparrow$ | $\uparrow$ |
| Base $a$ | Base $b$ |

The logarithm on the left side has a base of $a$, and both logarithms on the right side have a base of $b$. This allows us to change from base $a$ to any other base $b$ that is a positive number other than 1 . Here is a proof of property 4 for logarithms.
Proof We begin by writing the identity

$$
a^{\log _{a} x}=x
$$

Taking the logarithm base $b$ of both sides and writing the exponent $\log _{a} x$ as a coefficient, we have

$$
\begin{aligned}
\log _{b} a^{\log _{a} x} & =\log _{b} x \\
\log _{a} x \log _{b} a & =\log _{b} x
\end{aligned}
$$

Dividing both sides by $\log _{b} a$, we have the desired result:

$$
\begin{aligned}
\frac{\log _{a} x \log _{b} a}{\log _{b} a} & =\frac{\log _{b} x}{\log _{b} a} \\
\log _{a} x & =\frac{\log _{b} x}{\log _{b} a}
\end{aligned}
$$

We can use this property to find logarithms we could not otherwise compute on our calculators-that is, logarithms with bases other than 10 or $e$. The next example illustrates the use of this property.

## EXAMPLE 3 Find $\log _{8} 24$.

SOLUTION Since we do not have base-8 logarithms on our calculators, we can change this expression to an equivalent expression that contains only base10 logarithms:

$$
\log _{8} 24=\frac{\log 24}{\log 8} \quad \text { Property } 4
$$

Don't be confused. We did not just drop the base-we changed to base 10. We could have written the last line like this:

$$
\log _{8} 24=\frac{\log _{10} 24}{\log _{10} 8}
$$

From our calculators, we write

$$
\begin{aligned}
\log _{8} 24 & \approx \frac{1.3802}{0.9031} \\
& \approx 1.5283
\end{aligned}
$$

## Application

EXAMPLE 4 Suppose that the population in a small city is 32,000 in the beginning of 1994 and that the city council assumes that the population size $t$ years later can be estimated by the equation

$$
P=32,000 e^{0.05 t}
$$

Approximately when will the city have a population of 50,000 ?
SOLUTION We substitute 50,000 for $P$ in the equation and solve for $t$ :

$$
\begin{array}{ll}
50,000=32,000 e^{0.05 t} \\
1.5625 & =e^{0.05 t}
\end{array} \quad \frac{\mathbf{5 0 , 0 0 0}}{\mathbf{3 2 , 0 0 0}}=1.5625
$$

To solve this equation for $t$, we can take the natural logarithm of each side:

$$
\begin{aligned}
\ln 1.5625 & =\ln e^{0.05 t} & & \\
& =0.05 t \ln e & & \text { Property } \mathbf{3} \text { for logarithms } \\
& =0.05 t & & \text { Because } \ln e=1 \\
t & =\frac{\ln 1.5625}{0.05} & & \text { Divide each side by } 0.05 \\
& \approx 8.93 \text { years } & &
\end{aligned}
$$

We can estimate that the population will reach 50,000 toward the end of 2002.

## USING TECHNOLOGY

## Solving Exponential Equations

We can solve the equation $50,000=32,000 e^{0.05 t}$ from Example 4 with a graphing calculator by defining the expression on each side of the equation as a function. The solution to the equation will be the $x$-value of the point where the two graphs intersect.

First define functions $Y_{1}=50,000$ and $Y_{2}=32,000 e^{(0.05 x)}$ as shown in Figure 1. Set your window variables so that

$$
0 \leq x \leq 15 ; 0 \leq y \leq 70,000, \text { scale }=10,000
$$

Graph both functions, then use the appropriate command on your calculator to find the coordinates of the intersection point. From Figure 2 we see that the $x$-coordinate of this point is $x \approx 8.93$.


FIGURE 1


FIGURE 2

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1
B 3
C 2,4

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is an exponential equation?
2. How do logarithms help you solve exponential equations?
3. What is the change-of-base property?
4. Write an application modeled by the equation
$A=10,000\left(1+\frac{0.08}{2}\right)^{2 \cdot 5}$.

## Problem Set 11.6

Solve each exponential equation. Use a calculator to write the answer to four decimal places.

1. $3^{x}=5$
2. $4^{x}=3$
3. $5^{x}=3$
4. $3^{x}=4$
5. $5^{-x}=12$
6. $7^{-x}=8$
7. $12^{-x}=5$
8. $8^{-x}=7$
9. $8^{x+1}=4$
10. $9^{x+1}=3$
11. $4^{x-1}=4$
12. $3^{x-1}=9$
13. $3^{2 x+1}=2$
14. $2^{2 x+1}=3$
15. $3^{1-2 x}=2$
16. $2^{1-2 x}=3$
17. $15^{3 x-4}=10$
18. $10^{3 x-4}=15$
19. $6^{5-2 x}=4$
20. $9^{7-3 x}=5$

Use the change-of-base property and a calculator to find a decimal approximation to each of the following logarithms.
21. $\log _{8} 16$
22. $\log _{9} 27$
23. $\log _{16} 8$
24. $\log _{27} 9$
25. $\log _{7} 15$
26. $\log _{3} 12$
27. $\log _{15} 7$
28. $\log _{12} 3$
29. $\log _{8} 240$
30. $\log _{6} 180$
31. $\log _{4} 321$
32. $\log _{5} 462$

Find a decimal approximation to each of the following natural logarithms.
33. $\ln 345$
34. $\ln 3,450$
35. $\ln 0.345$
36. $\ln 0.0345$
37. $\ln 10$
38. $\ln 100$
39. $\ln 45,000$
40. $\ln 450,000$

## Applying the Concepts

41. Compound Interest How long will it take for $\$ 500$ to double if it is invested at $6 \%$ annual interest compounded twice a year?
42. Compound Interest How long will it take for $\$ 500$ to double if it is invested at $6 \%$ annual interest compounded 12 times a year?
43. Compound Interest How long will it take for $\$ 1,000$ to triple if it is invested at $12 \%$ annual interest compounded 6 times a year?
44. Compound Interest How long will it take for $\$ 1,000$ to become $\$ 4,000$ if it is invested at $12 \%$ annual interest compounded 6 times a year?
45. Doubling Time How long does it take for an amount of money $P$ to double itself if it is invested at $8 \%$ interest compounded 4 times a year?
46. Tripling Time How long does it take for an amount of money $P$ to triple itself if it is invested at $8 \%$ interest compounded 4 times a year?
47. Tripling Time If a $\$ 25$ investment is worth $\$ 75$ today, how long ago must that $\$ 25$ have been invested at $6 \%$ interest computed twice a year?
48. Doubling Time If a $\$ 25$ investment is worth $\$ 50$ today, how long ago must that $\$ 25$ have been invested at $6 \%$ interest computed twice a year?

Recall that if $P$ dollars are invested in an account with annual interest rate $r$, compounded continuously, then the amount of money in the account after $t$ years is given by the formula $A(t)=P e^{r t}$.
49. Continuously Compounded Interest Repeat Problem 41 if the interest is compounded continuously.
50. Continuously Compounded Interest Repeat Problem 44 if the interest is compounded continuously.
51. Continuously Compounded Interest How long will it take $\$ 500$ to triple if it is invested at $6 \%$ annual interest, compounded continuously?
52. Continuously Compounded Interest How long will it take $\$ 500$ to triple if it is invested at $12 \%$ annual interest, compounded continuously?

## Maintaining Your Skills

Find the vertex for each of the following parabolas, and then indicate if it is the highest or lowest point on the graph.
53. $y=2 x^{2}+8 x-15$
54. $y=3 x^{2}-9 x-10$
55. $y=12 x-4 x^{2}$
56. $y=18 x-6 x^{2}$
57. Maximum Height An object is projected into the air with an initial upward velocity of 64 feet per second. Its height $h$ at any time $t$ is given by the formula $h=64 t-16 t^{2}$. Find the time at which the object reaches its maximum height. Then, find the maximum height.
58. Maximum Height An object is projected into the air with an initial upward velocity of 64 feet per second from the top of a building 40 feet high. If the height $h$ of the object $t$ seconds after it is projected into the air is $h=40+64 t-16 t^{2}$, find the time at which the object reaches its maximum height. Then, find the maximum height it attains.

## Extending the Concepts

59. Exponential Growth Suppose that the population in a small city is 32,000 at the beginning of 1994 and that the city council assumes that the population size $t$ years later can be estimated by the equation

$$
P(t)=32,000 e^{0.05 t}
$$

Approximately when will the city have a population of 64,000 ?
60. Exponential Growth Suppose the population of a city is given by the equation

$$
P(t)=100,000 e^{0.05 t}
$$

where $t$ is the number of years from the present time. How large is the population now? (Now corresponds to a certain value of $t$. Once you realize what that value of $t$ is, the problem becomes very simple.)
61. Exponential Growth Suppose the population of a city is given by the equation

$$
P(t)=15,000 e^{0.04 t}
$$

where $t$ is the number of years from the present time. How long will it take for the population to reach 45,000 ?
62. Exponential Growth Suppose the population of a city is given by the equation

$$
P(t)=15,000 e^{0.08 t}
$$

where $t$ is the number of years from the present time. How long will it take for the population to reach 45,000 ?
63. Solve the formula $A=P e^{r t}$ for $t$.
64. Solve the formula $A=P e^{-r t}$ for $t$.
65. Solve the formula $A=P \cdot 2^{-k t}$ for $t$.
66. Solve the formula $A=P \cdot 2^{k t}$ for $t$.
67. Solve the formula $A=P(1-r)^{t}$ for $t$.
68. Solve the formula $A=P(1+r)^{t}$ for $t$.

## Chapter 11 sUMMARY

## Exponential Functions [11.1]

## EXAMPLES

1. For the exponential function $f(x)=2^{x}$,
$f(0)=2^{0}=1$
$f(1)=2^{1}=2$
$f(2)=2^{2}=4$
$f(3)=2^{3}=8$
2. The function $f(x)=x^{2}$ is not one-to-one because 9 , which is in the range, comes from both 3 and -3 in the domain.
3. The inverse of $f(x)=2 x-3$ is

$$
f^{-1}(x)=\frac{x+3}{2}
$$

4. The definition allows us to write expressions like

$$
y=\log _{3} 27
$$

equivalently in exponential form as

$$
3^{y}=27
$$

which makes it apparent that $y$ is 3 .

## One-to-One Functions [11.2]

A function is a one-to-one function if every element in the range comes from exactly one element in the domain.

## Inverse Functions [11.2]

The inverse of a function is obtained by reversing the order of the coordinates of the ordered pairs belonging to the function. Only one-to-one functions have inverses that are also functions.

## Definition of Logarithms [11.3]

If $b$ is a positive number not equal to 1 , then the expression

$$
y=\log _{b} x
$$

is equivalent to $x=b^{y}$; that is, in the expression $y=\log _{b} x, y$ is the number to which we raise $b$ in order to get $x$. Expressions written in the form $y=\log _{b} x$ are said to be in logarithmic form. Expressions like $x=b^{y}$ are in exponential form.

## Two Special Identities [11.3]

For $b>0, b \neq 1$, the following two expressions hold for all positive real numbers $x$ :
(1) $b^{\log _{b} x}=x$
(2) $\log _{b} b^{x}=x$
6. We can rewrite the expression

$$
\log _{10} \frac{45^{6}}{273}
$$

using the properties of logarithms, as
$6 \log _{10} 45-\log _{10} 273$
7. $\log _{10} 10,000=\log 10,000$

$$
=\log 10^{4}
$$

$$
=4
$$

## Properties of Logarithms [11.4]

If $x, y$, and $b$ are positive real numbers, $b \neq 1$, and $r$ is any real number, then:

1. $\log _{b}(x y)=\log _{b} x+\log _{b} y$
2. $\log _{b}\left(\frac{x}{y}\right)=\log _{b} x-\log _{b} y$

## Common Logarithms [11.5]

3. $\log _{b} x^{r}=r \log _{b} x$

Common logarithms are logarithms with a base of 10 . To save time in writing, we omit the base when working with common logarithms; that is,

$$
\log x=\log _{10} x
$$

## Natural Logarithms [11.5]

Natural logarithms, written $\ln x$, are logarithms with a base of $e$, where the number $e$ is an irrational number (like the number $\pi$ ). A decimal approximation for $e$ is 2.7183. All the properties of exponents and logarithms hold when the base is $e$.

## Change of Base [11.6]

If $x, a$, and $b$ are positive real numbers, $a \neq 1$ and $b \neq 1$, then

$$
\log _{a} x=\frac{\log _{b} x}{\log _{b} a}
$$

## COMMON MISTAKES

The most common mistakes that occur with logarithms come from trying to apply the three properties of logarithms to situations in which they don't apply. For example, a very common mistake looks like this:

$$
\frac{\log 3}{\log 2}=\log 3-\log 2 \quad \text { Mistake }
$$

This is not a property of logarithms. To write the equation $\log 3-\log 2$, we would have to start with

$$
\log \frac{3}{2} \quad \text { NOT } \quad \frac{\log 3}{\log 2}
$$

There is a difference.

## Chapter 11 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

Let $f(x)=2^{x}$ and $g(x)=\left(\frac{1}{3}\right)^{x}$, and find the following. [11.1]

1. $f(4)$
2. $f(-1)$
3. $g(2)$
4. $f(2)-g(-2)$
5. $f(-1)+g(1)$
6. $g(-1)+f(2)$
7. The graph of $y=f(x)$
8. The graph of $y=g(x)$

For each relation that follows, sketch the graph of the relation and its inverse, and write an equation for the inverse. [11.2]
9. $y=2 x+1$
10. $y=x^{2}-4$

For each of the following functions, find the equation of the inverse. Write the inverse using the notation $f^{-1}(x)$ if the inverse is itself a function. [11.2]
11. $f(x)=2 x+3$
12. $f(x)=x^{2}-1$
13. $f(x)=\frac{1}{2} x+2$
14. $f(x)=4-2 x^{2}$

Write each equation in logarithmic form. [11.3]
15. $3^{4}=81$
16. $7^{2}=49$
17. $0.01=10^{-2}$
18. $2^{-3}=\frac{1}{8}$

Write each equation in exponential form. [11.3]
19. $\log _{2} 8=3$
20. $\log _{3} 9=2$
21. $\log _{4} 2=\frac{1}{2}$
22. $\log _{4} 4=1$

Solve for $x$. [11.3]
23. $\log _{5} x=2$
24. $\log _{16} 8=x$
25. $\log _{x} 0.01=-2$

Graph each equation. [11.3]
26. $y=\log _{2} x$
27. $y=\log _{1 / 2} x$

Simplify each expression. [11.3]
28. $\log _{4} 16$
29. $\log _{27} 9$
30. $\log _{4}\left(\log _{3} 3\right)$

Use the properties of logarithms to expand each expression. [11.4]
31. $\log _{2} 5 x$
32. $\log _{10} \frac{2 x}{y}$
33. $\log _{a} \frac{y^{3} \sqrt{x}}{z}$
34. $\log _{10} \frac{x^{2}}{y^{3} Z^{4}}$

Write each expression as a single logarithm. [11.4]
35. $\log _{2} x+\log _{2} y$
36. $\log _{3} x-\log _{3} 4$
37. $2 \log _{a} 5-\frac{1}{2} \log _{a} 9$
38. $3 \log _{2} x+2 \log _{2} y-4 \log _{2} z$

Solve each equation. [11.4]
39. $\log _{2} x+\log _{2} 4=3$
40. $\log _{2} x-\log _{2} 3=1$
41. $\log _{3} x+\log _{3}(x-2)=1$
42. $\log _{4}(x+1)-\log _{4}(x-2)=1$
43. $\log _{6}(x-1)+\log _{6} x=1 \quad$ 44. $\log _{4}(x-3)+\log _{4} x=1$

Evaluate each expression. [11.5]
45. $\log 346$
46. $\log 0.713$

Find $x$. [11.5]
47. $\log x=3.9652$
48. $\log x=-1.6003$

Simplify. [11.5]
49. $\ln e$
50. $\ln 1$
51. $\ln e^{2}$
52. $\ln e^{-4}$

Use the formula $\mathrm{pH}=-\log \left[\mathrm{H}^{+}\right]$to find the pH of a solution with the given hydrogen ion concentration. [11.5]
53. $\left[\mathrm{H}^{+}\right]=7.9 \times 10^{-3}$
54. $\left[\mathrm{H}^{+}\right]=8.1 \times 10^{-6}$

Find $\left[\mathrm{H}^{+}\right]$for a solution when the given pH [11.5]
55. $\mathrm{pH}=2.7$
56. $\mathrm{pH}=7.5$

Solve each equation. [11.6]
57. $4^{x}=8$
58. $4^{3 x+2}=5$

Use the change-of-base property and a calculator to evaluate each expression. Round your answers to the nearest hundredth. [11.6]
59. $\log _{16} 8$
60. $\log _{12} 421$

Use the formula $A=P\left(1+\frac{r}{n}\right)^{n t}$ to solve each of the following problems. [11.6]
61. Investing How long does it take $\$ 5,000$ to double if it is deposited in an account that pays $16 \%$ annual interest compounded once a year?
62. Investing How long does it take $\$ 10,000$ to triple if it is deposited in an account that pays $12 \%$ annual interest compounded 6 times a year?

## Chapter 11 Projects

## Exponential and Logarithmic Functions

## GROUP PROJECT Two Depreciation Models

## Number of People 3

Time Needed 20 minutes
Equipment Paper, pencil, and graphing calculator
Background Recently, a consumer magazine contained an article on leasing a computer. The original price of the computer was $\$ 2,500$. The term of the lease was 24 months. At the end of the lease, the computer could be purchased for its residual value, $\$ 188$. This is enough information to find an equation that will give us the value of the computer $t$ months after it has been purchased.

| NEW Super Fast Gaming PC |
| :---: |
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Procedure We will find models for two types of depreciation: linear depreciation and exponential depreciation. Here are the general equations:

Linear Depreciation
Exponential Depreciation

$$
V=m t+b \quad V=V_{0} e^{-k t}
$$

1. Let $t$ represent time in months and $V$ represent the value of the computer at time $t$; then find two ordered pairs $(t, V)$ that correspond to the initial price of the computer and another ordered pair that corresponds to the residual value of $\$ 188$ when the computer is 2 years old.
2. Use the two ordered pairs to find $m$ and $b$ in the linear depreciation model; then write the equation that gives us linear depreciation.
3. Use the two ordered pairs to find $V_{0}$ and $k$ in the exponential depreciation model; then write the equation that gives us exponential depreciation.
4. Graph each of the equations on your graphing calculator; then sketch the graphs on the following templates.
5. Find the value of the computer after 1 year, using both models.
6. Which of the two models do you think best describes the depreciation of a computer?


FIGURE 2


T.he movie Heart Like a Wheel is based on the racing career of drag racer Shirley Muldowney. The movie includes a number of races. Choose four races as the basis for your report. For each race you choose, give a written description of the events leading to the race, along with the details of the race itself. Then draw a graph that shows the speed of the dragster as a function of time. Do this for each race. One such graph from earlier in the chapter is shown here as an example. (For each graph, label the horizontal axis from 0 to 12 seconds, and label the vertical axis from 0 to 200 miles per hour.) Follow each graph with a description of any significant events that happen during the race (such as a motor malfunction or a crash) and their correlation to the graph.

| Elapsed Time <br> (sec) | Speed <br> $(\mathbf{m p h})$ |
| :---: | ---: |
| 0 | 0.0 |
| 1 | 72.7 |
| 2 | 129.9 |
| 3 | 162.8 |
| 4 | 192.2 |
| 5 | 212.4 |
| 6 | 228.1 |



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## Conic Sections

## 12



## CHAPTER OUTLINE

### 12.1 The Circle

12.2 Ellipses and Hyperbolas
12.3 Second-Degree Inequalities and Nonlinear Systems

0ne of the curves we will study in this chapter has interesting reflective properties. Figure 1 (a) shows how you can draw one of these curves (an ellipse) using thumbtacks, string, pencil, and paper. Elliptical surfaces will reflect sound waves that originate at one focus through the other focus. This property of ellipses allows doctors to treat patients with kidney stones using a procedure called lithotripsy. A lithotripter is an elliptical device that creates sound waves that crush the kidney stone into small pieces, without surgery. The sound wave originates at one focus of the lithotripter. The energy from it reflects off the surface of the lithotripter and converges at the other focus, where the kidney stone is positioned. Figure 1 (b) shows a cross-section of a lithotripter, with a patient positioned so the kidney stone is at the other focus.


FIGURE 1

By studying the conic sections in this chapter, you will be better equipped to understand some of the more technical equipment that exists in the world outside of class.

## The Circle

## OBJECTIVES

A Use the distance formula.
B Write the equation of a circle, given its center and radius.

C Find the center and radius of a circle from its equation, and then sketch the graph.

Conic sections include ellipses, circles, hyperbolas, and parabolas. They are called conic sections because each can be found by slicing a cone with a plane as shown in Figure 1. We begin our work with conic sections by studying circles. Before we find the general equation of a circle, we must first derive what is known as the distance formula.


FIGURE 1
Suppose $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ are any two points in the first quadrant. (Actually, we could choose the two points to be anywhere on the coordinate plane. It is just more convenient to have them in the first quadrant.) We can name the points $P_{1}$ and $P_{2}$, respectively, and draw the diagram shown in Figure 2.


FIGURE 2

Notice the coordinates of point $Q$. The $x$-coordinate is $x_{2}$ since $Q$ is directly below point $P_{2}$. The $y$-coordinate of $Q$ is $y_{1}$ since $Q$ is directly across from point $P_{1}$.

It is evident from the diagram that the length of $P_{2} Q$ is $y_{2}-y_{1}$ and the length of $P_{1} Q$ is $x_{2}-x_{1}$. Using the Pythagorean theorem, we have

$$
\left(P_{1} P_{2}\right)^{2}=\left(P_{1} Q\right)^{2}+\left(P_{2} Q\right)^{2}
$$

or

$$
d^{2}=\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}
$$

Taking the square root of both sides, we have

$$
d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}
$$

We know this is the positive square root since $d$ is the distance from $P_{1}$ to $P_{2}$ and therefore must be positive. This formula is called the distance formula.

## EXAMPLE 1

Find the distance between $(3,5)$ and $(2,-1)$.
SOLUTION If we let $(3,5)$ be $\left(x_{1}, y_{1}\right)$ and $(2,-1)$ be $\left(x_{2}, y_{2}\right)$ and apply the distance formula, we have

$$
\begin{aligned}
d & =\sqrt{(2-3)^{2}+(-1-5)^{2}} \\
& =\sqrt{(-1)^{2}+(-6)^{2}} \\
& =\sqrt{1+36} \\
& =\sqrt{37}
\end{aligned}
$$

EXAMPLE 2 Find $x$ if the distance from $(x, 5)$ to $(3,4)$ is $\sqrt{2}$.
SOLUTION Using the distance formula, we have

$$
\begin{aligned}
\sqrt{2} & =\sqrt{(x-3)^{2}+(5-4)^{2}} \\
2 & =(x-3)^{2}+1^{2} \\
2 & =x^{2}-6 x+9+1 \\
0 & =x^{2}-6 x+8 \\
0 & =(x-4)(x-2) \\
x & =4 \text { or } x=2
\end{aligned}
$$

The two solutions are 4 and 2 , which indicates there are two points, $(4,5)$ and $(2,5)$, which are $\sqrt{2}$ units from $(3,4)$.

## Circles

Because of their perfect symmetry, circles have been used for thousands of years in many disciplines, including art, science, and religion. The photograph shown in Figure 3 is of Stonehenge, a 4,500-year-old site in England. The arrangement of the stones is based on a circular plan that is thought to have both religious and astronomical significance. More recently, the design shown in Figure 4 began appearing in agricultural fields in England in the 1990s. Whoever made these designs chose the circle as their basic shape.


FIGURE 3


FIGURE 4

We can model circles very easily in algebra by using equations that are based on the distance formula.

Circle Theorem The equation of the circle with center at $(a, b)$ and radius $r$ is given by

$$
(x-a)^{2}+(y-b)^{2}=r^{2}
$$

Proof By definition, all points on the circle are a distance $r$ from the center $(a, b)$. If we let $(x, y)$ represent any point on the circle, then $(x, y)$ is $r$ units from $(a, b)$. Applying the distance formula, we have

$$
r=\sqrt{(x-a)^{2}+(y-b)^{2}}
$$

Squaring both sides of this equation gives the equation of the circle:

$$
(x-a)^{2}+(y-b)^{2}=r^{2}
$$

We can use the circle theorem to find the equation of a circle, given its center and radius, or to find its center and radius, given the equation.

## EXAMPLE 3 Find the equation of the circle with center at $(-3,2)$ hav-

ing a radius of 5 .
SOLUTION We have $(a, b)=(-3,2)$ and $r=5$. Applying the circle theorem yields

$$
\begin{aligned}
{[x-(-3)]^{2}+(y-2)^{2} } & =5^{2} \\
(x+3)^{2}+(y-2)^{2} & =25
\end{aligned}
$$

## EXAMPLE 4

Give the equation of the circle with radius 3 whose cen-
ter is at the origin.
SOLUTION The coordinates of the center are ( 0,0 ), and the radius is 3 . The equation must be

$$
\begin{aligned}
(x-0)^{2}+(y-0)^{2} & =3^{2} \\
x^{2}+y^{2} & =9
\end{aligned}
$$

We can see from Example 4 that the equation of any circle with its center at the origin and radius $r$ will be

$$
x^{2}+y^{2}=r^{2}
$$

EXAMPLE 5 Find the center and radius, and sketch the graph, of the circle whose equation is

$$
(x-1)^{2}+(y+3)^{2}=4
$$

SOLUTION Writing the equation in the form

$$
(x-a)^{2}+(y-b)^{2}=r^{2}
$$

we have

$$
(x-1)^{2}+[y-(-3)]^{2}=2^{2}
$$

The center is at $(1,-3)$ and the radius is 2 . The graph is shown in Figure 5.


FIGURE 5

EXAMPLE $6 \quad$ Sketch the graph of $x^{2}+y^{2}=9$.
SOLUTION Since the equation can be written in the form

$$
(x-0)^{2}+(y-0)^{2}=3^{2}
$$

it must have its center at $(0,0)$ and a radius of 3 . The graph is shown in Figure 6.


FIGURE 6

## EXAMPLE $7 \quad$ Sketch the graph of $x^{2}+y^{2}+6 x-4 y-12=0$.

SOLUTION To sketch the graph, we must find the center and radius. The center and radius can be identified if the equation has the form

$$
(x-a)^{2}+(y-b)^{2}=r^{2}
$$

The original equation can be written in this form by completing the squares on $x$ and $y$ :

$$
\begin{aligned}
x^{2}+y^{2}+6 x-4 y-12 & =0 \\
x^{2}+6 x+y^{2}-4 y & =12 \\
x^{2}+6 x+\mathbf{9}+y^{2}-4 y+\mathbf{4} & =12+\mathbf{9}+\mathbf{4} \\
(x+3)^{2}+(y-2)^{2} & =25 \\
(x+3)^{2}+(y-2)^{2} & =5^{2}
\end{aligned}
$$

From the last line it is apparent that the center is at $(-3,2)$ and the radius is 5 . The graph is shown in Figure 7.


FIGURE 7

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1,2

B 3,4
C ${ }_{5-7}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Describe the distance formula in words, as if you were explaining to someone how they should go about finding the distance between two points.
2. What is the mathematical definition of a circle?
3. How are the distance formula and the equation of a circle related?
4. When graphing a circle from its equation, why is completing the square sometimes useful?

Find the distance between the following points.

1. $(3,7)$ and $(6,3)$
2. $(4,7)$ and $(8,1)$
3. $(0,9)$ and $(5,0)$
4. $(-3,0)$ and $(0,4)$
5. $(3,-5)$ and $(-2,1)$
6. $(-8,9)$ and $(-3,-2)$
7. $(-1,-2)$ and $(-10,5)$
8. $(-3,-8)$ and $(-1,6)$
9. Find $x$ so the distance between $(x, 2)$ and $(1,5)$ is $\sqrt{13}$.
10. Find $x$ so the distance between $(-2,3)$ and $(x, 1)$ is 3 .
11. Find $y$ so the distance between $(7, y)$ and $(8,3)$ is 1 .
12. Find $y$ so the distance between $(3,-5)$ and $(3, y)$ is 9 .

Write the equation of the circle with the given center and radius.
13. Center $(2,3) ; r=4$
14. Center $(3,-1) ; r=5$
15. Center $(3,-2) ; r=3$
16. Center $(-2,4) ; r=1$
17. Center $(-5,-1) ; r=\sqrt{5}$
18. Center $(-7,-6) ; r=\sqrt{3}$
19. Center $(0,-5) ; r=1$
20. Center $(0,-1) ; r=7$
21. Center $(0,0) ; r=2$
22. Center $(0,0) ; r=5$

Give the center and radius, and sketch the graph of each of the following circles.
23. $x^{2}+y^{2}=4$
24. $x^{2}+y^{2}=16$
25. $(x-1)^{2}+(y-3)^{2}=25$
26. $(x-4)^{2}+(y-1)^{2}=36$
27. $(x+2)^{2}+(y-4)^{2}=8$
28. $(x-3)^{2}+(y+1)^{2}=12$
29. $(x+1)^{2}+(y+1)^{2}=1$
30. $(x+3)^{2}+(y+2)^{2}=9$
31. $x^{2}+y^{2}-6 y=7$
32. $x^{2}+y^{2}+10 x=0$
33. $x^{2}+y^{2}-4 x-6 y=-4$
34. $x^{2}+y^{2}-4 x+2 y=4$
35. $x^{2}+y^{2}+2 x+y=\frac{11}{4}$
36. $x^{2}+y^{2}-6 x-y=-\frac{1}{4}$

Both of the following circles pass through the origin. In each case, find the equation.
37.

38.

39. Find the equations of circles $A, B$, and $C$ in the following diagram. The three points are the centers of the three circles.

40. Each of the following circles passes through the origin. The centers are as shown. Find the equation of each circle.

41. Find the equation of each of the three circles shown here.

42. A parabola and a circle each contain the points $(-4,0),(0,4)$, and $(4,0)$. Sketch the graph of each curve on the same coordinate system, then write an equation for each curve.
43. Ferris Wheel A giant Ferris wheel has a diameter of 240 feet and sits 12 feet above the ground. As shown in the diagram below, the wheel is 500 feet from the entrance to the park. The $x y$-coordinate system containing the wheel has its origin on the ground at the center of the entrance. Write an equation that models the shape of the wheel.

44. Magic Rings A magician is holding two rings that seem to lie in the same plane and intersect in two points. Each ring is 10 inches in diameter.
a. Find the equation of each ring if a coordinate system is placed with its origin at the center
 of the first ring and the $x$-axis contains the center of the second ring.
b. Find the equation of each ring if a coordinate system is placed with its origin at the center of the second ring and the $x$-axis contains the center of the first ring.

## Maintaining Your Skills

Find the equation of the inverse of each of the following functions. Write the inverse using the notation $f^{-1}(x)$, if the inverse is itself a function.
45. $f(x)=3^{x}$
46. $f(x)=5^{x}$
47. $f(x)=2 x+3$
48. $f(x)=3 x-2$
49. $f(x)=\frac{x+3}{5}$
50. $f(x)=\frac{x-2}{6}$

## Getting Ready for the Next Section

Solve.
51. $y^{2}=9$
52. $x^{2}=25$
53. $-y^{2}=4$
54. $-x^{2}=16$
55. $\frac{-x^{2}}{9}=1$
56. $\frac{y^{2}}{100}=1$
57. Divide $4 x^{2}+9 y^{2}$ by 36 .
58. Divide $25 x^{2}+4 y^{2}$ by 100 .

Find the $x$-intercepts and the $y$-intercepts.
59. $3 x-4 y=12$
60. $y=3 x^{2}+5 x-2$
61. If $\frac{x^{2}}{25}+\frac{y^{2}}{9}=1$, find $y$ when $x$ is 3 .
62. If $\frac{x^{2}}{25}+\frac{y^{2}}{9}=1$, find $y$ when $x$ is -4 .

## Extending the Concepts

A circle is tangent to a line if it touches, but does not cross, the line.
63. Find the equation of the circle with center at $(2,3)$ if the circle is tangent to the $y$-axis.
64. Find the equation of the circle with center at $(3,2)$ if the circle is tangent to the $x$-axis.
65. Find the equation of the circle with center at $(2,3)$ if the circle is tangent to the vertical line $x=4$.
66. Find the equation of the circle with center at $(3,2)$ if the circle is tangent to the horizontal line $y=6$.

Find the distance from the origin to the center of each of the following circles.
67. $x^{2}+y^{2}-6 x+8 y=144$
68. $x^{2}+y^{2}-8 x+6 y=144$
69. $x^{2}+y^{2}-6 x-8 y=144$
70. $x^{2}+y^{2}+8 x+6 y=144$
71. If we were to solve the equation $x^{2}+y^{2}=9$ for $y$, we would obtain the equation $y= \pm \sqrt{9-x^{2}}$. This last equation is equivalent to the two equations $y=\sqrt{9-x^{2}}$, in which $y$ is always positive, and $y=-\sqrt{9-x^{2}}$, in which $y$ is always negative. Look at the graph of $x^{2}+y^{2}=9$ in Example 6 of this section and indicate what part of the graph each of the two equations corresponds to.
72. Solve the equation $x^{2}+y^{2}=9$ for $x$, and then indicate what part of the graph in Example 6 each of the two resulting equations corresponds to.

## Ellipses and Hyperbolas

The photograph in Figure 1 shows Halley's comet as it passed close to earth in 1986. Like the planets in our solar system, it orbits the sun in an elliptical path (Figure 2). While it takes the earth 1 year to complete one orbit around the sun, it takes Halley's comet 76 years. The first known sighting of Halley's comet was in 239 b.c. Its most famous appearance occurred in 1066 A.D., when it was seen at the Battle of Hastings.


FIGURE 1


The Orbit of Halley's Comet
FIGURE 2

We begin this section with an introductory look at equations that produce ellipses. To simplify our work, we consider only ellipses that are centered about the origin.

Suppose we want to graph the equation

$$
\frac{x^{2}}{25}+\frac{y^{2}}{9}=1
$$

We can find the $y$-intercepts by letting $x=0$, and we can find the $x$-intercepts by letting $y=0$ :

$$
\begin{array}{rlrl}
\text { When } x & =0 & \text { When } y & =0 \\
\frac{0^{2}}{25}+\frac{y^{2}}{9} & =1 & \frac{x^{2}}{25}+\frac{0^{2}}{9} & =1 \\
y^{2} & =9 & x^{2} & =25 \\
y & = \pm 3 & x & = \pm 5
\end{array}
$$

The graph crosses the $y$-axis at $(0,3)$ and $(0,-3)$ and the $x$-axis at $(5,0)$ and $(-5,0)$. Graphing these points and then connecting them with a smooth curve gives the graph shown in Figure 3.


FIGURE 3

A graph of the type shown in Figure 3 is called an ellipse. If we were to find some other ordered pairs that satisfy our original equation, we would find that their graphs lie on the ellipse. Also, the coordinates of any point on the ellipse will satisfy the equation. We can generalize these results as follows.

The Ellipse The graph of any equation of the form

$$
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1 \quad \text { Standard form }
$$

will be an ellipse centered at the origin. The ellipse will cross the $x$-axis at $(a, 0)$ and $(-a, 0)$. It will cross the $y$-axis at $(0, b)$ and $(0,-b)$. When $a$ and $b$ are equal, the ellipse will be a circle.

The most convenient way to graph an ellipse is to locate the intercepts.

## Note

When the equation is written in standard form, the $x$-intercepts are the positive and negative square roots of the number below $X^{2}$. The $y$-intercepts are the square roots of the number below $y^{2}$.

EXAMPLE 1 Sketch the graph of $4 x^{2}+9 y^{2}=36$.
SOLUTION To write the equation in the form

$$
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1
$$

we must divide both sides by 36 :

$$
\begin{aligned}
& \frac{4 x^{2}}{36}+\frac{9 y^{2}}{36}=\frac{36}{36} \\
& \frac{x^{2}}{9}+\frac{y^{2}}{4}=1
\end{aligned}
$$

The graph crosses the $x$-axis at $(3,0),(-3,0)$ and the $y$-axis at $(0,2),(0,-2)$, as shown in Figure 4.


FIGURE 4

## Hyperbolas

Figure 5 shows Europa, one of Jupiter's moons, as it was photographed by the Galileo space probe in the late 1990s. To speed up the trip from Earth to Jupiter-nearly a billion miles-Galileo made use of the slingshot effect. This
involves flying a hyperbolic path very close to a planet, so that gravity can be used to gain velocity as the space probe hooks around the planet (Figure 6).


FIGURE 5


FIGURE 6

We use the rest of this section to consider equations that produce hyperbolas. Consider the equation

$$
\frac{x^{2}}{9}-\frac{y^{2}}{4}=1
$$

If we were to find a number of ordered pairs that are solutions to the equation and connect their graphs with a smooth curve, we would have Figure 7.


FIGURE 7

This graph is an example of a hyperbola. Notice that the graph has $x$-intercepts at $(3,0)$ and $(-3,0)$. The graph has no $y$-intercepts and hence does not cross the $y$-axis since substituting $x=0$ into the equation yields

$$
\begin{aligned}
\frac{0^{2}}{9}-\frac{y^{2}}{4} & =1 \\
-y^{2} & =4 \\
y^{2} & =-4
\end{aligned}
$$

for which there is no real solution. We can, however, use the number below $y^{2}$ to help sketch the graph. If we draw a rectangle that has its sides parallel to the $x$ - and $y$-axes and that passes through the $x$-intercepts and the points on the $y$-axis corresponding to the square roots of the number below $y^{2},+2$ and -2 , it
looks like the rectangle in Figure 8. The lines that connect opposite corners of the rectangle are called asymptotes. The graph of the hyperbola

$$
\frac{x^{2}}{9}-\frac{y^{2}}{4}=1
$$

will approach these lines. Figure 8 is the graph.


FIGURE 8

EXAMPLE $2 \quad$ Graph the equation $\frac{y^{2}}{9}-\frac{x^{2}}{16}=1$.


FIGURE 9

SOLUTION In this case the $y$-intercepts are 3 and -3 , and the $x$-intercepts do not exist. We can use the square roots of the number below $x^{2}$, however, to find the asymptotes associated with the graph. The sides of the rectangle used to draw the asymptotes must pass through 3 and -3 on the $y$-axis and 4 and -4 on the $x$-axis. (See Figure 9.)

Here is a summary of what we have for hyperbolas.

## The Hyperbola

The graph of the equation

$$
\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1
$$

will be a hyperbola centered at the origin. The graph will have $x$-intercepts at $-a$ and $a$.

The graph of the equation

$$
\frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1
$$

will be a hyperbola centered at the origin. The graph will have $y$-intercepts at $-a$ and $a$.

As an aid in sketching either of the preceding equations, the asymptotes can be found by drawing lines through opposite corners of the rectangle whose sides pass through $-a, a,-b$, and $b$ on the axes.

## Ellipses and Hyperbolas Not Centered at the Origin

The following equation is that of an ellipse with its center at the point (4, 1):

$$
\frac{(x-4)^{2}}{9}+\frac{(y-1)^{2}}{4}=1
$$

To see why the center is at $(4,1)$ we substitute $x^{\prime}$ (read " $x$ prime") for $x-4$ and $y^{\prime}$ for $y-1$ in the equation. That is,


This is the equation of an ellipse in a coordinate system with an $x^{\prime}$-axis and a $y^{\prime}$-axis. We call this new coordinate system the $x^{\prime} y^{\prime}$-coordinate system. The center of our ellipse is at the origin in the $x^{\prime} y^{\prime}$-coordinate system. The question is this: What are the coordinates of the center of this ellipse in the original $x y$-coordinate system? To answer this question, we go back to our original substitutions:

$$
\begin{aligned}
& x^{\prime}=x-4 \\
& y^{\prime}=y-1
\end{aligned}
$$

In the $x^{\prime} y^{\prime}$-coordinate system, the center of our ellipse is at $x^{\prime}=0, y^{\prime}=0$ (the origin of the $x^{\prime} y^{\prime}$ system). Substituting these numbers for $x^{\prime}$ and $y^{\prime}$, we have

$$
\begin{aligned}
& 0=x-4 \\
& 0=y-1
\end{aligned}
$$

Solving these equations for $x$ and $y$ will give us the coordinates of the center of our ellipse in the $x y$-coordinate system. As you can see, the solutions are $x=4$ and $y=1$ Therefore, in the $x y$-coordinate system, the center of our ellipse is at the point $(4,1)$. Figure 10 shows the graph.


FIGURE 10
The coordinates of all points labeled in Figure 10 are given with respect to the $x y$-coordinate system. The $x^{\prime}$ and $y^{\prime}$ are shown simply for reference in our discussion. Note that the horizontal distance from the center to the vertices is 3the square root of the denominator of the $(x-4)^{2}$ term. Likewise, the vertical distance from the center to the other vertices is 2 -the square root of the denominator of the $(y-1)^{2}$ term.

We summarize the information above with the following:

An Ellipse with Center at $(h, k)$ The graph of the equation

$$
\frac{(x-h)^{2}}{a^{2}}+\frac{(y-k)^{2}}{b^{2}}=1
$$

will be an ellipse with center at $(h, k)$. The vertices of the ellipse will be at the points, $(h+a, k),(h-a, k),(h, k+b)$, and $(h, k-b)$.

EXAMPLE $3 \quad$ Graph the ellipse: $x^{2}+9 y^{2}+4 x-54 y+76=0$.
SOLUTION To identify the coordinates of the center, we must complete the square on $x$ and also on $y$. To begin, we rearrange the terms so that those containing $x$ are together, those containing $y$ are together, and the constant term is on the other side of the equal sign. Doing so gives us the following equation:

$$
x^{2}+4 x+9 y^{2}-54 y=-76
$$

Before we can complete the square on $y$, we must factor 9 from each term containing $y$ :

$$
x^{2}+4 x \quad+9\left(y^{2}-6 y \quad\right)=-76
$$

To complete the square on $x$, we add 4 to each side of the equation. To complete the square on $y$, we add 9 inside the parentheses. This increases the left side of the equation by 81 since each term within the parentheses is multiplied by 9. Therefore, we must add 81 to the right side of the equation also.

$$
\begin{aligned}
x^{2}+4 x+\mathbf{4}+9\left(y^{2}-6 y+9\right) & =-76+\mathbf{4}+\mathbf{8 1} \\
(x+2)^{2}+9(y-3)^{2} & =9
\end{aligned}
$$

To identify the distances to the vertices, we divide each term on both sides by 9 :

$$
\begin{aligned}
& \frac{(x+2)^{2}}{9}+\frac{9(y-3)^{2}}{9}=\frac{9}{9} \\
& \frac{(x+2)^{2}}{9}+\frac{(y-3)^{2}}{1}=1
\end{aligned}
$$

The graph is an ellipse with center at $(-2,3)$, as shown in Figure 11.


FIGURE 11

The ideas associated with graphing hyperbolas whose centers are not at the origin parallel the ideas just presented about graphing ellipses whose centers have been moved off the origin. Without showing the justification for doing so, we state the following guidelines for graphing hyperbolas:

Hyperbolas with Centers at (h,k) The graphs of the equations

$$
\frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1 \text { and } \frac{(y-k)^{2}}{b^{2}}-\frac{(x-h)^{2}}{a^{2}}=1
$$

will be hyperbolas with their centers at $(h, k)$. The vertices of the graph of the first equation will be at the points $(h+a, k)$, and $(h-a, k)$, and the vertices for the graph of the second equation will be at $(h, k+b),(h, k-b)$. In either case, the asymptotes can be found by connecting opposite corners of the rectangle that contains the four points $(h+a, k),(h-a, k),(h, k+b)$, and $(h, k-b)$.


FIGURE 12

EXAMPLE $4 \quad$ Graph the hyperbola: $4 x^{2}-y^{2}+4 y-20=0$
SOLUTION To identify the coordinates of the center of the hyperbola, we need to complete the square on $y$. (Because there is no linear term in $x$, we do not need to complete the square on $x$. The $x$-coordinate of the center will be $x=0$.)

$$
\begin{array}{r}
4 x^{2}-y^{2}+4 y-20=0 \\
4 x^{2}-y^{2}+4 y=20 \\
4 x^{2}-1\left(y^{2}-4 y\right)=20
\end{array}
$$

Add 20 to each side
Factor - $\mathbf{1}$ from each term containing $y$
To complete the square on $y$, we add 4 to the terms inside the parentheses. Doing so adds -4 to the left side of the equation because everything inside the parentheses is multiplied by -1 . To keep from changing the equation we must add -4 to the right side also.

$$
\begin{aligned}
4 x^{2}-1\left(y^{2}-4 y+4\right) & =20-4 & & \text { Add }-4 \text { to each side } \\
4 x^{2}-1(y-2)^{2} & =16 & & y^{2}-4 y+4=(y-2)^{2} \\
\frac{4 x^{2}}{16}-\frac{(y-2)^{2}}{16} & =\frac{16}{16} & & \text { Divide each side by } 16 \\
\frac{x^{2}}{4}-\frac{(y-2)^{2}}{16} & =1 & & \text { Simplify each term }
\end{aligned}
$$

This is the equation of a hyperbola with center at $(0,2)$. The graph opens to the right and left as shown in Figure 12.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1,3
B 2,4

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How do we find the $x$-intercepts of a graph from the equation?
2. What is an ellipse?
3. How can you tell by looking at an equation if its graph will be an ellipse or a hyperbola?
4. Are the points on the asymptotes of a hyperbola in the solution set of the equation of the hyperbola? (That is, are the asymptotes actually part of the graph?) Explain.

## Problem Set 12.2

Graph each of the following. Be sure to label both the $x$ - and $y$-intercepts.

1. $\frac{x^{2}}{9}+\frac{y^{2}}{16}=1$
2. $\frac{x^{2}}{25}+\frac{y^{2}}{4}=1$
3. $\frac{x^{2}}{16}+\frac{y^{2}}{9}=1$
4. $\frac{x^{2}}{4}+\frac{y^{2}}{25}=1$
5. $\frac{x^{2}}{3}+\frac{y^{2}}{4}=1$
6. $\frac{x^{2}}{4}+\frac{y^{2}}{3}=1$
7. $4 x^{2}+25 y^{2}=100$
8. $4 x^{2}+9 y^{2}=36$
9. $x^{2}+8 y^{2}=16$
10. $12 x^{2}+y^{2}=36$

Graph each of the following. Show the intercepts and the asymptotes in each case.
11. $\frac{x^{2}}{9}-\frac{y^{2}}{16}=1$
12. $\frac{x^{2}}{25}-\frac{y^{2}}{4}=1$
13. $\frac{x^{2}}{16}-\frac{y^{2}}{9}=1$
14. $\frac{x^{2}}{4}-\frac{y^{2}}{25}=1$
15. $\frac{y^{2}}{9}-\frac{x^{2}}{16}=1$
16. $\frac{y^{2}}{25}-\frac{x^{2}}{4}=1$
17. $\frac{y^{2}}{36}-\frac{x^{2}}{4}=1$
18. $\frac{y^{2}}{4}-\frac{x^{2}}{36}=1$
19. $x^{2}-4 y^{2}=4$
20. $y^{2}-4 x^{2}=4$
21. $16 y^{2}-9 x^{2}=144$
22. $4 y^{2}-25 x^{2}=100$

Find the $x$ - and $y$-intercepts, if they exist, for each of the following. Do not graph.
23. $0.4 x^{2}+0.9 y^{2}=3.6$
24. $1.6 x^{2}+0.9 y^{2}=14.4$
25. $\frac{x^{2}}{0.04}-\frac{y^{2}}{0.09}=1$
26. $\frac{y^{2}}{0.16}-\frac{x^{2}}{0.25}=1$
27. $\frac{25 x^{2}}{9}+\frac{25 y^{2}}{4}=1$
28. $\frac{16 x^{2}}{9}+\frac{16 y^{2}}{25}=1$

Graph each of the following ellipses. In each case, label the coordinates of the center and the vertices.
29. $\frac{(x-4)^{2}}{4}+\frac{(y-2)^{2}}{9}=1$
30. $\frac{(x-2)^{2}}{4}+\frac{(y-4)^{2}}{9}=1$
31. $4 x^{2}+y^{2}-4 y-12=0$
32. $4 x^{2}+y^{2}-24 x-4 y+36=0$
33. $x^{2}+9 y^{2}+4 x-54 y+76=0$
34. $4 x^{2}+y^{2}-16 x+2 y+13=0$

Graph each of the following hyperbolas. In each case, label the coordinates of the center and the vertices and show the asymptotes.
35. $\frac{(x-2)^{2}}{16}-\frac{y^{2}}{4}=1$
36. $\frac{(y-2)^{2}}{16}-\frac{x^{2}}{4}=1$
37. $9 y^{2}-x^{2}-4 x+54 y+68=0$
38. $4 x^{2}-y^{2}-24 x+4 y+28=0$
$=$ Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
39. $4 y^{2}-9 x^{2}-16 y+72 x-164=0$
40. $4 x^{2}-y^{2}-16 x-2 y+11=0$
41. Find $y$ when $x$ is 4 in the equation $\frac{x^{2}}{25}+\frac{y^{2}}{9}=1$.
42. Find $x$ when $y$ is 3 in the equation $\frac{x^{2}}{4}+\frac{y^{2}}{25}=1$.
43. Find $y$ when $x$ is 1.8 in $16 x^{2}+9 y^{2}=144$.
44. Find $y$ when $x$ is 1.6 in $49 x^{2}+4 y^{2}=196$.
45. Give the equations of the two asymptotes in the graph you found in Problem 15.
46. Give the equations of the two asymptotes in the graph you found in Problem 16.

Find an equation for each graph.
47.

48.


## Applying the Concepts

The diagram shows the minor axis and the major axis for an ellipse.


In any ellipse, the length of the major axis is $2 a$, and the length of the minor axis is $2 b$ (these are the same $a$ and $b$ that appear in the general equation of an ellipse). Each of the two points shown on the major axis is a focus of the ellipse. If the distance from the center of the ellipse to each focus is $c$, then it is always true that $a^{2}=b^{2}+c^{2}$. You will need this information for some of the problems that follow.
49. Archway A new theme park is planning an archway at its main entrance. The arch is to be in the form of a semi-ellipse with the major axis as the span. If the span is to be 40 feet and the height at the center is to be 10 feet, what is the equation of the ellipse? How far left and right of center could a 6-foot man walk upright under the arch?

50. The Ellipse President's Park, located between the White House and the Washington Monument in Washington, DC, is also called the Ellipse. The park is enclosed by an elliptical path with major axis 458
meters and minor axis 390 meters. What is the equation for the path around the Ellipse?

51. Elliptical Pool Table A children's science museum plans to build an elliptical pool table to demonstrate that a ball rolled from a particular point (focus) will always go into a hole located at another particular point (the other focus). The focus needs to be 1 foot from the vertex of the ellipse. If the table is to be 8 feet long, how wide should it be? Hint: The distance from the center to each focus point is represented by c and is found by using the equation $a^{2}=b^{2}+c^{2}$.

52. Lithotripter A lithotripter similar to the one mentioned in the introduction of the chapter is based on the
ellipse $\frac{x^{2}}{36}+\frac{y^{2}}{25}=1$. Determine how many units the kidney stone and the wave source (focus points) must be placed from the center of the ellipse. Hint: The distance from the center to each focus point is represented by $c$ and is found by using the equation $a^{2}=b^{2}+c^{2}$.

## Maintaining Your Skills

Let $f(x)=\frac{2}{x-2}$ and $g(x)=\frac{2}{x+2}$. Find the following.
53. $f(4)$
54. $g(4)$
55. $f[g(0)]$
56. $g[f(0)]$
57. $f(x)+g(x)$
58. $f(x)-g(x)$

## Getting Ready for the Next Section

59. Which of the following are solutions to $x^{2}+y^{2}<16$ ? $(0,0) \quad(4,0) \quad(0,5)$
60. Which of the following are solutions to $y \leq x^{2}-2$ ? $(0,0) \quad(-2,0) \quad(0,-2)$

Expand and multiply.
61. $(2 y+4)^{2}$
62. $(y+3)^{2}$
63. Solve $x-2 y=4$ for $x$.
64. Solve $2 x+3 y=6$ for $y$.

Simplify.
65. $x^{2}-2\left(x^{2}-3\right)$
66. $x^{2}+\left(x^{2}-4\right)$

Factor.
67. $5 y^{2}+16 y+12$
68. $3 x^{2}+17 x-28$

Solve.
69. $y^{2}=4$
70. $x^{2}=25$
71. $-x^{2}+6=2$
72. $5 y^{2}+16 y+12=0$

## 12.3

## OBJECTIVES

A Graph second-degree inequalities.

B Solve systems of nonlinear
C Graph the solution sets to systems of inequalities.

## Second-Degree Inequalities and Nonlinear Systems

Previously we graphed linear inequalities by first graphing the boundary and then choosing a test point not on the boundary to indicate the region used for the solution set. The problems in this section are very similar. We will use the same general methods for graphing the inequalities in this section that we used when we graphed linear inequalities in two variables.

## EXAMPLE $1 \quad$ Graph $x^{2}+y^{2}<16$

SOLUTION The boundary is $x^{2}+y^{2}=16$, which is a circle with center at the origin and a radius of 4 . Since the inequality sign is $<$, the boundary is not included in the solution set and therefore must be represented with a broken line. The graph of the boundary is shown in Figure 1.


FIGURE 1
The solution set for $x^{2}+y^{2}<16$ is either the region inside the circle or the region outside the circle. To see which region represents the solution set, we choose a convenient point not on the boundary and test it in the original inequality. The origin $(0,0)$ is a convenient point. Since the origin satisfies the inequality $x^{2}+y^{2}<16$, all points in the same region will also satisfy the inequality. The graph of the solution set is shown in Figure 2.


## EXAMPLE 2 Graph the inequality $y \leq x^{2}-2$.

SOLUTION The parabola $y=x^{2}-2$ is the boundary and is included in the solution set. Using $(0,0)$ as the test point, we see that $0 \leq 0^{2}-2$ is a false statement, which means that the region containing $(0,0)$ is not in the solution set. (See Figure 3.)


FIGURE 3

## EXAMPLE $3 \quad$ Graph $4 y^{2}-9 x^{2}<36$.

SOLUTION The boundary is the hyperbola $4 y^{2}-9 x^{2}=36$ and is not included in the solution set. Testing $(0,0)$ in the original inequality yields a true statement, which means that the region containing the origin is the solution set. (See Figure 4.)


FIGURE 4

$$
\begin{gathered}
x^{2}+y^{2}=4 \\
x-2 y=4
\end{gathered}
$$

SOLUTION In this case the substitution method is the most convenient. Solving the second equation for $x$ in terms of $y$, we have

$$
\begin{aligned}
x-2 y & =4 \\
x & =2 y+4
\end{aligned}
$$

We now substitute $2 y+4$ for $x$ in the first equation in our original system and proceed to solve for $y$ :

$$
\begin{array}{rlrl}
(2 y+4)^{2}+y^{2} & =4 & & \\
4 y^{2}+16 y+16+y^{2} & =4 & & \text { Expand }(2 y+4)^{2} \\
5 y^{2}+16 y+16 & =4 & & \text { Simplify left side } \\
5 y^{2}+16 y+12 & =0 & & \text { Add }-\mathbf{4} \text { to each side } \\
(5 y+6)(y+2) & =0 & & \text { Factor } \\
5 y+6=0 & \text { or } y+2 & =0 & \\
\text { Set factors equal to } 0 \\
y=-\frac{6}{5} & \text { or } \quad y & =-2 & \text { Solve }
\end{array}
$$

These are the $y$-coordinates of the two solutions to the system. Substituting $y=-\frac{6}{5}$ into $x-2 y=4$ and solving for $x$ gives us $x=\frac{8}{5}$. Using $y=-2$ in the same equation yields $x=0$. The two solutions to our system are $\left(\frac{8}{5},-\frac{6}{5}\right)$ and $(0,-2)$. Although graphing the system is not necessary, it does help us visualize the situation. (See Figure 5.)


FIGURE 5

$$
\begin{aligned}
16 x^{2}-4 y^{2} & =64 \\
x^{2}+y^{2} & =9
\end{aligned}
$$

SOLUTION Since each equation is of the second degree in both $x$ and $y$, it is easier to solve this system by eliminating one of the variables by addition. To eliminate $y$, we multiply the bottom equation by 4 and add the result to the top equation:

$$
\begin{aligned}
16 x^{2}-4 y^{2} & =64 \\
4 x^{2}+4 y^{2} & =36 \\
\hline 20 x^{2} \quad & =100 \\
x^{2} & =5 \\
x & = \pm \sqrt{5}
\end{aligned}
$$

The $x$-coordinates of the points of intersection are $\sqrt{5}$ and $-\sqrt{5}$. We substitute each back into the second equation in the original system and solve for $y$ :

$$
\text { When } \begin{aligned}
x & =\sqrt{5} \\
(\sqrt{5})^{2}+y^{2} & =9 \\
5+y^{2} & =9 \\
y^{2} & =4 \\
y & = \pm 2 \\
\text { When } \quad x & =-\sqrt{5} \\
(-\sqrt{5})^{2}+y^{2} & =9 \\
5+y^{2} & =9 \\
y^{2} & =4 \\
y & = \pm 2
\end{aligned}
$$

The four points of intersection are $(\sqrt{5}, 2),(\sqrt{5},-2),(-\sqrt{5}, 2)$, and $(-\sqrt{5},-2)$. Graphically the situation is as shown in Figure 6.


FIGURE 6

## EXAMPLE 6 Solve the system.

$$
\begin{aligned}
x^{2}-2 y & =2 \\
y & =x^{2}-3
\end{aligned}
$$

SOLUTION We can solve this system using the substitution method. Replacing $y$ in the first equation with $x^{2}-3$ from the second equation, we have

$$
\begin{aligned}
x^{2}-2\left(x^{2}-3\right) & =2 \\
-x^{2}+6 & =2 \\
x^{2} & =4 \\
x & = \pm 2
\end{aligned}
$$

Using either +2 or -2 in the equation $y=x^{2}-3$ gives us $y=1$. The system has two solutions: $(2,1)$ and $(-2,1)$.

EXAMPLE 7 The sum of the squares of two numbers is 34 . The difference of their squares is 16 . Find the two numbers.
SOLUTION Let $x$ and $y$ be the two numbers. The sum of their squares is $x^{2}+y^{2}$, and the difference of their squares is $x^{2}-y^{2}$. (We can assume here that $x^{2}$ is the larger number.) The system of equations that describes the situation is

$$
\begin{aligned}
& x^{2}+y^{2}=34 \\
& x^{2}-y^{2}=16
\end{aligned}
$$

We can eliminate $y$ by simply adding the two equations. The result of doing so is

$$
\begin{aligned}
2 x^{2} & =50 \\
x^{2} & =25 \\
x & = \pm 5
\end{aligned}
$$

Substituting $x=5$ into either equation in the system gives $y= \pm 3$. Using $x=-5$ gives the same results, $y= \pm 3$. The four pairs of numbers that are solutions to the original problem are 5 and $3,-5$ and 3,5 and $-3,-5$ and -3 .

We now turn our attention to systems of inequalities. To solve a system of inequalities by graphing, we simply graph each inequality on the same set of axes. The solution set for the system is the region common to both graphs-the intersection of the individual solution sets.

EXAMPLE 8 Graph the solution set for the system.

$$
\begin{aligned}
x^{2}+y^{2} & \leq 9 \\
\frac{x^{2}}{4}+\frac{y^{2}}{25} & \geq 1
\end{aligned}
$$

SOLUTION The boundary for the top inequality is a circle with center at the origin and a radius of 3 . The solution set lies inside the boundary. The boundary for the second inequality is an ellipse. In this case the solution set lies outside the boundary. (See Figure 7.)


FIGURE 7

The solution set is the intersection of the two individual solution sets.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1-3

B 4-7
C 8

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the significance of a broken line when graphing inequalities?
2. Describe, in words, the set of points described by $(x-3)^{2}+(y-2)^{2}<9$.
3. When solving the nonlinear systems whose graphs are a line and a circle, how many possible solutions can you expect?
4. When solving the nonlinear systems whose graphs are both circles, how many possible solutions can you expect?

## Problem Set 12.3

Graph each of the following inequalities.

1. $x^{2}+y^{2} \leq 49$
2. $x^{2}+y^{2}<49$
3. $(x-2)^{2}+(y+3)^{2}<16$
4. $(x+3)^{2}+(y-2)^{2} \geq 25$
5. $y<x^{2}-6 x+7$
6. $y \geq x^{2}+2 x-8$
7. $4 x^{2}+25 y^{2} \leq 100$
8. $25 x^{2}-4 y^{2}>100$

Solve each of the following systems of equations.
9. $x^{2}+y^{2}=9$
$2 x+y=3$
10. $x^{2}+y^{2}=9$
$x+2 y=3$
11. $x^{2}+y^{2}=16$
$x+2 y=8$
12. $x^{2}+y^{2}=16$
$x-2 y=8$
= Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
13. $x^{2}+y^{2}=25$
$x^{2}-y^{2}=25$
14. $x^{2}+y^{2}=4$
$2 x^{2}-y^{2}=5$
15. $x^{2}+y^{2}=9$

$$
y=x^{2}-3
$$

16. $x^{2}+y^{2}=4$

$$
y=x^{2}-2
$$

17. $x^{2}+y^{2}=16$

$$
y=x^{2}-4
$$

18. $x^{2}+y^{2}=1$

$$
y=x^{2}-1
$$

19. $3 x+2 y=10$

$$
y=x^{2}-5
$$

21. $y=x^{2}+2 x-3$
$y=\quad-x+1$
22. $y=x^{2}-6 x+5$
$y=\quad x-5$
23. $4 x^{2}-9 y^{2}=36$
$4 x^{2}+9 y^{2}=36$
24. $x-y=4$
$x^{2}+y^{2}=16$
25. $y=x^{2}-2 x-4$
$y=\quad x-4$
26. $4 x+2 y=10$
$y=x^{2}-10$
27. $y=-x^{2}-2 x+3$
$y=\quad x-1$
28. $4 x^{2}+25 y^{2}=100$
$4 x^{2}-25 y^{2}=100$
29. $x+y=2$
$x^{2}-y^{2}=4$
30. In a previous problem set, you found the equations for the three circles below. The equations are

Circle $A$
Circle $B$
$(x+8)^{2}+y^{2}=64 \quad x^{2}+y^{2}=64 \quad(x-8)^{2}+y^{2}=64$

a. Find the points of intersection of circles $A$ and $B$.
b. Find the points of intersection of circles $B$ and $C$.
30. A magician is holding two rings that seem to lie in the same plane and intersect in two points. Each ring is 10 inches in diameter. If a coordinate system is placed with its origin at the center of the first ring and the $x$-axis contains the center of the second ring, then the equations are as follows:


$$
\text { First Ring } \quad \text { Second Ring }
$$

$$
x^{2}+y^{2}=25 \quad(x-5)^{2}+y^{2}=25
$$

Find the points of intersection of the two rings.

Graph the solution sets to the following systems.
31. $x^{2}+y^{2}<9$
$y \geq x^{2}-1$
32. $x^{2}+y^{2} \leq 16$

$$
y<x^{2}+2
$$

33. $\frac{x^{2}}{9}+\frac{y^{2}}{25} \leq 1$
34. $\frac{x^{2}}{4}+\frac{y^{2}}{16} \geq 1$
$\frac{x^{2}}{4}-\frac{y^{2}}{9}>1$

$$
\frac{x^{2}}{9}-\frac{y^{2}}{25}<1
$$

35. $4 x^{2}+9 y^{2} \leq 36$

$$
y>x^{2}+2
$$

36. $9 x^{2}+4 y^{2} \geq 36$

$$
y<x^{2}+1
$$

37. A parabola and a circle each contain the points $(-4,0),(0,4)$, and $(4,0)$, as shown below. The equations for each of the curves are

Circle Parabola


Write a system of inequalities that describes the regions in quadrants I and II that are between the two curves, if the boundaries are not included with the shaded region.
38. Find a system of inequalities that describes the shaded region in the figure below if the boundaries of the shaded region are included and equations of the circles are given by

## Circle A

Circle $B$
Circle C
$x^{2}+y^{2}=64$
$(x-8)^{2}+y^{2}=64$

## Applying the Concepts

39. Number Problems The sum of the squares of two numbers is 89 . The difference of their squares is 39 Find the numbers.
40. Number Problems The difference of the squares of two numbers is 35 . The sum of their squares is 37 . Find the numbers.
41. Number Problems One number is 3 less than the square of another. Their sum is 9 . Find the numbers.
42. Number Problems The square of one number is 2 less than twice the square of another. The sum of the squares of the two numbers is 25 . Find the numbers.

## Maintaining Your Skills

43. Let $g(x)=\left(x+\frac{2}{5}\right)^{2}$. Find all values for the variable $x$ for which $g(x)=0$.
44. Let $f(x)=(x+2)^{2}-25$. Find all values for the variable $x$ for which $f(x)=0$.

For the problems below, let $f(x)=x^{2}+4 x-4$. Find all values for the variable $x$ for which $f(x)=g(x)$.
45. $g(x)=1$
46. $g(x)=-7$
47. $g(x)=x-6$
48. $g(x)=x+6$

## Chapter 12 sUMMARY

## Distance Formula [12.1]

## EXAMPLES

1. The distance between $(5,2)$
and $(-1,1)$ is
$d=\sqrt{(5+1)^{2}+(2-1)^{2}}=\sqrt{37}$
2. The graph of the circle $(x-3)^{2}+(y+2)^{2}=25$ will have its center at $(3,-2)$ and the radius will be 5 .

The distance between the two points $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ is given by the formula

$$
d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}
$$

## The Circle [12.1]

The graph of any equation of the form

$$
(x-a)^{2}+(y-b)^{2}=r^{2}
$$

will be a circle having its center at $(a, b)$ and a radius of $r$.

## The Ellipse [12.2]

Any equation that can be put in the form

$$
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1
$$

will have an ellipse for its graph. The $x$-intercepts will be at $a$ and $-a$, and the $y$-intercepts will be at $b$ and $-b$.

## The Hyperbola [12.2]

The graph of an equation that can be put in either of the forms

$$
\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1 \quad \text { or } \quad \frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1
$$

will be a hyperbola. The $x$-intercepts for the first equation will be at $a$ and $-a$. The $y$-intercepts for the second equation will be at $a$ and $-a$. Two straight lines, called asymptotes, are associated with the graph of every hyperbola. Although the asymptotes are not part of the hyperbola, they are useful in sketching the graph.

## Second-Degree Inequalities in Two Variables [12.3]

We graph second-degree inequalities in two variables in much the same way that we graphed linear inequalities; that is, we begin by graphing the boundary, using a solid curve if the boundary is included in the solution (this happens when the inequality symbol is $\geq$ or $\leq$ ) or a broken curve if the boundary is not included in the solution (when the inequality symbol is $>$ or $<$ ). After we have graphed the boundary, we choose a test point that is not on the boundary and
therefore is shown with a broken curve.
try it in the original inequality. A true statement indicates we are in the region of the solution. A false statement indicates we are not in the region of the solution.

## Systems of Nonlinear Equations [12.3]

A system of nonlinear equations is two equations, at least one of which is not linear, considered at the same time. The solution set for the system consists of all ordered pairs that satisfy both equations. In most cases we use the substitution method to solve these systems; however, the addition method can be used if like variables are raised to the same power in both equations. It is sometimes helpful to graph each equation in the system on the same set of axes to anticipate the number and approximate positions of the solutions.

## Chapter 12 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the odd-numbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the evennumbered problems for a second practice test.

Find the distance between the following points. [12.1]

1. $(2,6),(-1,5)$
2. $(3,-4),(1,-1)$
3. $(0,3),(-4,0)$
4. $(-3,7),(-3,-2)$
5. Find $x$ so that the distance between $(x,-1)$ and $(2,-4)$ is 5. [12.1]
6. Find $y$ so that the distance between $(3,-4)$ and $(-3, y)$ is 10 . [12.1]

Write the equation of the circle with the given center and radius. [12.1]
7. Center $(3,1), r=2$
8. Center $(3,-1), r=4$
9. Center $(-5,0), r=3$
10. Center $(-3,4), r=3 \sqrt{2}$

Find the equation of each circle. [12.1]
11. Center at the origin, $x$-intercepts $\pm 5$
12. Center at the origin, $y$-intercepts $\pm 3$
13. Center at $(-2,3)$ and passing through the point $(2,0)$
14. Center at $(-6,8)$ and passing through the origin

Give the center and radius of each circle, and then sketch the graph. [12.1]
15. $x^{2}+y^{2}=4$
16. $(x-3)^{2}+(y+1)^{2}=16$
17. $x^{2}+y^{2}-6 x+4 y=-4$
18. $x^{2}+y^{2}+4 x-2 y=4$

Graph each of the following. Label the $x$ - and $y$-intercepts.
[12.2]
19. $\frac{x^{2}}{4}+\frac{y^{2}}{9}=1$
20. $4 x^{2}+y^{2}=16$

Graph the following. Show the asymptotes. [12.2]
21. $\frac{x^{2}}{4}-\frac{y^{2}}{9}=1$
22. $4 x^{2}-y^{2}=16$

Graph each equation. [12.2]
23. $\frac{(x+2)^{2}}{9}+\frac{(y-3)^{2}}{1}=1$
24. $\frac{(x-2)^{2}}{16}-\frac{y^{2}}{4}=1$
25. $9 y^{2}-x^{2}-4 x+54 y+68=0$
26. $9 x^{2}+4 y^{2}-72 x-16 y+124=0$

Graph each of the following inequalities. [12.3]
27. $x^{2}+y^{2}<9$
28. $(x+2)^{2}+(y-1)^{2} \leq 4$
29. $y \geq x^{2}-1$
30. $9 x^{2}+4 y^{2} \leq 36$

Graph the solution set for each system. [12.3]
31. $x^{2}+y^{2}<16$

$$
y>x^{2}-4
$$

32. $x+y \leq 2$

$$
\begin{aligned}
-x+y & \leq 2 \\
y & \geq-2
\end{aligned}
$$

Solve each system of equations. [12.3]
33. $x^{2}+y^{2}=16$
$2 x+y=4$
34. $x^{2}+y^{2}=4$

$$
y=x^{2}-2
$$

35. $9 x^{2}-4 y^{2}=36$
$9 x^{2}+4 y^{2}=36$
36. $2 x^{2}-4 y^{2}=8$
$x^{2}+2 y^{2}=10$

## Chapter 12 Projects



## Conic Sections

## GROUP PROJECT Constructing Ellipses

## Number of People 4

Time Needed 20 minutes
Equipment Graph paper, pencils, string, and thumbtacks

Background The geometric definition for an ellipse is the set of points the sum of whose distances from two fixed points (called foci) is a constant. We can use this definition to draw an ellipse using thumbtacks, string, and a pencil.


FIGURE 1


FIGURE 2

## Procedure

1. Start with a piece of string 7 inches long. Place thumbtacks through the string $\frac{1}{2}$ inch from each end, then tack the string to a pad of graph paper so that the tacks are 4 inches apart. Pull the string tight with the tip of a pencil, then trace all the way around the two tacks. (See Figure 1.) The resulting diagram will be an ellipse.
2. The line segment that passes through the tacks (these are the foci) and connects opposite ends of the ellipse is called the major axis. The line segment perpendicular to the major axis that passes through the center of the ellipse and connects opposite ends of the ellipse is called the minor axis. (See Figure 2.) Measure the length of the major axis and the length of the minor axis. Record your results in Table 1.
3. Explain how drawing the ellipse as you have in step 2 shows that the geometric definition of an ellipse given at the beginning of this project is, in fact, correct.
4. Next, move the tacks so that they are 3 inches apart. Trace out that ellipse. Measure the length of the major axis and the length of the minor axis, and record your results in Table 1.
5. Repeat step 4 with the tacks 2 inches apart.
6. If the length of the string between the tacks stays at 6 inches, and the tacks were placed 6 inches apart, then the resulting ellipse would be a $\qquad$ If the tacks were placed 0 inches apart, then the resulting ellipse would be a $\qquad$ -.

TABLE 1
Ellipses (All Lengths Are Inches)

| Length of <br> String | Distance <br> Between Foci | Length of <br> Major Axis | Length of <br> Minor Axis |
| :---: | :---: | :---: | :---: |
| 6 | 4 |  |  |
| 6 | 3 |  |  |
| 6 | 2 |  |  |

## RESEARCH PROJECT



## Sequences and Series

## 13


uppose you run up a balance of $\$ 1,000$ on a credit card that charges $1.65 \%$ interest each month (i.e., an annual rate of 19.8\%). If you stop using the card and make the minimum payment of $\$ 20$ each month, how long will it take you to pay off the balance on the card? The answer can be found by using the formula

$$
U_{n}=(1.0165) U_{n-1}-20
$$

where $U_{n}$ stands for the current unpaid balance on the card, and $U_{n-1}$ is the previous month's balance. The table below and Figure 1 were created from this formula and a graphing calculator. As you can see from the table, the balance on the credit card decreases very little each month.

| Monthly Credit Card Balances <br> Previous <br> Balance <br> $\boldsymbol{U}_{(n-1)}$ | Monthly <br> Interest <br> Rate | Payment <br> Number <br> $\boldsymbol{n}$ | Monthly <br> Payment | New <br> Balance <br> $\boldsymbol{U}_{(n)}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\$ 1,000.00$ | $1.65 \%$ | 1 | $\$ 20$ | $\$ 996.50$ |
| $\$ 996.50$ | $1.65 \%$ | 2 | $\$ 20$ | $\$ 992.94$ |
| $\$ 992.94$ | $1.65 \%$ | 3 | $\$ 20$ | $\$ 989.32$ |
| $\$ 989.32$ | $1.65 \%$ | 4 | $\$ 20$ | $\$ 985.64$ |
| $\$ 985.64$ | $1.65 \%$ | 5 | $\$ 20$ | $\$ 981.90$ |
| . | . | . | . | . |
| $\cdot$ | $\cdot$ | $\cdot$ | . | . |
| . | $\cdot$ | . | . | . |



In the group project at the end of the chapter, you will use a graphing calculator to continue this table and in so doing find out just how many months it will take to pay off this credit card balance.

## CHAPTER OUTLINE

### 13.1 Sequences

13.2 Series
13.3 Arithmetic Sequences
13.4 Geometric Sequences
13.5 The Binomial Expansion

- Improve your grade and save time! Go online to www.thomsonedu.com/login where you can
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## Sequences

## OBJECTIVES

A
Use a formula to find the terms of a sequence.

B Use a recursive formula. Find the general term of a sequence.

Many of the sequences in this chapter will be familiar to you on an intuitive level because you have worked with them for some time now. Here are some of those sequences:
The sequence of odd numbers

$$
1,3,5,7, \ldots
$$

The sequence of even numbers

$$
2,4,6,8, \ldots
$$

The sequence of squares

$$
1^{2}, 2^{2}, 3^{2}, 4^{2}, \ldots=1,4,9,16, \ldots
$$

The numbers in each of these sequences can be found from the formulas that define functions. For example, the sequence of even numbers can be found from the function

$$
f(x)=2 x
$$

by finding $f(1), f(2), f(3), f(4)$, and so forth. This gives us justification for the formal definition of a sequence.

DEFINITION A sequence is a function whose domain is the set of positive integers $\{1,2,3,4, \ldots\}$.

As you can see, sequences are simply functions with a specific domain. If we want to form a sequence from the function $f(x)=3 x+5$, we simply find $f(1)$, $f(2), f(3)$, and so on. Doing so gives us the sequence

$$
8,11,14,17, \ldots
$$

because $f(1)=3(1)+5=8, \quad f(2)=3(2)+5=11, \quad f(3)=3(3)+5=14, \quad$ and $f(4)=3(4)+5=17$.

Notation Because the domain for a sequence is always the set $\{1,2,3, \ldots\}$, we can simplify the notation we use to represent the terms of a sequence. Using the letter $a$ instead of $f$, and subscripts instead of numbers enclosed by parentheses, we can represent the sequence from the previous discussion as follows:

$$
a_{n}=3 n+5
$$

Instead of $f(1)$, we write $a_{1}$ for the first term of the sequence. Instead of $f(2)$, we write $a_{2}$ for the second term of the sequence. Instead of $f(3)$, we write $a_{3}$ for the third term of the sequence. Instead of $f(4)$, we write $a_{4}$ for the fourth term of the sequence. Instead of $f(n)$, we write $a_{n}$ for the $n$th term of the sequence.

The $n$th term is also called the general term of the sequence. The general term is used to define the other terms of the sequence. That is, if we are given the formula for the general term $a_{n}$, we can find any other term in the sequence. The following examples illustrate.

EXAMPLE 1 Find the first four terms of the sequence whose general term is given by $a_{n}=2 n-1$.
SOLUTION The subscript notation $a_{n}$ works the same way function notation works. To find the first, second, third, and fourth terms of this sequence, we simply substitute $1,2,3$, and 4 for $n$ in the formula $2 n-1$ :

$$
\begin{aligned}
& \text { If } \quad \begin{aligned}
\text { the general term is } a_{n} & =2 n-1 \\
\text { then first term is } a_{1} & =2(1)-1=1 \\
\text { the second term is } a_{2} & =2(2)-1=3 \\
\text { the third term is } a_{3} & =2(3)-1=5 \\
\text { the fourth term is } a_{4} & =2(4)-1=7
\end{aligned}
\end{aligned}
$$

The first four terms of the sequence in Example 1 are the odd numbers 1, 3, 5, and 7 . The whole sequence can be written as

$$
1,3,5, \ldots, 2 n-1, \ldots
$$

Because each term in this sequence is larger than the preceding term, we say the sequence is an increasing sequence.

## EXAMPLE 2 Write the first four terms of the sequence defined by

$$
a_{n}=\frac{1}{n+1}
$$

SOLUTION Replacing $n$ with $1,2,3$, and 4, we have, respectively, the first four terms:

$$
\begin{array}{r}
\text { First term }=a_{1}=\frac{1}{1+1}=\frac{1}{2} \\
\text { Second term }=a_{2}=\frac{1}{2+1}=\frac{1}{3} \\
\text { Third term }=a_{3}=\frac{1}{3+1}=\frac{1}{4} \\
\text { Fourth term }=a_{4}=\frac{1}{4+1}=\frac{1}{5}
\end{array}
$$

The sequence in Example 2 defined by

$$
a_{n}=\frac{1}{n+1}
$$

can be written as

$$
\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}, \ldots
$$

Because each term in the sequence is smaller than the term preceding it, the sequence is said to be a decreasing sequence.

EXAMPLE 3 Find the fifth and sixth terms of the sequence whose general term is given by $a_{n}=\frac{(-1)^{n}}{n^{2}}$.

SOLUTION For the fifth term, we replace $n$ with 5 . For the sixth term, we replace $n$ with 6:

$$
\begin{aligned}
& \text { Fifth term }=a_{5}=\frac{(-1)^{5}}{5^{2}}=\frac{-1}{25} \\
& \text { Sixth term }=a_{6}=\frac{(-1)^{6}}{6^{2}}=\frac{1}{36}
\end{aligned}
$$

The sequence in Example 3 can be written as

$$
-1, \frac{1}{4},-\frac{1}{9}, \frac{1}{16}, \ldots, \frac{(-1)^{n}}{n^{2}}, \ldots
$$

Because the terms alternate in sign-if one term is positive, then the next term is negative-we call this an alternating sequence. The first three examples all illustrate how we work with a sequence in which we are given a formula for the general term.

## USING TECHNOLOGY

## Finding Sequences on a Graphing Calculator

## Method 1: Using a Table

We can use the table function on a graphing calculator to view the terms of a sequence. To view the terms of the sequence $a_{n}=3 n+5$, we set $Y_{1}=3 X+5$. Then we use the table setup feature on the calculator to set the table minimum to 1 , and the table increment to 1 also. Here is the setup and result for a TI-83.

Table Setup
Table minimum $=1$
$Y$ Variables Setup
Table increment $=1$
Independent variable: Auto
Dependent variable: Auto
The table will look like this:

| $\mathbf{X}$ | $\mathbf{Y}$ |
| :---: | ---: |
| 1 | 8 |
| 2 | 11 |
| 3 | 14 |
| 4 | 17 |
| 5 | 20 |

To find any particular term of a sequence, we change the independent variable setting to Ask, and then input the number of the term of the sequence we want to find. For example, if we want term $a_{100}$, then we input 100 for the independent variable, and the table gives us the value of 305 for that term.

## Method 2: Using the Built-in seq( Function

Using this method, first find the seq( function. On a TI-83 it is found in the LIST OPS menu. To find terms $a_{1}$ through $a_{7}$ for $a_{n}=3 n+5$, we first bring up the seq( function on our calculator, then we input the following four items, in order, separated by commas: $3 X+5, X, 1,7$. Then we close the parentheses. Our screen will look like this:

$$
\operatorname{seq}(3 X+5, X, 1,7)
$$

Pressing ENTER displays the first five terms of the sequence. Pressing the right arrow key repeatedly brings the remaining members of the sequence into view.

## Method 3: Using the Built-in Seq Mode

Press the MODE key on your TI-83 and then select Seq (it's next to Func Par and Pol). Go to the Y variables list and set $n \mathrm{Min}=1$ and $u(n)=3 n+5$. Then go to the TBLSET key to set up your table like the one shown in Method 1. Pressing TABLE will display the sequence you have defined.

## Recursion Formulas

Let's go back to one of the first sequences we looked at in this section:
$8,11,14,17, \ldots$
Each term in the sequence can be found by simply substituting positive integers for $n$ in the formula $a_{n}=3 n+5$. Another way to look at this sequence, however, is to notice that each term can be found by adding 3 to the preceding term; so, we could give all the terms of this sequence by simply saying

Start with 8, and then add 3 to each term to get the next term.
The same idea, expressed in symbols, looks like this:

$$
a_{1}=8 \quad \text { and } \quad a_{n}=a_{n-1}+3 \quad \text { for } n>1
$$

This formula is called a recursion formula because each term is written recursively in terms of the term or terms that precede it.

EXAMPLE 4 Write the first four terms of the sequence given recur-
sively by

$$
a_{1}=4 \quad \text { and } \quad a_{n}=5 a_{n-1} \quad \text { for } n>1
$$

SOLUTION The formula tells us to start the sequence with the number 4 , and then multiply each term by 5 to get the next term. Therefore,

$$
\begin{aligned}
& a_{1}=4 \\
& a_{2}=5 a_{1}=5(4)=20 \\
& a_{3}=5 a_{2}=5(20)=100 \\
& a_{4}=5 a_{3}=5(100)=500
\end{aligned}
$$

The sequence is $4,20,100,500, \ldots$.

## USING TECHNOLOGY

Recursion Formulas on a Graphing Calculator
We can use a TI-83 graphing calculator to view the sequence defined recursively as

$$
a_{1}=8, a_{n}=a_{n-1}+3
$$

First, put your TI-83 calculator in sequence mode by pressing the MODE key, and then selecting Seq (it's next to Func Par and Pol). Go to the Y variables list and set $n \operatorname{Min}=1$ and $u(n)=u(n-1)+3$. (The $u$ is above the 7 , and the $n$ is on the $\mathrm{X}, \mathrm{T}, \theta, n$ and is automatically displayed if that key is pressed when the calculator is in the Seq mode.) Pressing TABLE will display the sequence you have defined.

## Finding the General Term

In the first four examples, we found some terms of a sequence after being given the general term. In the next two examples, we will do the reverse. That is, given some terms of a sequence, we will find the formula for the general term.

EXAMPLE 5 Find a formula for the $n$th term of the sequence $2,8,18,32, \ldots$

SOLUTION Solving a problem like this involves some guessing. Looking over the first four terms, we see each is twice a perfect square:

$$
\begin{aligned}
2 & =2(1) \\
8 & =2(4) \\
18 & =2(9) \\
32 & =2(16)
\end{aligned}
$$

If we write each square with an exponent of 2 , the formula for the $n$th term becomes obvious:

$$
\begin{aligned}
& a_{1}=2=2(1)^{2} \\
& a_{2}=8=2(2)^{2} \\
& a_{3}=18=2(3)^{2} \\
& a_{4}=32=2(4)^{2} \\
& \vdots \\
& a_{n}=\quad 2(n)^{2}=2 n^{2}
\end{aligned}
$$

The general term of the sequence $2,8,18,32, \ldots$ is $a_{n}=2 n^{2}$.

## Note

Finding the $n$th term of a sequence from the first few terms is not always automatic. That is, it sometimes takes a while to recognize the pattern. Don't be afraid to guess at the formula for the general term. Many times, an incorrect guess leads to the correct formula.

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A ${ }^{1-3}$
B 4
C 5,6

EXAMPLE 6 Find the general term for the sequence $2, \frac{3}{8}, \frac{4}{27}, \frac{5}{64}, \ldots$. SOLUTION The first term can be written as $\frac{2}{1}$. The denominators are all perfect cubes. The numerators are all 1 more than the base of the cubes in the denominators:

$$
\begin{aligned}
& a_{1}=\frac{2}{1}=\frac{1+1}{1^{3}} \\
& a_{2}=\frac{3}{8}=\frac{2+1}{2^{3}} \\
& a_{3}=\frac{4}{=}=\frac{3+1}{3^{3}} \\
& a_{4}=\frac{5}{64}=\frac{4+1}{4^{3}}
\end{aligned}
$$

Observing this pattern, we recognize the general term to be

$$
a_{n}=\frac{n+1}{n^{3}}
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. How are subscripts used to denote the terms of a sequence?
2. What is the relationship between the subscripts used to denote the terms of a sequence and function notation?
3. What is a decreasing sequence?
4. What is meant by a recursion formula for a sequence?

## Problem Set 13.1

Write the first five terms of the sequences with the fol-
lowing general terms.

1. $a_{n}=3 n+1$
2. $a_{n}=2 n+3$
3. $a_{n}=4 n-1$
4. $a_{n}=n+4$
5. $a_{n}=n$
6. $a_{n}=-n$
7. $a_{n}=n^{2}+3$
8. $a_{n}=n^{3}+1$
9. $a_{n}=\frac{n}{n+3}$
10. $a_{n}=\frac{n+1}{n+2}$
11. $a_{n}=\frac{1}{n^{2}}$
12. $a_{n}=\frac{1}{n^{3}}$
13. $a_{n}=2^{n}$
14. $a_{n}=3^{-n}$
15. $a_{n}=1+\frac{1}{n}$
$=$ Videos available by instructor request
b = Online student support materials available at www.thomsonedu.com/login
16. $a_{n}=n-\frac{1}{n}$
17. $a_{n}=(-2)^{n}$
18. $a_{n}=(-3)^{n}$
19. $a_{n}=4+(-1)^{n}$
20. $a_{n}=10+(-2)^{n}$
21. $a_{n}=(-1)^{n+1} \cdot \frac{n}{2 n-1}$
22. $a_{n}=(-1)^{n} \cdot \frac{2 n+1}{2 n-1}$
23. $a_{n}=n^{2} \cdot 2^{-n}$
24. $a_{n}=n^{n}$

Write the first five terms of the sequences defined by the following recursion formulas.
25. $a_{1}=3 \quad a_{n}=-3 a_{n-1} \quad n>1$
26. $a_{1}=3 \quad a_{n}=a_{n-1}-3 \quad n>1$
27. $a_{1}=1 \quad a_{n}=2 a_{n-1}+3 \quad n>1$
28. $a_{1}=1 \quad a_{n}=a_{n-1}+n \quad n>1$
29. $a_{1}=2 \quad a_{n}=2 a_{n-1}-1 \quad n>1$
30. $a_{1}=-4 \quad a_{n}=-2 a_{n-1} \quad n>1$
31. $a_{1}=5 \quad a_{n}=3 a_{n-1}-4 \quad n>1$
32. $a_{1}=-3 \quad a_{n}=-2 a_{n-1}+5 \quad n>1$
33. $a_{1}=4 \quad a_{n}=2 a_{n-1}-a_{1} \quad n>1$
34. $a_{1}=-3 \quad a_{n}=-2 a_{n-1}-n \quad n>1$

Determine the general term for each of the following sequences.
35. $4,8,12,16,20, \ldots$
36. $7,10,13,16, \ldots$
37. $1,4,9,16, \ldots$
38. $3,12,27,48$,
39. $4,8,16,32, \ldots$
40. $-2,4,-8,16, \ldots$
41. $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$
42. $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \ldots$
43. $5,8,11,14, \ldots$
44. $7,5,3,1, \ldots$
45. $-2,-6,-10,-14, \ldots$
46. $-2,2,-2,2, \ldots$
47. $1,-2,4,-8, \ldots$
48. $-1,3,-9,27, \ldots$
49. $\log _{2} 3, \log _{3} 4, \log _{4} 5, \log _{5} 6, \ldots$
50. $0, \frac{3}{5}, \frac{8}{10}, \frac{15}{17}, \ldots$

## Applying the Concepts

51. Salary Increase The entry level salary for a teacher is $\$ 28,000$ with $4 \%$ increases after every year of service.
a. Write a sequence for this teacher's salary for the first 5 years.
b. Find the general term of the sequence in part a.
52. Holiday Account To save money for holiday presents, a person deposits $\$ 5$ in a savings account on January 1, and then deposits an additional \$5 every week thereafter until Christmas.
a. Write a sequence for the money in that savings account for the first 10 weeks of the year.
b. Write the general term of the sequence in part a.
c. If there are 50 weeks from January 1 to Christmas, how much money will be available for spending on Christmas presents?
53. Akaka Falls If a boulder fell from the top of Akaka Falls in Hawaii, the distance, in feet, the boulder would fall in each consecutive second would be modeled by a sequence whose general term is $a_{n}=32 n-16$, where $n$ represents the number of seconds.

a. Write a sequence for the first 5 seconds the boulder falls.
b. What is the total distance the boulder fell in 5 seconds?
c. If Akaka Falls is approximately 420 feet high, will the boulder hit the ground within 5 seconds?
54. Health Expenditures The total national health expenditures, in billions of dollars, increased by approximately \$50 billion from 1995 to 1996 and from 1996 to 1997. The total national health expenditures for 1997 was approximately $\$ 1$,092 billion (U.S. Census Bureau, National Health Expenditures).
a. If the same approximate increase is assumed to continue, what would be the estimated total expenditures for 1998, 1999, and 2000?
b. Write the general term for the sequence, where $n$ represents the number of years since 1997.
c. If health expenditures continue to increase at this approximate rate, what would the expected expenditures be in 2010?
55. Polygons The formula for the sum of the interior angles of a polygon with $n$ sides is $a_{n}=180^{\circ}(n-2)$.
a. Write a sequence to represent the sum of the interior angles of a polygon with $3,4,5$, and 6 sides.
b. What would be the sum of the interior angles of a polygon with 20 sides?
c. What happens when $n=2$ to indicate that a polygon cannot be formed with only two sides?
56. Pendulum A pendulum swings 10 feet left to right on its first swing. On each swing following the first, the pendulum swings $\frac{4}{5}$ of the previous swing.
a. Write a sequence for the distance traveled by the pendulum on the first, second, and third swing.
b. Write a general term for the sequence, where $n$ represents the number of the swing.
c. How far will the pendulum swing on its tenth swing? (Round to the nearest hundredth.)

## Maintaining Your Skills

Find $x$ in each of the following.
57. $\log _{9} x=\frac{3}{2}$
58. $\log _{x} \frac{1}{4}=-2$

Simplify each expression.
59. $\log _{2} 32$
60. $\log _{10} 10,000$
61. $\log _{3}\left[\log _{2} 8\right]$
62. $\log _{5}\left[\log _{6} 6\right]$

## Getting Ready for the Next Section

Simplify.
63. $-2+6+4+22$
64. $9-27+81-243$
65. $-8+16-32+64$
66. $-4+8-16+32-64$
67. $(1-3)+(4-3)+(9-3)+(16-3)$
68. $(1-3)+(9+1)+(16+1)+(25+1)+(36+1)$
69. $-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\frac{1}{81}$
70. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$
71. $\frac{1}{3}+\frac{1}{2}+\frac{3}{5}+\frac{2}{3}$
72. $\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$

## Extending the Concepts

73. As $n$ increases, the terms in the sequence

$$
a_{n}=\left(1+\frac{1}{n}\right)^{n}
$$

get closer and closer to the number e (that's the same $e$ we used in defining natural logarithms). It takes some fairly large values of $n$, however, before we can see this happening. Use a calculator to find $a_{100}, a_{1,000}, a_{10,000}$, and $a_{100,000}$, and compare them to the decimal approximation we gave for the number $e$.
74. The sequence

$$
a_{n}=\left(1+\frac{1}{n}\right)^{-n}
$$

gets close to the number $\frac{1}{e}$ as $n$ becomes large. Use a calculator to find approximations for $a_{100}$ and $a_{1,000}$, and then compare them to $\frac{1}{2.7183}$.
75. Write the first ten terms of the sequence defined by the recursion formula

$$
a_{1}=1, a_{2}=1, a_{n}=a_{n-1}+a_{n-2} \quad n>2
$$

76. Write the first ten terms of the sequence defined by the recursion formula

$$
a_{1}=2, a_{2}=2, a_{n}=a_{n-1}+a_{n-2} \quad n>2
$$

## 13.2

## OBJECTIVES

A Expand and simplify an expression written with summation notation.

B Write a series using summation notation.

## Series

There is an interesting relationship between the sequence of odd numbers and the sequence of squares that is found by adding the terms in the sequence of odd numbers.

$$
\begin{array}{ll}
1 & =1 \\
1+3 & =4 \\
1+3+5 & =9 \\
1+3+5+7 & =16
\end{array}
$$

When we add the terms of a sequence the result is called a series.

DEFINITION The sum of a number of terms in a sequence is called a series.

A sequence can be finite or infinite, depending on whether the sequence ends at the $n$th term. For example,

$$
1,3,5,7,9
$$

is a finite sequence, but

$$
1,3,5, \ldots
$$

is an infinite sequence. Associated with each of the preceding sequences is a series found by adding the terms of the sequence:

$$
\begin{array}{ll}
1+3+5+7+9 & \text { Finite series } \\
1+3+5+\ldots & \text { Infinite series }
\end{array}
$$

In this section, we will consider only finite series. We can introduce a new kind of notation here that is a compact way of indicating a finite series. The notation is called summation notation, or sigma notation because it is written using the Greek letter sigma. The expression

$$
\sum_{i=1}^{4}(8 i-10)
$$

is an example of an expression that uses summation notation. The summation notation in this expression is used to indicate the sum of all the expressions $8 i-10$ from $i=1$ up to and including $i=4$. That is,

$$
\begin{aligned}
\sum_{i=1}^{4}(8 i-10) & =(8 \cdot 1-10)+(8 \cdot 2-10)+(8 \cdot 3-10)+(8 \cdot 4-10) \\
& =-2+6+14+22 \\
& =40
\end{aligned}
$$

The letter $i$ as used here is called the index of summation, or just index for short. Here are some examples illustrating the use of summation notation.

## EXAMPLE 1 Expand and simplify $\sum_{i=1}^{5}\left(i^{2}-1\right)$.

SOLUTION We replace $i$ in the expression $i^{2}-1$ with all consecutive integers from 1 up to 5 , including 1 and 5:

$$
\begin{aligned}
\sum_{i=1}^{5}\left(i^{2}-1\right) & =\left(1^{2}-1\right)+\left(2^{2}-1\right)+\left(3^{2}-1\right)+\left(4^{2}-1\right)+\left(5^{2}-1\right) \\
& =0+3+8+15+24 \\
& =50
\end{aligned}
$$

## EXAMPLE 2 Expand and simplify $\sum_{i=3}^{6}(-2)^{i}$.

SOLUTION We replace $i$ in the expression $(-2)^{i}$ with the consecutive integers beginning at 3 and ending at 6:

$$
\begin{aligned}
\sum_{i=3}^{6}(-2)^{i} & =(-2)^{3}+(-2)^{4}+(-2)^{5}+(-2)^{6} \\
& =-8+16+(-32)+64 \\
& =40
\end{aligned}
$$

## USING TECHNOLOGY

## Summing Series on a Graphing Calculator

A TI-83 graphing calculator has a built-in sum( function that, when used with the seq( function, allows us to add the terms of a series. Let's repeat Example 1 using our graphing calculator. First, we go to LIST and select MATH. The fifth option in that list is sum(, which we select. Then we go to LIST again and select OPS. From that list we select seq(. Next we enter $X^{\wedge} 2-1, X, 1,5$, and then we close both sets of parentheses. Our screen shows the following:

$$
\operatorname{sum}\left(\operatorname{seq}\left(X^{\wedge} 2-1, X, 1,5\right)\right) \quad \text { which will give us } \sum_{i=1}^{5}\left(i^{2}-1\right)
$$

When we press ENTER the calculator displays 50, which is the same result we obtained in Example 1.

## EXAMPLE 3 Expand $\sum_{i=2}^{5}\left(x^{i}-3\right)$.

SOLUTION We must be careful not to confuse the letter $x$ with $i$. The index $i$ is the quantity we replace by the consecutive integers from 2 to 5 , not $x$ :

$$
\sum_{i=2}^{5}\left(x^{i}-3\right)=\left(x^{2}-3\right)+\left(x^{3}-3\right)+\left(x^{4}-3\right)+\left(x^{5}-3\right)
$$

In the first three examples, we were given an expression with summation notation and asked to expand it. The next examples in this section illustrate how we can write an expression in expanded form as an expression involving summation notation.

## EXAMPLE 4 Write with summation notation: $1+3+5+7+9$.

SOLUTION A formula that gives us the terms of this sum is

$$
a_{i}=2 i-1
$$

where $i$ ranges from 1 up to and including 5 . Notice we are using the subscript $i$ in exactly the same way we used the subscript $n$ in the previous section-to indicate the general term. Writing the sum

$$
1+3+5+7+9
$$

with summation notation looks like this:

$$
\sum_{i=1}^{5}(2 i-1)
$$

## EXAMPLE 5 Write with summation notation: $3+12+27+48$.

SOLUTION We need a formula, in terms of $i$, that will give each term in the sum. Writing the sum as

$$
3 \cdot 1^{2}+3 \cdot 2^{2}+3 \cdot 3^{2}+3 \cdot 4^{2}
$$

we see the formula

$$
a_{i}=3 \cdot i^{2}
$$

where $i$ ranges from 1 up to and including 4 . Using this formula and summation notation, we can represent the sum

$$
3+12+27+48
$$

as

$$
\sum_{i=1}^{4} 3 i^{2}
$$

## EXAMPLE 6 Write with summation notation:

$$
\frac{x+3}{x^{3}}+\frac{x+4}{x^{4}}+\frac{x+5}{x^{5}}+\frac{x+6}{x^{6}}
$$

SOLUTION A formula that gives each of these terms is

$$
a_{i}=\frac{x+i}{x^{i}}
$$

where $i$ assumes all integer values between 3 and 6 , including 3 and 6 . The sum can be written as

$$
\sum_{i=3}^{6} \frac{x+i}{x^{i}}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.
$\begin{array}{ll}\text { A } & 1-3 \\ \text { B } & 4-6\end{array}$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is the difference between a sequence and a series?
2. Explain the summation notation $\sum_{i=1}^{4}$ in the series $\sum_{i=1}^{4}(2 i+1)$.
3. When will a finite series result in a numerical value versus an algebraic expression?
4. Determine for what values of $n$ the series $\sum_{i=1}^{n}(-1)^{i}$ will be equal to 0 .
Explain your answer.

## Problem Set 13.2

Expand and simplify each of the following.

1. $\sum_{i=1}^{4}(2 i+4)$
2. $\sum_{i=1}^{5}(3 i-1)$
3. $\sum_{i=2}^{3}\left(i^{2}-1\right)$
4. $\sum_{i=3}^{6}\left(i^{2}+1\right)$
5. $\sum_{i=1}^{4}\left(i^{2}-3\right)$
6. $\sum_{i=2}^{6}\left(2 i^{2}+1\right)$
7. $\sum_{i=1}^{4} \frac{i}{1+i}$
8. $\sum_{i=1}^{3} \frac{i^{2}}{2 i-1}$
9. $\sum_{i=1}^{4}(-3)^{i}$
10. $\sum_{i=1}^{4}\left(-\frac{1}{3}\right)^{i}$
11. $\sum_{i=3}^{6}(-2)^{i}$
12. $\sum_{i=4}^{6}\left(-\frac{1}{2}\right)^{i}$
13. $\sum_{i=2}^{6}(-2)^{i}$
14. $\sum_{i=2}^{5}(-3)^{i}$
15. $\sum_{i=1}^{5}\left(-\frac{1}{2}\right)^{i}$
16. $\sum_{i=3}^{6}\left(-\frac{1}{3}\right)^{i}$
17. $\sum_{i=2}^{5} \frac{i-1}{i+1}$
18. $\sum_{i=2}^{4} \frac{i^{2}-1}{i^{2}+1}$

Expand the following.
19. $\sum_{i=1}^{5}(x+i)$
20. $\sum_{i=2}^{7}(x+1)^{i}$
21. $\sum_{i=1}^{4}(x-2)^{i}$
22. $\sum_{i=2}^{5}\left(x+\frac{1}{i}\right)^{2}$
23. $\sum_{i=1}^{5} \frac{x+i}{x-1}$
24. $\sum_{i=1}^{6} \frac{x-3 i}{x+3 i}$
25. $\sum_{i=3}^{8}(x+i)^{i}$
26. $\sum_{i=1}^{5}(x+i)^{i+1}$
27. $\sum_{i=3}^{6}(x-2 i)^{i+3}$
28. $\sum_{i=5}^{8}\left(\frac{x-i}{x+i}\right)^{2 i}$

Write each of the following sums with summation notation. Do not calculate the sum.
29. $2+4+8+16$
30. $3+5+7+9+11$
31. $4+8+16+32+64$
32. $3+8+15+24$

Write each of the following sums with summation notation. Do not calculate the sum. Note: More than one answer is possible.
33. $5+9+13+17+21$
34. $3-6+12-24+48$
35. $-4+8-16+32$
36. $15+24+35+48+63$
37. $\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}+\frac{7}{8}$

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38. $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$
39. $\frac{1}{3}+\frac{2}{5}+\frac{3}{7}+\frac{4}{9}$
40. $\frac{3}{1}+\frac{5}{3}+\frac{7}{5}+\frac{9}{7}$
41. $(x-2)^{6}+(x-2)^{7}+(x-2)^{8}+(x-2)^{9}$
42. $(x+1)^{3}+(x+2)^{4}+(x+3)^{5}+(x+4)^{6}+(x+5)^{7}$
43. $\left(1+\frac{1}{x}\right)^{2}+\left(1+\frac{2}{x}\right)^{3}+\left(1+\frac{3}{x}\right)^{4}+\left(1+\frac{4}{x}\right)^{5}$
44. $\frac{x-1}{x+2}+\frac{x-2}{x+4}+\frac{x-3}{x+6}+\frac{x-4}{x+8}+\frac{x-5}{x+10}$
45. $\frac{x}{x+3}+\frac{x}{x+4}+\frac{x}{x+5}$
46. $\frac{x-3}{x^{3}}+\frac{x-4}{x^{4}}+\frac{x-5}{x^{5}}+\frac{x-6}{x^{6}}$
47. $x^{2}(x+2)+x^{3}(x+3)+x^{4}(x+4)$
48. $x(x+2)^{2}+x(x+3)^{3}+x(x+4)^{4}$
49. Repeating Decimals Any repeating, nonterminating decimal may be viewed as a series. For instance, $\frac{2}{3}=0.6+0.06+0.006+0.0006+\cdots$. Write the following fractions as series.
a. $\frac{1}{3}$
b. $\frac{2}{9}$
c. $\frac{3}{11}$
50. Repeating Decimals Refer to the previous exercise, and express the following repeating decimals as fractions.
a. $0.55555 \cdots$
b. 1.33333 .
c. 0.29292929 •

## Applying the Concepts

51. Skydiving A skydiver jumps from a plane and falls 16 feet the first second, 48 feet the second second, and 80 feet the third second. If he continues to fall in the same manner, how far will he fall the seventh second? What is the distance he falls in 7 seconds?

52. Bacterial Growth After 1 day, a colony of 50 bacteria reproduces to become 200 bacteria. After 2 days, they reproduce to become 800 bacteria. If they continue to reproduce at this rate, how many bacteria will be present after 4 days?

53. Akaka Falls In Section 11.1, when a boulder fell from the top of Akaka Falls in Hawaii, the sequence generated during the first 5 seconds the boulder fell was $16,48,80,112,144$.
a. Write a finite series that represents the sum of this sequence.
b. The general term of this sequence was given as $a_{n}=32 n-16$. Write the series produced in part $a$ in summation notation.
54. Pendulum A pendulum swings 12 feet left to right on its first swing, and on each swing following the first, swings $\frac{3}{4}$ of the previous swing. The distance the pendulum traveled in 5 seconds can be expressed in summation notation $\sum_{i=1}^{5} 12\left(\frac{3}{4}\right)^{i-1}$. Expand the summation notation and simplify. Round your final answer to the nearest tenth.

## Maintaining Your Skills

Use the properties of logarithms to expand each of the following expressions.
55. $\log _{2} x^{3} y$
56. $\log _{7} \frac{x^{2}}{y^{4}}$
57. $\log _{10} \frac{\sqrt[3]{x}}{y^{2}}$
58. $\log _{10} \sqrt[3]{\frac{x}{y^{2}}}$

Write each expression as a single logarithm.
59. $\log _{10} x-\log _{10} y^{2}$
60. $\log _{10} x^{2}+\log _{10} y^{2}$
61. $2 \log _{3} x-3 \log _{3} y-4 \log _{3} z$
62. $\frac{1}{2} \log _{6} x+\frac{1}{3} \log _{6} y+\frac{1}{4} \log _{6} z$

Solve each equation.
63. $\log _{4} x-\log _{4} 5=2$
64. $\log _{3} 6+\log _{3} x=4$
65. $\log _{2} x+\log _{2}(x-7)=3$
66. $\log _{5}(x+1)+\log _{5}(x-3)=1$

## Getting Ready for the Next Section

Simplify.
67. $2+9(8)$
68. $\frac{1}{2}+9\left(\frac{1}{2}\right)$
69. $\frac{10}{2}\left(\frac{1}{2}+5\right)$
70. $\frac{10}{2}(2+74)$
71. $3+(n-1) 2$
72. $7+(n-1) 3$

Solve each system of equations.
73. $x+2 y=7$
74. $x+3 y=14$
$x+7 y=17$
$x+9 y=32$

## Extending the Concepts

Solve each of the following equations for $x$.
75. $\sum_{i=1}^{4}(x-i)=16$
76. $\sum_{i=3}^{8}(2 x+i)=30$

## 13.3

## OBJECTIVES

A Find the common difference for an arithmetic sequence.

B Find the general term for an arithmetic sequence.

C Find the sum of the first $n$ terms of an arithmetic sequence.

## Arithmetic Sequences

In this and the following section, we will review and expand upon two major types of sequences, which we have worked with previously-arithmetic sequences and geometric sequences.

DEFINITION An arithmetic sequence is a sequence of numbers in which each term is obtained from the preceding term by adding the same amount each time. An arithmetic sequence is also called an arithmetic progression.

The sequence

$$
2,6,10,14, \ldots
$$

is an example of an arithmetic sequence, because each term is obtained from the preceding term by adding 4 each time. The amount we add each time-in this case, 4 -is called the common difference, because it can be obtained by sub-
tracting any two consecutive terms. (The term with the larger subscript must be written first.) The common difference is denoted by $d$.

EXAMPLE 1 Give the common difference $d$ for the arithmetic sequence $4,10,16,22, \ldots$.
SOLUTION Because each term can be obtained from the preceding term by adding 6 , the common difference is 6 . That is, $d=6$.

EXAMPLE 2 Give the common difference for $100,93,86,79, \ldots$.
SOLUTION The common difference in this case is $d=-7$, since adding -7 to any term always produces the next consecutive term.

EXAMPLE 3 Give the common difference for $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. SOLUTION The common difference is $d=\frac{1}{2}$.

## The General Term

The general term $a_{n}$ of an arithmetic progression can always be written in terms of the first term $a_{1}$ and the common difference $d$. Consider the sequence from Example 1:

$$
4,10,16,22, \ldots
$$

We can write each term in terms of the first term 4 and the common difference 6:

$$
\begin{array}{cccc}
4, & 4+(1 \cdot 6), & 4+(2 \cdot 6), & 4+(3 \cdot 6), \ldots \\
a_{1}, & a_{2}, & a_{3}, & a_{4}, \ldots
\end{array}
$$

Observing the relationship between the subscript on the terms in the second line and the coefficients of the 6 's in the first line, we write the general term for the sequence as

$$
a_{n}=4+(n-1) 6
$$

We generalize this result to include the general term of any arithmetic sequence.

Arithmetic Sequence The general term of an arithmetic progression with first term $a_{1}$ and common difference $d$ is given by

$$
a_{n}=a_{1}+(n-1) d
$$

SOLUTION The first term is $a_{1}=7$, and the common difference is $d=3$. Substituting these numbers into the formula for the general term, we have

$$
a_{n}=7+(n-1) 3
$$

which we can simplify, if we choose, to

$$
\begin{aligned}
a_{n} & =7+3 n-3 \\
& =3 n+4
\end{aligned}
$$

EXAMPLE 5 Find the general term of the arithmetic progression whose third term $a_{3}$ is 7 and whose eighth term $a_{8}$ is 17 .
SOLUTION According to the formula for the general term, the third term can be written as $a_{3}=a_{1}+2 d$, and the eighth term can be written as $a_{8}=a_{1}+7 d$. Because these terms are also equal to 7 and 17 , respectively, we can write

$$
\begin{aligned}
& a_{3}=a_{1}+2 d=7 \\
& a_{8}=a_{1}+7 d=17
\end{aligned}
$$

To find $a_{1}$ and $d$, we simply solve the system:

$$
\begin{aligned}
& a_{1}+2 d=7 \\
& a_{1}+7 d=17
\end{aligned}
$$

We add the opposite of the top equation to the bottom equation. The result is

$$
\begin{aligned}
5 d & =10 \\
d & =2
\end{aligned}
$$

To find $a_{1}$, we simply substitute 2 for $d$ in either of the original equations and get

$$
a_{1}=3
$$

The general term for this progression is

$$
a_{n}=3+(n-1) 2
$$

which we can simplify to

$$
a_{n}=2 n+1
$$

The sum of the first $n$ terms of an arithmetic sequence is denoted by $S_{n}$. The following theorem gives the formula for finding $S_{n}$, which is sometimes called the $n$th partial sum.

Theorem 1 The sum of the first $n$ terms of an arithmetic sequence whose first term is $a_{1}$ and whose $n$th term is $a_{n}$ is given by

$$
S_{n}=\frac{n}{2}\left(a_{1}+a_{n}\right)
$$

Proof We can write $S_{n}$ in expanded form as

$$
S_{n}=a_{1}+\left[a_{1}+d\right]+\left[a_{1}+2 d\right]+\cdots+\left[a_{1}+(n-1) d\right]
$$

We can arrive at this same series by starting with the last term $a_{n}$ and subtracting $d$ each time. Writing $S_{n}$ this way, we have

$$
S_{n}=a_{n}+\left[a_{n}-d\right]+\left[a_{n}-2 d\right]+\cdots+\left[a_{n}-(n-1) d\right]
$$

If we add the preceding two expressions term by term, we have

$$
\begin{aligned}
& 2 S_{n}=\left(a_{1}+a_{n}\right)+\left(a_{1}+a_{n}\right)+\left(a_{1}+a_{n}\right)+\cdots+\left(a_{1}+a_{n}\right) \\
& 2 S_{n}=n\left(a_{1}+a_{n}\right) \\
& S_{n}=\frac{n}{2}\left(a_{1}+a_{n}\right)
\end{aligned}
$$

EXAMPLE 6 Find the sum of the first 10 terms of the arithmetic progression $2,10,18,26, \ldots$

SOLUTION The first term is 2 , and the common difference is 8 . The tenth term is

$$
\begin{aligned}
a_{10} & =2+9(8) \\
& =2+72 \\
& =74
\end{aligned}
$$

Substituting $n=10, a_{1}=2$, and $a_{10}=74$ into the formula

$$
S_{n}=\frac{n}{2}\left(a_{1}+a_{n}\right)
$$

we have

$$
\begin{aligned}
S_{10} & =\frac{10}{2}(2+74) \\
& =5(76) \\
& =380
\end{aligned}
$$

The sum of the first 10 terms is 380 .

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1-3

B 4,5
C 6

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. Explain how to determine if a sequence is arithmetic.
2. What is a common difference?
3. Suppose the value of $a_{5}$ is given. What other possible pieces of information could be given to have enough information to obtain the first 10 terms of the sequence?
4. Explain the formula $a_{n}=a_{1}+(n-1) d$ in words so that someone who wanted to find the nth term of an arithmetic sequence could do so from your description.

Determine which of the following sequences are arithmetic progressions. For those that are arithmetic progressions, identify the common difference $d$.

1. $1,2,3,4, \ldots$
2. $4,6,8,10, \ldots$
3. $1,2,4,7, \ldots$
4. $1,2,4,8, \ldots$
5. $50,45,40, \ldots$
6. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
7. $1,4,9,16, \ldots$
8. $5,7,9,11, \ldots$
9. $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, \ldots$
10. $5,11,17, \ldots$

Each of the following problems refers to arithmetic sequences.
11. If $a_{1}=3$ and $d=4$, find $a_{n}$ and $a_{24}$.
12. If $a_{1}=5$ and $d=10$, find $a_{n}$ and $a_{100}$.
13. If $a_{1}=6$ and $d=-2$, find $a_{10}$ and $S_{10}$.
14. If $a_{1}=7$ and $d=-1$, find $a_{24}$ and $S_{24}$.
15. If $a_{6}=17$ and $a_{12}=29$, find the term $a_{1}$, the common difference $d$, and then find $a_{30}$.
16. If $a_{5}=23$ and $a_{10}=48$, find the first term $a_{1}$, the common difference $d$, and then find $a_{40}$.
17. If the third term is 16 and the eighth term is 26 , find the first term, the common difference, and then find $a_{20}$ and $S_{20}$.
18. If the third term is 16 and the eighth term is 51 , find the first term, the common difference, and then find $a_{50}$ and $S_{50}$.
19. If $a_{1}=3$ and $d=4$, find $a_{20}$ and $S_{20}$.
20. If $a_{1}=40$ and $d=-5$, find $a_{25}$ and $S_{25}$.
21. If $a_{4}=14$ and $a_{10}=32$, find $a_{40}$ and $S_{40}$.
22. If $a_{7}=0$ and $a_{11}=-\frac{8}{3}$, find $a_{61}$ and $S_{61}$.
23. If $a_{6}=-17$ and $S_{6}=-12$, find $a_{1}$ and $d$.
24. If $a_{10}=12$ and $S_{10}=40$, find $a_{1}$ and $d$.
25. Find $a_{85}$ for the sequence $14,11,8,5, \ldots$
26. Find $S_{100}$ for the sequence $-32,-25,-18,-11, \ldots$
27. If $S_{20}=80$ and $a_{1}=-4$, find $d$ and $a_{39}$.
28. If $S_{24}=60$ and $a_{1}=4$, find $d$ and $a_{116}$.
29. Find the sum of the first 100 terms of the sequence $5,9,13,17, \ldots$
30. Find the sum of the first 50 terms of the sequence $8,11,14,17, \ldots$
31. Find $a_{35}$ for the sequence $12,7,2,-3, \ldots$
32. Find $a_{45}$ for the sequence $25,20,15,10, \ldots$.
33. Find the tenth term and the sum of the first 10 terms of the sequence $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$.
34. Find the 15 th term and the sum of the first 15 terms of the sequence $-\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, \ldots$.

## Applying the Concepts

Straight-Line Depreciation Straight-line depreciation is an accounting method used to help spread the cost of new equipment over a number of years. The value at any time during the life of the machine can be found with a linear equation in two variables. For income tax purposes, however, it is the value at the end of the year that is most important, and for this reason sequences can be used.
35. Value of a Copy Machine A large copy machine sells for $\$ 18,000$ when it is new. Its value decreases
\$3,300 each year after that. We can use an arithmetic sequence to find the value of the machine at the end of each year. If we let $a_{0}$ represent the value when it is
 purchased, then $a_{1}$ is the value after 1 year, $a_{2}$ is the value after 2 years, and so on.
a. Write the first 5 terms of the sequence.
b. What is the common difference?
c. Construct a line graph for the first 5 terms of the sequence.
d. Use the line graph to estimate the value of the copy machine 2.5 years after it is purchased.
e. Write the sequence from part a using a recursive formula.
36. Value of a Forklift An electric forklift sells for $\$ 125,000$ when new. Each year after that, it decreases $\$ 16,500$ in value.
a. Write an arithmetic sequence that gives the value of the forklift at the end of each of the first 5 years after it is purchased.
b. What is the common difference for this sequence?
c. Construct a line graph for this sequence.
d. Use the line graph to estimate the value of the forklift 3.5 years after it is purchased.
e. Write the sequence from part (a) using a recursive formula.

37. Distance A rocket travels vertically 1,500 feet in its first second of flight, and then about 40 feet less each succeeding second. Use these estimates to answer the following questions.
a. Write a sequence of the vertical distance traveled by a rocket in each of its first 6 seconds.
b. Is the sequence in part (a) an arithmetic sequence? Explain why or why not.
c. What is the general term of the sequence in part (a)?
38. Depreciation Suppose an automobile sells for $N$ dollars new, and then depreciates 40\% each year.
a. Write a sequence for the value of this automobile (in terms of $N$ ) for each year.
b. What is the general term of the sequence in part (a)?
c. Is the sequence in part (a) an arithmetic sequence? Explain why it is or is not.
39. Triangular Numbers The first four triangular numbers are $\{1,3,6,10, \ldots\}$, and are illustrated in the following diagram.

a. Write a sequence of the first 15 triangular numbers.
b. Write the recursive general term for the sequence of triangular numbers.
c. Is the sequence of triangular numbers an arithmetic sequence? Explain why it is or is not.
40. Arithmetic Means Three (or more) arithmetic means between two numbers may be found by forming an arithmetic sequence using the original two numbers and the arithmetic means. For example, three arithmetic means between 10 and 34 may be found by examining the sequence $\{10, a, b, c, 34\}$. For the sequence to be arithmetic, the common difference must be 6 ; therefore, $a=16, b=22$, and $c=28$. Use this idea to answer the following questions.
a. Find four arithmetic means between 10 and 35 .
b. Find three arithmetic means between 2 and 62 .
c. Find five arithmetic means between 4 and 28 .

## Maintaining Your Skills

Find the following logarithms. Round to four decimal places.
41. $\log 576$
42. $\log 57,600$
43. $\log 0.0576$
44. $\log 0.000576$

Find $x$.
45. $\log x=2.6484$
46. $\log x=7.9832$
47. $\log x=-7.3516$
48. $\log x=-2.0168$

## Getting Ready for the Next Section

Simplify.
49. $\frac{1}{8}\left(\frac{1}{2}\right)$
50. $\frac{1}{4}\left(\frac{1}{2}\right)$
51. $\frac{3 \sqrt{3}}{3}$
52. $\frac{3}{\sqrt{3}}$
53. $2 \cdot 2^{n-1}$
54. $3 \cdot 3^{n-1}$
55. $\frac{a r^{6}}{a r^{3}}$
56. $\frac{a r^{7}}{a r^{4}}$
57. $\frac{\frac{1}{5}}{1-\frac{1}{2}}$
58. $\frac{\frac{9}{10}}{1-\frac{1}{10}}$
59. $\frac{-3\left[(-2)^{8}-1\right]}{-2-1}$
60. $\frac{4\left[\left(\frac{1}{2}\right)^{6}-1\right]}{\frac{1}{2}-1}$

## Extending the Concepts

61. Find $a_{1}$ and $d$ for an arithmetic sequence with $S_{7}=147$ and $S_{13}=429$.
62. Find $a_{1}$ and $d$ for an arithmetic sequence with $S_{12}=282$ and $S_{19}=646$.
63. Find $d$ for an arithmetic sequence in which $a_{15}-a_{7}=-24$.
64. Find $d$ for an arithmetic sequence in which $a_{25}-a_{15}=120$.
65. Find the sum of the first 50 terms of the sequence $1,1+\sqrt{2}, 1+2 \sqrt{2}$,
66. Given $a_{1}=-3$ and $d=5$ in an arithmetic sequence, and $S_{n}=890$, find $n$.

## TIVES

## OBJECTIVES

A Find the common ratio for a geometric sequence.

B Find the $n$th term for a geometric sequence.

C Find the general term for a geometric sequence.

D Find the sum of the first $n$ terms of a geometric sequence.

E Find the sum of all terms of an infinite geometric sequence.

## Geometric Sequences

This section is concerned with the second major classification of sequences, called geometric sequences. The problems in this section are similar to the problems in the preceding section.

DEFINITION A sequence of numbers in which each term is obtained from the previous term by multiplying by the same amount each time is called a geometric sequence. Geometric sequences are also called geometric progressions.

The sequence

$$
3,6,12,24, \ldots
$$

is an example of a geometric progression. Each term is obtained from the previous term by multiplying by 2 . The amount by which we multiply each time-in this case, 2 -is called the common ratio. The common ratio is denoted by $r$ and can be found by taking the ratio of any two consecutive terms. (The term with the larger subscript must be in the numerator.)

EXAMPLE 1 Find the common ratio for the geometric progression.

$$
\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots
$$

SOLUTION Because each term can be obtained from the term before it by multiplying by $\frac{1}{2}$, the common ratio is $\frac{1}{2}$. That is, $r=\frac{1}{2}$.

EXAMPLE 2 Find the common ratio for $\sqrt{3}, 3,3 \sqrt{3}, 9, \ldots$
SOLUTION If we take the ratio of the third term to the second term, we have

$$
\frac{3 \sqrt{3}}{3}=\sqrt{3}
$$

The common ratio is $r=\sqrt{3}$.

Geometric Sequences The general term $a_{n}$ of a geometric sequence with first term $a_{1}$ and common ratio $r$ is given by

$$
a_{n}=a_{1} r^{n-1}
$$

To see how we arrive at this formula, consider the following geometric progression whose common ratio is 3 :

$$
2,6,18,54, \ldots
$$

We can write each term of the sequence in terms of the first term 2 and the common ratio 3:

$$
\begin{array}{cccc}
2 \cdot 3^{0}, & 2 \cdot 3^{1}, & 2 \cdot 3^{2}, & 2 \cdot 3^{3}, \ldots \\
a_{1}, & a_{2}, & a_{3}, & a_{4}, \ldots
\end{array}
$$

Observing the relationship between the two preceding lines, we find we can write the general term of this progression as

$$
a_{n}=2 \cdot 3^{n-1}
$$

Because the first term can be designated by $a_{1}$ and the common ratio by $r$, the formula $a_{n}=2 \cdot 3^{n-1}$ coincides with the formula $a_{n}=a_{1} 1^{n-1}$

## EXAMPLE 3 Find the general term for the geometric progression

$$
5,10,20, \ldots
$$

SOLUTION The first term is $a_{1}=5$, and the common ratio is $r=2$. Using these values in the formula

$$
a_{n}=a_{1} r^{n-1}
$$

we have

$$
a_{n}=5 \cdot 2^{n-1}
$$

## EXAMPLE 4 Find the tenth term of the sequence $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots$.

SOLUTION The sequence is a geometric progression with first term $a_{1}=3$ and common ratio $r=\frac{1}{2}$. The tenth term is

$$
a_{10}=3\left(\frac{1}{2}\right)^{9}=\frac{3}{512}
$$

## EXAMPLE 5 Find the general term for the geometric progression

 whose fourth term is 16 and whose seventh term is 128 .SOLUTION The fourth term can be written as $a_{4}=a_{1} r^{3}$, and the seventh term can be written as $a_{7}=a_{1} r^{6}$.

$$
\begin{aligned}
& a_{4}=a_{1} r^{3}=16 \\
& a_{7}=a_{1} r^{6}=128
\end{aligned}
$$

We can solve for $r$ by using the ratio $\frac{a_{7}}{a_{4}}$.

$$
\begin{aligned}
\frac{a_{7}}{a_{4}} & =\frac{a_{1} r^{6}}{a_{1} r^{3}}=\frac{128}{16} \\
r^{3} & =8 \\
r & =2
\end{aligned}
$$

The common ratio is 2 . To find the first term, we substitute $r=2$ into either of the original two equations. The result is

$$
a_{1}=2
$$

The general term for this progression is

$$
a_{n}=2 \cdot 2^{n-1}
$$

which we can simplify by adding exponents, because the bases are equal:

$$
a_{n}=2^{n}
$$

As was the case in the preceding section, the sum of the first $n$ terms of a geometric progression is denoted by $S_{n}$, which is called the nth partial sum of the progression.

Theorem 2 The sum of the first $n$ terms of a geometric progression with first term $a_{1}$ and common ratio $r$ is given by the formula

$$
S_{n}=\frac{a_{1}\left(r^{n}-1\right)}{r-1}
$$

Proof We can write the sum of the first $n$ terms in expanded form:

$$
\begin{equation*}
S_{n}=a_{1}+a_{1} r+a_{1} r^{2}+\cdots+a_{1} r^{n-1} \tag{1}
\end{equation*}
$$

Then multiplying both sides by $r$, we have

$$
\begin{equation*}
r S_{n}=a_{1} r+a_{1} r^{2}+a_{1} r^{3}+\cdots+a_{1} r^{n} \tag{2}
\end{equation*}
$$

If we subtract the left side of equation (1) from the left side of equation (2) and do the same for the right sides, we end up with

$$
r S_{n}-S_{n}=a_{1} r^{n}-a_{1}
$$

We factor $S_{n}$ from both terms on the left side and $a_{1}$ from both terms on the right side of this equation:

$$
S_{n}(r-1)=a_{1}\left(r^{n}-1\right)
$$

Dividing both sides by $r-1$ gives the desired result:

$$
S_{n}=\frac{a_{1}\left(r^{n}-1\right)}{r-1}
$$

EXAMPLE 6 Find the sum of the first 10 terms of the geometric progression 5, 15, 45, 135,....
SOLUTION The first term is $a_{1}=5$, and the common ratio is $r=3$. Substituting these values into the formula for $S_{10}$, we have the sum of the first 10 terms of the sequence:

$$
\begin{aligned}
S_{10} & =\frac{5\left(3^{10}-1\right)}{3-1} \\
& =\frac{5\left(3^{10}-1\right)}{2}
\end{aligned}
$$

The answer can be left in this form. A calculator will give the result as 147,620 .

## Infinite Geometric Series

Suppose the common ratio for a geometric sequence is a number whose absolute value is less than 1 -for instance, $\frac{1}{2}$. The sum of the first $n$ terms is given by the formula

$$
S_{n}=\frac{a_{1}\left[\left(\frac{1}{2}\right)^{n}-1\right]}{\frac{1}{2}-1}
$$

As $n$ becomes larger and larger, the term $\left(\frac{1}{2}\right)^{n}$ will become closer and closer to 0 . That is, for $n=10,20$, and 30 , we have the following approximations:

$$
\begin{aligned}
& \left(\frac{1}{2}\right)^{10} \approx 0.001 \\
& \left(\frac{1}{2}\right)^{20} \approx 0.000001 \\
& \left(\frac{1}{2}\right)^{30} \approx 0.000000001
\end{aligned}
$$

so that for large values of $n$, there is little difference between the expression

$$
\frac{a_{1}\left(r^{n}-1\right)}{r-1}
$$

and the expression

$$
\frac{a_{1}(0-1)}{r-1}=\frac{-a_{1}}{r-1}=\frac{a_{1}}{1-r} \quad \text { if } \quad|r|<1
$$

In fact, the sum of the terms of a geometric sequence in which $|r|<1$ actually becomes the expression

$$
\frac{a_{1}}{1-r}
$$

as $n$ approaches infinity. To summarize, we have the following:

The Sum of an Infinite Geometric Series
If a geometric sequence has first term $a_{1}$ and common ratio $r$ such that $|r|<1$, then the following is called an infinite geometric series:

$$
S=\sum_{i=0}^{\infty} a_{1} r^{i}=a_{1}+a_{1} r+a_{1} r^{2}+a_{1} r^{3}+\cdots
$$

Its sum is given by the formula

$$
S=\frac{a_{1}}{1-r}
$$

## EXAMPLE 7 Find the sum of the infinite geometric series

$$
\frac{1}{5}+\frac{1}{10}+\frac{1}{20}+\frac{1}{40}+\cdots
$$

SOLUTION The first term is $a_{1}=\frac{1}{5}$, and the common ratio is $r=\frac{1}{2}$, which has an absolute value less than 1 . Therefore, the sum of this series is

$$
S=\frac{a_{1}}{1-r}=\frac{\frac{1}{5}}{1-\frac{1}{2}}=\frac{\frac{1}{5}}{\frac{1}{2}}=\frac{2}{5}
$$

## EXAMPLE 8 Show that $0.999 \ldots$ is equal to 1.

SOLUTION We begin by writing $0.999 \ldots$ as an infinite geometric series:

$$
\begin{aligned}
& 0.999 \ldots=0.9+0.09+0.009+0.0009+\cdots \\
& \quad=\frac{9}{10}+\frac{9}{100}+\frac{9}{1,000}+\frac{9}{10,000}+\cdots \\
& \quad=\frac{9}{10}+\frac{9}{10}\left(\frac{1}{10}\right)+\frac{9}{10}\left(\frac{1}{10}\right)^{2}+\frac{9}{10}\left(\frac{1}{10}\right)^{3}+\cdots
\end{aligned}
$$

## LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

## A 1, 2

B 4
C 3,5
D 6
E 7,8

As the last line indicates, we have an infinite geometric series with $a_{1}=\frac{9}{10}$ and $r=\frac{1}{10}$. The sum of this series is given by

$$
S=\frac{a_{1}}{1-r}=\frac{\frac{9}{10}}{1-\frac{1}{10}}=\frac{\frac{9}{10}}{\frac{9}{10}}=1
$$

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is a common ratio?
2. Explain the formula $a_{n}=a_{1} r^{n-1}$ in words so that someone who wanted to find the nth term of a geometric sequence could do so from your description.
3. When is the sum of an infinite geometric series a finite number?
4. Explain how a repeating decimal can be represented as an infinite geometric series.

## Problem Set 13.4

Identify those sequences that are geometric progressions. For those that are geometric, give the common ratio $r$.

1. $1,5,25,125, \ldots$
2. $6,12,24,48, \ldots$
3. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots$
4. $5,10,15,20, \ldots$
5. $4,9,16,25, \ldots$
6. $-1, \frac{1}{3},-\frac{1}{9}, \frac{1}{27}, \ldots$
7. $-2,4,-8,16, \ldots$
8. $1,8,27,64, \ldots$
9. $4,6,8,10, \ldots$
10. $1,-3,9,-27, \ldots$

Each of the following problems gives some information about a specific geometric progression.
11. If $a_{1}=4$ and $r=3$, find $a_{n}$.
12. If $a_{1}=5$ and $r=2$, find $a_{n}$.
13. If $a_{1}=-2$ and $r=-\frac{1}{2}$, find $a_{6}$.
14. If $a_{1}=25$ and $r=-\frac{1}{5}$, find $a_{6}$.
15. If $a_{1}=3$ and $r=-1$, find $a_{20}$.
16. If $a_{1}=-3$ and $r=-1$, find $a_{20}$.
17. If $a_{1}=10$ and $r=2$, find $S_{10}$.
18. If $a_{1}=8$ and $r=3$, find $S_{5}$.
19. If $a_{1}=1$ and $r=-1$, find $S_{20}$.
20. If $a_{1}=1$ and $r=-1$, find $S_{21}$.
21. Find $a_{8}$ for $\frac{1}{5}, \frac{1}{10}, \frac{1}{20}, \ldots$
22. Find $a_{8}$ for $\frac{1}{2}, \frac{1}{10}, \frac{1}{50}, \ldots$
23. Find $S_{5}$ for $-\frac{1}{2},-\frac{1}{4},-\frac{1}{8}, \ldots$
24. Find $S_{6}$ for $-\frac{1}{2}, 1,-2, \ldots$.
25. Find $a_{10}$ and $S_{10}$ for $\sqrt{2}, 2,2 \sqrt{2}, \ldots$.
$=$ Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
26. Find $a_{8}$ and $S_{8}$ for $\sqrt{3}, 3,3 \sqrt{3}, \ldots$.
27. Find $a_{6}$ and $S_{6}$ for $100,10,1, \ldots$.
28. Find $a_{6}$ and $S_{6}$ for $100,-10,1, \ldots$.
29. If $a_{4}=40$ and $a_{6}=160$, find $r$.
30. If $a_{5}=\frac{1}{8}$ and $a_{8}=\frac{1}{64}$, find $r$.
31. Given the sequence $-3,6,-12,24, \ldots$, find $a_{8}$ and $S_{8}$.
32. Given the sequence $4,2,1, \frac{1}{2}, \ldots$, find $a_{9}$ and $S_{9}$.
33. Given $a_{7}=13$ and $a_{10}=104$, find $r$.
34. Given $a_{5}=-12$ and $a_{8}=324$, find $r$.

Find the sum of each geometric series.
35. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$
36. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$
37. $4+2+1+\cdots$
38. $8+4+2+\cdots$
39. $2+1+\frac{1}{2}+\cdots$.
40. $3+1+\frac{1}{3}+\cdots$
41. $\frac{4}{3}-\frac{2}{3}+\frac{1}{3}+\cdots$
42. $6-4+\frac{8}{3}+\cdots$
43. $\frac{2}{5}+\frac{4}{25}+\frac{8}{125}+\cdots$
44. $\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\cdots$
45. $\frac{3}{4}+\frac{1}{4}+\frac{1}{12}+\cdots$
46. $\frac{5}{3}+\frac{1}{3}+\frac{1}{15}+\cdots$
47. Show that $0.444 \ldots$ is the same as $\frac{4}{9}$.
48. Show that $0.333 \ldots$ is the same as $\frac{1}{3}$.
49. Show that $0.272727 \ldots$ is the same as $\frac{3}{11}$.
50. Show that $0.545454 \ldots$ is the same as $\frac{6}{11}$.

## Applying the Concepts

Declining-Balance Depreciation The declining-balance method of depreciation is an accounting method businesses use to deduct most of the cost of new equipment during the first few years of purchase. The value at any time during the life of the machine can be found with a linear equation in two variables. For income tax purposes, however, it is the value at the end of the year that is most important, and for this reason sequences can be used.
51. Value of a Crane A construction crane sells for $\$ 450,000$ if purchased new. After that, the value decreases by $30 \%$ each year. We can use a geometric sequence to find the value of the crane at the end of each year. If we let $a_{0}$ represent the value when it is purchased, then $a_{1}$ is the value after 1 year, $a_{2}$ is the value after 2 years, and so on.
a. Write the first five terms of the sequence.
b. What is the common ratio?
c. Construct a line graph for the first five terms of the sequence.
d. Use the line graph to estimate the value of the crane 4.5 years after it is purchased.
e. Write the sequence from part a using a recursive formula.

52. Value of a Printing Press A large printing press sells for $\$ 375,000$ when it is new. After that, its value decreases $25 \%$ each year.
a. Write a geometric sequence that gives the value of the press at the end of each of the first 5 years after it is purchased.
b. What is the common ratio for this sequence?
c. Construct a line graph for this sequence.
d. Use the line graph to estimate the value of the printing press 1.5 years after it is purchased.
e. Write the sequence from part a using a recursive formula.

53. Adding Terms Given the geometric series $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$,
a. Find the sum of all the terms.
b. Find the sum of the first six terms.
c. Find the sum of all but the first six terms.
54. Perimeter Triangle $A B C$ has a perimeter of 40 inches. A new triangle $X Y Z$ is formed by connecting the midpoints of the sides of the first triangle, as shown in the following figure. Because midpoints are joined, the perimeter of triangle $X Y Z$ will be onehalf the perimeter of triangle $A B C$, or 20 inches.


Midpoints of the sides of triangle $X Y Z$ are used to form a new triangle RST. If this pattern of using midpoints to draw triangles is continued seven more times, so there are a total of 10 triangles drawn, what will be the sum of the perimeters of these ten triangles?
55. Bouncing Ball A ball is dropped from a height of 20 feet. Each time it bounces it returns to $\frac{7}{8}$ of the height it fell from. If the ball is allowed to bounce an infinite number of times, find the total vertical distance that the ball travels.

56. Stacking Paper Assume that a thin sheet of paper is 0.002 inch thick. The paper is torn in half, and the two halves placed together.
a. How thick is the pile of torn paper?
b. The pile of paper is torn in half again, and then the two halves placed together and torn in half again. The paper is large enough so this process may be performed a total of 5 times. How thick is the pile of torn paper?
c. Refer to the tearing and piling process described in part b. Assuming that somehow the original paper is large enough, how thick is the pile of torn paper if 25 tears are made?
57. Pendulum A pendulum swings 15 feet left to right on its first swing. On each swing following the first, the pendulum swings $\frac{4}{5}$ of the previous swing.
a. Write the general term for this geometric sequence.
b. If the pendulum is allowed to swing an infinite number of times, what is the total distance the pendulum will travel?
58. Bacteria Growth A bacteria colony of 50 bacteria doubles each day. The number of bacteria in the colony each day can be modeled as a geometric sequence.
a. What is the first term of the sequence? What is the common ratio?
b. Write the general term for this sequence.
c. What would be the total number of bacteria in the colony after 10 days?

## Maintaining Your Skills

Find each of the following to the nearest hundredth.
59. $\log _{4} 20$
60. $\log _{7} 21$
61. $\ln 576$
62. $\ln 5,760$
63. Solve the formula $A=10 e^{5 t}$ for $t$.
64. Solve the formula $A=P \cdot 2^{-5 t}$ for $t$.

## Getting Ready for the Next Section

65. $(x+y)^{0}$
66. $(x+y)^{1}$

Expand and multiply.
67. $(x+y)^{2}$
68. $(x+y)^{3}$

Simplify.
69. $\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1)} \quad$ 70. $\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)}$

## Extending the Concepts

71. Find the fraction form for 0.636363 . . . .
72. Find the fraction form for 0.123123123 ... .
73. The sum of an infinite geometric series is 6 and the first term is 4 . Find the common ratio.
74. The sum of an infinite geometric series is $\frac{5}{2}$ and the first term is 3 . Find the common ratio.
75. Sierpinski Triangle In the sequence that follows, the figures are moving toward what is known as the Sierpinski triangle. To construct the figure in stage 2 , we remove the triangle formed from the midpoints of the sides of the shaded region in stage 1 . Likewise, the figure in stage 3 is found by removing the triangles formed by connecting the midponts of the sides of the shaded regions in stage 2. If we repeat this process infinitely many times, we arrive at the Sierpinski triangle.
a. If the shaded region in stage 1 has an area of 1 , find the area of the shaded regions in stages 2 through 4.
b. Do the areas you found in part a form an arithmetic sequence or a geometric sequence?



Stage 3


Stage 4
c. The Sierpinski triangle is the triangle that is formed after the process of forming the stages shown in the figure is repeated infinitely many times. What do you think the area of the shaded region of the Sierpinski triangle will be?
d. Suppose the perimeter of the shaded region of the triangle in stage 1 is 1 . If we were to find the perimeters of the shaded regions in the other stages, would we have an increasing sequence or a decreasing sequence?

## 13.5

## OBJECTIVES

A Calculate binomial coefficients.

B Use the binomial formula to expand binomial powers.

C Find a particular term of a binomial expansion

## Note

The polynomials to the right have been found by expanding the binomials on the left-we just haven't shown the work.

## The Binomial Expansion

The purpose of this section is to write and apply the formula for the expansion of expressions of the form $(x+y)^{n}$, where $n$ is any positive integer. To write the formula, we must generalize the information in the following chart:

$$
\begin{aligned}
& (x+y)^{0}= \\
& (x+y)^{1}=\quad x \quad+\quad y \\
& (x+y)^{2}=\quad x^{2}+2 x y+y^{2} \\
& (x+y)^{3}=\quad x^{3}+3 x^{2} y+3 x y^{2}+3 y^{3} \\
& (x+y)^{4}=\quad x^{4}+4 x^{3} y+6 x^{2} y^{2}+4 x y^{3}+y^{4} \\
& (x+y)^{5}=x^{5}+5 x^{4} y+10 x^{3} y^{2}+10 x^{2} y^{3}+5 x y^{4}+y^{5}
\end{aligned}
$$

There are a number of similarities to notice among the polynomials on the right. Here is a list:

1. In each polynomial, the sequence of exponents on the variable $x$ decreases to 0 from the exponent on the binomial at the left. (The exponent 0 is not shown, since $x^{0}=1$.)
2. In each polynomial, the exponents on the variable $y$ increase from 0 to the exponent on the binomial at the left. (Because $y^{0}=1$, it is not shown in the first term.)
3. The sum of the exponents on the variables in any single term is equal to the exponent on the binomial at the left.

The pattern in the coefficients of the polynomials on the right can best be seen by writing the right side again without the variables. It looks like this:

Row 0
Row 1
Row 2
Row 3
Row 4
Row 5
1
14 4
1
1
$1 \quad 1$
2
1
31
6
4
1
5
1
This triangle-shaped array of coefficients is called Pascal's triangle. Each entry in the triangular array is obtained by adding the two numbers above it. Each row begins and ends with the number 1. If we were to continue Pascal's triangle, the next two rows would be
$\begin{array}{llllllll}\text { Row } 6 & 1 & 6 & 15 & 20 & 15 & 6 & 1\end{array}$
$\begin{array}{lllllllll}\text { Row } 7 & 1 & 7 & 21 & 35 & 35 & 21 & 7 & 1\end{array}$
The coefficients for the terms in the expansion of $(x+y)^{n}$ are given in the $n$th row of Pascal's triangle.
There is an alternative method of finding these coefficients that does not involve Pascal's triangle. The alternative method involves factorial notation.

DEFINITION The expression $\mathbf{n}$ ! is read " $n$ factorial" and is the product of all the consecutive integers from $n$ down to 1 . For example,

$$
\begin{aligned}
& 1!=1 \\
& 2!=2 \cdot 1=2 \\
& 3!=3 \cdot 2 \cdot 1=6 \\
& 4!=4 \cdot 3 \cdot 2 \cdot 1=24 \\
& 5!=5 \cdot 4 \cdot 3 \cdot 2 \cdot 1=120
\end{aligned}
$$

The expression 0 ! is defined to be 1 . We use factorial notation to define binomial coefficients as follows.

DEFINITION The expression $\binom{n}{r}$ is called a binomial coefficient and is
defined by

$$
\binom{n}{r}=\frac{n!}{r!(n-r)!}
$$

EXAMPLE 1 Calculate the following binomial coefficients:

$$
\binom{7}{5},\binom{6}{2},\binom{3}{0}
$$

SOLUTION We simply apply the definition for binomial coefficients:

$$
\begin{aligned}
\binom{7}{5} & =\frac{7!}{5!(7-5)!} \\
& =\frac{7!}{5!\cdot 2!} \\
& =\frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)} \\
& =\frac{42}{2} \\
& =21 \\
\binom{6}{2} & =\frac{6!}{2!(6-2)!} \\
& =\frac{6!}{2!\cdot 4!} \\
& =\frac{6 \cdot 5 \cdot 4-3 \cdot 2 \cdot 1}{(2 \cdot 1)(4-3 \cdot 2 \cdot 1)} \\
& =\frac{30}{2} \\
& =15 \\
& =\frac{3 \cdot 2 \cdot 1}{(1)(3-2 \cdot 1)} \\
& =1 \\
\binom{3}{0} & =\frac{3!}{0!(3-0)!} \\
& =\frac{3!}{2!} \\
& =1
\end{aligned}
$$

If we were to calculate all the binomial coefficients in the following array, we would find they match exactly with the numbers in Pascal's triangle. That is why
they are called binomial coefficients-because they are the coefficients of the expansion of $(x+y)^{n}$.
$\binom{1}{0}$
$\binom{1}{1}$
$\binom{2}{0}$
$\binom{2}{1}$
$\binom{2}{2}$
$\binom{3}{0}$
$\binom{3}{1}$
$\binom{3}{2}$
$\binom{3}{3}$
$\binom{5}{0}$
$\binom{4}{1}$
$\binom{4}{2}$
$\binom{4}{3}$
$\binom{4}{4}$

Using the new notation to represent the entries in Pascal's triangle, we can summarize everything we have noticed about the expansion of binomial powers of the form $(x+y)^{n}$.

## The Binomial Expansion

If $x$ and $y$ represent real numbers and $n$ is a positive integer, then the following formula is known as the binomial expansion or binomial formula:

$$
(x+y)^{n}=\binom{n}{0} x^{n} y^{0}+\binom{n}{1} x^{n-1} y^{1}+\binom{n}{2} x^{n-2} y^{2}+\cdots+\binom{n}{n} x^{0} y^{n}
$$

It does not make any difference, when expanding binomial powers of the form $(x+y)^{n}$, whether we use Pascal's triangle or the formula

$$
\binom{n}{r}=\frac{n!}{r!(n-r)!}
$$

to calculate the coefficients. We will show examples of both methods.

## EXAMPLE 2 Expand $(x-2)^{3}$.

SOLUTION Applying the binomial formula, we have

$$
(x-2)^{3}=\binom{3}{0} x^{3}(-2)^{0}+\binom{3}{1} x^{2}(-2)^{1}+\binom{3}{2} x^{1}(-2)^{2}+\binom{3}{3} x^{0}(-2)^{3}
$$

The coefficients

$$
\binom{3}{0},\binom{3}{1},\binom{3}{2}, \text { and }\binom{3}{3}
$$

can be found in the third row of Pascal's triangle. They are $1,3,3$, and 1 :

$$
\begin{aligned}
(x-2)^{3} & =1 x^{3}(-2)^{0}+3 x^{2}(-2)^{1}+3 x^{1}(-2)^{2}+1 x^{0}(-2)^{3} \\
& =x^{3}-6 x^{2}+12 x-8
\end{aligned}
$$

## EXAMPLE 3 Expand $(3 x+2 y)^{4}$.

SOLUTION The coefficients can be found in the fourth row of Pascal's triangle.

$$
1,4,6,4,1
$$

Here is the expansion of $(3 x+2 y)^{4}$ :

$$
\begin{aligned}
(3 x+2 y)^{4} & =1(3 x)^{4}+4(3 x)^{3}(2 y)+6(3 x)^{2}(2 y)^{2}+4(3 x)(2 y)^{3}+1(2 y)^{4} \\
& =81 x^{4}+216 x^{3} y+216 x^{2} y^{2}+96 x y^{3}+16 y^{4}
\end{aligned}
$$

## EXAMPLE 4 Write the first three terms in the expansion of $(x+5)^{9}$.

SOLUTION The coefficients of the first three terms are

$$
\binom{9}{0},\binom{9}{1}, \text { and }\binom{9}{2}
$$

which we calculate as follows:

$$
\begin{aligned}
& \binom{9}{0}=\frac{9!}{0!\cdot 9!}=\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(1)(9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}=\frac{1}{1}=1 \\
& \binom{9}{1}=\frac{9!}{1!\cdot 8!}=\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(1)(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}=\frac{9}{1}=9 \\
& \binom{9}{2}=\frac{9!}{2!\cdot 7!}=\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)}=\frac{72}{2}=36
\end{aligned}
$$

From the binomial formula, we write the first three terms:

$$
\begin{aligned}
(x+5)^{9} & =1 \cdot x^{9}+9 \cdot x^{8}(5)+36 x^{7}(5)^{2}+\cdots \\
& =x^{9}+45 x^{8}+900 x^{7}+\cdots
\end{aligned}
$$

## The $k$ th Term of a Binomial Expansion

If we look at each term in the expansion of $(x+y)^{n}$ as a term in a sequence, $a_{1}$, $a_{2}, a_{3}, \ldots$, we can write

$$
\begin{aligned}
& a_{1}=\binom{n}{0} x^{n} y^{0} \\
& a_{2}=\binom{n}{1} x^{n-1} y^{1} \\
& a_{3}=\binom{n}{2} x^{n-2} y^{2} \\
& a_{4}=\binom{n}{3} x^{n-3} y^{3} \quad \text { and so on }
\end{aligned}
$$

To write the formula for the general term, we simply notice that the exponent on $y$ and the number below $n$ in the coefficient are both 1 less than the term number. This observation allows us to write the following:

The General Term of a Binomial Expansion
The $k$ th term in the expansion of $(x+y)^{n}$ is

$$
a_{k}=\binom{n}{k-1} x^{n-(k-1)} y^{k-1}
$$

## EXAMPLE 5 Find the fifth term in the expansion of $(2 x+3 y)^{12}$.

SOLUTION Applying the preceding formula, we have

$$
\begin{aligned}
a_{5} & =\binom{12}{4}(2 x)^{8}(3 y)^{4} \\
& =\frac{12!}{4!\cdot 8!}(2 x)^{8}(3 y)^{4}
\end{aligned}
$$

Notice that once we have one of the exponents, the other exponent and the denominator of the coefficient are determined: The two exponents add to 12 and match the numbers in the denominator of the coefficient.
Making the calculations from the preceding formula, we have

$$
\begin{aligned}
a_{5} & =495\left(256 x^{8}\right)\left(81 y^{4}\right) \\
& =10,264,320 x^{8} y^{4}
\end{aligned}
$$

LINKING OBJECTIVES AND EXAMPLES

Next to each objective we have listed the examples that are best described by that objective.

A 1
B 2-4
C 5

## GETTING READY FOR CLASS

After reading through the preceding section, respond in your own words and in complete sentences.

1. What is Pascal's triangle?
2. Why is $\binom{n}{0}=1$ for any natural number?
3. State the binomial formula.
4. When is the binomial formula more efficient than multiplying to expand a binomial raised to a whole-number exponent?

## Problem Set 13.5

Use the binomial formula to expand each of the following.

1. $(x+2)^{4}$
2. $(x-2)^{5}$
3. $(x+y)^{6}$
4. $(2 x+1)^{5}$
5. $(2 x-1)^{4}$
6. $(x-2 y)^{5}$
7. $(2 x+y)^{5}$
8. $(3 x-2)^{4}$
9. $(2 x-3)^{4}$
10. $(4 x-3 y)^{3}$
11. $(3 x-4 y)^{3}$
12. $\left(x^{2}+2\right)^{4}$
13. $\left(x^{2}-3\right)^{3}$
14. $\left(x^{2}+y^{2}\right)^{3}$
15. $\left(x^{2}-3 y\right)^{4}$
16. $(2 x+3 y)^{4}$
17. $(2 x-1)^{5}$
18. $\left(\frac{x}{2}+\frac{y}{3}\right)^{3}$
19. $\left(\frac{x}{3}-\frac{y}{2}\right)^{4}$
20. $\left(\frac{x}{2}-4\right)^{3}$
21. $\left(\frac{x}{3}+6\right)^{3}$
22. $\left(\frac{x}{3}+\frac{y}{2}\right)^{4}$
23. $\left(\frac{x}{2}-\frac{y}{3}\right)^{4}$

Write the first four terms in the expansion of the following.
25. $(x+2)^{9}$ 26. $(x-2)^{9}$
27. $(x-y)^{10}$
28. $(x+2 y)^{10}$
29. $(x+3)^{25}$
31. $(x-2)^{60}$
30. $(x-1)^{40}$
32. $\left(x+\frac{1}{2}\right)^{30}$
$=$ Videos available by instructor request
= Online student support materials available at www.thomsonedu.com/login
33. $(x-y)^{18}$
34. $(x-2 y)^{65}$

Write the first three terms in the expansion of each of the following.
35. $(x+1)^{15}$
36. $(x-1)^{15}$
37. $(x-y)^{12}$
38. $(x+y)^{12}$
39. $(x+2)^{20}$
40. $(x-2)^{20}$

Write the first two terms in the expansion of each of the following.
41. $(x+2)^{100}$
42. $(x-2)^{50}$
43. $(x+y)^{50}$
44. $(x-y)^{100}$
45. Find the ninth term in the expansion of $(2 x+3 y)^{12}$.
46. Find the sixth term in the expansion of $(2 x+3 y)^{12}$.
47. Find the fifth term of $(x-2)^{10}$.
48. Find the fifth term of $(2 x-1)^{10}$.
49. Find the sixth term in the expansion of $(x-2)^{12}$.
50. Find the ninth term in the expansion of $(7 x-1)^{10}$.
51. Find the third term in the expansion of $(x-3 y)^{25}$.
52. Find the 24 th term in the expansion of $(2 x-y)^{26}$.
53. Write the formula for the 12th term of $(2 x+5 y)^{20}$. Do not simplify.
54. Write the formula for the eighth term of $(2 x+5 y)^{20}$. Do not simplify.
55. Write the first three terms of the expansion of $\left(x^{2} y-3\right)^{10}$.
56. Write the first three terms of the expansion of $\left(X-\frac{1}{X}\right)^{50}$.

## Applying the Concepts

57. Probability The third term in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^{7}$ will give the probability that in a family with 7 children, 5 will be boys and 2 will be girls. Find the third term.
58. Probability The fourth term in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^{8}$ will give the probability that in a family
with 8 children, 3 will be boys and 5 will be girls. Find the fourth term.

## Maintaining Your Skills

Solve each equation. Write your answers to the nearest hundreth.
59. $5^{x}=7$
60. $10^{x}=15$
61. $8^{2 x+1}=16$
62. $9^{3 x-1}=27$
63. Compound Interest How long will it take $\$ 400$ to double if it is invested in an account with an annual interest rate of $10 \%$ compounded four times a year?
64. Compound Interest How long will it take $\$ 200$ to become $\$ 800$ if it is invested in an account with an annual interest rate of $8 \%$ compounded four times a year?

## Extending the Concepts

65. Calculate both $\binom{8}{5}$ and $\binom{8}{3}$ to show that they are equal.
66. Calculate both $\binom{10}{8}$ and $\binom{10}{2}$ to show that they are equal.
67. Simplify $\binom{20}{12}$ and $\binom{20}{8}$.
68. Simplify $\binom{15}{10}$ and $\binom{15}{5}$.
69. Pascal's Triangle Copy the first eight rows of Pascal's triangle into the eight rows of the triangular array to the right. (Each number in Pascal's triangle will go into one of the hexagons in the array.) Next, color in each hexagon that contains an odd number. What pattern begins to emerge from this coloring process?


## Chapter 13 sUMMARY

## EXAMPLES

1. In the sequence
$1,3,5, \ldots, 2 n-1$,
$a_{1}=1$
$a_{2}=3$
$a_{3}=5$
and
$a_{n}=2 n-1$
2. $\sum_{i=3}^{6}(-2)^{i}$
$=(-2)^{3}+(-2)^{4}+(-2)^{5}+(-2)^{6}$
$=-8+16+(-32)+64$
$=40$
3. For the sequence
$3,7,11,15, \ldots$,
$a_{1}=3$ and $d=4$. The general term is

$$
\begin{aligned}
a_{n} & =3+(n-1) 4 \\
& =4 n-1
\end{aligned}
$$

Using this formula to find the tenth term, we have

$$
a_{10}=4(10)-1=39
$$

The sum of the first 10 terms is

$$
S_{10}=\frac{10}{2}(3+39)=210
$$

4. For the geometric progression 3, 6, 12, 24,
$a_{1}=3$ and $r=2$. The general term is

$$
a_{n}=3 \cdot 2^{n-1}
$$

The sum of the first 10 terms is

$$
S_{10}=\frac{3\left(2^{10}-1\right)}{2-1}=3,069
$$

## Sequences [13.1]

A sequence is a function whose domain is the set of positive integers. The terms of a sequence are denoted by

$$
a_{1}, a_{2}, a_{3}, \ldots, a_{n}, \ldots
$$

where $a_{1}$ (read "a sub 1 ") is the first term, $a_{2}$ the second term, and $a_{n}$ the $n$th or general term.

## Summation Notation [13.2]

The notation

$$
\sum_{i=1}^{n} a_{i}=a_{1}+a_{2}+a_{3}+\cdots+a_{n}
$$

is called summation notation or sigma notation. The letter $i$ as used here is called the index of summation or just index.

## Arithmetic Sequences [13.3]

An arithmetic sequence is a sequence in which each term comes from the preceding term by adding a constant amount each time. If the first term of an arithmetic sequence is $a_{1}$ and the amount we add each time (called the common difference) is $d$, then the $n$th term of the progression is given by

$$
a_{n}=a_{1}+(n-1) d
$$

The sum of the first $n$ terms of an arithmetic sequence is

$$
S_{n}=\frac{n}{2}\left(a_{1}+a_{n}\right)
$$

$S_{n}$ is called the $n$th partial sum.

## Geometric Sequences [13.4]

A geometric sequence is a sequence of numbers in which each term comes from the previous term by multiplying by a constant amount each time. The constant by which we multiply each term to get the next term is called the common ratio. If the first term of a geometric sequence is $a_{1}$ and the common ratio is $r$, then the formula that gives the general term $a_{n}$ is

$$
a_{n}=a_{1} r^{n-1}
$$

The sum of the first $n$ terms of a geometric sequence is given by the formula

$$
S_{n}=\frac{a_{1}\left(r^{n}-1\right)}{r-1}
$$

## The Sum of an Infinite Geometric Series [13.4]

5. The sum of the series

$$
\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\cdots
$$

is
$S=\frac{\frac{1}{3}}{1-\frac{1}{2}}=\frac{\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}$

If a geometric sequence has first term $a_{1}$ and common ratio $r$ such that $|r|<1$, then the following is called an infinite geometric series:

$$
S=\sum_{i=0}^{\infty} a_{1} r^{i}=a_{1}+a_{1} r+a_{1} r^{2}+a_{1} r^{3}+\cdots
$$

Its sum is given by the formula

$$
S=\frac{a_{1}}{1-r}
$$

## Factorials [13.5]

The notation $n!$ is called $n$ factorial and is defined to be the product of each consecutive integer from $n$ down to 1 . That is,

$$
\begin{aligned}
& 0!=1 \quad \text { (By definition) } \\
& 1!=1 \\
& 2!=2 \cdot 1 \\
& 3!=3 \cdot 2 \cdot 1 \\
& 4!=4 \cdot 3 \cdot 2 \cdot 1
\end{aligned}
$$

and so on.

## Binomial Coefficients [13.5]

The notation $\binom{n}{r}$ is called a binomial coefficient and is defined by

$$
\binom{n}{r}=\frac{n!}{r!(n-r)!}
$$

Binomial coefficients can be found by using the formula above or by Pascal's triangle, which is


## Binomial Expansion [13.5]

If $n$ is a positive integer, then the formula for expanding $(x+y)^{n}$ is given by

$$
(x+y)^{n}=\binom{n}{0} x^{n} y^{0}+\binom{n}{1} x^{n-1} y^{1}+\binom{n}{2} x^{n-2} y^{2}+\cdots+\binom{n}{n} x^{0} y^{n}
$$

## Chapter 13 Review Test

The problems below form a comprehensive review of the material in this chapter. They can be used to study for exams. If you would like to take a practice test on this chapter, you can use the odd-numbered problems. Give yourself an hour and work as many of the oddnumbered problems as possible. When you are finished, or when an hour has passed, check your answers with the answers in the back of the book. You can use the even-numbered problems for a second practice test.

Write the first four terms of the sequence with the following general terms. [13.1]

1. $a_{n}=2 n+5$
2. $a_{n}=3 n-2$
3. $a_{n}=n^{2}-1$
4. $a_{n}=\frac{n+3}{n+2}$
5. $a_{1}=4, a_{n}=4 a_{n-1}, n>1$
6. $a_{1}=\frac{1}{4}, a_{n}=\frac{1}{4} a_{n-1}, n>1$

Determine the general term for each of the following sequences. [13.1]
7. $2,5,8,11, \ldots$
8. $-3,-1,1,3,5, \ldots$
9. $1,16,81,256$,
10. $2,5,10,17, \ldots$
11. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
12. $2, \frac{3}{4}, \frac{4}{9}, \frac{5}{16}, \frac{6}{25}$,

Expand and simplify each of the following. [13.2]
13. $\sum_{i=1}^{4}(2 i+3)$
14. $\sum_{i=1}^{3}\left(2 i^{2}-1\right)$
15. $\sum_{i=2}^{3} \frac{i^{2}}{i+2}$
16. $\sum_{i=1}^{4}(-2)^{i-1}$
17. $\sum_{i=3}^{5}\left(4 i+i^{2}\right)$
18. $\sum_{i=4}^{6} \frac{i+2}{i}$

Write each of the following sums with summation notation. [13.2]
19. $3+6+9+12$
20. $3+7+11+15$
21. $5+7+9+11+13$
22. $4+9+16$
23. $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$
24. $\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+\frac{5}{243}$
25. $(x-2)+(x-4)+(x-6)$
26. $\frac{x}{x+1}+\frac{x}{x+2}+\frac{x}{x+3}+\frac{x}{x+4}$

Determine which of the following sequences are arithmetic progressions, geometric progressions, or neither. [13.3, 13.4]
27. $1,-3,9,-27, \ldots$
29. $5,11,17,23, \ldots$
31. $4,8,16,32, \ldots$
33. $12,9,6,3, \ldots$
28. $7,9,11,13, \ldots$
30. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
32. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
34. $2,5,9,14, \ldots$

Each of the following problems refers to arithmetic progressions. [13.3]
35. If $a_{1}=2$ and $d=3$, find $a_{n}$ and $a_{20}$.
36. If $a_{1}=5$ and $d=-3$, find $a_{n}$ and $a_{16}$.
37. If $a_{1}=-2$ and $d=4$, find $a_{10}$ and $S_{10}$.
38. If $a_{1}=3$ and $d=5$, find $a_{16}$ and $S_{16}$.
39. If $a_{5}=21$ and $a_{8}=33$, find the first term $a_{1}$, the common difference $d$, and then find $a_{10}$.
40. If $a_{3}=14$ and $a_{7}=26$, find the first term $a_{1}$, the common difference $d$, and then find $a_{9}$ and $S_{9}$.
41. If $a_{4}=-10$ and $a_{8}=-18$, find the first term $a_{1}$, the common difference $d$, and then find $a_{20}$ and $S_{20}$.
42. Find the sum of the first 100 terms of the sequence $3,7,11,15,19, \ldots$
43. Find $a_{40}$ for the sequence $100,95,90,85,80, \ldots$

Each of the following problems refers to infinite geometric progressions. [13.4]
44. If $a_{1}=3$ and $r=2$, find $a_{n}$ and $a_{20}$.
45. If $a_{1}=5$ and $r=-2$, find $a_{n}$ and $a_{16}$.
46. If $a_{1}=4$ and $r=\frac{1}{2}$, find $a_{n}$ and $a_{10}$.
47. If $a_{1}=-2$ and $r=\frac{1}{3}$, find the sum.
48. If $a_{1}=4$ and $r=\frac{1}{2}$, find the sum.
49. If $a_{3}=12$ and $a_{4}=24$, find the first term $a_{1}$, the common ratio $r$, and then find $a_{6}$.
50. Find the tenth term of the sequence $3,3 \sqrt{3}, 9$, $9 \sqrt{3}, \ldots$

Use the binomial formula to expand each of the following. [13.5]
51. $(x-2)^{4}$
52. $(2 x+3)^{4}$
53. $(3 x+2 y)^{3}$
54. $\left(x^{2}-2\right)^{5}$
55. $\left(\frac{x}{2}+3\right)^{4}$
56. $\left(\frac{x}{3}-\frac{y}{2}\right)^{3}$

Use the binomial formula to write the first three terms in the expansion of the following. [13.5]
57. $(x+3 y)^{10}$
58. $(x-3 y)^{9}$
59. $(x+y)^{11}$
60. $(x-2 y)^{12}$

Use the binomial formula to write the first two terms in the expansion of the following. [13.5]
61. $(x-2 y)^{16}$
62. $(x+2 y)^{32}$
63. $(x-1)^{50}$
64. $(x+y)^{150}$
65. Find the sixth term in $(x-3)^{10}$. [13.5]
66. Find the fourth term in $(2 x+1)^{9}$. [13.5]

## Chapter 13 Projects

## Sequences and Series

## GROUP PROJECT Credit Card Payments

## Number of People 2

Time Needed 20-30 minutes
Equipment Paper, pencil, and graphing calculator
Background In the beginning of this chapter, you were given a recursive function that you can use to find how long it will take to pay off a credit card. A graphing calculator can be used to model each payment by using the recall function on the graphing calculator. To set up this problem do the following:
(1) 1000
(2) Round (1.0165ANS—20,2)

ENTER
Note The Round function is under the MATH key in the NUM list.

Procedure Enter the preceding commands into your calculator. While one person in the group hits the ENTER key, another person counts, by keeping a tally
of the "payments" made. The credit card is paid off when the calculator displays a negative number.

1. How many months did it take to pay off the credit card?
2. What was the amount of the last payment?
3. What was the total interest paid to the credit card company?
4. How much would you save if you paid $\$ 25$ per month instead of $\$ 20$ ?
5. If the credit card company raises the interest rate to 21.5\% annual interest, how long will it take to pay off the balance? How much more would it cost in interest?
6. On larger balances, many credit card companies require a minimum payment of $2 \%$ of the outstanding balance. What is the recursion formula for this? How much would this save or cost you in interest?

| Detach here and return with check or money order. |  |
| :--- | :--- | :--- |
| Summary of Corporate Card Account <br> Retain this portion for your files |  |
| Corporate Cardmember Name Account Number <br> Leonardo Fibonacci Closing Date <br> $08-01-00$ <br> New Balance Other Debits Interest Rate Minimum Payment <br> $\$ 1,000.00$ $\$ .00$ $19.8 \%$ $\$ 20.00$   |  |

# Building Squares from Odd Numbers 



A
relationship exists between the sequence of squares and the sequence of odd numbers. In The Book of Squares, written in 1225, Leonardo Fibonacci has this to say about that relationship:

I thought about the origin of all square numbers and discovered that they arise out of the increasing sequence of odd numbers.

Work with the sequence of odd numbers until you discover how it can be used to produce the sequence of squares. Then write an essay in which you give a written description of the relationship between the two sequences, along with a diagram that illustrates the relationship. Then see if you can use summation notation to write an equation that summarizes the whole relationship. Your essay should be clear and concise and written so that any of your classmates can read it and understand the relationship you are describing.

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## Answers to Odd-Numbered Problems and Chapter Reviews

## CHAPTER 1

## Problem Set 1.1

1. $x+5=14$
2. $5 y<30$
3. $3 y \leq y+6$
4. $\frac{x}{3}=x+2$
5. 9 11. 49
6. 8
7. 64
8. 16
9. 100
10. 121
$\begin{array}{lllllllllll}\text { 23. a. } 11 & \text { b. } 16 & \text { 25. a. } 17 & \text { b. } 42 & \text { 27. a. } 30 & \text { b. } 30 & \text { 29. a. } 24 & \text { b. } 80 & \text { 31. } 27 & \text { 33. } 35 & 35.4\end{array}$ 37. 37
$\begin{array}{lllllllllll}\text { 39. a. } 16 & \text { b. } 16 & 41 . \text { a. } 81 & \text { b. } 41 & \text { 43. } 2,345 & 45.2 & 47.148 & 49.36 & 51.58 & 53.62 & 55.100 \\ 57.9 & 59.18\end{array}$
$\begin{array}{lllllllllll}61 . & 12 & 63.18 & 65.42 & 67.0 .17 & \text { 69. } 0.22 & \text { 71. } 2.3 & \text { 73. } 5.55 & \text { 75. } 240 & 77.10 & 79.3,000 \\ 81 . & 2,000\end{array}$
11. 35.33
12. $0.75 \quad$ 87. $0.33 \quad$ 89. 272.73
13. 10
14. 420
15. a. 600 mg
b. 231 mg
16. 

| Activity | Calories Burned in 1 Hour <br> by a $150-$ Pound Person |
| :--- | :---: |
| Bicycling | 374 |
| Bowling | 265 |
| Handball | 680 |
| Jogging | 680 |
| Skiing | 544 |

$99.5 \quad 101.10 \quad 103.25 \quad 105.10$

Problem Set 1.2
1-7.

9. $\frac{18}{24}$
11. $\frac{12}{24}$
13. $\frac{15}{24}$
15. $\frac{36}{60}$
17. $\frac{22}{60}$
19. 2
21. 25
23. 6
25. $-10, \frac{1}{10}, 10$
27. $3,-\frac{1}{3}, 3$
29. $-x, \frac{1}{x},|x|$
31.
33.
35.
37.
39.
41.
43. 6
45. 22
47. $3 \quad$ 49. 7
51. 3
53. $\frac{8}{15}$
55. $\frac{3}{2}$
57. 1
59. 1
61. a. 2
b. 4
c. 8
d. 0.03
63. a. 6
b. 12
c. 24
d. 0.09
65. $\frac{9}{16}$
67. $\frac{8}{27}$
69. a. 4
b. 14
c. 24
d. 34
71. a. 12
b. 20
c. 100
d. 1,024
73. a. 17
b. 25
c. 25
75. 4 inches; 1 inch $^{2}$
77. 4.5 inches; 1.125 inches $^{2}$
79. 10.25 centimeters; 5 centimeters ${ }^{2}$
81. $-8,-2$
83. $-64^{\circ} \mathrm{F} ;-54^{\circ} \mathrm{F}$
85. - 100 feet; -105 feet
87. Area $=93.5$ inches $^{2}$ : perimeter $=39$ inches
89. 1,387 calories
91. 654 more calories

Problem Set 1.3

1. $3+5=8,3+(-5)=-2$,
2. $15+20=35,15+(-20)=-5$,
$-3+5=2,-3+(-5)=-8$
$-15+20=5,-15+(-20)=-35$


$\begin{array}{llllllll}\text { 53. } 1.70 & \text { 55. } 23,28 & \text { 57. } 30,35 & \text { 59. } 0,-5 & \text { 61. }-12,-18 & \text { 63. }-4,-8 & \text { 65. Yes } & \text { 67. } 5+9=14\end{array}$
3. $[-7+(-5)]+4=-8 \quad$ 71. $[-2+(-3)]+10=5 \quad$ 73. $3 \quad$ 75. $-3 \quad$ 77. $-12+4 \quad$ 79. $10+(-6)+(-8)=-\$ 4$
4. $-30+40=10$

## Problem Set 1.4

| 1. -3 | 3. -6 | 5. 0 | 7. -10 | 9. -16 | 11. -12 | 13. -7 | 15. 35 | 17. 0 | 19. -4 | 21. 4 | 23. -24 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 27. 25 | 29. 4 | 31. 7 | 33. 17 | 35. 8 | 37. 4 | 39. 18 | 41. 10 | 43. 17 | 45. 1 | 47. 1 | 49. 27 |  |
| 53. -2 | 55. 68 | 57. | -11.3 | 59. -3.6 | 61. a. -9 b. -10 c. -3 |  |  | 63. $-7-4=-11$ |  | 65. $12-(-8)=20$ |  |  |
| 67. -5 | $(-7)=2$ | 69. | $4+(-5)]$ | $-17=-18$ | 71. 8 | $5=3$ | 73. $-8-5$ | $=-13$ | 75. $8-(-5)=13$ |  | 77. 10 |  |
| 81. 1,50 | -730 | 83. | + $15-2$ | $0=-\$ 40$ | 85. $98-65-53=-\$ 20$ |  |  | 87. \$4, | 0, \$3,950 | \$3,400 | ,850, \$2 | 0; |

89. 769 feet $\quad$ 91. 439 feet $\quad$ 93. 2 seconds $\quad 95.35^{\circ} \quad 97.60^{\circ} \quad$ 99. a.

| Day | Plant Height <br> (inches) |  | Plant Height <br> (inches) |  |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 |  | 6 |  |
| 1 | 0.5 |  |  |  |
|  | 1 |  | 9 |  |
| 2 | 1.5 |  | 13 |  |
| 3 | 3 |  | 10 | 18 |
| 4 | 4 |  | 2 |  |
| 5 |  |  |  |  |

Problem Set 1.5

1. Commutative
2. Multiplicative inverse 5. Commutative
3. Distributive
4. Commutative, associative
5. Commutative, associative 13. Commutative 15. Commutative, associative 17. Commutative 19. Additive inverse
6. $3 x+6$
7. $9 a+9 b$
$\begin{array}{lll}25.0 & 27.0 & 29.10\end{array}$
8. $(4+2)+x=6+x$
9. $x+(2+7)=x+9$
10. $(3 \cdot 5) x=15 x$
11. $(9 \cdot 6) y=54 y$
12. $\left(\frac{1}{2} \cdot 3\right) a=\frac{3}{2} a$
13. $\left(\frac{1}{3} \cdot 3\right) x=x$
14. $\left(\frac{1}{2} \cdot 2\right) y=y$
15. $\left(\frac{3}{4} \cdot \frac{4}{3}\right) x=x$
16. $\left(\frac{6}{5} \cdot \frac{5}{6}\right) a=a$
17. $8 x+16$
18. $8 x-16$
19. $4 y+4$
20. $18 x+15$
21. $6 a+14$
22. $54 y-72$
23. $\frac{3}{2} x-3$
24. $x+2$
25. $3 x+3 y$
26. $8 a-8 b$
27. $12 x+18 y$
28. $12 a-8 b$
29. $3 x+2 y$ 75. $4 a+25$
30. $6 x+12$
31. $14 x+38$
32. $2 x+1$
33. $6 x-3$
34. $5 x+10 \quad$ 87. $6 x+5 \quad$ 89. $5 x+6$
35. $6 m-5$
36. $7+3 x$ 95. $3 x-2 y$
37. $0.09 x+180$
$\begin{array}{llll}\text { 99. } 0.12 x+60 & \text { 101. } a+1 & \text { 103. } 1-a & \text { 105. No 107. No, not commutative 109. } 8 \div 4 \neq 4 \div 8\end{array}$
38. $12(2,400-480)=\$ 23,040 ; 12(2,400)-12(480)=\$ 23,040 \quad 113.3(650+225)=\$ 2,625 ; 3(650)+3(225)=\$ 2,625$

Problem Set 1.6

1. -42 3. -16
2. 3 7. 121
3. 6 11. -60
4. 24 15. 49 17. -27
5. 6
6. 10
7. 9
8. 45
$\begin{array}{ll}\text { 27. } 14 & \text { 29. }-2\end{array}$
9. 216 33. -2
10. $-18 \quad 37.29$
11. 38
12. $-5 \quad$ 43. 37
13. a. 60 b
14. $-\frac{10}{21}$
15. -4
16. $\frac{9}{16}$
17. a. 27 b. 3 c. -27 d. -27
18. 9
19. $\frac{25}{4}$
20. 4 61. $\frac{9}{4}$
21. $-8 x \quad$ 65. $42 x$
22. $x \quad$ 69. $-\frac{3}{2} x+3$
23. $-6 x+8$
24. $-15 x-30$
25. $-12 x-20 y$
26. $-6 x-10 y \quad$ 79. $-\frac{3}{2} x+3$
27. $-\frac{2}{3} x+2$
28. $\frac{2}{3} x-2$
29. $-2 x+y$
30. a. 2 b. 1 c. 3
31. a. 3 b. 3 c. -4
32. a. 12 b. 12
33. a. 44
b. 44
34. -25
35. $2(-4 x)=-8 x$
36. -26
37. 8
38. -80
39. $\frac{1}{8} \quad$ 107. -24
40. $\$ 60$
41. $1^{\circ} \mathrm{F}$
42. a. $\$ 544$
b. $-\$ 521$

## Problem Set 1.7

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1. -2 | 3. -3 | 5. $-\frac{1}{3}$ | 7. 3 | 9. $\frac{1}{7}$ | 11. 0 | 13.9 | 15. -15 | 17. -36 | 19. $-\frac{1}{4}$ | 21. $\frac{16}{15}$ | 23. $\frac{4}{3}$ | 25. $-\frac{8}{13}$ |
| 27. -1 | 29. 1 | 31. $\frac{3}{5}$ | 33. $-\frac{5}{3}$ | $35 . ~-2$ | $37 .-3$ | 39. Undefined | 41. Undefined | 43. 5 | 45. $-\frac{7}{3}$ | 47. -1 |  |  |

27. -1
28. 1
29. $\frac{3}{5}$
30. $-\frac{5}{3}$
31. -2 37. -3
32. Undefined
33. Undefined
34. 5
35. $-\frac{7}{3}$
36. -1
37. $-7 \quad$ 51. $\frac{15}{17} \quad$ 53. $-\frac{32}{17}$
38. a. $-\frac{5}{7}$ b. $-\frac{5}{7} \quad$ 77. a.
39. $\frac{1}{3} \quad$ 57. $1 \quad 59.1 \quad$ 61. -2
40. $\frac{9}{7} \quad 65 . \frac{16}{11}$
$\begin{array}{llll}\text { 67. }-1 & \text { 69. } \frac{7}{20}=0.35 & 71 .-1 & 73 . \\ \text { a. } \frac{3}{2} & \text { b. } \frac{3}{2}\end{array}$
$\begin{array}{llllllllll}\text { 83. } 5 x+6 & 85.3 x+20 & \text { 87. } 3+x & \text { 89. } 3 x-7 y & \text { 91. } 1 & 93.2-y & 95.3 & 97 .-10 & \text { 99. }-3 & \text { 101. }-8 \\ \text { 103. } \$ 350\end{array}$
41. Drops $3.5^{\circ} \mathrm{F}$ each hour 107. a. $\$ 20,000$ b. $\$ 50,000$ c. Yes, the projected revenue for 5,000 email addresses is $\$ 10,000$, which is $\$ 5,000$ more than the list costs.

Problem Set 1.8


## Problem Set 1.9

1. $\frac{2}{3} \quad$ 3. $-\frac{1}{4}$
2. $\frac{1}{2} \quad$ 7. $\frac{x-1}{3}$
3. $\frac{3}{2}$ 11. $\frac{x+6}{2}$
4. $-\frac{3}{5}$
5. $\frac{10}{a}$
$\begin{array}{ll}\text { 17. } \frac{7}{8} & \text { 19. } \frac{1}{10}\end{array}$
6. $\frac{7}{9}$
7. $\frac{7}{3}$
8. $\frac{1}{4}$
9. $\frac{7}{6}$
10. $\frac{19}{24}$
11. $\frac{13}{60}$
12. $\frac{29}{35}$
13. $\frac{949}{1,260}$
14. $\frac{13}{420}$
15. $\frac{41}{24}$
16. $\frac{5}{4}$
17. $-\frac{3}{2}$
18. $\frac{3}{2}$
19. $-\frac{2}{3}$
20. $\frac{7}{3}$
21. $\frac{1}{125}$
22. 4
23. -2
24. -18
25. -16
26. -2
27. 2
28. -144
29. -162
30. $\frac{5 x+4}{20}$
31. $\frac{a+4}{12}$
32. $\frac{9 x+4}{12}$
33. $\frac{10+3 x}{5 x} \quad 77 . \frac{3 y+28}{7 y}$
34. $\frac{60+19 a}{20 a}$
35. $\frac{2}{3} x$
36. $-\frac{1}{4} x$
37. $\frac{14}{15} x$
38. $\frac{11}{12} x$
39. $\frac{41}{40} x$
40. $\frac{x-1}{x}$
41. a. $\frac{1}{4}$
b. $\frac{5}{4}$
c. $-\frac{3}{8}$
d. $-\frac{3}{2}$
42. $\frac{1}{3}$
43. $\frac{3}{4} \quad$ 99. a. $\frac{3}{2}$
b. $\frac{4}{3}$ c. $\frac{5}{4}$
44. a. 8 b. 7 c. 8

## CHAPTER 1 REVIEW TEST

1. $-7+(-10)=-17$
2. $(-7+4)+5=2$
3. $(-3+12)+5=14$
4. $4-9=-5$
5. $9-(-3)=12$
6. $-7-(-9)=2$
7. $(-3)(-7)-6=15$
8. $5(-6)+10=-20$
9. $2[(-8)(3 x)]=-48 x$
10. $\frac{-25}{-5}=5$
11. 1.8
12. -10
13. $-6, \frac{1}{6}$
14. $\frac{12}{5},-\frac{5}{12}$
15. -5
16. $-\frac{5}{4}$
17. 3 18. -38
18. -25
19. 1
20. -3
21. 22 23. -4
22. -20
23. -30
24. -12
25. -24
26. 12
27. -4
28. $-\frac{2}{3}$
29. 23
30. 47
31. -3
32. -35
33. 32
34. 2
35. 20
36. -3
37. -98
38. 70
39. 2
40. $\frac{17}{2}$
41. Undefined
42. 3
43. Associative
44. Multiplicative identity
45. Commutative
46. Additive inverse
47. Commutative, associative
48. Distributive
49. $12+x$ 52. $28 a$
50. $x$
51. $y$ 55. $14 x+21$
52. $6 a-12$ 57. $\frac{5}{2} x-3$
53. $-\frac{3}{2} x+3$
54. $-\frac{1}{3}, 0,5,-4.5, \frac{2}{5},-3$
55. 0,5
56. $\sqrt{7}, \pi$
57. $0,5,-3$
58. $2 \cdot 3^{2} \cdot 5$
59. $2^{3} \cdot 3 \cdot 5 \cdot 7$
60. $\frac{173}{210}$
61. $\frac{2 x+7}{12}$
62. -2
63. 810
64. $8 \quad$ 70. 12
65. -1
66. $\frac{1}{16}$

## CHAPTER 2

Problem Set 2.1
 73. a.

| $n$ | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| $3 n$ | 3 | 6 | 9 | 12 |

b.

| $n$ | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| $n^{3}$ | 1 | 8 | 27 | 64 |

75. $1,4,7,10, \ldots$ an arithmetic sequence $\quad$ 77. $0,1,4,9, \ldots$ a sequence of squares 79. $-6 y+4 \quad$ 81. $0.17 x$
76. $2 x$ 85. $5 x-4$ 87. $7 x-5$ 89. $-2 x-9$ 91. $7 x+2$ 93. $-7 x+6$ 95. $7 x$ 97. $-y$ 99. $10 y$ 101. $0.17 x+180$
$\begin{array}{llllllllll}\text { 103. } 0.22 x+60 & \text { 105. } a=4 ; 13 x & \text { 107. } 49 & \text { 109. } 40 & \text { 111. a. } 42^{\circ} \mathrm{F} & \text { b. } 28^{\circ} \mathrm{F} & \text { c. }-14^{\circ} \mathrm{F} & \text { 113. a. } \$ 37.50 & \text { b. } \$ 40.00 & \text { c. } \$ 42.50\end{array}$ $\begin{array}{lllllllll}\text { 115. } 0.71 G ; ~ \\ 887.50 & \text { 117. }-\frac{1}{24} & \text { 119. }-\frac{7}{9} & \text { 121. }-\frac{23}{420} & \text { 123. } 12 & \text { 125. }-3 & \text { 127. }-9.7 & \text { 129. }-\frac{5}{4} & \text { 131. }-3\end{array}$ 133. $a-12$ $\begin{array}{lll}\text { 135. }-\frac{7}{2} & \text { 137. } \frac{51}{40} & 139.7\end{array}$

Problem Set 2.2

1. 11
2. $4 \quad$ 5. $-\frac{3}{4}$
3. $-5.8 \quad$ 9. -17
4. $-\frac{1}{8}$
5. -6
6. -3.6
7. -7
8. $-\frac{7}{45}$
9. 3
10. $\frac{11}{8}$
11. 21
12. 7
13. 3.5 31. 22
14. -2
15. -16
16. -3
17. 10
18. -12
19. 4
20. 2
21. -5 49. -1
22. -3
23. 8 55. -8
24. 11
25. a. $\frac{3}{2}$
b. 1 c. $-\frac{3}{2}$ d. -4 e. $\frac{8}{5}$
26. a. $6 \%$
b. $5 \%$ c. $2 \%$ d. $75 \%$
$\begin{array}{lllll}\text { 65. } x+55+55=180 ; 70^{\circ} & \text { 67. } & 18 x & \text { 69. } \\ \text { 87 }-15 & 89 .-18 & \text { 91. } & -\frac{5}{4} & \text { 93. } 3 x\end{array}$
$\begin{array}{llll}\text { 87. }-\frac{15}{8} & \text { 89. }-18 & \text { 91. }-\frac{5}{4} & \text { 93. } 3 x\end{array}$

Problem Set 2.3

1. 2 3. 4 5.
$-\frac{1}{2}$
2. -2
3. 3 11. 4 13. 0
4. 0
5. 6
6. -50
7. $\frac{3}{2}$
8. 12
9. -3
10. 32
11. -8
12. $\frac{1}{2}$
13. $-\frac{3}{2}$
61.1
14. -4
15. 4
16. -15
17. $-\frac{1}{2}$
18. 
19. 1
20. 1
21. $\frac{1}{4}$
22. -3
23. 3
24. 2
25. $-\frac{3}{2}$
26. 1
27. 1
28. -2
29. -2
30. $-\frac{4}{5}$
31. $1 \quad$ 73. $\frac{7}{2}$
32. $\frac{5}{4}$
33. 40
34. $\frac{7}{3}$
35. 200 tickets
36. $\$ 1,391$ per month
37. $6 x-10$
38. $\frac{3}{2} x+3$
39. $-x+2$
40. $10 x-43$
41. $-3 x-13$
42. $-6 y+4$
43. 2
99.6 101. 3,000
44. $3 x-11$
45. $0.09 x+180$
46. $-6 y+4$
47. $4 x-1$
48. $5 x$ 113. $0.17 x$

Problem Set 2.4

1. 3
2. -2
3. -1
4. 2 9. -4
5. -2
6. 0
7. 1
8. $\frac{1}{2}$
9. 7
10. 8
11. $-\frac{1}{3}$
12. $\frac{3}{4}$
13. 75
14. 2
15. 6
16. 8
17. 0
18. $\frac{3}{7}$
19. 1
20. 1
21. -1
22. 6
23. $\frac{3}{4}$
24. 3
25. $\frac{3}{4}$
26. 8
27. 6
28. -2
29. -2
30. 2
31. -6
32. 2
33. 4,000
34. 700
35. 11
36. 7
37. a. $\frac{5}{4}=1.25$
b. $\frac{15}{2}=7.5$ c. $6 x+20$
d. 15 e
f. $\frac{45}{2}=22.5$
38. $\frac{3}{2} \quad 81.4$
39. 1
40. 

| $\boldsymbol{x}$ | $\mathbf{3}(\boldsymbol{x}+\mathbf{2 )}$ | $\mathbf{3 x + 2}$ | $\mathbf{3 x + 6}$ |
| :---: | :---: | :---: | :---: |
| 0 | 6 | 2 | 6 |
| 1 | 9 | 5 | 9 |
| 2 | 12 | 8 | 12 |
| 3 | 15 | 11 | 15 |

87. 

| $\boldsymbol{a}$ | $(2 a+1)^{2}$ | $4 a^{2}+\mathbf{4 a + 1}$ |
| :---: | :---: | :---: |
| 1 | 9 | 9 |
| 2 | 25 | 25 |
| 3 | 49 | 49 |

89. 14
90. -3
91. $\frac{1}{4}$
92. $\frac{1}{3} \quad$ 97. $-\frac{3}{2} x+3$

## Problem Set 2.5

1. 100 feet
2. a. 2
b. $\frac{3}{2}$ c. 0
3. a. 2 b. -3 c. 1
4. -2
5. 1
6. a. 2
b. 4
7. a. 5 b. 18
8. a. 20
b. 75
9. $l=\frac{A}{w}$
10. $h=\frac{V}{l w}$
11. $a=P-b-c$
12. $x=3 y-1$
13. $y=3 x+6$
14. $y=-\frac{2}{3} x+2$
15. $w=\frac{P-2 l}{2}$
16. $v=\frac{h-16 t^{2}}{t}$
17. $h=\frac{A-\pi r^{2}}{2 \pi r}$
18. a. $y=-2 x-5$
b. $y=4 x-7$
19. a. $y=\frac{3}{4} x+\frac{1}{4}$
b. $y=\frac{3}{4} x-5$
20. a. $y=\frac{3}{5} x+1$
b. $y=\frac{1}{2} x+2$
c. $y=4 x+3$
21. $y=\frac{3}{7} x-3$
22. $y=2 x+8$
23. a. $\frac{15}{4}=3.75$
b. 17 c. $y=-\frac{4}{5} x+4 \quad$ d. $x=-\frac{5}{4} y+5$
24. $60^{\circ} ; 150^{\circ}$
25. $45^{\circ} ; 135^{\circ}$
26. 10
27. 240
28. $25 \%$
29. $35 \%$
30. 64
31. 2,000
32. $100^{\circ} \mathrm{C}$; yes
33. $20^{\circ} \mathrm{C}$; yes
34. $C=\frac{5}{9}(F-32)$
35. $60 \%$
36. $26.5 \%$
37. 7 meters
38. $\frac{3}{2}$ or 1.5 inches
39. 132 feet
40. $\frac{2}{9}$ centimeters
41. a. 95 b. -41 c. -95
d. 41
42. a. 9 b. -73
c. 9 d. -73
43. The sum of 4 and 1 is 5 .
44. The difference of 6 and 2 is 4 .
45. The difference of a number and 5 is -12 .
46. The sum of a number and 3 is four times the difference of that number and 3 .
47. $2(6+3)=18$
48. $2(5)+3=13$
49. $x+5=13$
50. $5(x+7)=30$

## Problem Set 2.6

Along with the answers to the odd-numbered problems in this problem set and the next, we are including some of the equations used to solve the problems. Be sure that you try the problems on your own before looking to see the correct equations.

1. 8 3. $2 x+4=14 ; 5$ 5. -1 7. The two numbers are $x$ and $x+2 ; x+(x+2)=8 ; 3$ and 5 9. 6 and 14
2. Shelly is 39, and Michele is 36 13. Evan is 11 ; Cody is 22 15. Barney is 27 ; Fred is 31 17. Lacy is 16 ; Jack is 32
3. Patrick is 18 ; Pat is 38 21. $s=9$ inches 23. $s=15$ feet 25. 11 feet, 18 feet, 33 feet 27. 26 feet, 13 feet, 14 feet
4. Length $=11$ inches; width $=6$ inches 31 . Length $=25$ inches; width $=9$ inches 33 . Length $=15$ feet; width $=3$ feet
5. 9 dimes; 14 quarters 37. 12 quarters; 27 nickels 39 . 8 nickels; 17 dimes 41. 7 nickels; 10 dimes; 12 quarters
6. 3 nickels; 9 dimes; 6 quarters 45. 4 is less than 10 . 47. 9 is greater than or equal to -5 . 49. $<$ 51. $<$
7. 12
8. $5 x$
9. $1.075 x$
10. $0.09 x+180$
11. 6,000
12. 30

## Problem Set 2.7

$\begin{array}{lllllll}\text { 1. } 5 \text { and } 6 & \text { 3. }-4 \text { and }-5 & \text { 5. } 13 \text { and } 15 & \text { 7. } 52 \text { and } 54 & \text { 9. }-14 \text { and }-16 & \text { 11. } 17,19 \text {, and } 21 & \text { 13. } 42,44 \text {, and } 46\end{array}$
15. $\$ 4,000$ invested at $8 \%, \$ 6,000$ invested at $9 \%$
19. $\$ 500$ at $8 \%, \$ 1,000$ at $9 \%, \$ 1,500$ at $10 \%$
21. $45^{\circ}, 45^{\circ}, 90^{\circ}$
23. $22.5^{\circ}, 45^{\circ}, 112.5^{\circ}$
25. $53^{\circ}, 90^{\circ}$
27. $80^{\circ}, 60^{\circ}, 40^{\circ}$
29. 16 adult and 22 children's tickets 31. 16 minutes
33. 39 hours
35. They are in offices 7329
37. Kendra is 8 years old and Marissa is 10 years old.
39. Jeff
41. $\$ 10.38$
43. Length $=12$ meters; width $=10$ meters
45. $59^{\circ}, 60^{\circ}, 61^{\circ}$ 47. $\$ 54.00$ 49. Yes 51. -3
53. -8
55. $0 \quad 57.8$
59. -24 61. 28
63. $-\frac{1}{2}$
65. -16
67. -25
69. 24
71. a. 9
b. 3 c. -9
d. -3
73. a. -8
b. 8 c. 8 d. -8
75. -2.3125
77. $\frac{10}{3}$

Problem Set 2.8


## CHAPTER 2 REVIEW TEST

$\begin{array}{ll}\text { 1. }-3 x & \text { 2. }-2 x-3\end{array}$
3. $4 a-7$
4. $-2 a+3$
5. $-6 y$ 6. $-6 x+1$
7. 19
8. -11 9. -18
10. -13
11. 8
$\begin{array}{lll}\text { 12. } 6 & \text { 13. }-8 & \text { 14. } \frac{15}{14}\end{array}$
15. 2
16. -5 17. -5
18. 0 19. 12
20. -8
21. -1
22. 1
23. 5
24. 2 25. 0
26. 0
27. 10
28. 2
29. 2
30. $-\frac{8}{3}$
31. 0
32. -5
33. 0
34. -4
35. -8
36. 4
37. $y=\frac{2}{5} x-2$
38. $y=\frac{5}{2} x-5$
39. $h=\frac{V}{\pi r^{2}}$
40. $w=\frac{P-2 l}{2}$
41. 206.4
42. $9 \%$
43. 11
44. 5 meters; 25 meters
45. $\$ 500$ at $9 \%$; $\$ 800$ at $10 \%$
46. 10 nickels; 5 dimes
47. $x>-2$
48. $x<2$
49. $a \geq 6$
50. $a<-15$
51.

53.


## CHAPTER 3

## Problem Set 3.1

1-17.

19. $(-4,4)$
21. $(-4,2)$
23. $(-3,0)$
25. $(2,-2)$
27. $(-5,-5)$
29. Yes
31. No

33. Yes
35. No

37. Yes
39. No

41. No
43. No

45. a. $(5,40),(10,80),(20,160)$, answers may vary b. $\$ 320$ c. 30 hours d. No, she should make $\$ 280$ for working 35 hours.
47.

49.
Wireless Talk

51. $(1981,5),(1985,20.2),(1990,34.4),(1995,44.8),(2000,65.4),(2004,99.4)$
53. $A=(1,2), \quad B=(6,7)$
55. $A=(2,2), \quad B=(2,5), C=(7,5)$
57. $\frac{x+3}{5}$
59. $\frac{2-a}{7}$
61. $\frac{1-2 y}{14}$
63. $\frac{x+6}{2 x}$
65. $\frac{x}{6}$
67. $\frac{8+x}{2 x}$
69. a. -3
b. 6
c. 0 d. -4
71. a. 4
b. 2 c. -1
d. 9

## Problem Set 3.2

1. $(0,6),(3,0),(6,-6)$
2. $(0,3),(4,0),(-4,6)$
3. $(1,1),\left(\frac{3}{4}, 0\right),(5,17)$
4. $(2,13),(1,6),(0,-1)$
5. $(-5,4),(-5,-3),(-5,0)$
6. 


13.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | ---: |
| 0 | 0 |
| $-\frac{1}{2}$ | -2 |
| -3 | -12 |
| 3 | 12 |

15. 

| $x$ | $y$ |
| :---: | ---: |
| 2 | 3 |
| 3 | 2 |
| 5 | 0 |
| 9 | -4 |

17. 

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | ---: |
| 2 | 0 |
| 3 | 2 |
| 1 | -2 |
| -3 | -10 |

19. 

| $x$ | $y$ |
| :---: | ---: |
| 0 | -1 |
| -1 | -7 |
| -3 | -19 |
| $\frac{3}{2}$ | 8 |

21. $(0,-2)$
22. $(1,5),(0,-2),(-2,-16)$
23. $(1,6),(-2,-12),(0,0)$
24. $(2,-2)$
25. $(3,0),(3,-3)$
26. 12 inches
27. a. Yes b. No, she should earn $\$ 108$ for working 9 hours. c. No, she should earn $\$ 84$ for working 7 hours. d. Yes
28. a. $\$ 375,000$ b. At the end of six years
29. 6
30. 3
31. 6
32. 4
33. -3
. No, the crane
34. 2
35. 0
36. $y=-5 x+4$
37. $y=\frac{3}{2} x-3$

## Problem Set 3.3

1. 


7.

3.

5.

9.

11.


A-8 Answers to Odd-Numbered Problems and Chapter Reviews
13.

19.

25.

31.

15.

21.

27.

33.

17.

23.

29.

35.

37.

39.

41.

Equation
H, V, and/or O

| $x=3$ | V |
| :--- | :---: |
| $y=3$ | H |
| $y=3 x$ | O |
| $y=0$ | O, H |

43. 

| Equation | $\mathbf{H}, \mathbf{V}$, and/or $\mathbf{O}$ |
| :--- | :---: |
| $x=-\frac{3}{5}$ | V |
| $y=-\frac{3}{5}$ | H |
| $y=-\frac{3}{5} x$ | O |
| $x=0$ | $\mathrm{O}, \mathrm{V}$ |

45. a. $\frac{5}{2}$ b. 5 c. 2 d.
e. $y=-\frac{2}{5} x+2$

46. $2 x+5$
47. $2 x-6$
48. $3 x+\frac{15}{2}$
49. $6 x+9$
50. $2 x+50$
51. $6 x-28$
52. $\frac{3}{2} x+9 y$
53. $\frac{5}{2} x-5 y+6$

## Problem Set 3.4

1. 


7.

3.

5.

11.

13.

19.

25.

31.

| Equation | $x$-intercept | $y$-intercept |
| :---: | :---: | :---: |
| $3 x+4 y=12$ | 4 | 3 |
| $3 x+4 y=4$ | $\frac{4}{3}$ | 1 |
| $3 x+4 y=3$ | 1 | $\frac{3}{4}$ |
| $3 x+4 y=2$ | $\frac{2}{3}$ | $\frac{1}{2}$ |

15. 


21.

27.

17.

23.

29.

33.

| Equation | $x$-intercept | $y$-intercept |
| :---: | :---: | :---: |
| $x-3 y=2$ | 2 | $-\frac{2}{3}$ |
| $y=\frac{1}{3} x-\frac{2}{3}$ | 2 | $-\frac{2}{3}$ |
| $x-3 y=0$ | 0 | 0 |
| $y=\frac{1}{3} x$ | 0 | 0 |

35. a. 0 b. $-\frac{3}{2}$ c. 1 d.

e. $y=\frac{2}{3} x+1$
36. $x$-intercept $=-1 ; y$-intercept $=-3$

37. $x$-intercept $=3 ; y$-intercept $=3$
38. 

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | ---: |
| -2 | 1 |
| 0 | -1 |
| -1 | 0 |
| 1 | -2 |

43. 


45.

47. a. $10 x+12 y=480$
b. $x$-intercept $=48 ; y$-intercept $=40$
C.

49. 12
51. -11
53. -6
55. 7
57. -2
59. -1
61. 14
63. -6
65. a. $\frac{3}{2}$
b. $\frac{3}{2}$
67. a. $\frac{3}{2}$
b. $\frac{3}{2}$

## Problem Set 3.5

1. 


7.

13.

19.

3.

9.

15.

21.

5.

11.

17.

23. slope $=3, y$-intercept $=2$
25. slope $=2, y$-intercept $=-2$
27. $m=2, b=4$

33. $m=\frac{4}{5}, b=-4$

37.

43.

29. $m=-3, b=3$

35. $m=-\frac{2}{5}, b=-2$

39.

45. 6
47.

| Equation | Slope |
| :--- | :---: |
| $y=3$ | 0 |
| $y=3 x$ | 3 |

31. $m=-\frac{3}{2}, b=3$

32. 


49.

Equation
Slope

| $y=-\frac{2}{3}$ | 0 |
| :---: | :---: |
| $y=-\frac{2}{3} x$ | $-\frac{2}{3}$ |

A-14 Answers to Odd-Numbered Problems and Chapter Reviews
51. a. $-\frac{5}{2}$
b. $y=2 x+6$
c. 6
d. 2 e.

53. a. $\$ 6,000$ b. 3 years c. Slope $=-3,000$ d. $\$ 3,000$ e. $V=-3,000 t+21,000$
55. a. $\$ 3,000$ b. 500 shirts c. Slope $=2$ d. $\$ 2$ e. $y=2 x+1,000 \quad 57.6$
59. -2
61. 4
63. -11
65. 10
67. -38
69. -12
71. -18
73.


Problem Set 3.6
1.

75.

3.

77.

5.

7.

13.

19.

25. $\varnothing$

9.

15.

21.

27. Any point on the line

11.

17.

23.

29. a. $4 x-5 y$
d.
b. 1 c. -2 e. $(0,-2)$

31.
35. a. 25 hours
b. Gigi's C. Marcy's

33.

37. $10 x+75$
39. $11 x-35$
41. $-x+58$
43. $-0.02 x+6$
45. $-0.003 x-28$
47. $1.31 x-125$
49. $2 x$
51. $7 x$
53. $13 x$
55. $-12 x-20 y$
57. $3 x+8 y$
59. $-4 x+2 y$
61. 1
63. 0
65. -5

Problem Set 3.7

1. $(2,1)$
2. $(3,7)$
3. $(2,-5)$
4. $(-1,0)$
5. Lines coincide
6. $(4,8)$
7. $\left(\frac{1}{5}, 1\right)$
8. $(1,0)$
9. $(-1,-2)$
10. $\left(-5, \frac{3}{4}\right)$
11. $(-4,5) \quad$ 23. $(-3,-10)$
12. $(3,2)$
13. $\left(5, \frac{1}{3}\right)$
14. $\left(-2, \frac{2}{3}\right)$
15. $(2,2)$ 33. Lines are parallel; $\varnothing$
16. $(1,1)$
17. Lines are parallel; $\varnothing$
18. $(10,12)$
19. $(6,8)$
20. $(7,000,8,000)$
21. $(3,000,8,000)$
22. $(11,12)$
23. $(10,12)$
24. slope $=3 ; y$-int $=-3$
25. slope $=\frac{2}{5} ; y$-int $=-5$
26. slope $=-1$
27. slope $=2$
28. 1
29. 2
30. 1
31. $x=3 y-1$
32. 1
33. 5
34. 34.5
35. 33.95

## Problem Set 3.8

1. $(4,7)$
2. $(3,17)$
3. $\left(\frac{3}{2}, 2\right)$
4. $(2,4)$
5. $(0,4)$
6. $(-1,3)$
7. $(1,1)$
8. $(2,-3)$
9. $\left(-2, \frac{3}{5}\right)$
10. $(-3,5)$
11. Lines are parallel; $\varnothing$
12. $(3,1) \quad$ 25. $\left(\frac{1}{2}, \frac{3}{4}\right)$
13. $(2,6)$
14. $(4,4)$
15. $(5,-2)$
16. $(18,10)$
17. Lines coincide
18. $(8,4)$
19. $(6,12)$
20. $(16,32)$
21. $(10,12)$
22. a. $\frac{25}{4}$
b. $y=\frac{4}{5} x-4$
c. $x=y+5$
d. $(5,0)$
23. a. 1,000 miles
b. Car c. Truck
d. We are only working with positive numbers.
24. 47
25. 14
26. 70
27. 35
28. 5
29. 6,540
30. 1,760
31. 20
32. 63
33. 53
34. 3 and 23
35. 15 and 24
36. Length $=23$ in.; Width $=6$ in.
37. 14 nickles; 10 dimes
38. $(3,17)$
39. $(7,000,8,000)$

## Problem Set 3.9

As you can see, in addition to the answers to the problems we sometimes have included the system of equations used to solve the problems. Remember, you should attempt the problem on your own before checking your answers or equations.

1. $x+y=25 \quad$ The two numbers
2. 3 and 12
3. $x-y=5$ The two numbers
4. 6 and 29
$y=x+5 \quad$ are 10 and 15 .
$x=2 y+1$ are 9 and 4 .
5. Let $x=$ the amount invested at $6 \%$ and $y=$ the amount invested at $8 \%$.
6. $\$ 2,000$ at $6 \%, \$ 8,000$ at $5 \%$

$$
x+y=20,000 \quad \text { He has } \$ 9,000 \text { at } 8 \%
$$

13. 6 nickels, 8 quarters
$0.06 x+0.08 y=1,380 \quad$ and $\$ 11,000$ at $6 \%$.
14. Let $x=$ the number of dimes and $y=$ the number of quarters.

$$
\begin{aligned}
x+y & =21 & & \text { He has } 12 \text { dimes } \\
0.10 x+0.25 y & =3.45 & & \text { and } 9 \text { quarters } .
\end{aligned}
$$

17. Let $x=$ the number of liters of $50 \%$ solution and $y=$ the number of liters of $20 \%$ solution.

$$
\begin{aligned}
x+y & =18 & & 6 \text { liters of } 50 \% \text { solution } \\
0.50 x+0.20 y & =0.30(18) & & 12 \text { liters of } 20 \% \text { solution }
\end{aligned}
$$

19. 10 gallons of $10 \%$ solution, 20 gallons of $7 \%$ solution
20. 20 adults, 50 kids
21. 16 feet wide, 32 feet long
22. $33 \$ 5$ chips, $12 \$ 25$ chips
23. 50 at $\$ 11,100$ at $\$ 20$
24. $(-2,2),(0,3),(2,4)$
25. 


33. 2

## CHAPTER 3 REVIEW TEST

1. $(4,-6),(0,6),(1,3),(2,0)$
2. $(5,-2),(0,-4),(15,2),(10,0)$
3. $(4,2),(2,-2),\left(\frac{9}{2}, 3\right)$
4. $(2,13),\left(-\frac{3}{5}, 0\right),\left(-\frac{6}{5},-3\right)$
5. $(2,-3),(-1,-3),(-3,-3)$
6. $(6,5),(6,0),(6,-1)$
7. $\left(2,-\frac{3}{2}\right)$
8. $\left(-\frac{8}{3},-1\right),(-3,-2)$
9.-14.

9. $(1,1),(0,-1),(-1,-3)$

10. $(-2,0),(0,-2),(1,-3)$

11. $(-3,0),(-3,5),(-3,-5)$

12. $(-1,-3),(1,3),(0,0)$

13. 


20.

23.

26.

21.

24.

27.

22.

25.

28.

29. $x$-intercept $=2 ; \quad y$-intercept $=-6$
32. $x$-intercept $=2 ; \quad y$-intercept $=-6$
30. $x$-intercept $=12 ; \quad y$-intercept $=-4$
33. no $x$-intercept; $y$-intercept $=-5$
$\begin{array}{lll}\text { 36. } m=-1 & \text { 37. } m=2 & \text { 38. } m=2\end{array}$
44. $m=-2, b=-5$
45. $m=-2, b=3$
39. $y=3 x+2 \quad$ 40. $y=-x+6$
31. $x$-intercept $=3 ; \quad y$-intercept $=-3$
46. $m=-\frac{5}{2}, b=$ 47. $(4,-2)$
34. $x$-intercept $=$
41. $y=-\frac{1}{3} x+\frac{3}{4}$
no $y$-intercept
35. $m=2$
52. $(-1,-2)$
53. $(1,-3)$
54. $(-2,5)$
55. Lines coincide
56. $(2,-2)$
48. $(-3,2)$
42. $y=0$
43. $m=4, b=-1$
60. $(5,-1)$
61. $(-2,7)$
62. $(-4,-2)$
63. $(-1,4)$
64. $(2,-6)$
71. $\$ 4,000$ at $4 \%, \$ 8,000$ at $5 \%$
57. $(1,1)$
49. $(3,-2)$
50. $(2,0)$
51. $(2,1)$
68. Lines coincide 69. 10,8 70. 24 ,
es are parallel: $\varnothing$
59. $(-2,-3)$
73. 10 dimes, 7 nickels 74. 9 dimes, 6 quarters
75. 40 liters of $10 \%$ solution, 10 liters of $20 \%$ solution
76. 20 liters of $25 \%$ solution, 20 liters of $15 \%$ solution

## CHAPTER 4

## Problem Set 4.1

1. Base 4; exponent 2; 16
2. Base 0.3; exponent $2 ; 0.09$
3. Base 4; exponent $3 ; 64$
4. Base -5 ; exponent $2 ; 25$
5. Base -2 ; exponent $3 ;-8$
6. Base 3; exponent $4 ; 81$
7. Base $\frac{2}{3}$; exponent $2 ; \frac{4}{9}$
8. Base $\frac{1}{2}$; exponent $4 ; \frac{1}{16}$
9. a.
b. Either larger or greater will work.
10. $x^{9}$ 21. $y^{30}$ 23. $2^{12}$
11. $x^{9}$
12. $y^{30}$
13. $2^{12}$
14. $x^{28}$ 27. $x^{10}$
15. $5^{12}$
16. $y^{9}$
17. $2^{50}$
18. $a^{3 x}$
19. $b^{x y}$
20. $16 x^{2}$
21. $32 y^{5}$
22. $81 x^{4}$
23. $0.25 a^{2} b^{2}$
24. $64 x^{3} y^{3} z^{3}$
25. $8 x^{12}$
26. $16 a^{6}$
27. $x^{14}$
28. $a^{11}$
29. $128 x$
30. $432 x^{10}$
31. $16 x^{4} y^{6}$
32. $\frac{8}{27} a^{12} b^{15}$
33. $x^{2}$
34. $4 x$
35. 2 71. $4 x$
36. a. 32
b. 64
c. 32
d. 64
37. 

| Number <br> $\boldsymbol{x}$ | Square <br> $\boldsymbol{x}^{\mathbf{2}}$ |
| :---: | :---: |
| -3 | 9 |
| -2 | 4 |
| -1 | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |


77.

| Number <br> $\boldsymbol{x}$ | Square <br> $\boldsymbol{x}^{2}$ |
| :---: | :---: |
| -2.5 | 6.25 |
| -1.5 | 2.25 |
| -0.5 | 0.25 |
| 0 | 0 |
| 0.5 | 0.25 |
| 1.5 | 2.25 |
| 2.5 | 6.25 |

79. $4.32 \times 10^{4}$
80. $5.7 \times 10^{2}$
81. $2.38 \times 10^{5}$
82. 2,490
83. 352
84. 28,000
85. 27 inches $^{3}$
86. 15.6 inches $^{3}$
87. 36 inches $^{3}$
88. Yes, the dimensions could be 2 feet times 3 feet times 7 feet.
89. $6.5 \times 10^{8}$ seconds $\quad$ 101. $\$ 740,000 \quad$ 103. $\$ 180,000 \quad$ 105. 219 inches $^{3} \quad$ 107. 182 inches $^{3} \quad$ 109. $2^{7} \quad$ 111. $2 \cdot 5^{3}$ 113. $2^{4} \cdot 3^{2} \cdot 5 \quad 115.2^{2} \cdot 5 \cdot 41 \quad 117.2^{3} \cdot 3^{3} \quad 119.2^{3} \cdot 3^{3} \cdot 5^{3} \quad 121.5^{6} \quad 123.2^{6} \cdot 3^{3} \quad 125 .-3 \quad 127.11 \quad 129 .-5$
90. $5 \quad 133.2 \quad 135.6 \quad 137.4 \quad 139.3$

## Problem Set 4.2

1. $\frac{1}{9}$
2. $\frac{1}{36}$
3. $\frac{1}{64}$
4. $\frac{1}{125}$
5. $\frac{2}{x^{3}}$
6. $\frac{1}{8 x^{3}}$
7. $\frac{1}{25 y^{2}}$
8. $\frac{1}{100}$
9. 

| Number <br> $\boldsymbol{x}$ | Square <br> $\boldsymbol{x}^{2}$ | Power of 2 <br> $\mathbf{2}^{\boldsymbol{x}}$ |
| :---: | :---: | :---: |
| -3 | 9 | $\frac{1}{8}$ |
| -2 | 4 | $\frac{1}{4}$ |
| -1 | 1 | $\frac{1}{2}$ |
| 0 | 0 | 1 |
| 1 | 4 | 2 |
| 2 | 9 | 4 |

35. $27 x^{3}$
36. $81 x^{4} y^{4}$
37. 1
38. $2 a^{2} b$
39. $\frac{1}{49 y^{6}}$
40. $\frac{1}{x^{8}}$
41. $\frac{1}{y^{3}}$
42. $x^{2}$
43. $a^{6}$
44. $\frac{1}{y^{9}}$
45. $y^{40}$
46. $\frac{1}{x}$
47. $x^{9}$
48. $a^{16}$
49. $\frac{1}{a^{4}}$
50. a. 32
b. 32
c. $\frac{1}{32}$
d. $\frac{1}{32}$
51. a. $\frac{1}{5}$
b. $\frac{1}{8}$
c. $\frac{1}{x}$
d. $\frac{1}{x^{2}}$
52. 

| Number <br> $\boldsymbol{x}$ | Power of 2 <br> $\mathbf{2}^{x}$ |
| :---: | :---: |
| -3 | $\frac{1}{8}$ |
| -2 | $\frac{1}{4}$ |
| -1 | $\frac{1}{2}$ |
| 0 | 1 |
| 1 | 2 |
| 2 | 8 |
| 3 |  |


71. $4.8 \times 10^{-3}$
73. $2.5 \times 10^{1}$
75. $9 \times 10^{-6}$
77.

| Expanded Form | Scientific Notation <br> $\boldsymbol{n} \times \mathbf{1 0}^{\boldsymbol{r}}$ |
| :---: | :---: |
| 0.000357 | $3.57 \times 10^{-4}$ |
| 0.00357 | $3.57 \times 10^{-3}$ |
| 0.0357 | $3.57 \times 10^{-2}$ |
| 0.357 | $3.57 \times 10^{-1}$ |
| 3.57 | $3.57 \times 10^{0}$ |
| 35.7 | $3.57 \times 10^{1}$ |
| 357 | $3.57 \times 10^{2}$ |
| 3,570 | $3.57 \times 10^{3}$ |
| 35,700 | $3.57 \times 10^{4}$ |

81. 0.00008
82. 4.2
83. 0.002
84. $0.0024=2.4 \times 10^{-3}, 0.0051=5.1 \times 10^{-3}, 0.0096=9.6 \times 10^{-3}, 0.0099=9.9 \times 10^{-3}, 0.0111=1.11 \times 10^{-2} \quad 89.2 .5 \times 10^{4}$
85. $2.35 \times 10^{5}$
86. $8.2 \times 10^{-4}$
87. 100 inches $^{2} ; 400$ inches $^{2} ; 4$
88. $x^{2}, 4 x^{2}, 4$
89. 216 inches $^{3} ; 1,728$ inches $^{3} ; 8$
90. $x^{3} ; 8 x^{3} ; 8$
91. $7 x$
92. $2 a$
93. 10y
94. 13.5
95. 8
96. 26.52
97. 12
98. $x^{8}$
99. $x$
100. $\frac{1}{y^{2}}$ 123. 340

Problem Set 4.3

33. $3 \times 10^{10}$
35. $5 \times 10^{-3}$
37. $8 x^{2}$
39. $-11 x^{5}$
41. 0
43. $4 x^{3}$
45. $31 a b^{2}$
49. $4 x^{3}$
51. $\frac{1}{b^{2}}$
53. $\frac{6 y^{10}}{x^{4}}$
55. $2 \times 10^{6}$
73. $x^{2}-x-6$
57. $1 \times 10^{1}$
59. $4.2 \times 10^{-6}$
61. $9 x^{3}$
63. $-20 a^{2}$
65. $6 x^{5} y^{2}$
81. a. $3 x y$
93. -5
89.


91.

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $\boldsymbol{a b}$ | $\boldsymbol{a}+\boldsymbol{b}$ |
| :---: | :---: | :---: | :---: |
| $5 x$ | $3 X$ | $15 X^{2}$ | $8 X$ |
| $4 X^{2}$ | $2 x^{2}$ | $8 X^{4}$ | $6 X^{2}$ |
| $3 X^{3}$ | $6 X^{3}$ | $18 x^{6}$ | $9 X^{3}$ |
| $2 x^{4}$ | $-3 x^{4}$ | $-6 X^{8}$ | $-X^{4}$ |
| $x^{5}$ | $7 x^{5}$ | $7 X^{10}$ | $8 X^{5}$ |

103. $-2 x-9$
104. 11

Problem Set 4.4

1. Trinomial, 3
2. Trinomial, 3
3. Binomial, 1
4. Binomial, 2
5. Monomial, 2
6. Monomial, 0
7. $5 x^{2}+5 x+9$
8. $5 a^{2}-9 a+7$
9. $x^{2}+6 x+8$
10. $6 x^{2}-13 x+5$
11. $x^{2}-9$
12. $3 y^{2}-11 y+10$
13. $6 x^{3}+5 x^{2}-4 x+3$
14. $2 x^{2}-x+1$
15. $2 a^{2}-2 a-2$
16. $-\frac{1}{9} x^{3}-\frac{2}{3} x^{2}-\frac{5}{2} x+\frac{7}{4}$
17. $-4 y^{2}+15 y-22$
18. $-2 x$
19. $4 x$
20. $x^{3}-27$
21. $x^{3}+6 x^{2}+12 x+8$
22. $4 x-8$
23. $x^{2}-33 x+63$
24. $8 y^{2}+4 y+26$
25. $75 x^{2}-150 x-75$
26. $12 x+2$
27. 4
28. a. 0
b. 0
29. a. 2,200
b. $-1,800$
30. $18 \pi$ inches $^{3}$
31. $-15 x^{2}$
32. $6 x^{3}$
33. $6 x^{4}$
34. 5
35. -6
36. $-20 x^{2}$
37. $-21 x$
38. $2 x$
39. $6 x-18$

Problem Set 4.5

1. $6 x^{2}+2 x$
2. $6 x^{4}-4 x^{3}+2 x^{2}$
3. $2 a^{3} b-2 a^{2} b^{2}+2 a b$
4. $3 y^{4}+9 y^{3}+12 y^{2}$
5. $8 x^{5} y^{2}+12 x^{4} y^{3}+32 x^{2} y^{4}$
6. $x^{2}+7 x+12$
7. $x^{2}+7 x+6$
8. $x^{2}+2 x+\frac{3}{4}$
9. $a^{2}+2 a-15$
10. $x y+b x-a y-a b$
11. $x^{2}-36$
12. $y^{2}-\frac{25}{36}$
13. $2 x^{2}-11 x+12$
14. $2 a^{2}+3 a-2$
15. $6 x^{2}-19 x+10$
16. $2 a x+8 x+3 a+12$
17. $25 x^{2}-16$
18. $2 x^{2}+\frac{5}{2} x-\frac{3}{4}$
19. $3-10 a+8 a^{2}$
20. 

| $x$ |  | 3 |
| :---: | :---: | :---: |
|  | $x^{2}$ | $3 x$ |
| 2 | $2 x$ | 6 |
|  |  |  |

$(x+2)(x+3)=x^{2}+2 x+3 x+6$

$$
=x^{2}+5 x+6
$$

## 41.

| $x$ | $x$ | 2 |
| :---: | :---: | :---: |
|  | $x^{2}$ | $x^{2}$ |
|  | $2 x$ |  |
| 1 | $x$ | $x$ |
|  |  | 2 |

$$
(x+1)(2 x+2)=2 x^{2}+4 x+2
$$

43. $a^{3}-6 a^{2}+11 a-6$ 45. $x^{3}+8$
44. $2 x^{3}+17 x^{2}+26 x+9$
45. $5 x^{4}-13 x^{3}+20 x^{2}+7 x+5$
46. $2 x^{4}+x^{2}-15$
47. $6 a^{6}+15 a^{4}+4 a^{2}+10 \quad$ 55. $x^{3}+12 x^{2}+47 x+60 \quad$ 57. $x^{2}-5 x+8 \quad$ 59. $8 x^{2}-6 x-5 \quad$ 61. $x^{2}-x-30 \quad$ 63. $x^{2}+4 x-6$
48. $x^{2}+13 x \quad$ 67. $x^{2}+2 x-3 \quad$ 69. $a^{2}-3 a+6 \quad$ 71. $a . x^{2}-1 \quad$ b. $x^{2}+2 x+1 \quad$ c. $x^{3}+3 x^{2}+3 x+1 \quad$ d. $x^{4}+4 x^{3}+6 x^{2}+4 x+1$
49. a. $x^{2}-1 \quad$ b. $x^{2}-x-2$
c. $x^{2}-2 x-3$ d. $x^{2}-3 x-4$
50. 0 77. -5
51. $-2,3$
52. Never 0
53. $a, b$
54. $A=x(2 x+5)=2 x^{2}+5 x$
55. $A=x(x+1)=x^{2}+x$
56. $R=(1,200-100 p) p=1,200 p-100 p^{2}$
57. 


93. $(5,-7)$
95. $(3,17)$
97. $\$ 800$ at $8 \%, \$ 400$ at $10 \%$
99. 169
101. $-10 x$
103. 0
105. 0 107. $-12 x+16$ 109. $x^{2}+x-2 \quad$ 111. $x^{2}+6 x+9$

## Problem Set 4.6

$\begin{array}{llll}\text { 1. } x^{2}-4 x+4 & \text { 3. } a^{2}+6 a+9 & \text { 5. } x^{2}-10 x+25 & \text { 7. } a^{2}-a+\frac{1}{4}\end{array}$
9. $x^{2}+20 x+100$
11. $a^{2}+1.6 a+0.64$
13. $4 x^{2}-4 x+1$
15. $16 a^{2}+40 a+25$
17. $9 x^{2}-12 x+4$
19. $9 a^{2}+30 a b+25 b^{2}$
21. $16 x^{2}-40 x y+25 y$
23. $49 m^{2}+28 m n+4 n^{2}$
25. $36 x^{2}-120 x y+100 y^{2}$
27. $x^{4}+10 x^{2}+25$
29. $a^{4}+2 a^{2}+1$
31. $y^{2}+3 y+\frac{9}{4}$
33. $a^{2}+a+\frac{1}{4}$
35. $x^{2}+\frac{3}{2} x+\frac{9}{16}$
37. $t^{2}+\frac{2}{5} t+\frac{1}{25}$
39.

| $\boldsymbol{x}$ | $(\boldsymbol{x}+3)^{2}$ | $x^{2}+\mathbf{9}$ | $\boldsymbol{x}^{2}+\mathbf{6 x}+\mathbf{9}$ |
| :---: | :---: | :---: | :---: |
| 1 | 16 | 10 | 16 |
| 2 | 25 | 13 | 25 |
| 3 | 36 | 18 | 36 |
| 4 | 49 | 25 | 49 |

41. 

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $(\boldsymbol{a}+\boldsymbol{b})^{2}$ | $\boldsymbol{a}^{2}+\boldsymbol{b}^{2}$ | $\boldsymbol{a}^{2}+\boldsymbol{a b}+\boldsymbol{b}^{\mathbf{2}}$ | $\boldsymbol{a}^{\mathbf{2}}+\mathbf{2 a b}+\boldsymbol{b}^{\mathbf{2}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 4 | 2 | 3 | 4 |
| 3 | 5 | 64 | 34 | 49 | 64 |
| 3 | 4 | 49 | 25 | 37 | 49 |
| 4 | 5 | 81 | 41 | 61 | 81 |

43. $a^{2}-25$
44. $y^{2}-1$
45. $81-x^{2}$
46. $4 x^{2}-25$
47. $16 x^{2}-\frac{1}{9}$
48. $4 a^{2}-49$
49. $36-49 x^{2}$
50. $x^{4}-9$
51. $a^{4}-16$
52. $25 y^{8}-64$
53. $2 x^{2}-34$
54. $-12 x^{2}+20 x+8$
55. $a^{2}+4 a+6$
56. $8 x^{3}+36 x^{2}+54 x+27$
57. a. 11 b. $1 \quad$ c. $11 \quad$ 73. a. $1 \quad$ b. 13 c.
58. $a^{2}+a b+b a+b^{2}=a^{2}+2 a b+b^{2}$

59. 


83.

85. $2 x^{2}$
87. $X^{2}$
89. $3 x$
91. $3 x y$
93. $5 x$
95. $\frac{a^{4}}{2 b^{2}}$

Problem Set 4.7

1. $x-2$
2. $3-2 x^{2}$
3. $5 x y-2 y$
4. $7 x^{4}-6 x^{3}+5 x^{2}$
5. $10 x^{4}-5 x^{2}+1$
6. $-4 a+2$
7. $-8 a^{4}-12 a^{3}$
8. $-4 b-5 a$
9. $-6 a^{2} b+3 a b^{2}-7 b^{3}$
10. $-\frac{a}{2}-b-\frac{b^{2}}{2 a}$
11. $3 x+4 y$
12. $-y+3$
13. $x^{2}+8 x-9$
14. $y^{2}-3 y+1$
15. $-1+x y$
16. $-a+1$
17. $x^{2}-3 x y+y^{2}$
18. $2-3 b+5 b^{2}$
19. $-2 x y+1$
20. $x y-\frac{1}{2}$
21. $\frac{1}{4 x}-\frac{1}{2 a}+\frac{3}{4}$
22. $\frac{4 x^{2}}{3}+\frac{2}{3 x}+\frac{1}{x^{2}}$
23. $3 a^{3 m}-9 a^{m}$
24. $2 x^{4 m}-5 x^{2 m}+7$
25. $3 x^{2}-x+6$
26. 4
27. $x+5$
28. a. 7
b. 7 c. 23
29. a. 19
b. 34 c. 19
30. a. 2 b. 2
31. $(7,-1)$
32. $(2,3)$
33. $(1,1)$
34. Lines coincide
35. $146 \frac{20}{27}$
36. $2 x+5$
37. $x^{2}-3 x$
38. $2 x^{3}-10 x^{2}$
39. $-2 x$
40. 2

## Problem Set 4.8

1. $x-2$
2. $a+4$
3. $x-3$
4. $x+3$
5. $a-5$
6. $x+2+\frac{2}{x+3}$
7. $a-2+\frac{12}{a+5}$
8. $x+4+\frac{9}{x-2}$
9. $x+4+\frac{-10}{x+1}$
10. $a+1+\frac{-1}{a+2} \quad$ 21. $x-3+\frac{17}{2 x+4}$
11. $3 a-2+\frac{7}{2 a+3}$
12. $2 a^{2}-a-3$
13. $x^{2}-x+5$
14. $x^{2}+x+1$
15. $x^{2}+2 x+4$
16. a. 19
b. 19
C. 5
17. a.
b. $\frac{25}{7}$
c. $\frac{25}{7}$
18. $\$ 430.17$
19. $\$ 290.17$
20. 5,20
21. $\$ 800$ at $8 \%, \$ 400$ at $9 \%$
22. Eight $\$ 5$ bills, twelve $\$ 10$ bills
23. 10 gallons of $20 \%, 6$ gallons of $60 \%$

## CHAPTER 4 REVIEW TEST

## 1. -1

2. -64
3. $\frac{9}{49}$
4. $y^{12}$
5. $X^{30}$
6. $X^{35}$
7. $2^{24}$
8. $27 y^{3}$
9. $-8 x^{3} y^{3} z^{3}$
10. $\frac{1}{49}$
11. $\frac{4}{X^{5}}$
12. $\frac{1}{27 y^{3}}$
13. $a^{6}$
14. $\frac{1}{x^{4}}$
15. $X^{15}$
16. $\frac{1}{X^{5}}$
17. 1
18. -3
19. $9 x^{6} y^{4}$
20. $64 a^{22} b^{20}$
21. $\frac{-1}{27 x^{3} y^{6}}$
22. $b$
23. $\frac{1}{x^{35}}$
24. $5 x^{7}$
25. $4 a^{6}$
26. $2 x^{5} y^{2}$
27. $6.4 \times 10^{7}$
28. $2.3 \times 10^{8}$
29. $8 \times 10^{3}$
30. $8 a^{2}-12 a-3$
31. $-12 x^{2}+5 x-14$
32. $-4 x^{2}-6 x$
33. 14
34. $12 x^{2}-21 x$
35. $24 x^{5} y^{2}-40 x^{4} y^{3}+32 x^{3} y^{4}$
36. $a^{3}+6 a^{2}+a-4$
37. $x^{3}+125$
38. $6 x^{2}-29 x+35$
39. $25 y^{2}-\frac{1}{25}$
40. $a^{4}-9$
41. $a^{2}-10 a+25$
42. $9 x^{2}+24 x+16$
43. $y^{4}+6 y^{2}+9$
44. $-2 b-4 a$
45. $-5 x^{4} y^{3}+4 x^{2} y^{2}+2 x$
46. $x+9$
47. $2 x+5$
48. $x^{2}-4 x+16$
49. $x-3+\frac{16}{3 x+2}$
50. $x^{2}-4 x+5+\frac{5}{2 x+1}$

## CHAPTER 5

## Problem Set 5.1

1. $5(3 x+5)$
2. $3(2 a+3)$
3. $4(x-2 y)$
4. $3\left(x^{2}-2 x-3\right)$
5. $3\left(a^{2}-a-20\right)$
6. $4\left(6 y^{2}-13 y+6\right)$
7. $x^{2}(9-8 x)$
8. $13 a^{2}(1-2 a)$
9. $7 x y$ ( $3 x$
10. $11\left(11 y^{4}-x^{4}\right)$
11. $25 x^{2}\left(4 x^{2}-2 x+1\right)$
12. $8\left(a^{2}+2 b^{2}+4 c^{2}\right)$
13. $4 a b(a-4 b+8 a b) \quad$ 31. $11 a^{2} b^{2}(11 a-2 b+3 a b)$
14. $12 x^{2} y^{3}\left(1-6 x^{3}-3 x^{2} y\right)$
15. $(x+3)(y+5)$
16. $(x+2)(y+6)$
17. $(a-3)(b+7)$
18. $(a-b)(x+y)$
19. $(2 x-5)(a+3)$
20. $(b-2)(3 x-4)$
21. $(x+2)(x+a)$
22. $(x-b)(x-a)$
23. $(x+y)(a+b+c)$
24. $(3 x+2)(2 x+3)$
25. $(10 x-1)(2 x+5)$
26. $(4 x+5)(5 x+1)$
27. $(x+2)\left(x^{2}+3\right)$
28. $(3 x-2)\left(2 x^{2}+5\right)$
29. 6
30. $3\left(4 x^{2}+2 x+1\right)$
31. $y^{2}-16 y+64$
32. $x^{2}-4 x-12$
33. $2 y^{2}-7 y+12$
34. $2 x^{2}+7 x-11$
35. $3 x+8 \quad$ 77. $3 x^{2}+4 x-5 \quad$ 79. $x^{2}-5 x-14$
36. $x^{2}-x-6$
37. $x^{3}+27$
38. $2 x^{3}+9 x^{2}-2 x-3$
39. $18 x^{7}-12 x^{6}+6 x^{5}$
40. $x^{2}+x+\frac{2}{9}$
41. $12 x^{2}-10 x y-12 y^{2}$
42. $81 a^{2}-1$
43. $x^{2}-18 x+81$
44. $x^{3}+8$

## Problem Set 5.2

1. $(x+3)(x+4)$
2. $(x+1)(x+2)$
3. $(a+3)(a+7)$
4. $(x-2)(x-5)$
5. $(y-3)(y-7)$
6. $(x-4)(x+3)$
7. $(y+4)(y-3)$
8. $(x+7)(x-2)$
9. $(r-9)(r+1)$
10. $(x-6)(x+5)$
11. $(a+7)(a+8)$
12. $(y+6)(y-7)$
13. $(x+6)(x+7)$
14. $2(x+1)(x+2)$
15. $3(a+4)(a-5)$
16. $100(x-2)(x-3)$
17. $100(p-5)(p-8)$
18. $x^{2}(x+3)(x-4)$
19. $2 r(r+5)(r-3)$
20. $2 y^{2}(y+1)(y-4)$
21. $x^{3}(x+2)(x+2)$
22. $3 y^{2}(y+1)(y-5)$
23. $4 x^{2}(x-4)(x-9)$
24. $(x+2 y)(x+3 y)$
25. $(x-4 y)(x-5 y)$
26. $(a+4 b)(a-2 b)$
27. $(a-5 b)(a-5 b)$
28. $(a+5 b)(a+5 b)$
29. $(x-6 a)(x+8 a)$
30. $(x+4 b)(x-9 b)$
31. $\left(x^{2}-3\right)\left(x^{2}-2\right)$
32. $(x-100)(x+20)$
33. $\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$
34. $(x+0.2)(x+0.4)$
35. $x+16$
36. $4 x^{2}-x-3$
37. $\frac{4}{25}$
38. $72 a^{12}$
39. $\frac{1}{16 x^{2}}$
40. $\frac{a^{3}}{6 b^{2}}$
41. $\frac{1}{3 x^{7} y^{2}}$
42. $-21 x^{4} y^{5}$
43. $-20 a b^{3}$
44. $-27 a^{5} b^{8}$
45. $6 a^{2}+13 a+2$
46. $6 a^{2}+7 a+2$
47. $6 a^{2}+8 a+2$

Problem Set 5.3

1. $(2 x+1)(x+3)$
2. $(2 a-3)(a+1)$
3. $(3 x+5)(x-1)$
4. $(3 y+1)(y-5)$
5. $(2 x+3)(3 x+2)$
6. $(2 x-3 y)(2 x-3 y)$
7. $(4 y+1)(y-3)$
8. $(4 x-5)(5 x-4) \quad$ 17. $(10 a-b)(2 a+5 b)$
9. $(4 x-5)(5 x+1)$
10. $(6 m-1)(2 m+3)$
11. $(4 x+5)(5 x+3)$
12. $(3 a-4 b)(4 a-3 b)$
13. $(3 x-7 y)(x+2 y)$
14. $(2 x+5)(7 x-3)$
15. $(3 x-5)(2 x-11)$
16. $(5 t-19)(3 t-2)$
17. $2(2 x+3)(x-1)$
18. $2(4 a-3)(3 a-4)$
19. $x(5 x-4)(2 x-3)$
20. $x^{2}(3 x+2)(2 x-5)$
21. $2 a(5 a+2)(a-1)$
22. $3 x(5 x+1)(x-7)$
23. $5 y(7 y+2)(y-2)$
24. $a^{2}(5 a+1)(3 a-1)$
25. $3 y(4 x+5)(2 x-3)$
26. $2 y(2 x-y)(3 x-7 y)$
27. Both equal 25 .
28. $4 x^{2}-9$ 59. $x^{4}-81$
29. $12 x-35$
30. $9 x+8$
31. $16(t-1)(t-3)$
32. $h=2(4-t)(1+8 t)$

| Time $t$ <br> (seconds) | Height $\boldsymbol{h}$ <br> (feet) |
| :---: | :---: |
| 0 | 8 |
| 1 | 54 |
| 2 | 68 |
| 3 | 0 |

69. $6 x^{3}+5 x^{2}-4 x+3 \quad$ 71. $x^{4}+11 x^{3}-6 x-9 \quad$ 73. $2 x^{5}+8 x^{3}-2 x-7$
70. $-8 x^{5}+2 x^{4}+2 x^{2}+12$ 77. $7 x^{5} y^{3} \quad$ 79. $3 a^{2} b^{3}$
71. $x^{2}-9$ 83. $x^{2}-25$
72. $x^{2}-49$ 87. $x^{2}-81 \quad$ 89. $4 x^{2}-9 y^{2}$
73. $x^{4}-16$
74. $x^{2}+6 x+9$
75. $x^{2}+10 x+25$ 97. $x^{2}+14 x+49 \quad$ 99. $x^{2}+18 x+81 \quad$ 101. $4 x^{2}+12 x+9 \quad$ 103. $16 x^{2}-16 x y+4 y^{2}$

Problem Set 5.4

1. $(x+3)(x-3)$
2. $(a+6)(a-6)$
3. $(x+7)(x-7)$ 7. $4(a+2)(a-2)$
4. Cannot be factored
5. $(5 x+13)(5 x-13)$
6. $(3 a+4 b)(3 a-4 b)$
7. $(3+m)(3-m)$
8. $(5+2 x)(5-2 x)$
9. $2(x+3)(x-3)$
10. $32(a+2)(a-2)$
11. $2 y(2 x+3)(2 x-3)$
12. $\left(a^{2}+b^{2}\right)(a+b)(a$
-b) 27. $\left(4 m^{2}+9\right)(2 m+3)(2 m-3)$
13. $3 x y(x+5 y)(x-5 y)$
14. $(x-1)^{2}$
15. $(x+1)^{2}$
16. $(a-5)^{2}$
17. $(y+2)^{2}$
18. $(x-2)^{2}$
19. $(m-6)^{2}$
20. $(2 a+3)^{2}$
21. $(7 x-1)^{2}$
22. $(3 y-5)^{2}$
23. $(x+5 y)^{2}$
24. $(3 a+b)^{2}$
25. $\left(y-\frac{3}{2}\right)^{2}$
26. $\left(a+\frac{1}{2}\right)^{2}$
27. $\left(x-\frac{7}{2}\right)^{2}$
28. $\left(x-\frac{3}{8}\right)^{2}$
29. $\left(t-\frac{1}{5}\right)^{2}$
30. $3(a+3)^{2}$
31. $2(x+5 y)^{2}$
32. $5 x(x+3 y)^{2}$
33. $(x+3+y)(x+3-y)$
34. $(x+y+3)(x+y-3)$
35. 14
36. 25
37. $4 y^{2}-6 y-3$
38. $12 x^{5}-9 x^{3}+3$
39. $-6 x^{3}-4 x^{2}+2 x$
40. $-7 x^{3}+4 x^{2}-11$
41. $x-2+\frac{2}{x-3}$
42. $3 x-2+\frac{9}{2 x+3}$
$\begin{array}{llllllllll}\text { 89. a. } 1 & \text { b. } 8 & \text { c. } 27 & \text { d. } 64 & \text { e. } 125 & \text { 91. a. } x^{3}-x^{2}+x & \text { b. } x^{2}-x+1 & \text { c. } x^{3}+1 & \text { 93. } \mathbf{a} \text {. } x^{3}-2 x^{2}+4 x & \text { b. } 2 x^{2}-4 x+8\end{array}$ c. $x^{3}+8$ 95. a. $x^{3}-3 x^{2}+9 x$ b. $3 x^{2}-9 x+27$ c. $x^{3}+27 \quad$ 97. a. $x^{3}-4 x^{2}+16 x \quad$ b. $4 x^{2}-16 x+64 \quad$ c. $x^{3}+64$
43. a. $x^{3}-5 x^{2}+25 x$
b. $5 x^{2}-25 x+125$
c. $x^{3}+125$

## Problem Set 5.5

1. $(x-y)\left(x^{2}+x y+y^{2}\right)$
2. $(a+2)\left(a^{2}-2 a+4\right)$
3. $(3+x)\left(9-3 x+x^{2}\right)$
4. $(y-1)\left(y^{2}+y+1\right)$
5. $(y-4)\left(y^{2}+4 y+16\right)$
6. $(5 h-t)\left(25 h^{2}+5 h t+t^{2}\right)$
7. $(x-6)\left(x^{2}+6 x+36\right)$
8. $2(y-3)\left(y^{2}+3 y+9\right)$
9. $2(a-4 b)\left(a^{2}+4 a b+16 b^{2}\right)$
10. $2(x+6 y)\left(x^{2}-6 x y+36 y^{2}\right)$
11. $10(a-4 b)\left(a^{2}+4 a b+16 b\right.$
12. $(2 x-3 y)\left(4 x^{2}+6 x y+9 y^{2}\right)$
13. $\left(t+\frac{1}{3}\right)\left(t^{2}-\frac{1}{3} t+\frac{1}{9}\right)$
14. $\left(3 x-\frac{1}{3}\right)\left(9 x^{2}+x+\frac{1}{9}\right)$
15. $(4 a+5 b)\left(16 a^{2}-20 a b+25 b^{2}\right)$
16. $\left(\frac{1}{2} x-\frac{1}{3} y\right)\left(\frac{1}{4} x^{2}+\frac{1}{6} x y+\frac{1}{9} y^{2}\right)$
17. $(a-b)\left(a^{2}+a b+b^{2}\right)(a+b)\left(a^{2}-a b+b^{2}\right)$
18. $(2 x-y)\left(4 x^{2}+2 x y+y^{2}\right)(2 x+y)\left(4 x^{2}-2 x y+y^{2}\right.$
19. $(x-5 y)\left(x^{2}+5 x y+25 y^{2}\right)(x+5 y)\left(x^{2}-5 x y+25 y^{2}\right)$
20. $x=3 y+4 \quad$ 45. $x=\frac{3}{2} y+2$
21. $y=\frac{1}{2} x+3$
22. $y=\frac{2}{3} x-4$
23. $2 x^{5}-8 x^{3}$
24. $3 x^{4}-18 x^{3}+27 x^{2}$
25. $y^{3}+25 y$
26. $15 a^{2}-a-2$
27. $4 x^{4}-12 x^{3}-40 x^{2}$
28. $2 a b^{5}-8 a b^{4}+2 a b^{3}$

## Problem Set 5.6

1. $(x+9)(x-9)$
2. $(x+5)(x-3)$
3. $(x+3)^{2}$
4. $(y-5)^{2}$
5. $2 a b\left(a^{2}+3 a+1\right)$
6. Cannot be factored
7. $3(2 a+5)(2 a-5)$
8. $(3 x-2 y)^{2}$
9. $4 x\left(x^{2}+4 y^{2}\right)$
10. $2 y(y+5)^{2}$
11. $a^{4}\left(a^{2}+4 b^{2}\right)$
12. $(x+4)(y+3)$
13. $(x-3)\left(x^{2}+3 x+9\right)$
14. $(x+2)(y-5)$
15. $5(a+b)^{2}$
16. Cannot be factored
17. $3(x+2 y)(x+3 y)$
18. $(2 x+19)(x-2)$
19. $100(x-2)(x-1)$
20. $(x+8)(x-8)$
21. $(x+a)(x+3)$
22. $a^{5}(7 a+3)(7 a-3)$
23. Cannot be factored
24. $a(5 a+1)(5 a+3)$
25. $(x+y)(a-b)$
26. $3 a^{2} b(4 a+1)(4 a-1)$
27. $5 x^{2}(2 x+3)(2 x-3)$
28. $(3 x+41 y)(x-2 y)$
29. $2 x^{3}(2 x-3)(4 x-5)$
30. $(2 x+3)(x+a)$
31. $\left(y^{2}+1\right)(y+1)(y-1)$
32. $3 x^{2} y^{2}(2 x+3 y)^{2}$
33. $16(t-1)(t-3)$
34. $(9 x+8)(6 x+7)$
35. 1 71. 30
36. -6
37. 3 77. 2
38. 5
39. $-\frac{3}{2}$
40. $-\frac{3}{4}$

Problem Set 5.7

1. $-2,1$
2. 4,5
3. $0,-1,3$
4. $-\frac{2}{3},-\frac{3}{2}$
5. $0,-\frac{4}{3}, \frac{4}{3}$
6. $0,-\frac{1}{3},-\frac{3}{5}$
7. $-1,-2$
8. 4,5
9. $6,-4$
10. 2,3
11. -3
12. $4,-4$
13. $\frac{3}{2},-4$
14. $-\frac{2}{3}$
15. 5
16. $\frac{5}{3},-4$
17. $\frac{7}{2},-\frac{7}{2}$
18. 0, -6
19. 0,3
20. 0,4
21. 0,5
22. 2,5
23. $\frac{1}{2},-\frac{4}{3}$
24. $4,-\frac{5}{2}$
25. $8,-10$
26. 5,8
27. 6,8
28. -4
29. 5,8
30. $6,-8$
31. $0,-\frac{3}{2},-4$
32. $0,3,-\frac{5}{2}$
33. $0, \frac{1}{2},-\frac{5}{2}$
34. $0, \frac{3}{5},-\frac{3}{2}$
35. $\frac{1}{2}, \frac{3}{2}$
36. $-5,4$
37. $-7,-6$
38. $-3,-1$
39. 2,3
40. $-15,10$
41. $-5,3$
42. $-3,-2,2$ 87. $-4,-1,4$
$\begin{array}{lllll}\text { 91. a. } x^{2}-5 x+6=0 & \text { b. } x^{2}-7 x+6=0 & \text { c. } x^{2}-x-6=0 & \text { 93. } \frac{1}{8} & \text { 95. } x^{8}\end{array}$
$\begin{array}{lllll}\text { 91. a. } x^{2}-5 x+6=0 & \text { b. } x^{2}-7 x+6=0 & \text { c. } x^{2}-x-6=0 & \text { 93. } \frac{1}{8} & \text { 95. } x^{8}\end{array}$
43. $x^{2}-8 x+15=0$
44. $x^{18} \quad 99.5 .6 \times 10^{-3}$
45. $5.67 \times 10^{9}$
46. $x(x+1)=72$
47. $x(x+2)=99$
48. $x(x+2)=5[x+(x+2)]-10$
49. Bicycle $\$ 75$, suit $\$ 15$
50. House $\$ 2,400$, lot $\$ 600$

## Problem Set 5.8

1. Two consecutive even integers are $x$ and $x+2 ; x(x+2)=80 ; 8,10$ and $-10,-8 \quad$ 3. 9,11 and $-11,-9$
2. $x(x+2)=5(x+x+2)-10 ; 8,10$ and $0,2 \quad$ 7. $8,6 \quad$ 9. The numbers are $x$ and $5 x+2 ; x(5 x+2)=24 ; 2,12$ and $-\frac{12}{5},-10$
3. 5,20 and 0,0
4. Let $x=$ the width; $x(x+1)=12$; width 3 inches, length 4 inches
5. Let $x=$ the base; $\frac{1}{2}(x)(2 x)=9$; base 3 inches
6. $x^{2}+(x+2)^{2}=10^{2} ; 6$ inches and 8 inches
7. 12 meters
8. $1,400=400+700 x-100 x^{2} ; 200$ or 500 items 23. 200 videotapes or 800 videotapes
9. $R=x p=$
$(1,200-100 p) p=3,200 ; \$ 4$ or $\$ 8$
10. $\$ 7$ or $\$ 10$
11. a. 5 feet b. 12 feet
12. a. 25 seconds later b.

| $\boldsymbol{t}$ (sec) | $\boldsymbol{h}$ (feet) |
| ---: | ---: |
| 0 | 100 |
| 5 | 1,680 |
| 10 | 2,460 |
| 15 | 2,440 |
| 20 | 1,620 |
| 25 | 0 |

33. $200 x^{24}$
34. $x^{7}$
35. $8 \times 10^{1}$
36. $10 a b^{2}$
37. $6 x^{4}+6 x^{3}-2 x^{2}$
38. $9 y^{2}-30 y+25$
39. $4 a^{4}-49$

## CHAPTER 5 REVIEW TEST

1. $10(x-2)$
2. $x^{2}(4 x-9)$
3. $5(x-y)$
4. $x\left(7 x^{2}+2\right)$
5. $4(2 x+1)$
6. $2\left(x^{2}+7 x+3\right)$
7. $8\left(3 y^{2}-5 y+6\right)$
8. $15 x y^{2}\left(2 y-3 x^{2}\right)$
9. $7\left(7 a^{3}-2 b^{3}\right)$
10. $6 a b\left(b+3 a^{2} b^{2}-4 a\right)$
11. $(x+a)(y+b)$
12. $(x-5)(y+4)$
13. $(2 x-3)(y+5)$
14. $(5 x-4 a)(x-2 b)$
15. $(y+7)(y+2)$
16. $(w+5)(w+10)$
17. $(a-6)(a-8)$
18. $(r-6)(r-12)$
19. $(y+9)(y+11)$
20. $(y+6)(y+2)$
21. $(2 x+3)(x+5)$
22. $(2 y-5)(2 y-1)$
23. $(5 y+6)(y+1)$
24. $(5 a-3)(4 a-3)$
25. $(2 r+3 t)(3 r-2 t)$
26. $(2 x-7)(5 x+3)$
27. $(n+9)(n-9)$
28. $(2 y+3)(2 y-3)$
29. Cannot be factored.
30. $(6 y+11 x)(6 y-11 x)$
31. $(8 a+11 b)(8 a-11 b)$
32. $(8+3 m)(8-3 m) \quad$ 33. $(y+10)^{2}$
33. $(m-8)^{2}$
34. $(8 t+1)^{2}$
35. $(4 n-3)^{2}$
36. $(2 r-3 t)^{2}$
37. $(3 m+5 n)^{2}$
38. $2(x+4)(x+6)$
39. $a(a-3)(a-7)$
40. $3 m(m+1)(m-7)$
41. $5 y^{2}(y-2)(y+4)$
42. $2(2 x+1)(2 x+3)$
43. $a(3 a+1)(a-5)$
44. $2 m(2 m-3)(5 m-1)$
45. $5 y(2 x-3 y)(3 x-y)$
46. $4(x+5)^{2} \quad$ 48. $x(2 x+3)^{2}$
47. $5(x+3)(x-3)$
48. $3 x(2 x+3 y)(2 x-3 y)$
49. $(2 a-b)\left(4 a^{2}+2 a b+b^{2}\right)$
50. $(3 x+2 y)\left(9 x^{2}-6 x y+4 y^{2}\right)$
51. $(5 x-4 y)\left(25 x^{2}+20 x y+16 y^{2}\right)$
52. $3 a b(2 a+b)(a+5 b)$
53. $x^{3}(x+1)(x-1)$
54. $y^{4}\left(4 y^{2}+9\right)$
55. $4 x^{3}(x+2 y)(3 x-y)$
56. $5 a^{2} b(3 a-b)(2 a+3 b)$
57. $3 a b^{2}(2 a-b)(3 a+2 b)$
58. $-2,5$
59. $-\frac{5}{2}, \frac{5}{2}$
60. $2,-5$
61. $7,-7$
62. 0,9
63. $-\frac{3}{2},-\frac{2}{3}$
64. $0,-\frac{5}{3}, \frac{2}{3}$
65. 10,12 and $-12,-10$
66. 10,11 and $-11,-10$
67. 5,7 , and $-1,1$
68. 5,15
69. 6,11 and $-\frac{11}{2},-12$
70. 2 inches

## CHAPTER 6

## Problem Set 6.1

1. a.
b. $\frac{1}{x-1}, x \neq-1, x \neq 1 \quad$ c. $\frac{x}{x+1}, x \neq-1, x \neq 1$
d. $\frac{x^{2}+x+1}{x+1}, x \neq-1, x \neq 1$
$\frac{x^{2}+x+1}{x^{2}}, x \neq 0, x \neq 1$
2. $\frac{1}{a+3}, a \neq-3, a \neq 3$
3. $\frac{1}{x-5}, x \neq-5, x \neq 5$
4. $\frac{(x+2)(x-2)}{2}$
5. $\frac{2(x-5)}{3(x-2)}, x \neq 2$
6. 2
7. $\frac{5(x-1)}{4}$
8. $\frac{1}{x-3}$
9. $\frac{1}{x+3}$
10. $\frac{1}{a-5}$
11. $\frac{1}{3 x+2}$
12. $\frac{x+5}{x+2}$
13. $\frac{2 m(m+2)}{m-2}$
14. $\frac{x-1}{x-4}$
15. $\frac{2(2 x-3)}{2 x+3}$
16. $\frac{2 x-3}{2 x+3}$
17. $\frac{1}{\left(x^{2}+9\right)(x-3)}$
18. $\frac{3 x-5}{\left(x^{2}+4\right)(x-2)}$
19. $\frac{2 x(7 x+6)}{2 x+1}$
20. $\frac{x^{2}+x y+y^{2}}{x+y}$
21. $\frac{x^{2}-2 x+4}{x-2}$
22. $\frac{x^{2}-2 x+4}{x-1}$
23. $\frac{x+2}{x+5}$
24. $\frac{x+a}{x+b}$
25. a. $x^{2}-16$
b. $x^{2}-8 x+16$
c. $4 x^{3}-32 x^{2}+64 x$
d. $\frac{x}{4} \quad$ 51. $\frac{4}{3}$
26. $\frac{4}{5}$
27. $\frac{8}{1}$

| Checks Written $x$ | $\begin{aligned} & \text { Total Cost } \\ & \mathbf{2 . 0 0}+\mathbf{0 . 1 5 x} \end{aligned}$ | Cost per Check $\frac{2.00+0.15 x}{x}$ |
| :---: | :---: | :---: |
| 0 | \$2.00 | Undefined |
| 5 | \$2.75 | \$0.55 |
| 10 | \$3.50 | \$0.35 |
| 15 | \$4.25 | \$0.28 |
| 20 | \$5.00 | \$0.25 |

59. 40.7 miles/hour
60. 39.25 feet/minute
61. Johnson: 1.23 Wood: 1.16 Martinez: 1.16 Nomo: 0.98
62. Level ground: 10 minutes/mile or 0.1 mile/minute; downhill: $\frac{20}{3}$ minutes/mile or $\frac{3}{20}$ mile/minute
63. 48 miles/gallon
64. $5+4=9$
65. 

| $\boldsymbol{x}$ | $\frac{\boldsymbol{x}-\mathbf{3}}{\mathbf{3}-\boldsymbol{x}}$ |
| :---: | :---: |
| -2 | -1 |
| -1 | -1 |
| 0 | -1 |
| 1 | -1 |
| 2 | -1 |

73. 

| $\boldsymbol{x}$ | $\frac{\boldsymbol{x}-\mathbf{5}}{\boldsymbol{x}^{2}-\mathbf{2 5}}$ | $\frac{\mathbf{1}}{\boldsymbol{x}+\mathbf{5}}$ |
| :---: | :---: | :---: |
| 0 | $\frac{1}{5}$ | $\frac{1}{5}$ |
| 2 | $\frac{1}{7}$ | $\frac{1}{7}$ |
| -2 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 5 | Undefined | $\frac{1}{10}$ |
| -5 | Undefined | Undefined |

The entries are all -1 because the numerator and denominator are opposites. Or, $\frac{x-3}{3-x}=\frac{-1(x-3)}{3-x}=-1$.
75. Disney, Nike.

| Company | P/E |
| :--- | :---: |
| Yahoo | 35.33 |
| Google | 84.53 |
| Disney | 18.63 |
| Nike | 17.51 |
| eBay | 56.11 |

77. 0 79. $16 a^{2} b^{2}$
78. $19 x^{4}+21 x^{2}-42$
79. $7 a^{4} b^{4}+9 b^{3}-11 a^{3}$
80. $\frac{5}{14}$
81. $\frac{9}{10}$
82. $(x+3)(x-3)$
83. $3(x-3)$
84. $(x-5)(x+4)$
85. $a(a+5)$
86. $(a-5)(a+4)$
87. 5.8

## Problem Set 6.2

1. 2
2. $\frac{x}{2}$
3. $\frac{3}{2}$
4. $\frac{1}{2(x-3)}$
5. $\frac{4 a(a+5)}{7(a+4)}$
6. $\frac{y-2}{4}$
7. $\frac{2(x+4)}{x-2}$
8. $\frac{x+3}{(x-3)(x+1)}$
9. 1
10. $\frac{y-5}{(y+2)(y-2)}$
11. $\frac{x^{2}+5 x+25}{x-5}$
12. 1
13. $\frac{a+3}{a+4}$
14. $\frac{2 y-3}{y-6}$
15. $\frac{x-1}{(x-2)\left(x^{2}-x+1\right)}$
16. $\frac{3}{2}$
17. a. $\frac{4}{13}$
b. $\frac{x+1}{x^{2}+x+1}$
c. $\frac{x-1}{x^{2}+x+1}$
d. $\frac{x+1}{x-1}$
18. $2(x-3)$
19. $(x+2)(x+1)$
20. $-2 x(x-5)$
21. $\frac{2(x+5)}{x(y+5)}$
22. $\frac{2 x}{x-y}$
23. $\frac{1}{x-2}$
24. $\frac{1}{5}$
25. $\frac{1}{100}$
26. 2.7 miles
27. 742 miles/hour
28. 0.45 mile/hour
29. 8.8 miles/hour
30. a. 21.3 kilometers/hour
b. 13.2 miles/hour
31. 3
32. $\frac{11}{4}$
33. $9 x^{2}$
34. $6 a b^{2}$
35. $\frac{4}{5}$
36. $\frac{11}{35}$
37. $-\frac{4}{35}$
38. $2 x-6$
39. $x^{2}-x-20$
40. $\frac{1}{x-3}$
41. $\frac{x-6}{2(x-5)}$
42. $x^{2}-x-30$

Problem Set 6.3

1. $\frac{7}{x}$
2. $\frac{4}{a}$
3. 1
4. $y+1$
5. $x+2$
6. $x-2$
7. $\frac{6}{x+6}$
8. $\frac{(y+2)(y-2)}{2 y}$
9. $\frac{3+2 a}{6}$
10. $\frac{7 x+3}{4(x+1)}$
11. $\frac{1}{5}$
12. $\frac{1}{3}$
13. $\frac{3}{x-2}$
14. $\frac{4}{a+3}$
15. $\frac{2 x-20}{(x-5)(x+5)}$
16. $\frac{x+2}{x+3}$
17. $\frac{a+1}{a+2}$
18. $\frac{1}{(x+3)(x+4)}$
19. $\frac{y}{(y+5)(y+4)}$
20. $\frac{3(x-1)}{(x+4)(x+1)}$
21. $\frac{1}{3}$
22. a. $\frac{2}{27}$
b. $\frac{8}{3}$
c. $\frac{11}{18}$
d. $\frac{3 x+10}{(x-2)^{2}}$
e. $\frac{(x+2)^{2}}{3 x+10}$
f. $\frac{x+3}{x+2}$
23. 

| Number | Reciprocal | Sum | Sum |
| :---: | :---: | :---: | :---: |
| $\boldsymbol{x}$ | $\frac{1}{\boldsymbol{x}}$ | $\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{x}}$ | $\frac{\boldsymbol{x + 1}}{\boldsymbol{x}}$ |
| 1 | 1 | 2 | 2 |
| 2 | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{3}{2}$ |
| 3 | $\frac{1}{3}$ | $\frac{4}{3}$ | $\frac{4}{3}$ |
| 4 | $\frac{1}{4}$ | $\frac{5}{4}$ | $\frac{5}{4}$ |

47. 

| $x$ | $x+\frac{4}{x}$ | $\frac{x^{2}+4}{x}$ | $x+4$ |
| :---: | :---: | :---: | :---: |
| 1 | 5 | 5 | 5 |
| 2 | 4 | 4 | 6 |
| 3 | $\frac{13}{3}$ | $\frac{13}{3}$ | 7 |
| 4 | 5 | 5 | 8 |

49. $\frac{x+3}{x+2}$
50. $\frac{x+2}{x+3}$
51. 3
52. -2
53. $\frac{1}{5}$
54. $-2,-3$
55. $3,-2$
56. 0,5
57. 3
58. 0
59. Undefined
60. $-\frac{3}{2}$
61. $2 x+15$
62. $x^{2}-5 x$
63. -6
64. -2

## Problem Set 6.4

1. -3
2. 20
3. -1
4. 5
5. -2
6. 4
7. 3,5
8. $-8,1$
9. 2
10. 1
11. 8
12. 5
13. Possible solution 2 , which does not check; $\varnothing$ 27.

35
29. Possible solution -3 , which does not check; $\varnothing$
31. 0
33. Possible solutions 2 and 3 , but only 2 checks; 2
35. -4
37. -1
39. $-6,-7$
41. Possible solutions -3 and -1 , but only -1 checks; -1
43. a.
b. 5 c. $\frac{25}{3}$
d. 3 e. $\frac{1}{5}, 5$
45. a. $\frac{7}{(a-6)(a+2)}$
b. $\frac{a-5}{a-6}$
c. Possible answers $7,-1$; only 7 checks
47. 7
49. Length 13 inches, width 4 inches
51. 6,8 , or $-8,-6$
53. 6 inches, 8 inches
55. $\frac{1}{3}$
57. 30
59. 1
61. -3

## Problem Set 6.5

1. $\frac{1}{4}, \frac{3}{4}$
2. $x+\frac{1}{x}=\frac{13}{6} ; \frac{2}{3}$ and $\frac{3}{2}$
3. -2
4. $\frac{1}{x}+\frac{1}{x+2}=\frac{5}{12} ; 4$ and 6
5. Let $x=$ the speed of the boat in still water.

|  | $\boldsymbol{d}$ | $\boldsymbol{r}$ | $\boldsymbol{t}$ |
| :---: | :---: | :---: | :---: |
| Upstream | 26 | $x-3$ | $\frac{26}{x-3}$ |
| Downstream | 38 | $x+3$ | $\frac{38}{x+3}$ |

The equation is $\frac{26}{x-3}=\frac{38}{x+3} ; x=16$ miles per hour.
11. $300 \mathrm{miles} /$ hour
13. 170 miles/hour, 190 miles/hour
15. 9 miles/hour
17. 8 miles/hour
19. $\frac{1}{12}-\frac{1}{15}=\frac{1}{x} ; 60$ hours
21. $\frac{60}{11}$ minutes
23. 12 minutes
25.

27.

29.

31. $5 a b\left(3 a^{2} b^{2}-4 a-7 b\right)$
33. $(x-6)(x+2)$
35. $\left(x^{2}+4\right)(x+2)(x-2)$
37. $5 x(x-6)(x+1)$
39. 0,6
41. $-10,8$
43. 9 inches and 12 inches
45. $\frac{4}{9}$ 47. $\frac{5}{3}$
49. $3 x^{5} y^{3}$
51. $\frac{x^{3} y^{3}}{2}$
53. $y(x y+1)$
55. $\frac{x^{3} y^{3}}{2}$
57. $\frac{x+3}{x-2}$

Problem Set 6.6

1. 6
2. $\frac{1}{6}$
3. $x y^{2}$
4. $\frac{x y}{2}$
5. $\frac{y}{x}$
6. $\frac{a+1}{a-1}$
7. $\frac{x+1}{2(x-3)}$
8. $a-3$
9. $\frac{y+3}{y+2}$
10. $x+y$
11. $\frac{a+1}{a}$
12. $\frac{1}{x}$
13. $\frac{2 a+3}{3 a+4}$
14. $\frac{x-1}{x+2}$
15. $\frac{x+4}{x+1}$
16. $\frac{7}{3}, \frac{17}{7}, \frac{41}{17}$
17. 

| Number | Reciprocal | Quotient | Square |
| :---: | :---: | :---: | :---: |
| $\boldsymbol{x}$ | $\frac{1}{x}$ | $\frac{x}{\frac{1}{x}}$ | $\boldsymbol{x}^{\mathbf{2}}$ |
| 1 | 1 | 1 | 1 |
| 2 | $\frac{1}{2}$ | 4 | 4 |
| 3 | $\frac{1}{3}$ | 9 | 9 |
| 4 | $\frac{1}{4}$ | 16 | 16 |

35. 

\(\left.\left.$$
\begin{array}{cccc}\text { Number } & \text { Reciprocal } & \text { Sum } & \begin{array}{c}\text { Quotient } \\
\boldsymbol{x}\end{array}
$$ <br>

\hline \frac{\mathbf{1}}{\boldsymbol{x}} \& \mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{x}}\end{array}\right] $$
\begin{array}{c}\frac{\mathbf{x}}{\boldsymbol{x}} \boldsymbol{x}\end{array}
$$\right]\)| 1 | 1 | $\frac{3}{2}$ | 3 |
| :---: | :---: | :---: | :---: |
| 2 | $\frac{1}{2}$ | $\frac{4}{3}$ | 4 |
| 3 | $\frac{1}{3}$ | $\frac{5}{4}$ | 5 |
| 4 | $\frac{1}{4}$ |  |  |

37. $x<1$
38. $x \geq-7$
39. $x<6$
40. $x \leq 2$
41. $\frac{7}{2}$
42. $-3,2$

Problem Set 6.7

1. 1
2. 10
3. 40
4. $\frac{5}{4} \quad$ 9. $\frac{7}{3}$
5. $3,-4$
6. $4,-4$
7. $2,-4$
8. $6,-1$
9. 15 hits
10. 21 milliliters
11. 45.5 grams
12. 14.7 inches
13. 12.5 miles
14. 17 minutes
15. 343 miles
16. $\frac{y+5}{x+a}$
17. $\frac{3}{x+1}$
18. $\frac{-3}{(x+1)(x+4)}$
19. 15
20. 4
21. $-6,6$
22. 18
23. 5

## CHAPTER 6 REVIEW TEST

1. $\frac{1}{2(x-2)}, x \neq 2$
2. $\frac{1}{a-6}, a \neq-6,6$
3. $\frac{2 x-1}{x+3}, x \neq-3$
4. $\frac{1}{x+4}, x \neq-4$
5. $\frac{3 x-2}{2 x-3}, x \neq \frac{3}{2},-6,0$
6. $\frac{1}{(x-2)\left(x^{2}+4\right)}, x \neq \pm 2$
7. $x-2, x \neq-7$
8. $a+8, a \neq-8$
9. $\frac{y+b}{y+5}, x \neq-a, y \neq-5$
10. $\frac{x}{2}$
11. $\frac{x+2}{x-4}$
12. $(a-6)^{2}$
13. $\frac{x-1}{x+2}$
14. 1
15. $x-9$
16. $\frac{13}{a+8}$
17. $\frac{x^{2}+5 x+45}{x(x+9)}$
18. $\frac{4 x+5}{4(x+5)}$
19. $\frac{1}{(x+6)(x+2)}$
20. $\frac{a-6}{(a+5)(a+3)}$
21. 4
22. 9
23. 1,6
24. Possible solution 2 , which does not check; $\varnothing$
25. 5;-4 does not check
26. 3 and $\frac{7}{3}$
27. 15 miles/hour
28. 84 hours
29. $\frac{1}{2}$
30. $\frac{y+3}{y+7}$
31. $\frac{4 a-7}{a-1}$
32. $\frac{2}{5}$
33. $\frac{2}{9}$
34. 12
35. $-6,6$
36. $-6,8$

## CHAPTER 7

## Problem Set 7.1

$\begin{array}{lll}\text { 1. } 5 & \text { 3. }-\frac{4}{3} & \text { 5. }-2\end{array}$
7. $\frac{3}{4} \quad$ 9. -10
11. -7
13. $\frac{4}{5}$
15. -3
17. 4
19. 17
21. 6
23. 12
25. 6,000
27. $\frac{3}{2}$
29. $6,-1$
31. $0,2,3$
33. $\frac{1}{3},-4$
35. $5,-5$
37. $-4, \frac{5}{2}$
39. $0, \frac{4}{3}$
41. $-\frac{1}{5}, \frac{1}{3}$
43. $0,-\frac{4}{3}, \frac{4}{3}$
45. $-10,0$
47. $-5,1$
49. $-2,3$
51. $-3,-2,2$
53. $-\frac{3}{2},-2,2$
55. No solution
57. All real numbers are solutions.
59. No solution
61. $-\frac{7}{640}$
63. 2
65. 20
67. $\frac{7}{10}$
69. 2,400
71. 24
73. 6
75. 3
77. $\frac{46}{15}$
79. $-\frac{73}{33}$
81. $-2, \frac{5}{3}$
83. $-1,9$
85. 10 feet by 20 feet
87. \$92.00
89. $\$ 54$
91.

| Age <br> (years) | Maximum Heart Rate <br> (beats per minute) |
| :---: | :---: |
| 18 | 202 |
| 19 | 201 |
| 20 | 200 |
| 21 | 199 |
| 22 | 198 |
| 23 | 197 |

93. 

| Resting Heart Rate <br> (beats per minute) | Training Heart Rate <br> (beats per minute) |
| :---: | :---: |
| 60 | 144 |
| 62 | 145 |
| 64 | 146 |
| 68 | 147 |
| 70 | 148 |
| 72 | 149 |

95. $-5,-3$ or 3,5
96. 3
97. -3
98. The distance between $x$ and 0 on the number line
99. -3
100. 15
101. $\varnothing$
102. -1

Problem Set 7.2

1. $-4,4$
2. $-2,2$
3. $\varnothing \quad$ 7. $-1,1$
4. $\varnothing$
5. $-6,6$
6. $-3,7$
7. $\frac{17}{3}, \frac{7}{3}$
8. 2, 4
9. $-\frac{5}{2}, \frac{5}{6}$
10. $-1,5$
11. $\varnothing$
12. $-4,20$
13. $-4,8$
14. $-\frac{10}{3}, \frac{2}{3}$
15. $\varnothing$
16. $-1, \frac{3}{2}$
17. 5,25
18. $-30,26$
19. $-12,28$
20. $-5, \frac{3}{5}$
21. $1, \frac{1}{9}$
22. $-\frac{1}{2}$
23. $0 \quad$ 49. $-\frac{1}{2}$
24. $-\frac{1}{6},-\frac{7}{4}$
25. All real numbers
26. All real numbers
27. a. $\frac{5}{4}=1.25$ b. . $\frac{5}{4}=1.25$ c. 2 d
d. $\frac{1}{2}, 2$ e. $\frac{1}{3}, 4$
28. 1987 and 1995
29. $\frac{19}{15}$
30. 40
31. -2
32. 0
33. -5
34. 6
35. $x=a-b$ or $x=a+b$
36. $x=\frac{-b-c}{a}$ or $x=\frac{-b+c}{a}$
37. $x=-\frac{a}{b} y-a$ or $x=-\frac{a}{b} y+a$

## Problem Set 7.3

1. $x \leq \frac{3}{2}$

2. $x>4$

3. $x<4$

4. $x \geq-6$

5. $x \geq 4$

6. $(-\infty,-2]$
7. $[1, \infty)$
8. $(-\infty, 3)$
9. $(-\infty,-1]$
10. $[-17, \infty)$
11. $x>435$

12. $(-4,2)$

13. $[4,6]$

14. $(-4,2)$

15. $(-3,3)$

16. $(-\infty,-7] \cup[-3, \infty)$

17. $(-\infty,-1] \cup\left[\frac{3}{5}\right.$,

18. $(-\infty,-10) \cup(6, \infty)$

19. $-2<x \leq 4$
20. $x<-4$ or $x \geq 1$
21. a. $x>0$ b. $x \geq 0$ c. $x \geq 0$
22. $\varnothing$
23. all real numbers
24. $\varnothing$
25. a. 1 b. 16 c. no d. $x>16$
26. $p \leq 2$; set the price at $\$ 2.00$ or less per pad $\quad$ 69. $p>1.25$; charge more than $\$ 1.25$ per pad
27. During the years 1990 through 1994 73. Adults: $0.61 \leq r \leq 0.83 ; 61 \%$ to $83 \%$; Juveniles: $0.06 \leq r \leq 0.20 ; 6 \%$ to $20 \%$
28. a. $35^{\circ}$ to $45^{\circ}$ Celsius; $35^{\circ} \leq C \leq 45^{\circ}$ b. $20^{\circ} \leq C \leq 30^{\circ}$ c. $-25^{\circ}$ to $-10^{\circ}$ Celsius; $-25^{\circ} \leq C \leq-10^{\circ}$ d. $-20^{\circ} \leq C \leq-5^{\circ}$
29. $x-4$
30. $\frac{2}{a-2}$
31. $\frac{2 x+1}{x}$
32. $\frac{x+2}{x+a}$
33. $x<4$
34. $-\frac{11}{3} \leq a$
35. $t \leq-\frac{3}{2}$

Problem Set 7.4

1. $-3<x<3$

2. $x \leq-2$ or $x \geq 2$

3. $-3<x<3$

4. $t<-7$ or $t>7$

5. $\varnothing$
6. All real numbers
7. $-4<x<10$

8. $\varnothing \quad$ 19. $-1<x<5$

9. $a \leq-9$ or $a \geq-1$

10. $y \leq-5$ or $y \geq-1$

11. $k \leq-5$ or $k \geq 2 \longleftarrow \underset{-5}{\square}$
12. $-1<x<7$

13. $a \leq-2$ or $a \geq 1$

14. $-6<x<\frac{8}{3}$

15. $x<2$ or $x>8$

16. $x \leq-3$ or $x \geq 12$

17. $x<2$ or $x>6$

18. $0.99<x<1.01$
19. $x \leq-\frac{3}{5}$ or $x \geq-\frac{2}{5}$
20. $-\frac{1}{6} \leq x \leq \frac{3}{2}$
21. $-0.05<x<0.25$
22. $|x| \leq 4$
23. $|x-5| \leq 1$
24. a. 3 b. $\frac{4}{5}$ or -2 c. No d. $x<-2$ or $x>\frac{4}{5}$
25. $|x-65| \leq 10$
26. $\frac{2(x+5)}{x(x+1)}$
27. $\frac{x+1}{x-1}$
28. $(x+6)(x+3)$
29. $(1,3)$
30. $(1,3)$
31. $(1,1)$

## Problem Set 7.5

1. $(4,3)$

2. $(-5,-6)$

3. $(4,2)$

4. Lines are parallel; there is no solution.

5. Lines coincide; any solution to one of the equations is a solution to the other

6. $(2,3)$
7. $(1,1)$
8. Lines coincide: $\{(x, y) \mid 3 x-2 y=6\}$
9. $\left(1,-\frac{1}{2}\right)$
10. $\left(\frac{1}{2},-3\right)$
11. $\left(-\frac{8}{3}, 5\right)$
12. $(2,2)$
13. Parallel lines; $\varnothing$
14. $(12,30)$
15. $(10,24)$
16. $(3,-3)$
17. $\left(\frac{4}{3},-2\right)$
18. $y=5, z=2 \quad$ 37. $(2,4)$
19. Lines coincide: $\{(x, y) \mid 2 x-y=5\}$
20. Lines coincide: $\left\{(x, y) \left\lvert\, x=\frac{3}{2} y\right.\right\}$
21. $\left(-\frac{15}{43},-\frac{27}{43}\right)$
22. $\left(\frac{60}{43}, \frac{46}{43}\right)$
23. $\left(\frac{9}{41},-\frac{11}{41}\right)$
24. $\left(-\frac{11}{7},-\frac{20}{7}\right)$
25. $\left(2, \frac{4}{3}\right)$
26. $(-12,-12)$
27. Parallel, $\varnothing$
28. $y=5, z=2$
29. $\left(\frac{3}{2}, \frac{3}{8}\right)$
30. $\left(-4,-\frac{8}{3}\right)$
31. a. $-y$
b. -2 c. -2 d.

e. $(0,-2)$
32. 5,13
33. 225 adult and 700 children's tickets
34. 5 gallons of $20 \%, 10$ gallons of $14 \%$
35. speed of boat, 9 mph ; speed of current, 3 mph
36. speed of airplane, 270 mph ; speed of wind, 30 mph
37. $x+5$
38. $\frac{5 a+6}{15}$
39. $\frac{1}{(a+2)(a+3)}$
40. $3 y+2 z$
41. 1
42. 3
43. $10 x-2 z$
44. $9 x+3 y-6 z$

## Problem Set 7.6

1. $(1,2,1)$
2. $(2,1,3)$
3. $(2,0,1)$
4. $\left(\frac{1}{2}, \frac{2}{3},-\frac{1}{2}\right)$
5. No solution, inconsistent system
6. $(4,-3,-5)$
7. No unique solution
8. $(4,-5,-3)$
9. No unique solution
10. $\left(\frac{1}{2}, 1,2\right)$
11. $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right)$
12. $\left(\frac{10}{3},-\frac{5}{3},-\frac{1}{3}\right)$
13. $\left(\frac{1}{4},-\frac{1}{3}, \frac{1}{8}\right)$
14. $(6,8,12)$
15. $(3,6,4)$
16. $(1,3,1)$
17. $(-1,2,-2)$
18. $4 \mathrm{amp}, 3 \mathrm{amp}, 1 \mathrm{amp}$
19. $\$ 200$ at $6 \% ; \$ 1,400$ at $8 \% ; \$ 600$ at $9 \%$
20. 3 of each
21. -5
22. 5
23. 7
24. 9
25. 6
26. $(0,0),(4,0)$
27. $(0,0),(2,0)$

A-32 Answers to Odd-Numbered Problems and Chapter Reviews

## Problem Set 7.7

1. 


7.

13.

19.

3.

9.

15.

21.

5.

11.

17.

23.

25.

27.

33.

29.

35.

37. $x+y>4$
39. $-x+2 y \leq 4$
41. a. $y<\frac{4}{3}$
b. $y>-\frac{4}{3}$
c. $y=-\frac{2}{3} x+2$ d.

47. $y \leq 7$
49. $t>-2$
51. $-1<t<2$
53. No
55. (4, 0)
57. $x>435$
45.


## Problem Set 7.8

1. 


3.

5.

7.

9.

11.

13.

15.

17.

19. $x+y \leq 4$
$-x+y<4$
23. a. $0.55 x+0.65 y \leq 40$
21. $x+y \geq 4$
$-x+y<4$

$$
\begin{aligned}
& x \geq 2 y \\
& x>15 \\
& y \geq 0
\end{aligned}
$$

b. 1065 -cent stamps
25. $\frac{x+2}{x+3}$
27. $\frac{2(x+5)}{x-4}$
29. $\frac{1}{x-4}$
31. $\frac{x+5}{x-3}$
33. -2

## CHAPTER 7 REVIEW TEST

1. -3
2. $\frac{10}{13}$
3. $-\frac{5}{11}$
4. $-\frac{4}{9}$
5. $-\frac{3}{2}, 4$
6. $-\frac{1}{9}, \frac{1}{9} \quad$ 7. 1,4
7. $-2,-\frac{2}{3}, \frac{2}{3}$
8. 4 feet by 12 feet
9. $3,4,5$
10. 2,4 12. $-1,5$
11. $-1,4$
12. $-\frac{5}{3}, 3$
13. $-\frac{3}{2}, 3$
14. $-\frac{5}{3}, \frac{7}{3}$
15. $(-\infty, 12]$
16. $(-1, \infty)$
17. $[2,6]$
18. $[-3,2]$
19. $\left(-\infty,-\frac{3}{2}\right] \cup[3, \infty)$
20. $\left(-\infty, \frac{4}{5}\right] \cup[2, \infty)$
21. 


24.

25.

27. $\varnothing$
28. All real numbers
29. $(0,1)$
30. Lines coincide
31. $(3,5)$
32. $(4,8)$
33. $\left(\frac{3}{2}, \frac{1}{2}\right)$
34. $(4,1)$
35. $(-5,3)$
36. Parallel lines
37. $(3,-1,4)$
38. $\left(2, \frac{1}{2},-3\right)$
39. Dependent system
40. Dependent system
42. Dependent system
43.

44.

45.

46.

47.

48.


## CHAPTER 8

## Problem Set 8.1

1. $\frac{3}{2}$
2. No slope
3. $\frac{2}{3}$
4. 


9.

11.

13.

15.

17.

19. $x=3$
21. $a=5$
23. $b=2$
25.

| $x$ | $y$ |
| :--- | ---: |
| 0 | 2 |
| 3 | 0 |
| Slope $=-\frac{2}{3}$ |  |

27. 

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :--- | ---: |
| 0 | -5 |
| 3 | -3 |
| Slope $=\frac{2}{3}$ |  |

29. 


31. $\frac{1}{5}$
33. 0
35. -1
37. $-\frac{3}{2}$
39. a. yes b. no
41. a. 17.5 mph
b. $40 \mathrm{~km} / \mathrm{h}$
c. $120 \mathrm{ft} / \mathrm{sec}$
d. $28 \mathrm{~m} / \mathrm{min}$
43. Slopes: $A,-50 ; B-75 ; C,-25$
45. a. 15,800 feet b. $-\frac{7}{100}$
47. 6.5 percent/year. Over the 10 years from 1995 to 2005 , the percent of adults that access the Internet has increased at an average rate of $6.5 \%$ each year.
49. a. $-4,768.6$ cases/year. New cases of rubella decreased at an average of 4,768.6 cases per year from 1969 to 1979 .
b. -999.1 cases/year. New cases of rubella decreased at an average of 999.1 cases per year from 1979 to 1989.
c. The slope is 0, meaning that the number of new cases of German Measles is constant at 9 cases/year.
51. 0
53. $y=-\frac{3}{2} x+6 \quad$ 55. $t=\frac{A-P}{P r}$
57. -1
59. -2
61. $y=m x+b$
63. $y=-2 x-5$
65. 5
67. $\left(\frac{5}{4}, \frac{7}{4}\right)$

## Problem Set 8.2

1. $y=2 x+3$
2. $y=x-5$
3. $y=\frac{1}{2} x+\frac{3}{2}$
4. $y=4$
5. a. 3 b. $-\frac{1}{3}$
6. a. -3 b. $\frac{1}{3}$
7. a. $-\frac{2}{5}$
b. $\frac{5}{2}$
8. Slope $=3$,
$y$-intercept $=-2$,
perpendicular slope $=-\frac{1}{3}$
9. slope $=\frac{2}{3}$,
$y$-intercept $=-4$,
perpendicular slope $=-\frac{3}{2}$

10. Slope $=\frac{1}{2}, y$-intercept $=-4, y=\frac{1}{2} x-4$
11. $y=-3 x+1$
12. $y=\frac{2}{3} x+\frac{2}{3}$
13. $y=-\frac{1}{4} x+\frac{3}{4}$
14. $x-y=2$
15. $2 x-y=3$
16. $y=2 x-1$
17. $y=-\frac{1}{2} x-1$
18. a. $x$-intercept $=\frac{10}{3} ; y$-intercept $=-5$ b. $(4,1)$ c. $y=\frac{3}{2} x-5$ d. No
19. a. $x$-intercept $=\frac{4}{3} ; y$-intercept $=-2$ b. $(2,1)$ c. $y=\frac{3}{2} x-2$ d. No
20. a. 2 b. $y=2 x-3$ c. $y=-3$ d. 2 e.

21. $(0,-4),(2,0) ; y=2 x-4$
22. $(-2,0)(0,4) ; y=2 x+4$
23. Slope $=0$, $y$-intercept $=-2$

24. $y=3 x+7$
25. $y=-\frac{5}{2} x-13$
26. $y=\frac{1}{4} x+\frac{1}{4}$
27. $y=-\frac{2}{3} x+2$
28. b. $86^{\circ} \mathrm{F}$
29. a. $\$ 190,000$
b. $\$ 19$
c. $\$ 6.50$
30. a. $y=\frac{69}{2} x-68,994$
b. 213 books
31. a. $y=6.5 x-12,958.5$
b. In 2006 80.5\% of adults will be online.
c. When we try to estimate the number of adults online in 2010 we get a number larger than $100 \%$, which is impossible.
32. 23 inches, 5 inches 71. $\$ 46.5073$.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| ---: | ---: |
| 0 | 0 |
| 10 | 75 |
| 20 | 150 |

75. 

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :--- | ---: |
| 0 | 0 |
| 1 | 1 |
| 1 | -1 |

77. $y=-\frac{3}{2} x+12$
78. $y=-\frac{4}{3} x-\frac{3}{2}$

## Problem Set 8.3

1. Domain $=\{1,2,4\}$; Range $=\{3,5,1\}$; a function
2. Domain $=\{7,3\}$; Range $=\{-1,4\}$; not a function
3. Domain $=\{a\}$; Range $=\{1,2,3,4\}$; not a function
4. Domain $=\{a, b, c, d\}$; range $=\{3,4,5\}$; a function
5. Yes
6. No
7. No
8. Yes
9. Yes
10. Domain $=\{x \mid-5 \leq x \leq 5\}$; Range $=\{y \mid 0 \leq y \leq 5\}$
11. Domain $=\{x \mid-5 \leq x \leq 3\}$; Range $=\{y \mid y=3\}$
12. Domain $=$ All real numbers

Range $=\{y \mid y \geq-1\}$;
a function

31. Domain $=\{x \mid x \geq 4\}$; Range $=$ All real numbers; not a function

27. Domain $=$ All real numbers; Range $=\{y \mid y \geq 4\} ;$ a function

33. Domain $=$ All real numbers; Range $=\{y \mid y \geq 0\}$;
a function

29. Domain $=\{x \mid x \geq-1\}$; Range $=$ All real numbers; not a function

35. Domain $=$ All real numbers;

Range $=\{y \mid y \geq-2\}$;
a function

37. a. $y=8.5 x$ for $10 \leq x \leq 40$
b.

| Hours Worked | Function Rule <br> $\boldsymbol{y}=\mathbf{8 . 5 \boldsymbol { x }}$ | Gross Pay (\$) <br> $\boldsymbol{x}$ |
| :---: | :---: | :---: |
| 10 | $y=8.5(10)=85$ | $\boldsymbol{y}$ |
| 20 | $y=8.5(20)=170$ | 85 |
| 30 | $y=8.5(30)=255$ | 170 |
| 40 | $y=8.5(40)=340$ | 255 |
|  |  | 340 |

c.

d. Domain $=\{x \mid 10 \leq x \leq 40\} ;$ Range $=\{y \mid 85 \leq y \leq 340\}$
e. Minimum $=\$ 85$; Maximum $=\$ 340$
39. a.

| Time (sec) <br> $\boldsymbol{t}$ | Function Rule <br> $\boldsymbol{h}=\mathbf{1 6} \boldsymbol{t}-\mathbf{1 6} \boldsymbol{t}^{\mathbf{2}}$ | Distance (ft) <br> $\boldsymbol{h}$ |
| :---: | :--- | :---: |
| 0 | $h=16(0)-16(0)^{2}$ | 0 |
| 0.1 | $h=16(0.1)-16(0.1)^{2}$ | 1.44 |
| 0.2 | $h=16(0.2)-16(0.2)^{2}$ | 2.56 |
| 0.3 | $h=16(0.3)-16(0.3)^{2}$ | 3.36 |
| 0.4 | $h=16(0.4)-16(0.4)^{2}$ | 3.84 |
| 0.5 | $h=16(0.5)-16(0.5)^{2}$ | 4 |
| 0.6 | $h=16(0.6)-16(0.6)^{2}$ | 3.84 |
| 0.7 | $h=16(0.7)-16(0.7)^{2}$ | 3.36 |
| 0.8 | $h=16(0.8)-16(0.8)^{2}$ | 2.56 |
| 0.9 | $h=16(0.9)-16(0.9)^{2}$ | 1.44 |
| 1 | $h=16(1)-16(1)^{2}$ | 0 |

41. a.

b. Domain $=\{t \mid 0 \leq t \leq 1\}$; Range $=\{h \mid 0 \leq h \leq 4\}$
C.

b. Domain $=\{r \mid 0 \leq r \leq 3\}$

Range $=\{A \mid 0 \leq A \leq 9 \pi\}$
43. a. Yes b. Domain $=\{t \mid 0 \leq t \leq 6\}$; Range $=\{h \mid 0 \leq h \leq 60\}$ c. $t=3$ d. $h=60$ e. $t=6$
49. -14
51. 1
53. -3
55. $-\frac{6}{5}$
69. Domain $=$ All real numbers; Range $=\{y \mid y \leq 5\}$; a function

57. $-\frac{7}{640}$
59. 150
61. 113
71. Domain $=\{x \mid x \geq 3\}$;

Range $=$ All real numbers; not a function

45. a. III b. I c. II d. IV
47. 10
65. a. 6 b. 7.5
67. a. 27 b. 6
73. Domain $=\{x \mid-4 \leq x \leq 4\}$; Range $=\{y \mid-4 \leq y \leq 4\}$; not a function


## Problem Set 8.4

1. -1
2. $-11 \quad$ 5. 2
3. 4
4. 35
5. -13
6. 1
7. -9
8. 8
9. 0
19 21. 16
10. 0
11. $3 a^{2}-4 a+1$
12. 4 29. 0
13. 2
14. -8
15. -1
16. $2 a^{2}-8$
17. $2 b^{2}-8$
18. -2
19. -3


20. $V(3)=300$, the painting is worth $\$ 300$ in 3 years; $V(6)=600$, the painting is worth $\$ 600$ in 6 years.
21. a. True b. True c. True d. False e. True
22. a. $\$ 5,625$ b. $\$ 1,500 \quad$ c. $\{t \mid 0 \leq t \leq 5\} \quad$ d.
e. $\{V(t) \mid 1,500 \leq V(t) \leq 18,000\} \quad$ f. 2.42 years

23. $-\frac{2}{3}, 4$
24. $-2,1$
25. $\varnothing$
26. $-500+27 x-0.1 x^{2}$
27. $6 x^{2}-2 x-4$
28. $2 x^{2}+8 x+8$
29. $0.6 m-42$
30. $4 x^{2}-7 x+3$
31. a. 2 b. 0
32. a.

| Weight (ounces) | 0.6 | 1.0 | 1.1 | 2.5 | 3.0 | 4.8 | 5.0 | 5.3 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Cost (cents) | 39 | 39 | 63 | 87 | 87 | 135 | 135 | 159 |

b. More than 2 ounces, but not more than 3 ounces; $2<x \leq 3$
c. domain: $\{x \mid 0<x \leq 6\}$
d. range: $\{c \mid c=39,63,87,111,135,159\}$

## Problem Set 8.5

1. $6 x+2$
2. $-2 x+8$
3. $4 x-7$
4. $3 x^{2}+x-2$
5. $-2 x+3$
6. $9 x^{3}-15 x^{2}$
7. $\frac{3 x^{2}}{3 x-5}$
8. $\frac{3 x-5}{3 x^{2}}$
9. $3 x^{2}+4 x-7$
10. 15
11. 98
12. $\frac{3}{2}$
13. 1
14. 40
15. 147
16. a. 81 b. 29 c. $(x+4)^{2}$ d. $x^{2}+4$
17. a. -2 b. -1 c. $4 x^{2}+12 x-1 \quad 37.6$
18. 2
19. 3 43. -8
20. 6
21. 5
22. 3 51. -6
23. a. $R(x)=11.5 x-0.05 x^{2}$ b. $C(x)=2 x+200$ c. $P(x)=-0.05 x^{2}+9.5 x-200$ d. $\bar{C}(x)=2+\frac{200}{x}$
24. a. $M(x)=220-x$
6 c. 142 d. 135 e. 128
25. $2<x<4$
26. $x<4$ or $x>8$
27. $-\frac{5}{7} \leq x \leq 1$
28. 196
29. 4
30. 1.6
31. 3 71. 2,400

Problem Set 8.6

1. Direct
2. Direct
3. Direct
4. Direct
5. Direct
6. Inverse
7. 30
8. 5
9. -6
10. $\frac{1}{2}$
11. 40
12. 225
13. 2 27. 10
14. $\frac{81}{5}$
15. 40.5
16. 64
17. 8
18. $\frac{50}{3}$ pounds
19. a. $T=4 P$ c. 70 pounds per square inch
b.

20. 12 pounds per square inch
21. $\frac{1,504}{15}$ square inches
22. 1.5 ohms
23. 1.28 meters
24. $F=G \frac{m_{1} m_{2}}{d^{2}}$
25. 12
26. 28
27. $-\frac{7}{4}$
28. $w=\frac{P-21}{2}$
29. $t \geq-6$
30. $x<6$
31. 6,2
32. $\varnothing$
33. $x \leq-9$ or $x \geq-1$

34. All real numbers
35. a. Square of speed b. $d=0.0675 s^{2}$ c. About 204 feet from the cannon d. About 7.5 feet further away

## CHAPTER 8 REVIEW TEST

1. 


2.

3.

12. $m=3, b=-6$
13. $m=\frac{2}{3}, b=-3$
11. $y=-2 x$
19. $y=2 x-7$
20. $y=\frac{1}{3} x-\frac{2}{3}$
14. $y=2 x$
15. $y=-\frac{1}{3} x$
16. $y=2 x+1$
17. $y=7$
18. $y=-\frac{3}{2} x-\frac{17}{2}$
22. Domain $=\{6,-4,-2\}$; Range $=\{3,0\}$; a function
23. 2
24. 4
21. Domain $=\{2,3,4\}$; Range $=\{4,3,2\}$; a function
30. 31
31. 24
32. 6
33. 4
34. 25
35. 84 pounds
36. 16 footcandles

## CHAPTER 9

## Problem Set 9.1

1. 12
2. Not a real number
3. -7
4. -3
5. 2
6. Not a real number
7. 0.2
8. 0.2
9. 5
10. -6
11. $\frac{1}{6}$
12. $\frac{2}{5}$
13. $6 a^{4}$
14. $3 a^{4}$
15. $2 x^{2} y$
16. $2 a^{3} b^{5}$
17. 6
18. -3
19. 2 39. -2
20. 2
21. $\frac{9}{5}$
22. 9
23. 125
24. $\frac{1}{3}$
25. $\frac{1}{27}$
26. $\frac{6}{5}$
27. $\frac{8}{27}$
28. $7 \quad$ 59. $\frac{3}{4}$
29. $x^{4 / 5}$
30. $a$
31. $\frac{1}{x^{2 / 5}}$
32. $x^{1 / 6}$
33. $x^{9 / 25} y^{1 / 2} Z^{1 / 5}$
34. $\frac{b^{7 / 4}}{a^{1 / 8}}$
35. $y^{3 / 10}$
36. $\frac{1}{a^{2} b^{4}}$
37. $\frac{S^{1 / 2}}{r^{20}}$
38. $10 b^{3}$
39. 25 mph
40. a. 424 picometers b. 600 picometers c. $6 \times 10^{-10}$ meters
41. a. B b. A c. C d. $(0,0)$ and $(1,1)$
42. $x^{6}-x^{3}$
43. $x^{2}+2 x-15$
44. $x^{4}-10 x^{2}+25$
45. $x^{3}-27$
46. $x^{6}-x^{5}$
47. $12 a^{2}-11 a b+2 b^{2}$
48. $x^{6}-4$ 101. $3 x-4 x^{3} y$
49. $(x-5)(x+2)$
50. $(3 x-2)(2 x+5)$
51. $x^{2}$
52. $t$ 111. $x^{1 / 3}$
53. $\left(9^{1 / 2}+4^{1 / 2}\right)^{2}=(3+2)^{2}=5^{2}=25 \neq 9+4$
54. $\sqrt{\sqrt{a}}=\left(a^{1 / 2}\right)^{1 / 2}=a^{1 / 4}=\sqrt[4]{a}$

## Problem Set 9.2

1. $x+x^{2} \quad$ 3. $a^{2}-a$
2. $6 x^{3}-8 x^{2}+10 x$
3. $12 x^{2}-36 y^{2} \quad$ 9. $x^{4 / 3}-2 x^{2 / 3}-8$
4. $a-10 a^{1 / 2}+21$
5. $20 y^{2 / 3}-7 y^{1 / 3}-6$
6. $10 x^{4 / 3}+21 x^{2 / 3} y^{1 / 2}+9 y$
7. $t+10 t^{1 / 2}+25$
8. $x^{3}+8 x^{3 / 2}+16$
9. $a-2 a^{1 / 2} b^{1 / 2}+b$
10. $4 x-12 x^{1 / 2} y^{1 / 2}+9 y \quad$ 25. $a-3$
11. $x^{3}-y^{3}$
12. $t-8$
13. $4 x^{3}-3$
14. $x+y$ 35. $a-8$
15. $8 x+1$ 39. $t-1$
16. $2 x^{1 / 2}+3 \quad$ 43. $3 x^{1 / 3}-4 y^{1 / 3}$
17. $3 a-2 b$
18. $3(x-2)^{1 / 2}(4 x-11)$
19. $5(x-3)^{7 / 5}(x-6)$
20. $3(x+1)^{1 / 2}\left(3 x^{2}+3 x+2\right)$
21. $\left(x^{1 / 3}-2\right)\left(x^{1 / 3}-3\right)$
22. $\left(a^{1 / 5}-4\right)\left(a^{1 / 5}+2\right)$ 57. $\left(2 y^{1 / 3}+1\right)\left(y^{1 / 3}-3\right)$ 59. $\left(3 t^{1 / 5}+5\right)\left(3 t^{1 / 5}-5\right)$
23. $\left(2 x^{1 / 7}+5\right)^{2}$
24. -8
25. 90 67. -3
26. 0
27. $\frac{3+x}{x^{1 / 2}}$
28. $\frac{x+5}{x^{1 / 3}}$
29. $\frac{x^{3}+3 x^{2}+1}{\left(x^{3}+1\right)^{1 / 2}}$
30. $\frac{-4}{\left(x^{2}+4\right)^{1 / 2}}$
31. $15.8 \%$
32. 3
33. $m=\frac{2}{3}, b=-2$
34. $y=\frac{2}{3} x+6$
35. $y=\frac{2}{3} x-2$
36. 5
37. 6
38. $4 x^{2} y$
39. $5 y$
40. 3
41. 2 101. $2 a b$
42. 25
43. $48 x^{4} y^{2}$
44. $4 x^{6} y^{6}$

Problem Set 9.3

1. $2 \sqrt{2}$
2. $7 \sqrt{2}$
3. $12 \sqrt{2}$
4. $4 \sqrt{5} \quad$ 9. $4 \sqrt{3}$
5. $15 \sqrt{3}$
6. $3 \sqrt[3]{2} \quad$ 15. $4 \sqrt[3]{2}$
7. $6 \sqrt[3]{2}$
8. $2 \sqrt[5]{2}$
9. $3 x \sqrt{2 x}$
10. $2 y \sqrt[4]{2 y^{3}}$
11. $2 x y^{2} \sqrt[3]{5 x y}$
12. $4 a b c^{2} \sqrt{3 b} \quad$ 29. $2 b c \sqrt[3]{6 a^{2} c}$
13. $2 x y^{2} \sqrt[5]{2 x^{3} y^{2}}$
14. $3 x y^{2} z \sqrt[5]{x^{2}}$
15. $2 \sqrt{3}$
16. $\sqrt{-20}$, which is not a real number
17. $\frac{\sqrt{11}}{2} \quad$ 41. $\frac{2 \sqrt{3}}{3}$
18. $\frac{5 \sqrt{6}}{6}$
19. $\frac{\sqrt{2}}{2}$
20. $\frac{\sqrt{5}}{5}$
21. $2 \sqrt[3]{4}$
22. $\frac{2 \sqrt[3]{3}}{3}$
23. $\frac{\sqrt[4]{24 x^{2}}}{2 x}$
24. $\frac{\sqrt[4]{8 y^{3}}}{y}$
25. $\frac{\sqrt[3]{36 x y^{2}}}{3 y}$
26. $\frac{\sqrt[3]{6 x y^{2}}}{3 y}$
27. $\frac{\sqrt[4]{2 x}}{2 x}$
28. $\frac{3 x \sqrt{15 x y}}{5 y}$
29. $\frac{5 x y \sqrt{6 x z}}{2 z}$
30. $\frac{2 a b \sqrt[3]{6 a c^{2}}}{3 c}$
31. $\frac{2 x y^{2} \sqrt[3]{3 z^{2}}}{3 z}$
32. $\sqrt{x}$
33. $\sqrt[6]{x y}$
34. $\sqrt[12]{a}$
35. $x \sqrt[9]{6 x}$
36. $a b^{2} c \sqrt[6]{c}$
37. $a b c^{2} \sqrt[15]{3 a^{2} b}$
38. $2 b^{2} \sqrt[3]{a}$
39. $5|x|$
40. $3|x y| \sqrt{3 x}$
41. $|x-5|$
42. $|2 x+3|$
43. $2|a(a+2)|$
44. $2|x| \sqrt{x-2}$
45. $\sqrt{9+16}=\sqrt{25}=5 ; \sqrt{9}+\sqrt{16}=3+4=7$
46. $5 \sqrt{13}$ feet
47. a. 13 feet b. $2 \sqrt{14} \approx 7.5$ feet
48. 3
49. 0
50. 1
51. 3
52. $27 x y^{2}$
53. $\frac{5}{6} x$
54. $3 \sqrt{2}$
55. $5 y \sqrt{3 x y}$
56. $2 a \sqrt[3]{a b^{2}}$
57. $12 \sqrt[3]{5}$
58. $6 \sqrt[3]{49}$
59. $\frac{\sqrt[10]{a^{7}}}{a}$
60. $\frac{\sqrt[20]{a^{9}}}{a}$
61. 



## Problem Set 9.4

1. $7 \sqrt{5}$
2. $-x \sqrt{7}$
3. $\sqrt[3]{10}$
4. $9 \sqrt[5]{6}$
5. 0
6. $\sqrt{5}$
7. $-32 \sqrt{2}$
8. $-3 x \sqrt{2}$
9. $-2 \sqrt[3]{2}$
10. $8 x \sqrt[3]{x y^{2}}$
11. $3 a^{2} b \sqrt{3 a b}$
12. $11 a b \sqrt[3]{3 a^{2} b}$
13. $10 x y \sqrt[4]{3 y}$
14. $\sqrt{2}$
15. $\frac{8 \sqrt{5}}{15}$
16. $\frac{(x-1) \sqrt{x}}{x}$
17. $\frac{3 \sqrt{2}}{2}$
18. $\frac{5 \sqrt{6}}{6}$
19. $\frac{8 \sqrt[3]{25}}{5}$
20. a. $8 \sqrt{2 x}$ b. $-4 \sqrt{2 x}$
21. a. $4 \sqrt{2 x}$ b. $2 \sqrt{2 x}$
22. a. $3 x \sqrt{2}$ b. $-x \sqrt{2}$
23. a. $3 \sqrt{2 x}+3$ b. $-\sqrt{2 x}-7$
24. $\sqrt{12} \approx 3.464 ; 2 \sqrt{3}=2(1.732) \approx 3.464$
25. $\sqrt{8}+\sqrt{18} \approx 2.828+4.243=7.071 ; \sqrt{50} \approx 7.071 ; \sqrt{26} \approx 5.099$
26. $\frac{\sqrt{3}}{2}$
27. 


59.

61.

63. $-\frac{1}{5} \leq x \leq \frac{3}{5}$
65. 6
67. $4 x^{2}+3 x y-y^{2}$
69. $x^{2}+6 x+9$
71. $x^{2}-4$
73. $6 \sqrt{2}$
75. 6
77. $9 x$
79. $\frac{\sqrt{6}}{2}$
81. $-x y$
83. $-x y^{2} z^{3}$
85. $5 b^{2} c \sqrt[4]{2 a^{3} b}$
87. $-3 a b \sqrt[3]{2 a}$

## Problem Set 9.5

1. $3 \sqrt{2}$
2. $10 \sqrt{21}$
3. 720
4. 54 9. $\sqrt{6}-9$
5. $24+6 \sqrt[3]{4}$
6. $2+2 \sqrt[3]{3}$
7. $x y \sqrt[3]{y}+x^{2} \sqrt[3]{y}$
8. $2 x^{2} \sqrt[4]{x}+2 x^{3} \sqrt[4]{2}$
9. $7+2 \sqrt{ }$ 21. $x+2 \sqrt{x}-15$
10. $34+20 \sqrt{3}$
11. $19+8 \sqrt{3}$
12. $x-6 \sqrt{x}+9$
13. $4 a-12 \sqrt{a b}+9 b \quad$ 31. $x+4 \sqrt{x-4}$
14. $x-6 \sqrt{x-5}+4$
15. 1
$\begin{array}{lll}\text { 37. } a-49 & \text { 39. } 25-x & \text { 41. } x-8\end{array}$
16. $10+6 \sqrt{3}$
17. $5+\sqrt[3]{12}+\sqrt[3]{18}$
18. $x^{2}+x \sqrt[3]{x^{2} y^{2}}+\sqrt[3]{x y}+y$
19. $\frac{\sqrt{2}}{2} \quad$ 51. $\frac{\sqrt{x}}{x}$
20. $\frac{4 \sqrt{3}}{3}$
21. $\frac{x \sqrt{6}}{3}$
22. $\frac{2 \sqrt{10 x}}{5 x}$
23. $\frac{\sqrt{2}}{2}$
24. $\frac{2 x \sqrt{2 y}}{y}$
25. $\frac{a \sqrt{6 c}}{2 b c}$
26. $\frac{2 a c \sqrt[3]{b^{2}}}{b}$
27. $\frac{\sqrt{3}+1}{2}$
28. $\frac{5-\sqrt{5}}{4}$
29. $\frac{x+3 \sqrt{x}}{x-9}$
30. $\frac{10+3 \sqrt{5}}{11}$
31. $\frac{3 \sqrt{x}+3 \sqrt{y}}{x-y}$
32. $2+\sqrt{3}$
33. $\frac{11-4 \sqrt{7}}{3}$
34. $\frac{a+2 \sqrt{a b}+b}{a-b}$
35. $\frac{x+4 \sqrt{x}+4}{x-4}$
36. $\frac{5-\sqrt{21}}{4}$
37. $\frac{\sqrt{x}-3 x+2}{1-x}$
38. $\frac{8 \sqrt{3}}{3}$
39. $\frac{11 \sqrt{5}}{5}$
40. $-4 \sqrt{3}$
41. $5 \sqrt{3}$
42. $(\sqrt[3]{2}+\sqrt[3]{3})(\sqrt[3]{4}-\sqrt[3]{6}+\sqrt[3]{9})$
$=\sqrt[3]{8}-\sqrt[3]{12}+\sqrt[3]{18}+\sqrt[3]{12}-\sqrt[3]{18}+\sqrt[3]{27}$

$$
=2+3=5
$$

$\begin{array}{lll}\text { 99. } 10 \sqrt{3} & \text { 101. } x+6 \sqrt{x}+9 & \text { 103. } 75\end{array}$
105. $\frac{5 \sqrt{2}}{4}$ seconds; $\frac{5}{2}$ seconds
111. 147
113. 50
115. 100
117. $t^{2}+10 t+25$
119. $x$
121. 7
123. $-4,-3$
125. $-6,-3$
127. $-5,-2$
129. $\frac{x(\sqrt{x-2}-4)}{x-18}$
131. $\frac{x(\sqrt{x+5}+5)}{x-20}$
133. $\frac{3(\sqrt{5 x}-x)}{5-x}$

## Problem Set 9.6

1. $4 \quad 3 . \varnothing 5$
2. Possible solutions $\frac{1}{2}$ and 11 ; only 11 checks; 11
3. 5 15. 3
4. $-\frac{32}{3}$
5. 3,4
6. $-1,-2$
7. $3,7 \quad$ 27. $-\frac{1}{4}, 14$
8. -1
9. $\varnothing$ 33. 7
10. 0,3
11. -4
12. 8
13. 0
14. 9
15. 0
16. 8 49. Possible solution 9 , which does not check; $\varnothing$
17. Possible solutions 0 and 32; only 0 checks; 0
18. Possible solutions -2 and 6 ; only 6 checks; 6
19. $\frac{1}{2}$
20. $\frac{1}{2}, 1$
21. 5, 13
22. $-\frac{3}{2}$
23. Possible solution 1, which does not check; $\varnothing$
24. 1
25. -11
26. 


75.

77.

79.

85. $h=100-16 t^{2}$
87. $\frac{392}{121} \approx 3.24$ feet
89. 5 meters
81.

83.

99. 5 101. $2 \sqrt{3}$
103. -1
105. 1
107. 4
109. 2
91. 2,500 meters
93. $\sqrt{6}-2$
95. $x+10 \sqrt{x}+25$
97. $\frac{x-3 \sqrt{x}}{x-9}$
119. $4+12 x+9 x^{2}$
121. $4-9 x^{2}$

Problem Set 9.7

1. $6 i \quad$ 3. $-5 i$
2. $6 i \sqrt{2} \quad$ 7. $-2 i \sqrt{3}$
3. 1 11. -1
4. $-i$
5. $x=3, y=-1$
6. $x=-2, y=-\frac{1}{2}$
7. $x=-8, y=-5$
8. $x=7, y=\frac{1}{2}$
$\begin{array}{ll}\text { 23. } x=\frac{3}{7}, y=\frac{2}{5} & \text { 25. } 5+9 i\end{array}$
9. $5-i$
10. $2-4 i$
11. $1-6 i$
12. $2+2 i \quad 35 .-1-7 i \quad$ 37. $6+8 i$
13. $2-24 i$
14. $-15+12 i \quad$ 43. $18+24 i$
15. $10+11 i$
16. $21+23 i \quad$ 49. $-2+2 i$
17. 2
$\begin{array}{rr}11 i & \text { 53. }-21 \\ \text { 73. } & -\frac{18}{13}-\frac{12}{13} i\end{array}$
$+20 i$ 55. $-2 i$
18. $-7-24 i$
19. $5 \quad 61.40$
20. 13
21. 16
22. domain
; is a function
$\begin{array}{lll}\text { 67. }-3-2 i & \text { 69. }-2+5 i & \text { 71. } \frac{8}{13}+\frac{12}{13} i\end{array}$
23. $-\frac{5}{13}+\frac{12}{13} i$
24. $\frac{13}{15}-\frac{2}{5} i$
25. $R=-11-7 i$ ohms
26. is a function
27. not a function

## CHAPTER 9 REVIEW TEST

1. 7
2. -3
3. 2
4. 27
5. $2 x^{3} y^{2}$
6. $\frac{1}{16}$
7. $X^{2}$
8. $a^{2} b^{4}$
9. $a^{7 / 20}$
10. $a^{5 / 12} b^{8 / 3}$
11. $12 x+11 x^{1 / 2} y^{1 / 2}-15 y$
12. $a^{2 / 3}-10 a^{1 / 3}+25$
13. $4 x^{1 / 2}+2 x^{5 / 6}$
14. $2(x-3)^{1 / 4}(4 x-13)$
15. $\frac{x+5}{x^{1 / 4}}$
16. $2 \sqrt{3}$
17. $5 \sqrt{2}$
18. $2 \sqrt[3]{2}$
19. $3 \times \sqrt{2}$
20. $4 a b^{2} c \sqrt{5 a}$
21. $2 a b c \sqrt[4]{2 b c^{2}}$
22. $\frac{3 \sqrt{2}}{2}$
23. $3 \sqrt[3]{4}$
24. $\frac{4 x \sqrt{21 x y}}{7 y}$
25. $\frac{2 y \sqrt[3]{45 x^{2} Z^{2}}}{3 z}$
26. $-2 x \sqrt{6}$
27. $3 \sqrt{3}$
28. $\frac{8 \sqrt{5}}{5}$
29. $7 \sqrt{2}$
30. $11 a^{2} b \sqrt{3 a b}$
31. $-4 x y \sqrt[3]{x z^{2}}$
32. $\sqrt{6}-4$
33. $x-5 \sqrt{x}+6$
34. $3 \sqrt{5}+6$
35. $6+\sqrt{35}$
36. $\frac{63+12 \sqrt{7}}{47}$
37. 0
38. 3
39. 5
40. $\varnothing$
41. 


42.

43. 1 44. $-i$
45. $x=-\frac{3}{2}, y=-\frac{1}{2}$
46. $x=-2, y=-4$
47. $9+3 i$
48. $-7+4 i$
49. $-6+12 i$
50. $5+14 i$
51. $12+16 i$
52. 25
53. $1-3 i^{2}$
54. $-\frac{6}{5}+\frac{3}{5} i$
55. 22.5 feet
56. 65 yards

## CHAPTER 10

## Problem Set 10.1

1. $\pm 5$
2. $\pm \frac{\sqrt{3}}{2}$
3. $\pm 2 i \sqrt{3}$
4. $\pm \frac{3 \sqrt{5}}{2}$
5. $-2,3$
6. $\frac{-3 \pm 3 i}{2}$
7. $-4 \pm 3 i \sqrt{3}$
8. $\frac{3 \pm 2 i}{2}$
9. $x^{2}+12 x+36=(x+6)^{2}$
10. $x^{2}-4 x+4=(x-2)^{2}$
11. $a^{2}-10 a+25=(a-5)^{2}$
12. $x^{2}+5 x+\frac{25}{4}=\left(x+\frac{5}{2}\right)^{2}$
13. $y^{2}-7 y+\frac{49}{4}=\left(y-\frac{7}{2}\right)^{2}$
14. $x^{2}+\frac{1}{2} x+\frac{1}{16}=\left(x+\frac{1}{4}\right)^{2}$
15. $x^{2}+\frac{2}{3} x+\frac{1}{9}=\left(x+\frac{1}{3}\right)^{2}$
16. $-3,-9$
17. $1 \pm 2 i$
18. $4 \pm \sqrt{15}$
19. $\frac{5 \pm \sqrt{37}}{2}$
20. $1 \pm \sqrt{5}$
21. $\frac{4 \pm \sqrt{13}}{3}$
22. $\frac{3 \pm i \sqrt{71}}{8}$
23. a. No
b. $\pm 3 i$
24. a. 0,6 b. 0,6
25. a. $-7,5$ b. $-7,5$
26. No
27. a. $\frac{7}{5}$
b. 3 c. $\frac{7 \pm 2 \sqrt{2}}{5}$
d. $\frac{71}{5}$ e. 3
28. $\pm 1$
29. $\pm 1$
30. $\pm 1$
31. 

| $\boldsymbol{x}$ | $\boldsymbol{f ( x )}$ | $\boldsymbol{g ( x )}$ | $\boldsymbol{h}(\boldsymbol{x})$ |
| ---: | ---: | ---: | ---: |
| -2 | 49 | 49 | 25 |
| -1 | 25 | 25 | 13 |
| 0 | 9 | 9 | 9 |
| 1 | 1 | 1 | 13 |
| 2 | 1 | 1 | 25 |

63. 3
64. a. $-1,6$ b. $-1,6$ c. -6 d. -10
65. $\frac{\sqrt{3}}{2}$ inch, 1 inch
66. $\sqrt{2}$ inches
67. 781 feet
68. $7.3 \%$ to the nearest tenth
69. $20 \sqrt{2} \approx 28$ feet
70. $3 \sqrt{5}$
71. $3 y^{2} \sqrt{3 y}$
72. $3 x^{2} y \sqrt[3]{2 y^{2}}$
73. $\frac{3 \sqrt{2}}{2}$
74. $\sqrt[3]{2}$
75. 13
76. 7
77. $\frac{1}{2}$
78. 13
79. $(3 t-2)\left(9 t^{2}+6 t+4\right)$
80. $x= \pm 2 a$
81. $x=\frac{-p \pm \sqrt{p^{2}-4 q}}{2}$
82. $x=\frac{-p \pm \sqrt{p^{2}-12 q}}{6}$
83. $(x-5)^{2}+(y-3)^{2}=2^{2}$

## Problem Set 10.2

1. a. $\frac{-2 \pm \sqrt{10}}{3}$
b. $\frac{2 \pm \sqrt{10}}{3}$
c. $\frac{-2 \pm i \sqrt{2}}{3}$
d. $\frac{-2 \pm \sqrt{10}}{2}$
e. $\frac{2 \pm i \sqrt{2}}{2}$
2. a. $1 \pm i$ b. $1 \pm 2 i$ c. $-1 \pm i$
3. 1,2
4. $\frac{2 \pm i \sqrt{14}}{3}$
5. 0,5
6. $0,-\frac{4}{3}$
7. $\frac{3 \pm \sqrt{5}}{4}$
8. $-3 \pm \sqrt{17}$
9. $\frac{-1 \pm i \sqrt{5}}{2}$
10. 1
11. $\frac{1 \pm i \sqrt{47}}{6}$
12. $4 \pm \sqrt{2}$
13. $\frac{1}{2}, 1$
14. $-\frac{1}{2}, 3$
15. $\frac{-1 \pm i \sqrt{7}}{2}$
16. $1 \pm \sqrt{2}$
17. $\frac{-3 \pm \sqrt{5}}{2}$
18. $-5,3$
19. $2,-1 \pm i \sqrt{3}$
20. $-\frac{3}{2}, \frac{3 \pm 3 i \sqrt{3}}{4}$
21. $\frac{1}{5}, \frac{-1 \pm i \sqrt{3}}{10}$
22. $0, \frac{-1 \pm i \sqrt{5}}{2}$
23. $0,1 \pm i$
24. $0, \frac{-1 \pm i \sqrt{2}}{3}$
25. $a$ and $b$
26. a. $\frac{5}{3}, 0$ b. $\frac{5}{3}, 0$
27. No, $2 \pm i \sqrt{3}$
28. Yes
29. a. $-1,3$ b. $1 \pm i \sqrt{7}$ c. $-2,2$ d. $2 \pm 2 \sqrt{2}$
30. a. $-2, \frac{5}{3}$
b. $\frac{3}{2}, \frac{-3 \pm 3 i \sqrt{3}}{4}$
c. $\varnothing$ d. $1,-1$
31. 2 seconds
32. $40 \pm 20=20$ or 60 items
33. 0.49 centimeter ( 8.86 cm is impossible)
34. a. $\ell+w=10, \ell w=15$
b. 8.16 yards, 1.84 yards
c. Two answers are possible because either dimension (long or short) may be considered the length.
35. $4 y+1+\frac{-2}{2 y-7}$
36. $x^{2}+7 x+12$
37. 5
38. $\frac{27}{125}$
39. $\frac{1}{4}$
40. $21 x^{3} y$
41. 169
42. 0
43. $\pm 12$
44. $x^{2}-x-6$
45. $x^{3}-4 x^{2}-3 x+18$
46. $-2 \sqrt{3}, \sqrt{3}$
47. $\frac{-1 \pm \sqrt{3}}{\sqrt{2}}=\frac{-\sqrt{2} \pm \sqrt{6}}{2}$
48. $-2 i, i$

## Problem Set 10.3

1. $D=16$, two rational
2. $D=0$, one rational
3. $D=5$, two irrational
4. $D=17$, two irrational
5. $D=36$, two rational
6. $D=116$, two irrational
7. $\pm 10$ 15. $\pm 12$
8. 9 19. -16
9. $\pm 2 \sqrt{6}$
10. $x^{2}-7 x+10=0$
11. $t^{2}-3 t-18=0$
12. $y^{3}-4 y^{2}-4 y+16=0$
13. $2 x^{2}-7 x+3=0$
14. $4 t^{2}-9 t-9=0$
15. $6 x^{3}-5 x^{2}-54 x+45=0$
16. $10 a^{2}-a-3=0$
17. $9 x^{3}-9 x^{2}-4 x+4=0$
18. $x^{4}-13 x^{2}+36=0$
19. $x^{2}-5=0$
20. $x^{2}+(\sqrt{7}-\sqrt{3}) x-\sqrt{21}=0$
21. $x^{2}+16=0$
$\begin{array}{lllll}\text { 47. } x^{2}-5 i x-6=0 & \text { 49. } x^{2}-4 x+5=0 & \text { 51. } x^{2}-x-1=0 & \text { 53. a. } y=g(x) \quad \text { b. } y=h(x) \quad \text { c. } y=f(x) \quad \text { 55. } a-2 a^{1 / 2}-15\end{array}$
22. $x-64 \quad$ 59. $5 x y^{2}-4 x^{2} y \quad$ 61. $2(x+1)^{1 / 3}(4 x+3) \quad$ 63. $\left(3 x^{1 / 3}+2\right)^{2} \quad$ 65. $x^{2}-7 x \quad$ 67. $9 a^{2}-6 a-3$
23. $24 a^{2}+94 a+90$
24. $\pm i \sqrt{2}$
25. 4
26. -5
27. $-3,2$
28. $-\frac{1}{3}, \frac{5}{2}$
29. $-\frac{1}{2}, \frac{5}{2}$
30. $\frac{1+\sqrt{2}}{2} \approx 1.21, \frac{1-\sqrt{2}}{2} \approx-0.21$
31. $\frac{2}{3}, 1,-1$
32. $\frac{2}{3}, \sqrt{3}, \approx 1.73,-\sqrt{3}, \approx-1.73$

## Problem Set 10.4

1. 1,2
2. $-8,-\frac{5}{2}$
3. $\pm 3, \pm 1$
4. $\pm 2, \pm \sqrt{3}$
5. $\pm 3, \pm i \sqrt{3}$
6. $\pm 2 i, \pm i \sqrt{5}$
7. $\frac{7}{2}, 4$
8. $-\frac{9}{8}, \frac{1}{2}$
9. $\pm \frac{\sqrt{30}}{6}, \pm i$
10. $\pm \frac{\sqrt{21}}{3}, \pm \frac{i \sqrt{21}}{3}$
11. 4,25
12. Possible solutions 25 and 9; only 25 checks; 25
13. Possible solutions $\frac{25}{9}$ and $\frac{49}{4}$; only $\frac{25}{9}$ checks; $\frac{25}{9}$
14. Possible solutions 4 and 16; only 16 checks; 16
15. Possible solutions 9 and 36 ; only 9 checks; 9
16. Possible solutions $\frac{1}{4}$ and 25 ; only $\frac{1}{4}$ checks; $\frac{1}{4}$
17. 27,38
18. 4,12
19. $\pm 2, \pm 2 i$
20. $\pm 2, \pm 2 i \sqrt{2}$
21. $\frac{1}{2}, 1$
22. $\pm \frac{\sqrt{5}}{5}$
23. $-216,8$
24. $\pm 3 i \sqrt{3}$
25. $t=\frac{v \pm \sqrt{v^{2}+64 h}}{32}$
26. $x=\frac{-4 \pm 2 \sqrt{4-k}}{k}$
27. $x=-y$
28. $t=\frac{1 \pm \sqrt{1+h}}{4}$
29. a.

30. Let $x=B C$. Then $\frac{4}{x}=\frac{x}{x+4}$, or $4(x+4)=x^{2}$.

$$
x=\frac{4+4 \sqrt{5}}{2}=4\left(\frac{1+\sqrt{5}}{2}\right)
$$

61. $3 \sqrt{7}$
62. $5 \sqrt{2}$
63. $39 x^{2} y \sqrt{5 x}$
64. $-11+6 \sqrt{5}$
65. $x+4 \sqrt{x}+4$
66. $\frac{7+2 \sqrt{7}}{3}$
67. -2
68. $1,322.5$
69. $-\frac{7}{640}$
70. 1,5
71. $-3,1$
72. $\frac{3}{2} \pm \frac{1}{2} i$
73. 9; 3
74. $1 ; 1$
75. $x$-intercepts $=-2,0,2 ; y$-intercept $=0$
76. $x$-intercepts $=-3,-\frac{1}{3}, 3 ; y$-intercept $=-9$ 93. $\frac{1}{2}$ and -1

## Problem Set 10.5

1. $x$-intercepts $=-3,1$;

Vertex $=(-1,-4)$

3. $x$-intercepts $=-5,1$; Vertex $=(-2,9)$

5. $x$-intercepts $=-1,1$; Vertex $=(0,-1)$

7. $x$-intercepts $=3,-3$;

Vertex $=(0,9)$

13. Vertex $=(2,-8)$

9. $x$-intercepts $=-1,3$;

Vertex $=(1,-8)$

15. Vertex $=(1,-4)$

11. $x$-intercepts $=1+\sqrt{5}, 1-\sqrt{5}$; Vertex $=(1,-5)$

17. Vertex $=(0,1)$

19. Vertex $=(0,-3)$

r
21. $(3,-4)$ lowest 23. $(1,9)$ highest
25. $(2,16)$ highest $\quad$ 27. $(-4,16)$ highest $\quad$ 29. 875 patterns; maximum profit $\$ 731.25$
25. $(2,16)$ highest $\quad$ 27. $(-4,16)$ highest $\quad$ 29. 875 patterns; maximum profit $\$ 731.25$
31. The ball is in her hand when $h(t)=0$, which means $t=0$ or $t=2$ seconds. Maximum height is $h(1)=16$ feet.
33. Maximum $R=\$ 3,600$ when $p=\$ 6.00$

35. Maximum $R=\$ 7,225$ when $p=\$ 8.50$

37. $y=-\frac{1}{135}(x-90)^{2}+60$

39. $1-i$
41. $27+5 i$
43. $\frac{1}{10}+\frac{3}{10} i$
45. $-2,4$
47. $-\frac{1}{2}, \frac{2}{3}$
49. $3 \quad$ 51. $y=(x-2)^{2}-4$

## Problem Set 10.6


3.

5.

9.

11.

13.

15. All real numbers
17. No solution; $\varnothing$
19.

21.

23.

25.

27.

29.

31.

33. a. $x-1>0$ b. $x-1 \geq 0$ c. $x-1 \geq 0$
35. $\varnothing$
37. $x=1$
39. all real numbers
41. $x>1$ or $x<1$
43. $\varnothing$
45. $x>3$ or $x<3$; all real numbers except 3
47. a. $-2<x<2$ b. $x<-2$ or $x>2$
c. $x=-2$ or $x=2$
49. a. $-2<x<5$ b. $x<-2$ or $x>5$ c. $x=-2$ or $x=5$
51. a. $x<-1$ or $1<x<3$ b. $-1<x<1$ or $x>3$ c. $x=-1$ or $x=1$ or $x=3$
53. $x \geq 4$; the width is at least 4 inches
55. $5 \leq p \leq 8$; charge at least $\$ 5$ but no more than $\$ 8$ for each radio
57. 1.5625
59. 0.6549
61. $\frac{5}{3}$
63. Possible solutions 1 and 6 ; only 6 checks; 6
65.

67.

69.


## CHAPTER 10 REVIEW TEST

1. 0,5
2. $0, \frac{4}{3}$
3. $\frac{4 \pm 7 i}{3}$
4. $-3 \pm \sqrt{3}$
5. $-5,2$
6. $-3,-2$
7. 3
8. 2
9. $\frac{-3 \pm \sqrt{3}}{2}$
10. $\frac{3 \pm \sqrt{5}}{2}$
11. $-5,2$
12. $0, \frac{9}{4}$
13. $\frac{2 \pm i \sqrt{15}}{2}$
14. $1 \pm \sqrt{2}$
15. $-\frac{4}{3}, \frac{1}{2}$
16. 3
17. $5 \pm \sqrt{7}$
18. $0, \frac{5 \pm \sqrt{21}}{2}$
19. $-1, \frac{3}{5}$
20. $3, \frac{-3 \pm 3 i \sqrt{3}}{2}$
21. $\frac{1 \pm i \sqrt{2}}{3}$
22. $\frac{3 \pm \sqrt{29}}{2}$
23. 100 or 170 items
24. 20 or 60 items
25. $D=0 ; 1$ rational
26. $D=0 ; 1$ rational
27. $D=25 ; 2$ rational
28. $D=361 ; 2$ rational
29. $D=5 ; 2$ irrational
30. $D=5 ; 2$ irrational
31. $D=-23 ; 2$ complex
32. $D=-87 ; 2$ complex
33. $\pm 20$
34. $\pm 20$
35. $4 \quad 36.4 \quad 37.25$
36. 49 39. $x^{2}-8 x+15=0$
37. $x^{2}-2 x-8=0$
38. $2 y^{2}+7 y-4=0$
39. $t^{3}-5 t^{2}-9 t+45=0$
40. $-4,12$
41. $-\frac{3}{4},-$
42. $\pm 2, \pm i \sqrt{3}$
43. Possible solutions 4 and 1, only 4 checks; 4
44. $\frac{9}{4}, 16$
45. 4
46. 4
47. $7 \quad$ 51. $t=\frac{5 \pm \sqrt{25+16 h}}{16}$
48. $t=\frac{v \pm \sqrt{v^{2}+640}}{32}$
49. 


54.

55.

56.

57.

58.


## CHAPTER 11

## Problem Set 11.1

1. 1
2. 2
3. $\frac{1}{27}$
4. 13
5. 


11.

13.

15.

21. $h=6 \cdot\left(\frac{2}{3}\right)^{n} ; 5^{\text {th }}$ bounce: $6\left(\frac{2}{3}\right)^{5} \approx 0.79$ feet
17.

19.

23. After 8 days, 700 micrograms; After 11 days, $1,400 \cdot 2^{-11 / 8} \approx 539.8$ micrograms
25. a. $A(t)=1,200\left(1+\frac{.06}{4}\right)^{4 t}$ b. $\$ 1,932.39$ c. About 11.64 years d. $\$ 1,939.29$
27. a. The function underestimated the expenditures by $\$ 69$ billion.
b. $\$ 4,123$ billion
$\$ 4,577$ billion $\$ 5,080$ billion
29.

31. After 6 years
33. a. $\$ 129,138.48$
b. $\{t \mid 0 \leq t \leq 6\}$
c.

d. $\{V(t) \mid 52,942.05 \leq V(t) \leq 450,000\}$
e. After approximately 4 years and 8 months
35. $D=\{1,3,4\}, R=\{2,4,1\}$, a function
37. $\left\{x \left\lvert\, x \geq-\frac{1}{3}\right.\right\}$
39. -18
41. 2
43. $y=\frac{x+3}{2}$
45. $y= \pm \sqrt{x+2}$
47. $y=\frac{2 x-4}{x-1}$
49. $y=x^{2}+3$
51.

| $\boldsymbol{t}$ | $\boldsymbol{s}(\boldsymbol{t})$ |
| :--- | ---: |
| 0 | 0 |
| 1 | 83.3 |
| 2 | 138.9 |
| 3 | 175.9 |
| 4 | 200.6 |
| 5 | 217.1 |
| 6 | 228.1 |



Problem Set 11.2

1. $f^{-1}(x)=\frac{x+1}{3}$
2. $f^{-1}(x)=\sqrt[3]{x}$
3. $f^{-1}(x)=\frac{x-3}{x-1}$
4. $f^{-1}(x)=4 x+3$
5. $f^{-1}(x)=2(x+3)=2 x+6$
6. $f^{-1}(x)=\frac{1-x}{3 x-2}$
7. 


15.

17.

19.

21.

23.

25.

27.

29. a. Yes b. No c. Yes
31. a. 4 b. $\frac{4}{3}$ c. 2 d. 2
33. $f^{-1}(x)=\frac{1}{x}$
35. a. -3 b. -6 c. 2 d. 3 e. -2 f. 3 g. They are inverses of each other.
37. $-2,3$
39. 25
41. $5 \pm \sqrt{17}$
43. $1 \pm i$
45. $\frac{1}{9}$
47. $\frac{2}{3}$
49. $\sqrt[3]{4}$
51. 3
53. 4
55. 4
57. 1
59. $f^{-1}(x)=\frac{x-5}{3}$
61. $f^{-1}(x)=\sqrt[3]{x-1}$
63. $f^{-1}(x)=\frac{2 x-4}{x-1}$
65. a. 1 b. 2 c. 5 d. 0 e. 1 f. 2 g. 2 h. 5

Problem Set 11.3

1. $\log _{2} 16=4$
2. $\log _{5} 125=3$
3. $\log _{10} 0.01=-2$
4. $\log _{2} \frac{1}{32}=-5$
5. $\log _{1 / 2} 8=-3$
6. $\log _{3} 27=3$
7. $10^{2}=100$
8. $2^{6}=64 \quad$ 17. $8^{0}=1$
9. 
10. 



41.

43.

55. 0
57. $\frac{1}{2}$
59. 7
61. $10^{-6}$
63. 2
65. $10^{8}$ times as large
67. $25 ; 5$ 69. $1 ; 1$
71. $\pm 3 i$
73. $\pm 2 i$
75. $\frac{-2 \pm \sqrt{10}}{2}$
77. $\frac{2 \pm i \sqrt{3}}{2}$
79. $5, \frac{-5 \pm 5 i \sqrt{3}}{2}$
81. $0,5,-1$
83. $\frac{3 \pm \sqrt{29}}{2}$
85. 4
87. $-4,2$
89. $-\frac{11}{8}$
91. $2^{3}=(x+2)(x)$
93. $3^{4}=\frac{x-2}{x+1}$
95. a.

| $\boldsymbol{x}$ | $\boldsymbol{f}(\boldsymbol{x})$ |
| ---: | ---: |
| -1 | $\frac{1}{8}$ |
| 0 | 1 |
| 1 | 8 |
| 2 | 64 |

b.

| $\boldsymbol{x}$ | $\boldsymbol{f}^{-1}(\boldsymbol{x})$ |
| ---: | :---: |
| $\frac{1}{8}$ | -1 |
| 1 | 0 |
| 8 | 1 |
| 64 | 2 |

c. $f(x)=8^{x}$
d. $f^{-1}(x)=\log _{8} x$

Problem Set 11.4

1. $\log _{3} 4+\log _{3} x$
2. $\log _{6} 5-\log _{6} x$
3. $5 \log _{2} y$
4. $\frac{1}{3} \log _{9} Z$
5. $2 \log _{6} x+4 \log _{6} y$
6. $\frac{1}{2} \log _{5} x+4 \log _{5} y$
7. $\log _{b} x+\log _{b} y-\log _{b} z$
8. $\log _{10} 4-\log _{10} x-\log _{10} y$
9. $2 \log _{10} x+\log _{10} y-\frac{1}{2} \log _{10} z$
10. $3 \log _{10} x+\frac{1}{2} \log _{10} y-4 \log _{10} z$
11. $\frac{2}{3} \log _{b} x+\frac{1}{3} \log _{b} y$
12. $\log _{10} \frac{x^{3 / 2}}{y^{3 / 4} z^{4 / 5}}$
13. Possible solutions
14. Possible solutions
15. $D=-7$; two com
Problem Set 11.5
16. $\log _{b} x z$
17. $\log _{3} \frac{x^{2}}{y^{3}}$
18. $\log _{10} \sqrt{x} \sqrt[3]{y}$
19. $\log _{2} \frac{x^{3} \sqrt{y}}{z}$
20. $\log _{2} \frac{\sqrt{x}}{y^{3} Z^{4}}$
21. $\frac{2}{3}$
22. 18
23. Possible solutions -1 and 3; only 3 checks; 3
24. 3
. 4 . 4 ; only 4 checks; 4
25. Possible solutions -1 and 4; only 4 checks; 4
26. Possible solutions $-\frac{5}{2}$ and $\frac{5}{3}$; only $\frac{5}{3}$ checks; $\frac{5}{3}$
27. $N$ is 2 in both cases
28. $\frac{x^{2}}{3}+\frac{y^{2}}{36}$
29. $\frac{x^{2}}{4}+\frac{y^{2}}{25}$
30. $3 y^{2}-11 y+6=0$
31. 1
32. 1
33. 4
34. $2.5 \times 10^{-6}$
35. 51
36. 2.5775
37. 1.5775
38. 3.5775
39. -1.4225
40. 4.5775
41. 2.7782
42. 3.3032
43. -2.0128
44. -1.5031
45. -0.3990
46. 1,000 41. 1
47. 5
48. 1,430
49. 0.00000447
50. 0.0000000918
51. $10^{10}$
52. $10^{-10}$
53. $10^{20}$
54. $\frac{1}{100}$
55. a. 1.1886 b. 0.2218
56. $\ln A-2 t \quad$ 51. 2.7080
57. $-1.0986 \quad 55.2 .1972$
58. 2.7724
59. 10 times larger
60. 
61. $3.19 \quad 65.1 .78 \times 10^{-5}$
62. $3.16 \times 10^{5}$
63. $2.00 \times 10^{8}$

| Location | Date | Magnitude, $\boldsymbol{M}$ | Shock Wave, $\boldsymbol{T}$ |
| :--- | :--- | :---: | :---: |
| Moresby Island | January 23 | 4.0 | $1.00 \times 10^{4}$ |
| Vancouver Island | April 30 | 5.3 | $1.99 \times 10^{5}$ |
| Quebec City | June 29 | 3.2 | $1.58 \times 10^{3}$ |
| Mould Bay | November 13 | 5.2 | $1.58 \times 10^{5}$ |
| St. Lawrence | December 14 | 3.7 | $5.01 \times 10^{3}$ |


81. $0,-3$
83. $-4,-2$
85. $\pm 2, \pm i \sqrt{2}$
87. $1, \frac{9}{4}$
89. 0.7
91. 3.125
93. 1.2575
95. $t \cdot \log (1.05)$
97. $0.05 t$

Problem Set 11.6

1. 1.4650
2. 2.1131
3. 0.6826
4. -1.5440
5. -0.6477
6. $-\frac{1}{3}$
7. 2 13. -0.1845
8. 0.1845
9. 1.6168
10. 2.3026
11. 1.3333
12. 0.7500
13. 1.3917
14. 0.7186 29. 2.6356
15. 4.1632
16. $5.8435 \quad 35 .-1.0642$
17. 18.31 years
18. lowest point: $(-2,-23)$
19. 9.25 years
20. 8.75 years
21. 18.58 years
22. 11.55 years
23. 13.9 years later or toward the end of 2007
24. 27.5 years
25. $t=\frac{1}{r} \ln \frac{A}{P}$
26. $t=\frac{1}{k} \frac{\log P-\log A}{\log 2}$
27. $t=\frac{\log A-\log P}{\log (1-r)}$

## CHAPTER 11 REVIEW TEST

1. 16
2. $\frac{1}{2}$
3. $\frac{1}{9}$
4. -5
5. $\frac{5}{6}$
6. 7
7. 


8.

9.

10.

12. $y= \pm \sqrt{x+1}$
13. $f^{-1}(x)=2 x-4$
19. $2^{3}=8$
18. $\log _{2} \frac{1}{8}=-3$
26.

14. $y= \pm \sqrt{\frac{4-x}{2}}$
21. $4^{1 / 2}=2$
27.

32. $\log _{10} 2+\log _{10} x-\log _{10} y$
33. $\frac{1}{2} \log _{a} x+3 \log _{a} y-\log _{a} z$
34. $2 \log _{10} x-3 \log _{10} y-4 \log _{10} z$
35. $\log _{2} x y$
36. $\log _{3} \frac{x}{4}$
37. $\log _{a} \frac{25}{3}$
38. $\log _{2} \frac{x^{3} y^{2}}{z^{4}}$
39. 2
40. 6
41. Possible solutions -1 and 3 ; only 3 checks; 3
42. 3
43. Possible solutions -2 and 3 ; only 3 checks; 3
44. Possible solutions -1 and 4; only 4 checks; 4
45. 2.5391 46. -0.1469
47. 9,230
48. 0.0251
49. $1 \quad 50.0$
51. 2
52. -4
53. 2.1
54. 5.1
55. $2.0 \times 10^{-3}$
56. $3.2 \times 10^{-8}$
57. $\frac{3}{2}$
58. $x=\frac{1}{3}\left(\frac{\log 5}{\log 4}-2\right) \approx-0.28$
59. 0.75
60. 2.43
61. About 4.67 years
62. About 9.25 years

## CHAPTER 12

## Problem Set 12.1

1. 5 3. $\sqrt{106}$
2. $\sqrt{61}$
3. $\sqrt{130}$
4. 3 or -1
5. 3
6. $(x-2)^{2}+(y-3)^{2}=16$
7. $(x-3)^{2}+(y+2)^{2}=9$
8. $(x+5)^{2}+(y+1)^{2}=5$
9. $x^{2}+(y+5)^{2}=1$
10. $x^{2}+y^{2}=4$
11. Center $=(0,0)$,

Radius $=2$


29. Center $=(-1,-1)$,

Radius $=1$
31. $\begin{aligned} \text { Center } & =(0,3), \\ \text { Radius } & =4\end{aligned}$

33. Center $=(2,3)$,

Radius $=3$

35. Center $=\left(-1,-\frac{1}{2}\right)$,
Radius $=2$

37. $(x-3)^{2}+(y-4)^{2}=25$

41. A: $(x+8)^{2}+y^{2}=64$

B: $x^{2}+y^{2}=64$
C: $(x-8)^{2}+y^{2}=64$
43. $(x-500)^{2}+(y-132)^{2}=120^{2}$
45. $f^{-1}(x)=\log _{3} x$
47. $f^{-1}(x)=\frac{x-3}{2}$
49. $f^{-1}(x)=5 x-3$
51. $y= \pm 3$
53. $y= \pm 2 i$, no real solution
55. $x= \pm 3 i$, no real solution
57. $\frac{x^{2}}{9}+\frac{y^{2}}{4}$
59. $x$-intercept $4, y$-intercept -3
61. $\pm 2.4$
63. $(x-2)^{2}+(y-3)^{2}=4$
65. $(x-2)^{2}+(y-3)^{2}=4$
67.5
69.5
71. $y=\sqrt{9-x^{2}}$ corresponds to the top half; $y=-\sqrt{9-x^{2}}$ to the bottom half.

## Problem Set 12.2

1. 


7.

13.

3.

9.

15.

5.

11.

17.

19.

23. $x$-intercepts $= \pm 3, y$-intercepts $= \pm 2$
29.

35.

21.

25. $x$-intercepts $= \pm 0.2$, no $y$-intercepts
31.

27. $x$-intercepts $= \pm \frac{3}{5}, y$-intercepts $= \pm \frac{2}{5}$
33.

37.

39.

41. $\pm \frac{9}{5}$
43. $\pm 3.2$
45. $y=\frac{3}{4} x, y=-\frac{3}{4} x$
47. $\frac{x^{2}}{16}+\frac{y^{2}}{4}=1$
49. The equation is $\frac{x^{2}}{20^{2}}+\frac{y^{2}}{10^{2}}=1$. A 6 -foot man could walk upright under the arch anywhere between 16 feet to the left and 16 feet the
right of the center.
51. About 5.3 feet wide
53. 1 55. -2
57. $\frac{4 x}{(x-2)(x+2)}$
59. $(0,0)$
61. $4 y^{2}+16 y+16$
63. $x=2 y+4$
65. $-x^{2}+6$ 67. $(5 y+6)(y+2)$
69. $\pm 2$
71. $\pm 2$

## Problem Set 12.3

1. 


3.

5.

7.

9. $(0,3),\left(\frac{12}{5},-\frac{9}{5}\right)$
11. $(0,4),\left(\frac{16}{5}, \frac{12}{5}\right)$
13. $(5,0),(-5,0)$
15. $(0,-3),(\sqrt{5}, 2),(-\sqrt{5}, 2)$
17. $(0,-4),(\sqrt{7}, 3),(-\sqrt{7}, 3)$
19. $(-4,11),\left(\frac{5}{2}, \frac{5}{4}\right)$
21. $(-4,5),(1,0)$
23. $(2,-3),(5,0)$
25. $(3,0),(-3,0)$
27. $(4,0),(0,-4)$
29. a. $(-4,4 \sqrt{3})$ and $(-4,-4 \sqrt{3})$ b. $(4,4 \sqrt{3})$ and $(4,-4 \sqrt{3})$
31.

33.

35. No intersection

37. $x^{2}+y^{2}<16$
$y>4-\frac{1}{4} x^{2}$
$\begin{array}{llllll}\text { 39. } 8,5 \text { or }-8,-5 \text { or } 8,-5 \text { or }-8,5 & \text { 41. } 6,3 \text { or } 13,-4 & \text { 43. }-\frac{2}{5} & \text { 45. }-5,1 & \text { 47. }-2,-1\end{array}$

## CHAPTER 12 REVIEW TEST

1. $\sqrt{10}$
2. $\sqrt{13}$
3. 5 4. 9
4. $-2,6$
5. $-12,4$
6. $(x-3)^{2}+(y-1)^{2}=4$
7. $(x-3)^{2}+(y+1)^{2}=16$
8. $(x+5)^{2}+y^{2}=9$
9. $(x+3)^{2}+(y-4)^{2}=18$
10. $x^{2}+y^{2}=25$
11. $x^{2}+y^{2}=9$
12. $(x+2)^{2}+(y-3)^{2}=25$
13. $(x+6)^{2}+(y-8)^{2}=100$
14. $(0,0) ; r=2$

15. $(-2,1) ; r=3$

16. 


16. $(3,-1) ; r=4$

19.

17. $(3,-2) ; r=3$

20.

22.

23.

24.

25.

26.

29.

32.

33. $(0,4),\left(\frac{16}{5},-\frac{12}{5}\right)$
34. $(0,-2),(\sqrt{3}, 1),(-\sqrt{3}, 1)$
35. $(-2,0),(2,0)$
36. $\left(-\sqrt{7},-\frac{\sqrt{6}}{2}\right),\left(-\sqrt{7}, \frac{\sqrt{6}}{2}\right),\left(\sqrt{7},-\frac{\sqrt{6}}{2}\right),\left(\sqrt{7}, \frac{\sqrt{6}}{2}\right)$

## CHAPTER 13

## Problem Set 13.1

1. $4,7,10,13,16$
2. $3,7,11,15,19$
3. $1,2,3,4,5$
4. $4,7,12,19,28$
5. $\frac{1}{4}, \frac{2}{5}, \frac{1}{2}, \frac{4}{7}, \frac{5}{8}$
6. $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$
7. $2,4,8,16,32$
8. $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$
9. $-2,4,-8,16,-32$
10. $3,5,3,5,3$
11. $1,-\frac{2}{3}, \frac{3}{5},-\frac{4}{7}, \frac{5}{9}$
12. $\frac{1}{2}, 1, \frac{9}{8}, 1, \frac{25}{32}$
13. $3,-9,27,-81,243$
14. $1,5,13,29,61$
15. $2,3,5,9,17$
16. $5,11,29,83,245$
17. $4,4,4,4,4$
18. $a_{n}=4 n$
19. $a_{n}=n^{2}$
20. $a_{n}=2^{n+}$
21. $a_{n}=\frac{1}{2^{n+1}}$
22. $a_{n}=3 n+2$
23. $a_{n}=-4 n+2$
24. $a_{n}=(-2)^{n-1}$
25. $a_{n}=\log _{n+1}(n+2) \quad$ 51. a. $\$ 28000, \$ 29120, \$ 30284.80, \$ 31,496.19, \$ 32756.04$ b. $a_{n}=28000(1.04)^{n-1}$
26. a. $16 \mathrm{ft}, 48 \mathrm{ft}, 80 \mathrm{ft}, 112 \mathrm{ft}, 144 \mathrm{ft}$ b. 400 ft c. No 55 . a. $180^{\circ}, 360^{\circ}, 540^{\circ}, 720^{\circ}$ b. $3,240^{\circ}$ c. Sum of interior angles would be $0^{\circ}$
27. 27
28. 5
29. 1
30. 30
31. 40
32. 18
33. $-\frac{20}{81}$
34. $\frac{21}{10}$
35. $a_{100} \approx 2.7048, a_{1,000} \approx 2.7169, a_{10,000} \approx 2.7181, a_{100,000} \approx 2.7183$
36. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

## Problem Set 13.2

1. 36
2. 11
3. 18
4. $\frac{163}{60}$
5. 60
6. 40
7. 44
8. $-\frac{11}{32}$
9. $\frac{21}{10}$
10. $5 x+15$
11. $(x-2)+(x-2)^{2}+(x-2)^{3}+(x-2)^{4}$
12. $\frac{x+1}{x-1}+\frac{x+2}{x-1}+\frac{x+3}{x-1}+\frac{x+4}{x-1}+\frac{x+5}{x-1}$
13. $(x+3)^{3}+(x+4)^{4}+(x+5)^{5}+(x+6)^{6}+(x+7)^{7}+(x+8)^{8}$
14. $(x-6)^{6}+(x-8)^{7}+(x-10)^{8}+(x-12)^{9}$
15. $\sum_{i=1}^{4} 2^{i}$
16. $\sum_{i=2}^{6} 2^{i}$
17. $\sum_{i=1}^{5}(4 i+1)$
18. $\sum_{i=2}^{5}-(-2)^{i}$
19. $\sum_{i=3}^{7} \frac{i}{i+1}$
20. $\sum_{i=1}^{4} \frac{i}{2 i+1}$
21. $\sum_{i=6}^{9}(x-2)^{i}$
22. $\sum_{i=1}^{4}\left(1+\frac{i}{x}\right)^{i+1}$
23. $\sum_{i=3}^{5} \frac{x}{x+i}$
24. $\sum_{i=2}^{4} x^{i}(x+i)$
25. a. $0.3+0.03+0.003+0.0003+$
b. $0.2+0.02+0.002+0.0002+.$.
c. $0.27+0.0027+0.000027+\ldots$
26. seventh second: 208 feet; total: 784 feet
27. a. $16+48+80+112+144$
b. $\sum_{i=1}^{5}(32 i-16)$
28. $3 \log _{2} x+\log _{2} y$
29. $\frac{1}{3} \log _{10} x-2 \log _{10} y$
30. $\log _{10} \frac{x}{y^{2}}$
31. $\log _{3} \frac{x^{2}}{y^{3} z^{4}}$
32. 80
33. Possible solutions -1 and 8 ; only 8 .
34. 74
35. $\frac{55}{2}$
36. $2 n+1$
37. $(3,2)$
38. $\frac{13}{2}$

## Problem Set 13.3

1. arithmetic; $d=1$
2. not arithmetic
3. arithmetic; $d=-5$
4. not arithmetic
5. arithmetic; $d=\frac{2}{3}$
6. $a_{n}=4 n-1 ; a_{24}=95$
7. $a_{10}=-12 ; S_{10}=-30$
8. $a_{1}=7 ; d=2 ; a_{30}=65$
9. $a_{1}=12 ; d=2 ; a_{20}=50 ; S_{20}=620$
10. $a_{20}=79 ; S_{20}=820$
11. $a_{40}=122 ; S_{40}=2,540$
12. $a_{1}=13 ; d=-6$
13. $a_{85}=-238$
14. $d=\frac{16}{19} ; a_{85}=28$
15. 20,300
16. $a_{35}=-158$
17. $a_{10}=5 ; S_{10}=\frac{55}{2}$
18. a. $\$ 18,000, \$ 14,700, \$ 11,400, \$ 8,100, \$ 4,800$ b. $-\$ 3,300$ c.

d. $\$ 9,750$ e. $a_{0}=18000 ; a_{n}=a_{n-1}-3300$ for $n \geq 1 \quad 37$. a. $1500 \mathrm{ft}, 1460 \mathrm{ft}, 1420 \mathrm{ft}, 1380 \mathrm{ft}, 1340 \mathrm{ft}, 1300 \mathrm{ft}$ b. It is arithmetic because the same amount is subtracted from each succeeding term. c. $a_{n}=1,540-40 n$
19. a. $1,3,6,10,15,21,28,36,45,55,66,78,91,105,120$ b. $a_{1}=1 ; a_{n}=n+a_{n-1}$ for $n \geq 2$
c. It is not arithmetic because the same amount is not added to each term.
20. 2.7604
21. -1.2396
22. 445
23. $4.45 \times 10^{-8}$
24. $\frac{1}{16}$
25. $\sqrt{3}$
26. $2^{n}$
27. $r^{3}$
28. $\frac{2}{5}$
29. 255
30. $a_{1}=9, d=4$
31. $d=-3$
32. $S_{50}=50+1225 \sqrt{2}$

## Problem Set 13.4

1. geometric; $r=5$
2. geometric; $r=\frac{1}{3}$
3. not geometric
4. geometric; $r=-2$
5. not geometric
6. $a_{n}=4 \cdot 3^{n-1}$
7. $a_{6}=\frac{1}{16}$
8. $a_{20}=-3$
9. $S_{10}=10,230$
10. $S_{20}=0$
11. $a_{8}=\frac{1}{640}$
12. $S_{5}=-\frac{31}{32}$
13. $a_{10}=32 ; S_{10}=62+31 \sqrt{2}$
14. $a_{6}=\frac{1}{1000} ; S_{6}=111.111$
15. $\pm 2$
16. $a_{8}=384 ; S_{8}=255$
17. $r=2$
18. 1
$37.8 \quad 39.4 \quad 41 . \frac{8}{9}$
19. $\frac{2}{3}$
20. $\frac{9}{8}$
21. a. $\$ 450,000, \$ 315,000, \$ 220,500, \$ 154,350, \$ 108,045$
b. 0.7
c.

d. Approximately $\$ 90,000$ e. $a_{0}=450000 ; a_{n}=0.7 a_{n-1}$ for $n \geq 1$
22. a. $\frac{1}{2}$
b. $\frac{364}{729}$ c. $\frac{1}{1458}$
23. 300 feet
24. a. $a_{n}=15\left(\frac{4}{5}\right)^{n-1}$ b. 75 feet
25. 2.16
26. 6.36
27. $t=\frac{1}{5} \ln \frac{A}{10}$
28. 1
29. $x^{3}+2 x y+y^{2}$
30. 15
31. $\frac{7}{11}$
32. $r=\frac{1}{3}$
33. a. stage $1: 1$; stage $2: \frac{3}{4}$; stage $3: \frac{9}{16}$; stage $4: \frac{27}{64}$
b. geometric sequence c. 0 d. increasing sequence

## Problem Set 13.5

1. $x^{4}+8 x^{3}+24 x^{2}+32 x+16$
2. $x^{6}+6 x^{5} y+15 x^{4} y^{2}+20 x^{3} y^{3}+15 x^{2} y^{4}+6 x y^{5}+y^{6}$
3. $32 x^{5}+80 x^{4}+80 x^{3}+40 x^{2}+10 x+1$
4. $x^{5}-10 x^{4} y+40 x^{3} y^{2}-80 x^{2} y^{3}+80 x y^{4}-32 y^{5}$
5. $81 x^{4}-216 x^{3}+216 x^{2}-96 x+16$
6. $64 x^{3}-144 x^{2} y+108 x y^{2}-27 y^{3}$
7. $x^{8}+8 x^{6}+24 x^{4}+32 x^{2}+16$
8. $x^{6}+3 x^{4} y^{2}+3 x^{2} y^{4}+y^{6}$
9. $16 x^{4}+96 x^{3} y+216 x^{2} y^{2}+216 x y^{3}+81 y^{4}$
10. $\frac{x^{3}}{8}+\frac{x^{2} y}{4}+\frac{x y^{2}}{6}+\frac{y^{3}}{27}$
11. $\frac{x^{3}}{8}-3 x^{2}+24 x-64$
12. $\frac{x^{4}}{81}+\frac{2 x^{3} y}{27}+\frac{x^{2} y^{2}}{6}+\frac{x y^{3}}{6}+\frac{y^{4}}{16}$
13. $x^{9}+18 x^{8}+144 x^{7}+672 x^{6}$
14. $x^{10}-10 x^{9} y+45 x^{8} y^{2}-120 x^{7} y^{3}$
15. $x^{25}+75 x^{24}+2,700 x^{23}+62,100 x^{22}$
16. $x^{60}-120 x^{59}+7,080 x^{58}-273,760 x^{57}$
17. $x^{18}-18 x^{17} y+153 x^{16} y^{2}-816 x^{15} y^{3}$
18. $x^{15}+15 x^{14}+105 x^{13}$
19. $x^{12}-12 x^{11} y+66 x^{10} y^{2}$
20. $x^{20}+40 x^{19}+760 x^{18}$
21. $x^{100}+200 x^{99}$
22. $x^{50}+50 x^{49} y$
23. $51,963,120 x^{4} y^{8}$
24. $3,360 x^{6}$
25. $-25,344 x^{7}$
26. $2,700 x^{23} y^{2}$
27. $\binom{20}{11}(2 x)^{9}(5 y)^{11}=\frac{20!}{11!9!}(2 x)^{9}(5 y)^{11}$
28. $x^{20} y^{10}-30 x^{18} y^{9}+405 x^{16} y^{8}$
29. $\frac{21}{128}$
30. $x \approx 1.21$
31. $\frac{1}{6}$
32. $\approx 17.5$ years
33. 56
34. 125,970

## CHAPTER 13 REVIEW TEST

1. $7,9,11,13$
2. $1,4,7,10$
3. $0,3,8,15$
4. $\frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}$
5. $4,16,64,256$
6. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}$ 7. $a_{n}=3 n-1$
7. $a_{n}=2 n-5$
8. $a_{n}=n^{4}$
9. $a_{n}=n^{2}+1$
10. $a_{n}=\left(\frac{1}{2}\right)^{n}=2^{-n}$
11. $a_{n}=\frac{n+1}{n^{2}}$
12. 32
13. 25
14. $\frac{14}{5}$
15. -5
16. 98
17. $\frac{127}{30}$
18. $\sum_{i=1}^{4} 3 i$
19. $\sum_{i=1}^{4}(4 i-1)$
20. $\sum_{i=1}^{5}(2 i+3)$
21. $\sum_{i=2}^{4} i^{2}$
22. $\sum_{i=1}^{4} \frac{1}{i+2}$
23. $\sum_{i=1}^{5} \frac{i}{3 i}$
24. $\sum_{i=1}^{3}(x-2 i)$
25. $\sum_{i=1}^{4} \frac{x}{x+i}$
26. geometric
27. arithmetic
28. arithmetic
29. neither
30. geometric
31. geometric
32. arithmetic
33. neither
34. $a_{n}=3 n-1 ; a_{20}=59$
35. $a_{n}=8-3 n ; a_{16}=-40$
36. $a_{10}=34 ; S_{10}=160$
37. $a_{16}=78 ; S_{16}=648$
38. $a_{1}=5 ; d=4 ; a_{10}=41$
39. $a_{1}=8 ; d=3 ; a_{9}=32 ; S_{9}=180$
40. $a_{1}=-4 ; d=-2 ; a_{20}=-42 ; S_{20}=-460$
41. 20,100
42. $a_{40}=-95$
43. $a_{n}=3(2)^{n-1} ; a_{20}=1,572,864$
44. $a_{n}=5(-2)^{n-1} ; a_{16}=-163,840$
45. $a_{n}=4\left(\frac{1}{2}\right)^{n-1} ; a_{10}=\frac{1}{128}$
46. -3
47. 8
48. $a_{1}=3 ; r=2 ; a_{6}=96$
49. $243 \sqrt{3}$
50. $x^{4}-8 x^{3}+24 x^{2}-32 x+16$
51. $16 x^{4}+96 x^{3}+216 x^{2}+216 x+81$
52. $27 x^{3}+54 x^{2} y+36 x y^{2}+8 y^{3}$
53. $x^{10}-10 x^{8}+40 x^{6}-80 x^{4}+80 x^{2}-32$
54. $\frac{1}{16} x^{4}+\frac{3}{2} x^{3}+\frac{27}{2} x^{2}+54 x+81$
55. $\frac{1}{27} x^{3}-\frac{1}{6} x^{2} y+\frac{1}{4} x y^{2}-\frac{1}{8} y^{3}$
56. $x^{10}+30 x^{9} y+405 x^{8} y^{2}$
57. $x^{9}-27 x^{8} y+324 x^{7} y^{2}$
58. $x^{11}+11 x^{10} y+55 x^{9} y^{2}$
59. $x^{12}-24 x^{11} y+264 x^{10} y^{2}$
60. $x^{16}-32 x^{15} y$
61. $x^{32}+64 x^{31} y$
62. $x^{50}-50 x^{49}$
63. $x^{150}+150 x^{149} y$
64. $-61,236 x^{5}$
65. $5376 x^{6}$

## Answers to Selected

 Even-Numbered Problems
## CHAPTER 2

Problem Set 2.8
2.

6.

10.

14.

18.

22.

26.

30.

34.

38.

42.

46.

50.

4.

8.

12.

16.

20.

24.

28.

32.

36.

40.

44.

48.


B-2 Answers to Selected Even-Numbered Problems

## CHAPTER 3

## Problem Set 3.1

## Even-numbered problems 2-18. <br> 

50. 


48.


## Problem Set 3.3

2. 


4.

6.

8.

10.

12.

14.

20.

26.

32.

16.

22.

28.

34.

18.

24.

30.

36.


B-4 Answers to Selected Even-Numbered Problems
38.

40.

46c.


Problem Set 3.4
2.

4.

6.

8.

10.

12.

14.

16.

18.

20.

22.

24.

26.

28.

30.

36.

44.

46.


Problem Set 3.5
2.

4.

6.

8.

10.

14.

20.

30.

16.

22.

32.

12.

18.

28.

34.

36.

38.

42.

44.


Problem Set 3.6
2.

4.

6.

8.

10.

12.


B-8 Answers to Selected Even-Numbered Problems
14.

20.

26.

32.

16.

22.

28.

42.

30.

18.

24.


## Problem Set 3.9

30. 



## CHAPTER 4

Problem Set 4.3
90.

32.

92.


Problem Set 4.6
82.

84.


## CHAPTER 6

## Problem Set 6.5

26. 


28.

30.


## CHAPTER 7

Problem Set 7.3
2.
4.

6.

8.
10. $\qquad$ 12. $\qquad$
14
16.
18. $\qquad$
20.
22. $\qquad$ 24. $\qquad$
38 $\qquad$ 40. $\qquad$ 42. $\qquad$
44. $\qquad$ 46. $\qquad$ 48.
50.

52. $\qquad$

Problem Set 7.4
2. $\longleftarrow$
4.
6.
8. $\qquad$
14.
22.
16. $\qquad$ 20. $\qquad$
24.

26. $\qquad$
28.
30. $\qquad$ 32. $\qquad$
34.
36.


Problem Set 7.5
2.

4.

8.

10.

6.


Problem Set 7.7
2.

4.

6.

8.

10.

12.

14.

20.

26.

28.

30.

32.

44.

34.

36.


## Problem Set 7.8

2. 


4.

6.

8.

10.

12.

14.

16.

24. a.


## CHAPTER 8

## Problem Set 8.1

8. 


10.

12.

14.

16.

18.

26.

70.

Problem Set 8.2
52.

30.

72.
74.


Problem Set 8.3
26.

28.

30.

32.

34.

36.

38.

40.

42.


## Problem Set 8.4

48. 


52.

58.


## Problem Set 8.6

40. 


44.

72. $\qquad$

## CHAPTER 9

Problem Set 9.3
132.

Problem Set 9.6
74.

76.

78.

80.

82.

84.


## CHAPTER 10

Problem Set 10.5
2.

4.

6.

8.

10.

12.

14.

20.

16.

18.


## Problem Set 10.6

2. 


4.
8. $\qquad$ 10.

12.

14.

20. $\qquad$ 22.

28.

30.

32.

66.

68.

70.

## CHAPTER 11

Problem Set 11.1
10.

12.

14.

16.


Problem Set 11.2
14.

30.

18.

34.

20.

16.

18.


B-22 Answers to Selected Even-Numbered Problems
20.

26.


Problem Set 11.3
38.

44.

22.

28.

40.

42.

24.


## CHAPTER 12

## Problem Set 12.1

24. 


30.

36.

26.

32.

28.

34.


B-24 Answers to Selected Even-Numbered Problems

## Problem Set 12.2


10.

14.

16.

20.

4.

8.

22.

6.

12.

18.

30.

32.

38.


Problem Set 12.3
2.

4.

8.

6.

32.

34.

34.

40.

36.

36.


## CHAPTER 13

Problem Set 13.3
36c.


Problem Set 13.4
52c.


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## Functions: A Summary

## Graphing Ordered Pairs [Section 3.1]

To graph the ordered pair $(a, b)$ on a rectangular coordinate system, we start at the origin and move $a$ units right or left (right if $a$ is positive and left if $a$ is negative). We then move $b$ units up or down (up if $b$ is positive and down if $b$ is negative). The point where we end up is the graph of the ordered pair $(a, b)$. Figure 1 shows the graphs of the ordered pairs $(2,5),(-2,5)$, $(-2,-5)$, and $(2,-5)$.


FIGURE 1 The graphs of four ordered pairs.

## Functions [Section 8.3]

Definition A function is a rule that pairs each element in one set, called the domain, with exactly one element from a second set, called the range.

With functions, we take numbers from the domain and put them into the function rule. The function rule then outputs values in the range of the function. The only restriction on this process is that each input can have only one output. Here is a diagram that will help visualize the relationship between the function rule and the domain and range of the function.


Example If you work from 0 to 40 hours per week at a job that pays $\$ 7.50$ per hour, then the amount of money you earn each week is a function of the number of hours you work. If $x$ is hours worked and $y$ is weekly pay, then the function rule is

$$
y=7.5 x \text { for } 0 \leq x \leq 40
$$

## Graphing Functions [Section 8.3]

There are many ways to graph functions. Figure 2 is a graph constructed by substituting 0,10,20,30, and 40
for $x$ in the formula $f(x)=7.5 x$, and then using the formula to find the corresponding values of $y$. Each pair of numbers corresponds to a point in Figure 2. The points are connected with straight lines to form the graph.


FIGURE 2 The graph of $f(x)=7.5 x, 0 \leq x \leq 40$.

## Function Notation [Section 8.4]

The notation $f(x)$ is used to denote elements in the range of a function. For example, if the rule for a function is given by $y=7.5 x$, then the rule can also be written as $f(x)=7.5 x$. If we ask for $f(20)$, we are asking for the value of $y$ that comes from $x=20$. That is

$$
\begin{array}{ll}
\text { if } & f(x)=7.5 x \\
\text { then } & f(20)=7.5(20)=150
\end{array}
$$

Figure 3 is a diagram called a function map that gives a visual representation of this particular function:


FIGURE 3 A function map.
A function can also be thought of as a machine. We put values of $x$ into the machine which transforms them into values of $f(x)$, which are then output by the machine. The diagram is shown in Figure 4.


FIGURE 4 A function machine.
We can organize our work with paired data, graphing, and functions into a more standard form by using ordered pairs and the rectangular coordinate system.

## More about Functions [Section 8.3]

Since functions are rules that assign values from one set (the domain) to a second set (the range), the result is a set of ordered pairs, where the first coordinate in each ordered pair comes from the domain and the second coordinate of each ordered pair comes from the range. This fact allows us to write an alternate definition for functions.
Alternate Definition A function is a set of ordered pairs in which no two different ordered pairs have the same first coordinate. The set of all first coordinates is called the domain of the function. The set of all second coordinates is called the range of the function.

In order to classify together all sets of ordered pairs, whether they are functions or not, we include the following definitions.

## Relations [Section 8.3]

A relation is a rule that pairs each element in one set, called the domain, with one or more elements from a second set, called the range.

Alternate Definition A relation is a set of ordered pairs. The set of all first coordinates is the domain of the relation. The set of all second coordinates is the range of the relation.

Example Figure 5 is a scatter diagram showing the advertised prices of used Ford Mustangs appearing in a Los Angeles newspaper one day in 1995. The set of data from which the scatter diagram was constructed is not a function because some values of $x$ are paired with more than one value of $y$.


FIGURE 5 Scatter diagram of a relation that is not also a function.

As a consequence of the definitions for functions and relations, we have the following two facts: Every function is also a relation, but not every relation is a function.

## Vertical Line Test [Section 8.3]

If a vertical line crosses the graph of a relation in more than one place, the relation cannot be a function. If no vertical line can be found that crosses a graph in more than one place, then the graph is the graph of a function.

Examples The curve shown in Figure 6 does not represent a function since there are many vertical lines that cross this graph in more than one place.


FIGURE 6 The graph of the relation $x=y^{2}-2$.

## Algebra with Functions [Section 8.5]

If we are given functions $f$ and $g$ with a common domain, we can define four other functions as follows.
$(f+g)(x)=f(x)+g(x)$ The function $f+g$ is the sum of the functions $f$ and $g$.
$(f-g)(x)=f(x)-g(x) \quad$ The function $f-g$ is the difference of the functions $f$ and $g$.
$(f g)(x)=f(x) g(x)$
$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$ The function $f g$ is the product of the functions $f$ and $g$.
The function $f / g$ is the quotient of the functions $f$ and $g$, where $g(x) \neq 0$.

## Composition of Functions [Section 8.5]

To find the composition of two functions $f$ and $g$, we first require that the range of $g$ have numbers in com-
mon with the domain of $f$. Then the composition of $f$ with $g, f \circ g$, is defined this way:

$$
(f \circ g)(x)=f(g(x))
$$

Composition of functions is not commutative. The composition of $f$ with $g, f \circ g$, may therefore be different from the composition of $g$ with $f, g \circ f$.

$$
(g \circ f)(x)=g(f(x))
$$

## One-to-One Functions [Section 11.2]

Definition A one-to-one function is a function in which no element in the range comes from more than one element in the domain.

This definition gives two important results:

1. The result is a set of ordered pairs for which no first or second coordinates are repeated.
2. The graph of a one-to-one function is such that no horizontal line can be found that crosses the graph in more than one place.

## Inverse Functions [Section 11.2]

For every one-to-one function, $f$, there exists another function, $f^{-1}$, such that

$$
f\left(f^{-1}(x)\right)=f^{-1}(f(x))=x
$$

For inverse functions $f$ and $f^{-1}$, if $(x, y)$ belongs to $f$, then $(y, x)$ belongs to $f^{-1}$. Visually, this means that the graph of a function and its inverse function will have symmetry about the line $y=x$.

Examples Figure 7 shows the graph of a function and its inverse on the same coordinate system. Notice how the graph of the inverse can be obtained by reflecting the graph of the original function about the line $y=x$.


FIGURE 7 The graph of a function and its inverse.

$$
\begin{array}{ll}
\text { Function } & f(x)=2 x-3 \\
\text { Inverse } & f^{-1}(x)=\frac{x+3}{2}
\end{array}
$$

Finding Formulas for Inverse Functions [Section 11.2] If $y=f(x)$ is a one-to-one function, we can find the formula for $f^{-1}(x)$ by interchanging $x$ and $y$ in the formula for $f(x)$ and then solve for $y$.

Example To find the formula for the inverse of $y=2 x-3$, we interchange $x$ and $y$ and then solve for $y$ :

$$
\begin{aligned}
& \text { Function } \quad y=2 x-3 \\
& \text { Its inverse } \quad x=2 y-3 \quad \text { (exchange } x \text { and } y \text { ) } \\
& \text { Solve for } y \quad 2 y-3=x \\
& 2 y=x+3 \\
& y=\frac{x+3}{2}
\end{aligned}
$$

The inverse is $f^{-1}(x)=\frac{x+3}{2}$

## Exponential Functions [Section 11.1]

If $b$ is a positive number other than 1 , then the function below is an exponential function.

$$
f(x)=b^{x}
$$

## Logarithmic Functions [Section 11.3]

If $b$ is a positive number other than 1 , and $x$ is a positive real number, then the function below is a logarithmic function.

$$
f(x)=\log _{b} x
$$

The inverse of every exponential function is a logarithmic function. Likewise, the inverse of every logarithmic function is an exponential function.

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