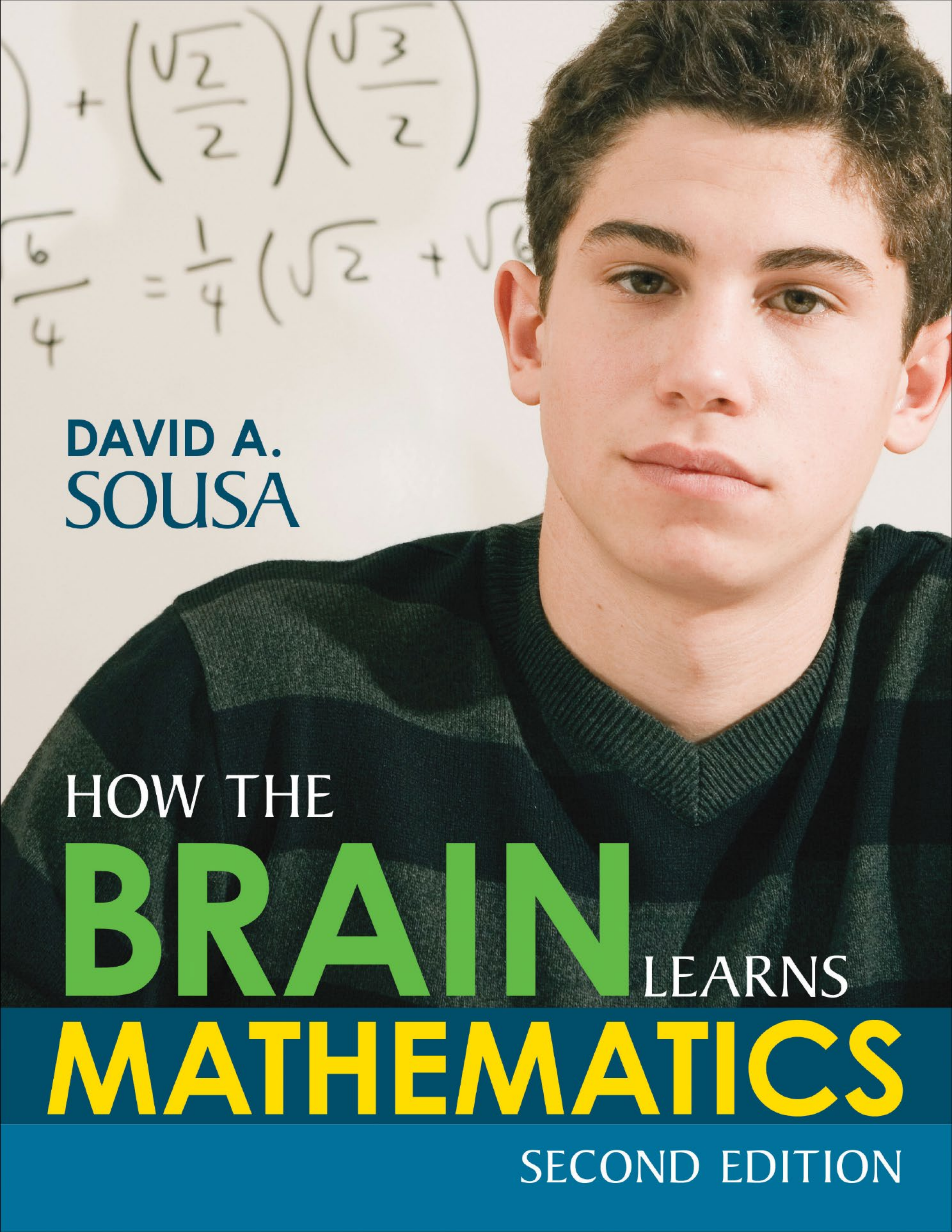

DAVID A. SOUSA

HOW THE BRAIN LEARNS MATHEMATICS

SECOND EDITION

HOW THE BRAIN LEARNS MATHEMATICS

A mathematician, like a painter or a poet, is a maker of patterns.
If his patterns are more permanent than theirs, it is because they are made with ideas.

—Godfrey Harold Hardy
A Mathematician's Apology

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About the Author

David A. Sousa, EdD, is an international consultant in educational neuroscience and author of 16 books that suggest ways educators and parents can translate current brain research into strategies for improving learning. A member of the Cognitive Neuroscience Society, he has conducted workshops in hundreds of school districts on brain research, instructional skills, and science education at the preK–12 and university levels. He has made presentations to more than 200,000 educators at national conventions of educational organizations and to regional and local school districts across the United States, Canada, Europe, Australia, New Zealand, and Asia.

Dr. Sousa has a bachelor's degree in chemistry from Bridgewater State University in Massachusetts, a Master of Arts in Teaching degree in science from Harvard University, and a doctorate from Rutgers University. His teaching experience covers all levels. He has taught senior high school science and served as a K–12 director of science, a supervisor of instruction, and a district superintendent in New Jersey schools. He was an adjunct professor of education at Seton Hall University for 10 years and a visiting lecturer at Rutgers University.

Prior to his career in New Jersey, Dr. Sousa taught at the American School of Paris (France) and served for 5 years as a Foreign Service Officer and science advisor at the USA diplomatic missions in Geneva (Switzerland) and Vienna (Austria).

Dr. Sousa has edited science books and published dozens of articles in leading journals on staff development, science education, and educational research. His most popular books for educators include *How the Brain Learns,* now in its fourth edition; *How the Special Needs Brain Learns,* second edition; *How the Gifted Brain Learns; How the Brain Learns to Read,* second edition; *How the Brain Influences Behavior; How the ELL Brain Learns; Differentiation and the Brain* (with Carol Tomlinson); and *How the Brain Learns Mathematics,* which was selected by the Independent Publishers' Association as one of the best professional development books of 2008. *The Leadership Brain* suggests ways for educators to lead today's schools more effectively. *Brainwork: The Neuroscience of How We Lead Others* was written for business and organizational leaders. His books have been published in French, Spanish, Chinese, Arabic, Korean, Russian, and several other languages.

Dr. Sousa is past president of the National Staff Development Council (now called Learning Forward). He has received numerous awards from professional associations, school districts, and educational foundations for his commitment to research, staff development, and science education. He received the Distinguished Alumni Award and an honorary doctorate from Bridgewater (Mass.) State University, and an honorary doctorate in humane letters from Gratz College in Philadelphia.

Dr. Sousa has been interviewed by Matt Lauer on NBC's *Today* show, on other television programs, and on National Public Radio about his work with schools using brain research. He makes his home in South Florida.

Introduction

Numbers rule the universe.

—Pythagoras

EVERYONE CAN DO MATHEMATICS

Human beings are born with some remarkable capabilities. One is language. In just a few years after birth, toddlers are carrying on running conversations without the benefit of direct instruction. Over the next few years, their sentences become more complex and their vocabulary grows exponentially. By the age of 10, they understand about 10,000 words and speak their native language with 95 percent accuracy.

Another innate talent is number sense—the ability to determine the number of objects in a small collection, to count, and to perform simple addition and subtraction, also without direct instruction. Yet by the age of 10, some of these children are already saying, "I can't do math!" But you never hear them saying, "I can't do language!" Why this difference?

> *Children often say, "I can't do math!" But you never hear them say, "I can't do language!" Why this difference?*

One reason is that spoken language and number sense are survival skills; abstract mathematics is not. In elementary schools we present complicated notions and procedures to a brain that was first designed for survival in the African savanna. Human culture and society have changed a lot in the past 5,000 years, but the human brain has not. So how does the brain cope when faced with a task, such as multiplying a pair of two-digit numbers, for which it was not prepared? Thanks to modern imaging devices that can look inside the living brain, we can see which cerebral circuits are called into play when the brain tackles a task for which it has limited innate capabilities. That the human brain *can* rise to this challenge is testimony to its remarkable ability to assess its environment and make calculations that can safely land humans on the moon and send a space probe into orbit around a planet hundreds of millions of miles away.

WHAT IS MATHEMATICS?

To most people, mathematics is about calculating numbers. Some may even expand the definition to include the study of quantity (arithmetic), space (plane and solid geometry), and change (calculus). But even this definition does not encompass the many areas where mathematics and mathematicians are found. A broader definition of mathematics comes from W. W. Sawyer (1982). In the 1950s, he described mathematics as the "classification and study of all possible patterns" (p. 12). He explained that *pattern* was meant "to cover almost any kind of regularity that can be recognized by the mind" (p. 12).

Other mathematicians who share Sawyer's view have shortened the definition even further: Mathematics is the science of patterns. Devlin (2000) not only agrees with this definition but has used it as the title of one of his books. He explains that patterns include order, structure, and logical relationships, and go beyond the visual patterns found in tiles and wallpaper to those that occur everywhere in nature. For example, patterns can be found in the orbits of the planets, the symmetry of flowers, how people vote, the spots on a leopard, the outcomes of games of chance, the relationship between the words that make up a sentence, and the sequence of sounds we recognize as music. Some patterns are numerical and can be described with numbers, such as voting patterns of a nation or the odds of winning the lottery. Other patterns, such as a leopard's spots, are visual designs not connected to numbers at all.

Mathematics can be defined simply as the science of patterns.

Devlin (2000) further points out that mathematics can help make the invisible visible. Two-thousand years ago, the Greek mathematician Eratosthenes was able to calculate the diameter of Earth with considerable accuracy and without ever stepping foot off the planet. The equations developed by 18th-century mathematician Daniel Bernoulli explain how a jet plane flying overhead stays aloft. Thanks to Isaac Newton, we can calculate the effects of the unseen force of gravity. More recently, linguist Noam Chomsky has used mathematics to explain the invisible and abstract patterns of words that we recognize as a grammatical sentence.

If mathematics is the science of patterns and if visible and invisible patterns exist all around us, then mathematics is not just about numbers but about the world we live in. If that is the case, then why are so many students turned off by mathematics before they leave high school? What happens in those classrooms that gives students the impression that mathematics is a sterile subject filled with meaningless abstract symbols? Clearly, educators have to work harder at planning a mathematics curriculum that is exciting and relevant and at designing lessons that carry this excitement into every day's instruction.

I will leave the discussion of what content to include in a preK–12 mathematics curriculum to experts in that area, especially now that the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) have been released and adopted by many states. My purpose here is to suggest how the research in cognitive neuroscience can be used to plan lessons in mathematics that are more likely to result in learning and retention.

Why Is Learning Mathematics So Hard?

Succeeding in high school mathematics is still no easy feat. Take a look at Table I.1. The results of the 2013 National Assessment of Educational

Table I.1 Proficiency Levels for Grades 4, 8, and 12 in Mathematics on NAEP, 2005–2013

Year	*Grade 4*				*Grade 8*				*Grade 12*			
	Below Basic	*At Basic*	*Proficient*	*Advanced*	*Below Basic*	*At Basic*	*Proficient*	*Advanced*	*Below Basic*	*At Basic*	*Proficient*	*Advanced*
2013	17	41	34	8	26	38	27	9	35	39	23	3
2011	18	42	34	7	27	39	26	8	—	—	—	—
2009	18	43	33	6	27	39	26	8	36	38	23	3
2007	18	43	34	6	29	39	25	7	—	—	—	—
2005	20	44	31	5	31	39	24	6	39	38	21	2

SOURCE: NAEP (2013).

Progress (NAEP) mathematics tests of twelfth-grade students revealed that 23 percent were considered proficient in mathematics skills. This was the same percentage found in the 2009 assessment. No educator or parent can feel reassured by results showing such a low percentage of high school seniors performing at this proficiency level in mathematics. For fourth graders, 34 percent were proficient, compared with 33 percent for 2009 and 34 percent for 2011. As for the eighth graders, 27 percent scored proficient in 2013, compared with 26 percent for both the 2009 and 2011 assessments. The improvement was not significant (NAEP, 2013). Despite all the attention and high-stakes testing focused on mathematics instruction in recent years, achievement results have barely moved.

Explanations for this lackluster performance abound. Some say that learning mathematics is difficult because it is so abstract and requires more logical and ordered thinking. Others say that the various symbols used in mathematics make it similar to tackling a foreign language. Education critics maintain that only a few students are really developmentally incapable of handling mathematics and that the poor performance stems mainly from inadequate instruction. They cite the so-called "math wars" as hindering major progress in mathematics curriculum development, similar to what the "reading wars" did to reading instruction during the 1990s.

Impact of Teacher Preparation

Another potential factor affecting students' success in mathematics is the content knowledge of their teachers. Numerous studies have shown that middle and high school students learn more when their teachers have certifications or degrees in mathematics (e.g., Wayne & Youngs, 2003). Although states have been increasing the course requirements for individuals to be licensed to teach mathematics in secondary schools, problems persist. A recent study of 115 prospective middle school mathematics teachers at a large U.S. public university revealed that a substantial number of them had a limited knowledge of algebra for teaching (Huang & Kulm, 2012). The students made numerous mistakes when solving quadratic equations and in algebraic reasoning and manipulation. They also had difficulties with connecting algebraic and graphical representations.

A survey by the U.S. Department of Education showed that only 63 percent of the nation's nearly 144,000 high school mathematics teachers have both a college major and certification in mathematics (Hill, 2011). Nearly 26 percent have only a major or only certification in mathematics, and about 11 percent have neither a major nor certification in mathematics. This last group is referred to as out-of-field teachers. Other surveys find that out-of-field teachers are more likely to be found in high-poverty schools or in schools where they are assigned to the most challenging students (e.g., Kalogrides, Loeb, & Betelle, 2011).

Responses From Mathematics Educators

The National Council of Teachers of Mathematics (NCTM) published the *Principles and Standards for School Mathematics* in 2000, proposing five process standards and five content standards for preK–12 mathematics instruction. Since then, interpretation of the standards in the elementary and middle school grades became so broad that NCTM decided to refocus the curriculum at those grade levels.

In 2006, NCTM released *Curriculum Focal Points for Mathematics in Prekindergarten Through Grade 8,* which identified three important mathematical topics at each level, described as "cohesive clusters of related knowledge, skills, and concepts" that form the necessary foundation for understanding concepts in higher-level mathematics. The publication was intended to bring more coherence to the very diverse mathematics curricula currently in use. It provided a framework for states and districts to design more focused curricular expectations and assessments for preK–8 mathematics curriculum development. Shortly thereafter, the National Mathematics Advisory Panel (2008) published its final report, making recommendations for curriculum changes, teacher education, instructional practices and materials, and assessment. In the meantime, the National Governors Association and the Council of Chief State School Officers launched an effort to develop standards in mathematics and English/language arts. They were finally published in 2010 as the Common Core State Standards for Mathematics, and have been adopted by most of the states. We will discuss these standards further in Chapters 4 and 5.

Whether this new effort will succeed in improving student achievement in mathematics remains to be seen. In the meantime, teachers enter classrooms every day prepared to help their students feel confident enough to master mathematics principles and operations. One thing seems certain: Students who are poor in mathematics in their early years remain poor in mathematics in their later years.

■ ABOUT THIS BOOK

I am often asked to give specific examples of how the fruits of scientific research have made an impact on educational practice. That question is a lot easier to answer now than it was 20 years ago because recent discoveries in cognitive neuroscience have given us a deeper understanding of the brain. Thanks to brain-scanning technology we now have more knowledge about our short-term and long-term memory systems, the impact of emotions on learning, how we acquire language and motor skills, and how the brain learns to read. But only more recently have researchers begun to examine closely the neural mechanisms involved in processing arithmetic and mathematical operations.

Questions This Book Will Answer

This book will answer questions such as these:

- What innate number capabilities are we born with?
- How much number manipulation and basic arithmetic can young children learn without direct instruction?
- Why are native speakers of Asian languages able to learn counting sooner and faster than are English-speaking children?
- What kind of number word system could help English-speaking children learn to count easier and faster?
- Why is learning mathematics so difficult for so many students?
- For everyday classroom practice, what are the implications of the current research on how we learn to calculate?
- How do the mind-sets of teachers and students affect mathematics instruction and learning?
- How does the brain manage to deal with abstract mathematics concepts?
- How is the omnipresent technology affecting students' attention and memory systems?
- What strategies are effective in teaching students with reading difficulties to learn mathematics?
- How can we tell if a student's difficulties in mathematics are the result of environmental factors or developmental deficits?
- What strategies should teachers of mathematics consider when planning lessons?
- What have brain-imaging studies revealed about the nature of dyscalculia?
- How can elementary and secondary school teachers successfully detect mathematics difficulties?
- What instructional strategies work best with students who have difficulties in mathematics?
- What instructional strategies are successful with English language learners who are having difficulties in mathematics?
- How can teachers use research on how the brain learns mathematics to design an instructional model for teaching preK–12 mathematics?
- How can integrating the arts into mathematics lessons improve instruction and learning?

Chapter Contents

Chapter 1—Developing Number Sense. Children's ability to determine quantities begins soon after birth. This chapter examines the components of this innate number sense and how it leads to counting and basic arithmetic operations. It looks at the regions of the brain that work together and manipulate numbers and the ways language affects how quickly and easily children learn to count.

Chapter 2—Learning to Calculate. Because counting large numbers is not a survival skill, the brain must learn mathematical concepts and procedures. This chapter explores the various developmental stages the brain must go through to understand number relationships and manipulations, such as in multiplication. It discusses why the brain views learning to multiply as an unnatural act, and it suggests some other ways to look at teaching multiplication that may be easier for students to learn.

Chapter 3—Reviewing the Elements of Learning. This chapter presents some of the recent findings from cognitive neuroscience, including discoveries about the power of feedback, memory systems and how technology affects them, the nature and value of practice and rehearsal, lesson timing, formative assessments, and the benefits of writing in mathematics classes. Gender differences and fixed and growth mindsets in mathematics, as well as learning and teaching styles, are also discussed. The chapter concludes with a section on strategies to motivate students in mathematics.

Chapter 4—Teaching Mathematics to the Preschool and Kindergarten Brain. Although young children have an innate number sense, certain instructional strategies can enhance those capabilities and prepare children to be more successful in learning arithmetic operations. This chapter suggests some of those strategies.

Chapter 5—Teaching Mathematics to the Preadolescent Brain. Here we look at the development and characteristics of the preadolescent brain and how they affect the individual's emotional and rational behavior. The chapter offers suggestions on how lesson plans can be modified, from the primary grades up through middle school, to take into account the nature of this developing brain so more of these students will be successful in learning mathematics. Also included is a discussion of how the Common Core State Standards for Mathematics were developed and what they mean for mathematics instruction.

Chapter 6—Teaching Mathematics to the Adolescent Brain. Similar to the previous chapter, we review the nature of the adolescent brain and suggest what considerations need to be made to adapt lessons to meet these students' needs. Included here are discussions of mathematical reasoning and instructional choices—such as layering the curriculum, the flipped classroom, and graphic organizers—that can be very effective strategies for making mathematics relevant and interesting to today's students.

Chapter 7—Recognizing and Addressing Mathematics Difficulties. Numerous suggestions are offered in this chapter to enable teachers to identify and help students experiencing difficulties in learning mathematics, including math anxiety. This chapter discusses the major differences between the environmental and developmental factors that contribute to mathematics difficulties. It presents some tested strategies that teachers of all grade levels can use with students who are poor in mathematics to help them understand number operations and gain a more accurate and deeper understanding of mathematical concepts. A new section discusses strategies for English language learners who are having difficulties in mathematics.

Chapter 8—Putting It All Together: Planning Lessons in PreK–12 Mathematics. How do we use in daily practice the important findings discussed in the previous chapters? This chapter suggests ways to incorporate this research into the planning of mathematics lessons and presents a four-step instructional model for teaching preK–12 mathematics. It also discusses the positive impact that integrating the arts can have on mathematics instruction and student motivation, creativity, and achievement.

Other Helpful Tools

At the end of each chapter, you will find a section called **Questions and Reflections**, an organizing tool for helping you remember important ideas, strategies, and resources you may wish to consider later. The information here would be useful in professional development and book study activities that are using this book as a guide.

I have included some information on the history of mathematics that I thought might be interesting and attach a human aspect to this topic. As in all my books, I have referred to the original scientific research and listed those citations whenever possible.

Look for the ✓. Most of the chapters contain suggestions for translating the research on learning mathematics into instructional practice. These suggestions are indicated with a checkmark (✓). Any time you see this symbol it means: "Here is a strategy to consider!"

At the back of the book is an extensive listing of Internet **Resources** that offer a wide range of activities for teachers and students at all grade levels.

This is not a book of activities in mathematics. Rather, this book is designed to help teachers decide which books, resources, and activities are likely to be effective in light of current research on how the brain learns mathematics.

Who Should Use This Book?

Classroom teachers who teach mathematics at any grade level will find this book useful because it presents a research-based rationale for why and when certain instructional strategies should be considered. It focuses on the brain as the organ of thinking and learning, and it assumes that the more teachers know about how the brain learns mathematics, the greater the number of instructional options will be available to them. Increasing the options increases the likelihood that successful learning will occur.

Professional developers continually need to update their own knowledge bases to include research and research-based strategies and support systems as part of their repertoires. Professional developers will find suggestions throughout the book that should help them design and implement meaningful coaching in mathematics instruction.

Principals and head teachers should find here a substantial source of topics for discussion in faculty and department meetings. In raising these topics, they will support the attitude that professional growth for teaching mathematics is an ongoing school responsibility and not an occasional event. More important, being familiar with what brain research says about learning mathematics enhances the principal's credibility as the school's instructional leader and promotes the notion that the school is a learning organization for all its occupants.

College and university instructors should also find merit in the research and applications presented here, both as suggestions to improve their own teaching and as information to pass on to prospective elementary and mathematics teachers.

Parents will also find some of the information in this book useful, especially since parents are, after all, their children's first teachers.

Indeed, the ideas in this book provide the research support for a variety of initiatives, such as cooperative learning groups, differentiated instruction, integrated thematic units, and the interdisciplinary approach to curriculum. This book is *not* meant to be a sourcebook for preK–12 mathematics activities. Rather, it is meant to suggest instructional approaches that are compatible with what cognitive neuroscience is telling us about how the brain deals with numbers and mathematical relationships. Of course, some suggested activities represent my view of how these research findings can be translated into effective classroom practice, but these are meant to suggest the *type* of activity rather than to be the definitive activity. There are hundreds of books and computer programs on the market, as well as Internet resources, loaded with mathematics activities, games, and worksheets. This book is designed to help the teacher decide which of those books and activities are likely to be effective in light of current research.

The information presented here was current at the time of publication. However, as scientists continue to explore the inner workings of the brain, they will likely discover more about the cerebral mechanisms involved in learning mathematics. These discoveries should help parents and educators understand more about the nature of mathematics, mathematics difficulties, and effective mathematics instruction. Stay tuned!

ASSESSING YOUR CURRENT KNOWLEDGE OF HOW WE LEARN MATHEMATICS

The value of this book can be measured in part by how much it enhances your knowledge of how humans learn mathematics. This might be a good time for you to take the following true/false test and assess your current understanding of some concepts related to mathematics and mathematics instruction. Decide whether the statements are *generally* true or false, and circle T or F. Explanations for the answers are found throughout the book in special boxes.

1. T F Children do not understand that number words are different from those that describe the size, shape, or color of objects.
2. T F The brain's ability to detect patterns and make associations is often referred to as *dissociative memory*.
3. T F Teachers should assume that students who have difficulty with language processing will definitely encounter difficulties in arithmetic computation.

4. T F Working memory can deal with unlimited items for an unlimited amount of time.

5. T F Taking notes on a laptop will allow for greater learning and a better review of that learning at a later date.

6. T F A young child's social and emotional functioning will have no impact on the development of mathematical competence.

7. T F Emotional attention comes after cognitive recognition.

8. T F Using technology for routine calculations leads to greater understanding and achievement in mathematics.

9. T F Students without cognitive deficits do not display difficulties with arithmetic and mathematical operations.

10. T F Mathematics and the arts are not related.

WHAT'S COMING?

What innate number capabilities are we born with? Are schools taking advantage of these capabilities when teaching arithmetic operations? How does our native language affect our ability to learn to count? The answers to these intriguing questions are found in the next chapter.

1

Developing Number Sense

Wherever there is a number, there is beauty.

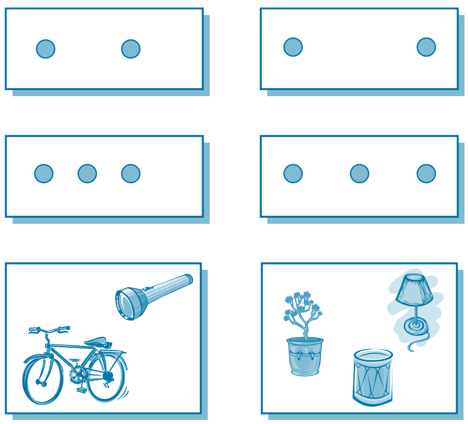
—Proclus (AD 410–485)

BABIES CAN COUNT

In 1980, Prentice Starkey persuaded 72 mothers to bring their young babies to his laboratory at the University of Pennsylvania for a novel experiment (Starkey & Cooper, 1980). While seated on his or her mother's lap, each baby, aged between 16 and 30 weeks, observed slides projected onto a screen. The slides contained two or three large black dots spread out horizontally. Starkey varied the spacing between the dots so that neither the total length of the line nor the density of the dots could be used to discriminate their number. After numerous trials, Starkey noticed that the average fixation time of 1.9 seconds for a two-dot slide jumped to an average of 2.5 seconds (a 32 percent increase) for a three-dot slide. Thus, the babies detected the change from two to three dots.

In a follow-up experiment, Strauss and Curtis (1981) at the University of Pittsburgh repeated this format but used colored photographs of common objects instead of dots. The objects varied in size and alignment so that the only constant was their number. The babies continued to notice the difference between slides of two and three objects (Figure 1.1). Similar experiments with infants have been conducted by various researchers (Berger, Tzur, & Posner, 2006; Brez, Colombo, & Cohen, 2012; vanMarle, 2013). They all yield the same finding: In the first few months of life, babies notice the constancy of objects and detect differences in their numerical quantities. Babies, of course, do not have a sophisticated concept of counting, but they

Figure 1.1 Researchers used slides similar to these to prove that infants can discriminate between the numerosities of 2 and 3. The slides with dots are similar to those used by Starkey and Cooper (1980), and the slides with objects are representative of the experiments conducted by Strauss and Curtis (1981).



do have a conception of quantity, or what scientists call *numerosity.*

Is numerosity innate, or is it something the babies were able to learn in their first few months? Newborns can distinguish two objects from three, and perhaps three from four. Their ears notice the difference between two sounds and three. It seems unlikely that newborns could gather enough information from the environment to learn the numbers 1, 2, and 3 in just the few months after birth. Thus, this ability seems to have a strong genetic component.

More support for the notion that numerosity is prewired in the brain comes from case studies of patients who have lost or never had a sense of numbers. Butterworth (1999), for example, describes a patient who had a stroke that left her language and reasoning abilities intact but destroyed her ability to estimate or determine the number of objects in any collection. After another patient had an operation to remove a tumor from the left side of her brain, numbers had no meaning for her. Once again, this patient's language ability and general intelligence were unaffected, but she could not even be taught finger addition. The multiplication tables were just a nonsense poem to her. Butterworth also describes a man who apparently never had number sense, although he earned a college degree in psychology. He had to use his fingers for simple arithmetic and resorted to a calculator for other computations, but the answer had no meaning for him. He was unable to tell the larger of two numbers or quickly count just three items in a collection.

Dehaene (1997) examined how one's sense of numbers can be disrupted after a stroke. One patient counted about half the items in a collection and then stopped counting because she thought she had counted them all. Another patient would count the same items over and over again, insisting there were 12 items when there were only 4. Here, too, language ability and general intelligence were not affected. These are just a few examples from a large collection of case studies that lead to one conclusion: We are born with a built-in number sense!

What Is Number Sense?

Tobias Danzig (1967) introduced the term *number sense* in 1954, describing it as a person's ability to recognize that something has changed in a small collection when, without that person's knowledge, an object has been added or removed from the collection. We have number sense because numbers have *meaning* for us, just like words and music. And as in the case of learning words, we were born with number sense or, at the very least, the ability to acquire it at a very young age without effort. Mathematician Keith Devlin (2000, 2010) refined the definition by suggesting that number sense consists of two important components: the ability to compare the sizes of two collections shown simultaneously and the ability to remember numbers of objects presented successively in time.

That we are born with number sense does not necessarily mean we all can become great mathematicians, but it does mean that most of us have the potential to be a lot better at arithmetic and mathematics than we think. If this is true, then why do so many students and adults say they "can't do math"? We will answer this fascinating question later.

Because we are born with number sense, most of us have the potential to be a lot better at arithmetic and mathematics than we think.

Animals Also Have Number Sense

The discovery that infants have number sense came as no surprise to researchers who work with animals. For more than 50 years, experiments have shown that birds (Koehler, 1951; Roberts, 2010), rats (Mechner & Guevrekian, 1962), lions (McComb, Packer, & Pusey, 1994), and chimpanzees (Beran, McIntyre, Garland, & Evans, 2013; Woodruff & Premack, 1981) possess both of the number sense abilities described by Devlin.

Of course, different species of animals exhibit their number sense at varying levels of sophistication. Many birds, for example, display a sense of numerosity in the number of times they repeat a particular note in their song. Even members of the same species will learn the number of repetitions common to their location, which may be six repetitions in one woodland area and seven in another.

Rats and lions seem to have the ability to *estimate* and *compare* numbers. The ability of animals to compare the numbers of objects in different collections has an obvious survival advantage. It would help a group of animals know whether to defend their territory if the defenders outnumber the attackers, or to retreat if there are more attackers. Note that I am referring to the animals as "estimating" and "comparing" numbers, not "counting." No one believes that animals actually count by number, as in 1–2–3 or 11–12–13. Rather, most researchers accept that many animals recognize the difference between one and two objects; after that, it is probably just "more than two."

Rhesus monkeys display the ability to tally across different senses—that is, match a sequence of shapes or sounds (Jordan, MacLean, & Brannon, 2008)—and they can choose the lesser of two sets of objects, even when their shape, size, and color are changed (Cantlon & Brannon, 2007). Curiously, in the latter experiment, the monkeys were only slightly less accurate than college students performing the same task, but the monkeys reacted faster. Why was that? The researchers suggested that the monkeys did not mind making an occasional mistake and moved on to the next task. The college students, however, hesitated and worried over guessing wrong. What about chimpanzees, our closest relatives on the evolutionary tree? Experiments show that chimpanzees can do basic arithmetic. For instance, in a classic experiment using chocolate bars, chimpanzees recognized that 3 bars + 4 bars = 7 bars and 5 bars + 1 bar = 6 bars. They also recognized that 6 is less than 7. Again, the chimpanzees were not actually counting to 6 or 7 but most likely were comparing visual scans to recognize that one sum was *greater* than the other (Woodruff & Premack, 1981).

The numerical estimating ability shown in rats and chimpanzees resembles the innate number sense of human infants. Animals can count in that they can increase an internal counter each time an external stimulus occurs—for example, a rat pressing a lever to get food. But their representation of numbers is fuzzy. Humans can do much more. After just a few months of age, toddlers discover numbers and number words in a precise sequence, and they quickly begin to extend their innate ability to the point where they can eventually measure exact quantities, even into the billions.

Why Do We Have Number Sense?

Number sense became an innate ability in humans and other animals most likely because it contributed to their survival. Animals in the wild must constantly assess dangers and opportunities in their environment. To do so, they need cerebral systems that can rapidly compute the magnitude of any challenge. As primitive humans went searching for food, they also had to determine quickly whether the number of animals they spotted represented an opportunity or a danger, whether they were moving too fast, were too big to capture, or were just too far away. A mistake in these calculations could be fatal. Consequently, individuals who were good at determining these magnitudes survived and contributed to strengthening their species' genetic capabilities in number sense.

Humans and other animals developed an innate number sense because it contributed to their survival.

LEARNING TO COUNT

Although infants are born with the same rudimentary number sense observed in rats and chimpanzees, they possess two arithmetic capabilities that quickly separate them from other animals: One is the ability to count; the other is to use and manipulate symbols that represent numeric quantities.

Recognizing the number of objects in a small collection is part of innate number sense. It requires no counting, because the numerosity is identified in an instant. Researchers call this process *subitizing* (from the Latin for "sudden"). But when the number in a collection exceeds the limits of subitizing, counting becomes necessary.

Subitizing

Our innate visual processing system allows us to comprehend the numerosity of a collection. It works instantly and accurately to quantify groups of four or fewer objects without actually counting them. But subsidization loses accuracy as the number in the collection increases. With more objects, the process slows down as we abandon subitizing and resort to counting or estimation based on visual patterns we discern in the collection. Why is that? It is likely that subitizing is a primitive cerebral process while counting involves more sophisticated operations.

Indeed, recent studies using PET (positron emission tomography) scans seem to indicate just that. When participants in the studies were subitizing one to four items, areas in the visual cortex were activated and areas involving attention were quiet. While participants were counting five to eight items, however, numerous brain networks were recruited, including those involved in visual attention in the top area of the brain and cognitive processing in the front regions of the brain. These results suggest that subitizing is a low-key subconscious (neuroscientists call it *pre-attentive*) operation, while counting provokes significant cerebral activity (Piazza, Mechelli, Butterworth, & Price, 2002; Sathian et al., 1999; Vuokko, Niemivirta, & Helenius, 2013).

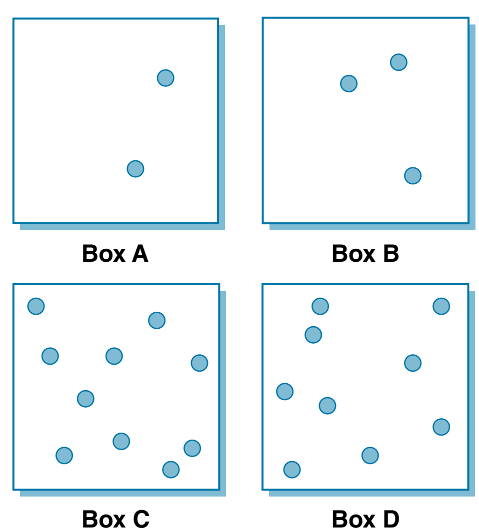
Figure 1.2 shows the difference between subitizing and counting. Look at Boxes A and B. The eyes can immediately detect the difference between two and three items in these boxes without counting. How many dots are there in Box C? And how many in Box D? Chances are you had to resort to counting to determine the number of dots in each box. This, along with research studies (e.g., Kroesbergen, Van Luit, Van Lieshout, Van Loosbroek, & Van de Rijt, 2009), suggests that subitizing may well be the developmental prerequisite skill for learning to count. If that is the case, then we should examine subitizing more closely and determine if reinforcing this skill in children will help them learn counting more easily.

Types of Subitizing

Clements (1999) describes two types of subitizing: perceptual and conceptual. *Perceptual subitizing* involves recognizing a number without using other mathematical processes, just as you did when looking at Boxes A and B in Figure 1.2. This innate cerebral mechanism is very likely the same used by animals and accounts for some of the surprising capabilities of infants described earlier in this chapter. Perceptual subitizing also helps children separate collections of objects into single units and connect each unit with only one number word, thus developing the process of counting.

Conceptual subitizing allows one to know the number of a collection by recognizing a familiar pattern, such as the spatial arrangement of dots on the faces of dice or on domino tiles. Other patterns may be kinesthetic, such as using finger patterns to figure out addition problems, or rhythmic patterns, such as gesturing out one "beat" with each count. Creating and using conceptual subitizing patterns help young children develop the abstract number and arithmetic strategies they will need to master counting (Clements, 1999; Steffe & Cobb, 1988). Those children who cannot conceptually subitize are likely to have problems learning basic arithmetic processes. Can this innate ability of subitizing be strengthened through practice? The answer is yes. You will find suggestions for how to teach subitizing in Chapter 4.

Figure 1.2 We can easily perceive the difference between two and three items through subitizing. But as the number of items increases, we resort to counting to arrive at an accurate total.



Creating and using conceptual subitizing patterns help young children develop the abstract number and arithmetic strategies they will need to master counting.

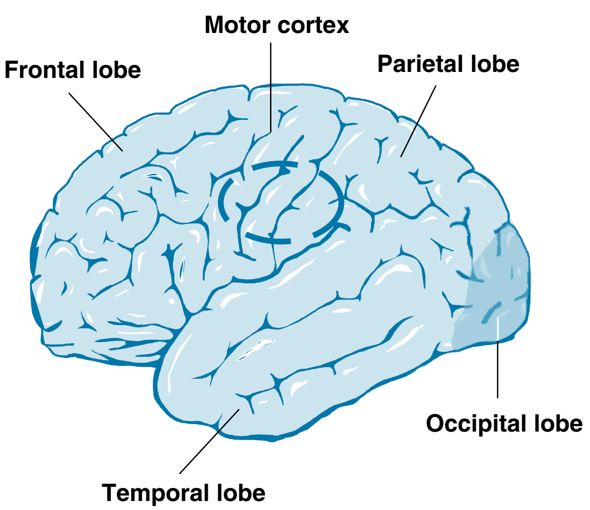
Counting

Origins of Counting

No one knows when and how humans first developed the idea of counting beyond the innate sequence of "one, two, and many." Perhaps they began the way young children do today: using their fingers. (This system is so reliable that many adults also do arithmetic with their fingers.) Our base-10 number system suggests that counting began as finger enumeration. The Latin word *digit* is used to mean both *numeral* and *finger.* Even evidence from brain scans lends further support to this number-to-finger connection.

When a person is performing basic arithmetic, the greatest brain activity is in the left parietal lobe and in the region of the motor cortex that controls the fingers (Dehaene, Molko, Cohen, & Wilson, 2004). Figure 1.3 shows the four major lobes of the brain and the motor cortex. The area within the dotted oval is highly activated when a person is doing arithmetic. This area includes both a part of the parietal lobe and the section of the motor cortex that controls finger movement.

Figure 1.3 This view of the left side of the brain shows the four lobes and the motor cortex. The area within the oval is highly activated when a person is doing arithmetic. This area includes part of the parietal lobe and the section of the motor cortex that controls the fingers.



This raises an interesting question: Is it just a coincidence that the region of the brain we use for counting includes the same portion that controls our fingers? Or is it possible that counting began with our fingers and the brain later learned to do counting without manipulating them? Some researchers speculate that if our human ancestors' first experience with numbers involved using their fingers, then the region of the brain that controls the fingers would be the area where more abstract mental arithmetic would be located in their descendants (Devlin, 2000).

Assuming fingers were our first counting tools, we obviously ran into a problem when counting collections of more than 10 objects. Some cultures resorted to using other body parts to increase the total. Even today, the natives of the Torres Straits Islands in New Guinea denote numbers up to 33 by pointing to different parts of their bodies, including fingers, arms, shoulders, chest, legs, and toes. Naming the body part evokes the corresponding number. Thus, the word *six* is literally "wrist," and *nine* is "left breast." They use sticks for numbers larger than 33 (Ifrah, 1985). But this process is hopeless for numbers beyond 30 or so. Eventually, some cultures used a physical tally system, such as making notches on a bone or stick. Notched bones that date back about 40,000 years have been discovered. According to the fossil record, this is about the same time that humans started to use symbolic representations in rock carvings and cave paintings (Devlin, 2000).

Finger counting and physical tallies show that these cultures understood the concept of numerosity, but that does not imply they understood the abstract concept of *number.* Archeologists, such as Denise Schmandt-Besserat (1985), suspect that *abstract* counting numbers, as opposed to markings, appeared around 8000 BC and were used by the highly advanced Sumerian society that flourished in the Fertile Crescent of what is now Iraq and Syria. They used tokens of different shapes to represent the specific quantity of a trade item, such as a jar of oil or loaf of bread. They used symbolic markings on clay tablets to keep running totals of items in commerce. It was not really a separate number system, but it was the first use of a symbol system that set the stage for the functional, abstract numbers we use today.

Our present numbering system was developed more than 2,000 years ago by the Hindus and attained its present form in about the 6th century. In the 7th century, it was introduced to Europe by Persian mathematicians and thus became known as the "Arabic system." This ingenious invention is now accepted worldwide for several reasons:

- Each number has its own word, and the number words can be read aloud. Saying a number, such as 1,776 *(one thousand, seven hundred and seventy-six),* clearly reveals the numeric structure of units—tens, hundreds, and thousands.
- The numerical system is not just symbols but also a *language,* thereby allowing humans to use their innate language fluency to handle numbers.
- It is concise and easily learned.
- We can use it to represent numbers of unlimited magnitude and apply them to measurements and collections of all types.
- It reduces computation with numbers to the routine manipulation of symbols on a page.

In fairness, I should mention that the original idea of denoting numbers by stringing together a small collection of basic symbols to form number words came from the Babylonians around 2000 BC. But the system was cumbersome to use because it was built on the base 60; thus, it did not gain wide acceptance. Nonetheless, we still use it in our measurements of time (60 seconds make 1 minute, etc.) and geography (60 seconds make 1 degree of latitude and longitude).

Beginning to Count

Wynn (1990) was among the first researchers to examine how young children conceptualize the how and why of counting. She discovered that by the age of 30 months, most children have seen someone counting on numerous occasions. They also demonstrate the ability to count different types of sounds on a videotape. So, quite early on and without explicit teaching, they understand that counting is an abstract procedure that applies to all kinds of visual and auditory objects.

By the age of 3, most children recognize that there are separate words to describe the quantity of something; that is, they answer the question of "how many." Children also know that number words are different from those that describe the size, shape, or color of objects and that they hold a specific place in the sequence of describing words. They learn to say "three big dogs" but never "big three dogs." At this stage, they know that "three" is a number, but they may not know the precise value it represents. That will come to them later with experience and practice.

For the young mind, counting is a complex process that uses a one-to-one principle. It involves saying number words in the correct sequence while systematically assigning a number word to each object being counted. Eventually, children recognize that the last number in the counting sequence tells them the total number of objects in the collection, a concept known as the *cardinal principle.* Students who do not attain the cardinal principle will be delayed in their ability to add and subtract with meaning. As a result, these students always recount each item when adding. They recognize addition as an increase in number but do not start from the last number counted. In Chapter 4, you will find some suggestions on how to help children learn to count.

Answer to Question 1. False: *Children know that number words are different from words describing the size, shape, or color of objects.*

How Language Affects Counting

Cultural Variations in Working-Memory Capacity

Every time the results of international test scores in mathematics are released, the performance of children from the United States is usually dismal compared with children from other nations, particularly Asia. Differences in classroom instruction and curriculum may be partly to blame, but cultural differences in computational ability may have their roots in the words that different cultures use to represent numbers.

Read the following list of numbers aloud: 7, 5, 9, 11, 8, 3, 7, 2. Cover the list and take about 20 seconds to try to memorize it. Now recite the numbers again without looking at the list. Did you get them all correct? If your native language is English, you might have gotten only about four or five numbers in the correct order. But if you are Chinese, you may have gotten all of them correct. Why is that? When you try to remember a list of numbers by saying them aloud, you are using a verbal memory loop, a part of immediate memory that can hold information for only about 2 seconds. This forces you to rehearse the words to refresh them in the loop. As a result, your memory span is limited by how many number words you can say in less than 2 seconds. That time span is too short for most people to say aloud the 12 syllables contained in the eight numbers you were trying to remember. Of course, if you can recite faster, you will remember more.

Chinese numbers are very brief. Most of them can be recited in less than one fourth of a second. Pronouncing their English equivalents takes about one third of a second. This difference might seem trivial to you, but it is significant to researchers. Studies of languages as diverse as English, Hebrew, Arabic, Chinese, and Welsh show a correlation between the time required to pronounce numbers in a given language and the memory span of numbers in its speakers. People in Hong Kong, where the Cantonese dialect of Chinese is spoken, have a number memory span of about 10 digits, as opposed to 7 digits for speakers of English and other Western languages.

One factor contributing to this difference is the finding from brain-imaging studies that native Chinese speakers process arithmetic manipulation in areas of the brain different from those of native English speakers. Researchers speculate that the biological encoding of numbers may differ in the two cultures because their languages are written so differently, resulting in vastly dissimilar visual reading experiences (Tang et al., 2006). Other characteristics of the Chinese language and culture allow children to acquire and practice concepts relating to numbers more easily and logically than in other languages. For example, the days of the week and months are named by their number—Weekday Number 1, Weekday Number 2, Month Number 1, Month Number 2, and so on. (The names of the weekdays and months in English, on the other hand, are derived mainly from the names of ancient Roman gods.) The consistencies in numerical patterns in different aspects of Chinese culture facilitate the child's learning of numerical concepts.

Surprisingly, the magical number of seven items, long considered the fixed span of working memory, is just the standard span for a special population of humans—namely, Western adults on whom about 90 percent of psychological studies have been focused. No doubt, there is a biological limit to the capacity of working memory, but that limit also appears to be affected by culture and training. The cultural variations in memory span suggest that Asian numerical notations, such as in Chinese and Japanese, are more easily memorized than our Western notations because they are more compact (Miller, Smith, Zhu, & Zhang, 1995).

There are some tricks that adults can use to increase digit memory span. These tricks can also be taught to young students at the appropriate age.

- ✓ Memorize numbers by saying them aloud and using the shortest words possible. The number 76,391 is easier to remember as "seven-six-three-nine-one" (6 syllables) rather than "seventy-six thousand, three hundred and ninety-one" (13 syllables).
- ✓ Chunking numbers into groups is another useful strategy. Ten-digit telephone numbers are easier to remember when they are divided into the three-digit area code, followed by two groups of three and four digits.
- ✓ Look for ways to tie parts of the number you are memorizing to other numbers that are familiar to you, such as your area code, postal zip code, address, or Social Security number.

English Words Make Learning Arithmetic Harder

Although the base-10 system has taken over most languages, how we say numbers in different languages runs the gamut from simple to complex. English has many inconsistencies in its number words. Ten has three forms: *ten, -teen,* and *-ty. Eleven* and *twelve* fit no pattern, and the ones are stated before the tens in the numbers 13 through 19. Chinese and Japanese hold the prize for simplicity. Not only is their number syntax easy to learn and remember, but it perfectly reflects the decimal structure. English syntax does not. As a result, Asian children learn to count earlier and higher than their American and Western peers and can do simple addition and subtraction sooner as well. By age 4, Chinese children

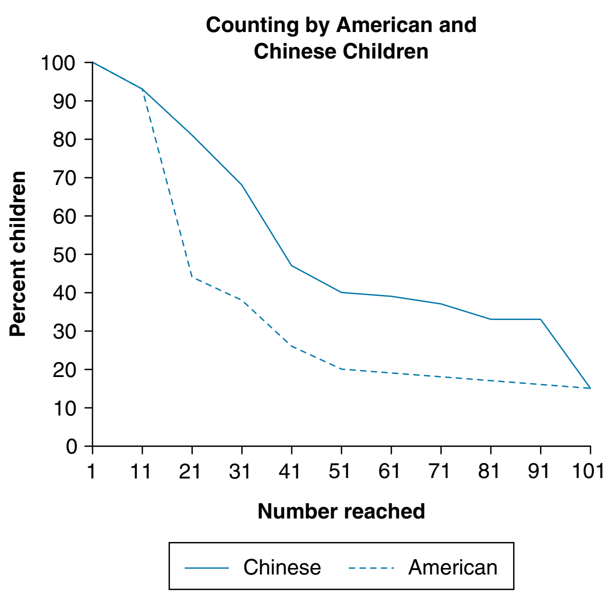
Because of language differences, Asian children learn to count earlier and higher than their Western peers.

can generally count up to 40, while American children of the same age can barely get to 15, and it takes them another year to get to 40.

How do we know the difference is due to language? Because children in the two countries show no age differences in their ability to count from 1 to 12. Take a look at Figure 1.4. The curves represent the percentage of children who could correctly count up to a certain number. Note the marked separation of the counting curves just past the number 12. Differences appear only when English-speaking children encounter the special rules for forming number words.

Here's why: In Chinese, for example, the nine short names for the numbers 1 through 9 respectively are *yi, èr, san, si, wu, liù, qi, ba,* and *jiu*. The four multipliers are 10 *(shi),* 100 *(bai),* 1,000 *(qian),* and 10,000 *(wàn).* Composing a number past 10 is simple: 11 is *ten one (shi yi),* 12 is *ten two (shi èr),* 13 is *ten three (shi san),* and so on up to 20, which is *two ten (èr shi).* This logical system continues: 21 is *two ten one (èr shi yi),* 22 is *two ten two (èr shi èr),* 30 is *three ten (san shi),* and 40 is *four ten (si shi).* For numbers past 12, Chinese children just keep applying the simple rules that worked for 1 through 12. (Japanese has an almost identical counting system.) Chinese needs only 11 words to count from 1 to 100, but English requires 28.

Figure 1.4 The chart shows the percentage of American and Chinese children reciting numbers as far as they could. More Chinese children could count much further than their American peers. (Adapted from Miller et al., 1995, with permission of the publisher and author)



American children often try to apply logical number rules but find that after correctly reciting *twenty-eight* and *twenty-nine,* they have made a mistake when they continue with words like *twenty-ten* and *twenty-eleven.* These types of grammatical errors in number syntax are almost nonexistent in Asian countries.

The number word differences affect the experiences that Asian and American children will have with arithmetic in their early school years. Because the system of spoken Chinese numerals directly parallels the structure of written Arabic numerals, Chinese children have much less difficulty than their American peers in learning the principles of place value notation in base 10. For instance, when asked to form the number 25 with unit cubes and bars of 10, Chinese children readily select two bars of 10 and five units. American children, however, laboriously count out 25 units, and fail to take advantage of the shortcut provided by groups of 10. If given a bar of 20 units, they use it more frequently than two bars of 10. This indicates that they seem to give attention to the surface meaning of 25, while Chinese children are exhibiting a deeper understanding of the base-10 structure (Dehaene, 1997; Ng & Rao, 2010).

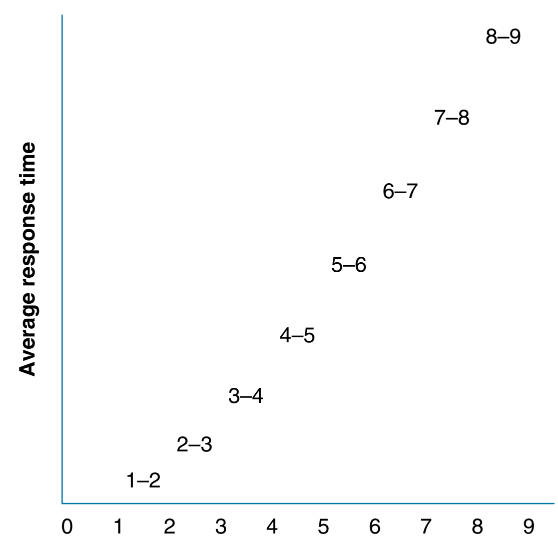
I should also note that the French and German languages have their own peculiarities. For instance, 70 in French is *soixante-dix* (*sixty-ten*), and 97 is an awkward *quatre-vingt-dix-sept* (*four-twenty-ten-seven*). German has its unique reversal of decades and units in its number words. The number 542 is said as *funf hundert zwei und vierzig* (*five hundred two and forty*).

To summarize, the Western language systems for saying numbers pose more problems for children learning to count than do Asian languages. The Western systems are harder to keep in temporary memory, make the acquisition of counting and the conception of base 10 more difficult, and slow down calculation. Unfortunately, no one realistically expects that Western counting systems will be modified to resemble the Asian model. But educators should at least be aware of these significant language problems, especially when comparing the test results in mathematics of Asian and English-speaking elementary students.

The Mental Number Line

During the past 40 years or so, numerous experimenters have made some intriguing discoveries when they have asked people to compare numbers. One of the earliest experiments consisted of measuring the time it took for adults to decide which was the larger of two Arabic digits. When two digits were far apart in value, such as 2 and 9, the adults responded quickly and almost without error. But when the digits were closer in value, such as 5 and 6, the response time increased significantly and the error rate rose dramatically. Furthermore, responses for an equal distance between numbers slowed down as the number pairs became increasingly larger. That is, the response time was greater when comparing digits 3 and 4 than for digits 2 and 3, and greater still for digits 8 and 9 (Moyer & Landauer, 1967). Subsequent experiments have consistently yielded similar results (see Figure 1.5).

Figure 1.5 This qualitative chart illustrates how the average response time increases for adults deciding which is the larger of the two digits in each pair as the value of the digits increases.



Another experiment measured the time it took adults to decide whether a two-digit numeral was larger or smaller than 65. Once again, the response time grew longer as the numerals got closer in value to 65 (e.g., "Is 71 larger or smaller than 65?") and, conversely, became progressively shorter as the value of the numerals became more distant from 65 (e.g., "Is 43 larger or smaller than 65?") (Dehaene, Dupoux, & Mehler, 1990). Similar results were found in other studies in which response times were measured for number comparisons among kindergartners (Temple & Posner, 1998), second graders (Nuerk, Kaufmann, Zoppoth, & Willmes, 2004), and adults (Brannon, 2003).

These experiments led to two conclusions:

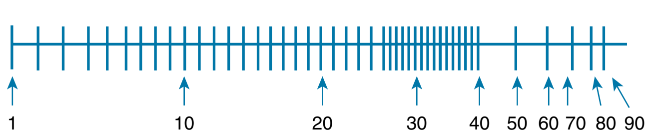
- The speed with which we compare two numbers depends not just on the distance between them but on their size as well. It takes far longer to decide that 9 is larger than 8 than to decide that 2 is larger than 1. For numbers of equal distance apart, larger numbers are more difficult to compare than smaller ones.
- It takes much longer to decide on the larger of two numbers that are a small distance apart than to decide on the larger of two numbers that are a greater distance apart. It is easier to recognize that 74 is larger than 37 than to decide that 74 is larger than 73.

What can explain these findings? Researchers suggest that the brain comprehends each numeral and transforms it quickly into an internal quantity, ignoring the digit symbols representing that quantity. How easily the brain distinguishes two numbers depends not so much on their absolute numerical distance as on their distance relative to their size. In other words, it appears that humans possess a mental number line, where we envision numbers as points on a line, with 1 on the left, 2 to its right, then 3, and so on. When we have to decide which of two numbers is larger, we mentally view them on our internal line and determine which one is on the right.

The mental number line is similar to the standard one we learn in elementary school but with one important difference. On our mental number line, the numbers are not spaced out evenly as they are on the standard number line. Instead, the farther we go along the mental number line, the closer together the numbers appear to be. This explains the results of the number-comparison experiments described earlier. The increasing compression of numbers makes it more difficult to distinguish the larger of a pair of numbers as their values grow. We can decide which is the larger of 6 and 5 much faster than for the pair 65 and 64. Although both pairs have the same numerical difference of 1, the larger pair appear closer together on our mental number line than do the smaller pair. As a result, the speed and accuracy with which we carry out calculations decrease as the numbers get larger. Figure 1.6 illustrates this phenomenon. (Incidentally, experiments with people whose native language is read from right to left, such as Arabic and Hebrew, possess mental number lines that also run from right to left. Apparently, our mental number line generally runs in the same direction as our reading.)

Why are these findings important? The internal number line offers us a limited degree of intuition about numbers. It deals with only positive integers and their quantitative relationship to each other (there were no negative numbers in our ancestral environment). This probably explains why we have no intuition regarding other numbers that modern mathematicians use, such as negative integers, fractions, and irrational numbers. Yet all these entities posed significant challenges to the mathematicians of the past and still present great difficulties to

Figure 1.6 This illustration of the mental number line shows why the brain can decide that 10 is larger than 1 faster than it can decide that 80 is larger than 70.



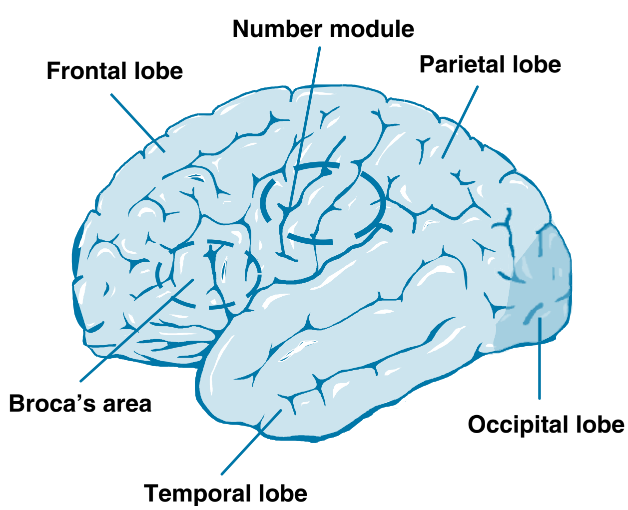
the students of today. They remain difficult for the average person because they do not correspond to any natural category in our brain. Small positive integers make such sense to our innate sense of numerosity that even 4-year-olds can comprehend them. But the other entities make no such natural connection. To understand them, we have to construct mental models that provide understanding. Teachers do this when they discuss these topics. For example, when introducing negative numbers, teachers resort to metaphors such as money borrowed from a bank, temperatures below zero, or simply an extension of the number line to the left of zero.

Number Symbols Are Different From Number Words

One fascinating discovery about numerical symbols and number words is that the brain processes them in different locations. Brain-imaging experiments and clinical case studies have convinced researchers that number symbols are hardwired in our intuitive number module in the left parietal lobe. Ordinary-language number words, however, are stored in Broca's area, located in the left frontal lobe (Figure 1.7). Broca's area is where our language vocabulary is processed. Of course, these two regions communicate with each other during numerical operations (Lachmair, Dudschig, de la Vega, & Kaup, 2014).

Clinical studies describe people who are unable to read words due to damage in Broca's area but can read aloud single or multidigit numbers presented to them using numerals. Other patients with severe language impairments can hardly read or write but do just fine on a standard arithmetic test if the questions are presented in a purely numerical form (Butterworth, 1999).

Figure 1.7 Broca's area in the left frontal lobe processes our language vocabulary, including number words. Number symbols, however, are hardwired in the number module located in the left parietal lobe.



Our number system may indeed be a language, but it is a very special one that is handled in a different region of the brain from normal language. Devlin (2000, 2010) suggests that this separation of number symbols from number words is just what would be expected if our number symbols were derived from the use of our fingers (a parietal lobe process) and number words from ordinary language (a frontal lobe process).

The major implication here is that the human brain comprehends numerals first as a quantity, not as words. Automatically and unconsciously, numerical symbols are converted almost instantly to an internal quantity. Moreover, the conversion includes an automatic orientation of numbers in space—small ones to the left and large ones to the right. Comprehending numbers, then, is a reflex action that is deeply rooted in our brains, resulting in an immediate attribution of meaning to numbers.

Expanded Notions of Number Sense

Mathematics educators have a much broader view of number sense than do cognitive neuroscientists. We have already noted that cognitive neuroscientists view number sense as a biologically based innate quality that is limited to simple intuitions about quantity, including the rapid and accurate perception of small numerosities (subitizing) and the ability to count, compare numerical magnitudes, and comprehend simple arithmetic operations. Dehaene (2001) is a major proponent of a single number sense—namely, the basic representation of quantity—rather than a patchwork of representations and abilities. He does suggest, however, that this core number sense becomes connected to other cognitive systems as a consequence of both cognitive development and education.

When Berch (2005) reviewed the literature in cognitive development, mathematics cognition, and mathematics education, he found that mathematics educators consider number sense to be much more complex and multifaceted in nature. They expand this concept to include skill sets that develop as a result of involvement with learning activities in mathematics. According to Berch, these abilities include the following:

- Recognizing something has changed in a small collection when, without direct knowledge, an object has been removed or added to the collection
- Elementary abilities or intuitions about numbers and arithmetic
- A mental number line on which analog representations of numerical quantities can be manipulated
- An innate capacity to process approximate numerosities
- Making numerical magnitude comparisons
- Decomposing numbers naturally
- Developing useful strategies for solving complex problems
- Using the relationships among arithmetic operations to understand the base-10 number system
- Using numbers and quantitative methods to communicate, process, and interpret information
- Awareness of levels of accuracy and sensitivity for the reasonableness of calculations
- Desire to make sense of numerical situations by looking for links between new information and previously acquired knowledge
- Knowledge of the effects of operations on numbers
- Fluency and flexibility with numbers and understanding of number meanings
- Recognition of gross numerical errors
- Understanding of numbers as tools to measure things in the real world
- Inventing procedures for conducting numerical operations
- Thinking or talking in a sensible way about the general properties of a numerical problem or expression, without doing any precise computation

Portions of this more expansive view of number sense already appear

- in the Common Core State Standards in Mathematics,
- in contemporary mathematics textbooks, and

- as a distinct set of test items included in the mathematics portions of the National Assessment of Educational Progress, the Trends in International Mathematics and Science Study, and the Program for International Student Assessment.

Can We Teach Number Sense?

Those who view number sense as an intrinsic ability will argue that the elementary components are genetically programmed, have a long evolutionary history, and develop spontaneously without explicit instruction as a young human interacts with the environment. However, most of these researchers do not view number sense as a fixed or immutable entity. Rather, they suggest that the neurocognitive systems supporting these elementary numerical abilities provide just the foundational structure needed for acquiring the expanded abilities cited by mathematics educators. And they recognize that both formal and informal instruction can enhance number sense development prior to entering school.

Berch (2005) notes that the abilities and skills associated with the expanded view of number sense cannot be isolated into special textbook chapters or instructional units, and that their development does not result from a set of activities designed specifically for this purpose. He agrees with those mathematics educators who contend that number sense constitutes a way of thinking that should permeate all aspects of mathematics teaching and learning. It may be more beneficial to view number sense as a by-product of other learning than as a specific goal of direct instruction.

Just as phonemic awareness is a prerequisite for learning phonics and becoming a successful reader, developing number sense is a prerequisite for succeeding in mathematics.

Gersten and Chard (1999) suggest that the innate qualities of number sense may be similar to phonemic awareness in reading development, especially for early experiences in arithmetic. Just as phonemic awareness is a prerequisite for learning phonics and becoming a successful reader, developing number sense is a prerequisite for succeeding in mathematics. They further propose that number sense is the missing component in the learning of early arithmetic facts and explain why rote drill and practice do not lead to significant improvement in mathematics ability.

Because Gersten and Chard (1999) believe that number sense is so critical to success in learning mathematics, they have identified five stepping-stones that allow teachers to assess a child's understanding of number sense:

- **Level 1.** Children have not yet developed number sense beyond their innate notions of numerosity. They have no sense of relative quantity and may not know the difference between "less than" and "more than" or "fewer" and "greater."
- **Level 2.** Children are starting to acquire number sense. They can understand terms such as "lots of," "six," and "nine," and are beginning to understand the concepts of "less than" and "more than." They also understand lesser or greater amounts but do not yet have basic computation skills.
- **Level 3.** Children fully understand "less than" and "more than." They have a concept of computation and may use their fingers or

objects to apply the "count up from one" strategy to solve problems. Errors occur when children are calculating numbers higher than 5, because this requires using the fingers of both hands.

- **Level 4.** Children are now relying on the "count up" or "counting on" process instead of the "counting all" process they used at the previous level. They understand the conceptual reality of numbers in that they do not have to count to 5 to know that 5 exists. Assuming they can count accurately, children at this level are able to solve any digit problem.
- **Level 5.** Children demonstrate retrieval strategies for solving problems. They have already automated addition facts and are acquiring basic subtraction facts.

Teaching Number Sense at All Grade Levels

Gurganus (2004) agrees that number sense is analogous to phonemic awareness. However, she takes a broader view and notes that, unlike phonemic awareness, number sense develops throughout a student's mathematics education and applies to a wide range of concepts. Here are her suggestions to teachers for promoting number sense across the grade levels:

- ✓ **Pair numbers with meaningful objects.** To help young students view numbers as values rather than labels, associate numbers with concrete objects. For example, there are two wheels on a bicycle, three wheels on a tricycle, and four wheels on a car.
- ✓ **Use language to gradually match numbers with objects and symbols.** Model using talk to create sentences about number activities so that students can use self-talk to describe these relationships. For instance, "Two blocks and three more blocks give us five blocks."
- ✓ **Incorporate counting activities.** Ask younger students to count to 10 and back. Challenge older students to count by 2s, 5s, or 10s, and even 3s, 4s, or 7s. Counting up and back builds understanding of number relationships and magnitudes. Have students challenge each other to guess a counting pattern. For example: "500, 525, 550, 575—What is my pattern?"
- ✓ **Provide experiences with number lines.** Create a large number line across the classroom floor using colored tape (Figure 1.8). Have students move from number to number to show counting, operations, or even rounding. Draw number lines using whole numbers, integers, or decimals.
- ✓ **Plan meaningful estimation experiences.** Students need to recognize that many things cannot and need not be measured precisely. Provide lots of practice with estimation. Stress that estimation is not guessing but that there should be a reasonable range for the estimation based on experience. For example, "How many students do you think ate in the cafeteria today?"
- ✓ **Measure and then make measurement estimates.** Have students use measurement tools to measure length, area, volume, mass, temperature, and other attributes of meaningful things in their environment. Young students can start with measuring the teacher's desk or distances on the classroom floor. After some practice, ask students to estimate before they measure. This builds a stronger sense of measurement units and what they represent.

Figure 1.8 Different number lines placed on the floor of the classroom with tape can help students understand number relationships.

- ✓ **Use number charts.** Charts in different arrangements (e.g., 1–100) offer many opportunities for students to explore number patterns. Cover up specific numbers on the charts and challenge students to discover the underlying relationships of difficult concepts such as factors and primes.
- ✓ **Introduce materials that involve numbers or number representations.** Ask students to examine items such as dice, dominoes, playing cards, coins, clocks, and rulers. Ask them to search for ways they can adapt these items for counting, pattern making, number operations, and number comparisons.
- ✓ **Read literature that involves numbers.** Books such as *The Mud Flat Olympics,* by James Stevenson (1994), or *Anno's Counting Book,* by Anno Mitsumasa (1977), provide a different way to take a mathematical journey.
- ✓ **Create magic number squares.** Show students how to determine the missing numbers, and have them create new squares to challenge their classmates (Figure 1.9).
- ✓ **Manipulate different representations of the same quantity.** Model moving back and forth between decimals, fractions, and percentages (e.g., 0.25 = 1/4 = 25%). Point out the same length in millimeters, centimeters, and meters (e.g., 35 mm = 3.5 cm = 0.035 m).
- ✓ **Explore very large numbers and their representations.** Students love the sound of large numbers, such as billion and trillion, but often have difficulty conceptualizing them. Use calculators to investigate the effects of squaring and other exponents. Where appropriate, express large numbers with scientific notation (e.g., 500,000 can be written as 5×10^5).
- ✓ **Collect and chart data.** At every grade level, students can collect meaningful data. Ask the students to use concrete objects whenever possible, such as counting each type of bean in a mixture or the number of marbles of each color in a collection. Also ask the students to examine the data using graphs, formulas, and other comparisons.

Figure 1.9 Number squares come in many different configurations and are enjoyable ways to learn addition. In these examples, rows, columns, and diagonals must add up to 34.

	2	3	13
5		10	
	7		12
4			

1			
	6		9
8		11	
	3	2	16

- ✓ **Compare number representations in other cultures.** Students can gain insights into number relationships by exploring how other cultures count, use symbols for numbers, and solve algorithms. Students often find these activities fascinating. They can read about gesture counting and the various symbols and systems that different cultures have used to represent numbers (see Zaslavsky, 2001).
- ✓ **Set up spreadsheets.** Commercial spreadsheets are a great tool for teaching students how to encode formulas for cells that will compute and compare values within other cells. Ask "what if" questions and manipulate values within the spreadsheet.
- ✓ **Solve problems and consider the reasonableness of the solution.** Remind students that the last step in problem solving should be to ask, "Does this answer make sense?" Have them practice selecting solutions by estimation without actually working out the problems.
- ✓ **Find everyday, functional uses of numbers.** Explore every opportunity for students to see the practical applications of mathematics. For example, they could follow their favorite sports team's averages, track a company on the stock market, look for sales at the department store, or determine distances on a road map for the school field trip. Whenever possible, ask the students to graph, compare, predict, and discuss their data and measurements.
- ✓ **Explore unusual numbers.** Older students might find adventure in special numbers with intriguing patterns. Examples are Fibonacci and the golden ratio; abundant, perfect, and weird numbers; and number patterns that form palindromes.
- ✓ **Model the enjoyment of numbers and number patterns.** Research studies repeatedly show that the teacher is the most critical factor in establishing a climate for curiosity and enjoyment of mathematics. Keep learning and searching for new ways to have fun with numbers. The Internet is a valuable resource for number games. See the **Resources** section of this book for some ideas.

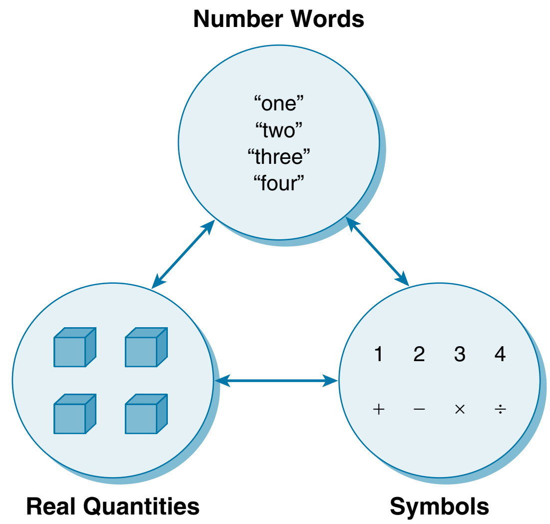
More suggestions for teaching number sense to students in the primary grades are provided in Chapter 5.

Quantities to Words to Symbols

In an effort to describe how number sense emerges, researcher Sharon Griffin (2002) created a model showing that the development of number sense goes through three major phases. First, the visual processing system recognizes objects in a collection. For small collections, the numerosity can be determined quickly and without counting through our innate capacity to subitize. As the quantity of objects in a collection grows larger, we move to the second phase and create number words to communicate to others an exact count in our native language.

The third phase emerges when we realize that writing number words for large quantities is tedious and that they do not lend themselves to mathematical manipulation. Therefore, we create numerical symbols and operational signs. At the beginning, the flow from one stage to the next is linear. But with practice, all three phases interact whenever the brain performs mathematical operations (Figure 1.10). Griffin's model has been supported by other cognitive neuroscientists using brain-imaging methods (e.g., Dehaene, Piazza, Pinel, & Cohen, 2003; Holloway, Price, & Ansari, 2010).

Figure 1.10 Researcher Sharon Griffin created a model that shows the development of number sense from recognizing real-world quantities to creating number words to describe those quantities and, finally, creating symbols and operational signs to represent and manipulate quantities. With practice, all three interact when the brain processes mathematical operations. (Adapted with permission from Griffin, 2002)



Gardner's Logical/Mathematical Intelligence

Many readers may be familiar with the theory proposed in 1983 by Howard Gardner of Harvard University that humans are born with a variety of capabilities that allow them to succeed in their environment. His idea—known as the theory of multiple intelligences—was that we possess at least seven (now up to 10) different intelligences. The original seven intelligences he proposed are musical, logical/mathematical, spatial, bodily/kinesthetic, linguistic, interpersonal, and intrapersonal. Soon after, he added naturalist, and several years later, spiritualist and emotionalist. Gardner defined intelligence as an individual's ability to use a learned skill, create products, or solve problems in a way that is valued by the society of that individual. This novel approach expanded our understanding of intelligence to include divergent thinking and interpersonal expertise. He further differentiated between the terms *intelligence* and *creativity,* and suggested that in everyday life people can display intelligent originality in any of the intelligences (Gardner, 1993).

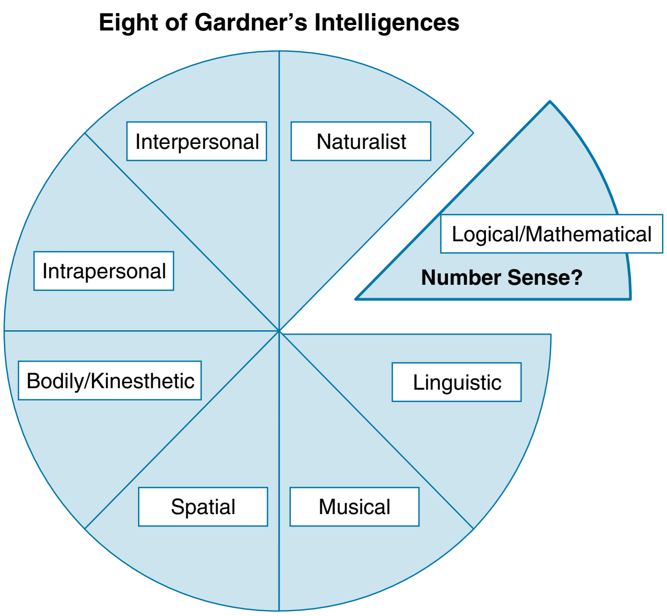
This theory suggests that at the core of each intelligence is an information-processing system unique to that intelligence. The intelligence of an athlete is different from that of a musician or physicist. Gardner also suggests that each intelligence is a continuum and semiautonomous. A person who has abilities in athletics but does poorly in music has enhanced athletic intelligence. The presence or absence of musical capabilities exists separately from the individual's athletic prowess.

Is Logical/Mathematical Intelligence the Same as Number Sense?

According to Gardner, the logical/mathematical intelligence uses numbers, sequencing, and patterns to solve problems (Figure 1.11). Thus, it deals with the ability to think logically, systematically, inductively, and to some degree deductively. It also includes the ability to recognize both geometric and numerical patterns, and to see and work with abstract concepts. Students strong in this intelligence

- can easily compute numbers mentally,
- like to be organized,
- are very precise,

Figure 1.11 This chart shows 8 of Gardner's 10 intelligences (Gardner, 1993). Is it possible that the logical/mathematical intelligence is the same as number sense?



- have a systematic approach to problem solving,
- recognize numerical and geometric patterns,
- like computer games and puzzles,
- like to explore and experiment in a logical way,
- are able to move easily from the concrete to the abstract, and
- think conceptually.

Gardner made clear that intelligence is not just how a person thinks but also includes the materials and values of the situation where and when the thinking occurs. The availability of appropriate materials and the values of any particular culture will thus have a significant impact on the degree to which specific intelligences are activated, developed, or discouraged. A person's combined intellectual capability, then, is the result of innate tendencies (the genetic contribution) and the society in which that individual develops (the environmental contribution).

Are there genes that enhance mathematical ability? Very likely. Studies of identical twins (they share the same genes) and fraternal twins (they share half of their genes) suggest how much a certain trait is inherited. Several studies over the past decades have found that identical twins usually exhibit similar levels of mathematical performance. In fraternal twins, however, one may be an excellent performer in mathematics while the other is just mediocre (Alarcón, Knopik, & DeFries, 2000).

Number sense, then, can be considered the innate beginnings of mathematical intelligence. But the extent to which it becomes an individual's major talent still rests with the type and strength of the genetic input and the environment in which the individual grows and learns.

We will discuss more about Gardner's theory and its application to classroom instruction in Chapters 3 and 7.

WHAT'S COMING?

Number sense provides students with a limited ability to subitize and determine the numerosity of small groups of objects. As the number of objects increases, the brain must resort to a more exact system of enumeration that we call counting. Simultaneously, the skills necessary to do exact addition and subtraction emerge. But as students need to manipulate larger and larger numbers, addition is no longer an efficient process. They must now learn to calculate through multiplication. Why is learning multiplication so difficult, even for adults? Are we teaching multiplication in the most effective way? Do we really even need to learn the multiplication tables? The answers to these and other interesting questions about how we learn to calculate are found in the next chapter.

Chapter 1—Developing Number Sense

QUESTIONS AND REFLECTIONS

Respond to the following questions, and jot down on this page key points, ideas, strategies, and resources you want to consider later. This sheet is your personal journal summary and will help jog your memory.

What is meant by number sense? ______________________________

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How does our native language affect our ability to learn to count?

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How does learning to count in some Asian languages differ from learning to count in English? ______________________________

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What are some ways to teach number sense at various grade levels?

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2

Learning to Calculate

Mathematics possesses not only truth, but some supreme beauty—a beauty cold and austere, like that of sculpture.

—Bertrand Russell

Counting up to small quantities comes naturally to children. Either spontaneously or by imitating their peers, they begin to solve simple arithmetic problems based on counting, with or without words. Their first excursion into calculation occurs when they add two sets by counting them both on their fingers. Gradually, they learn to add without using their fingers and, by the age of 5, demonstrate an understanding of *commutativity* of addition (the rule that $a + b$ is always equal to $b + a$). But as calculations become more difficult, errors abound, even for adults. One thing is certain: the human brain has serious problems with calculations. Nothing in its evolution prepared it for the task of memorizing dozens of multiplication facts or for carrying out the multistep operations required for two-digit subtraction. Our ability to approximate numerical quantities may be embedded in our genes, but dealing with exact symbolic calculation can be an error-prone ordeal.

DEVELOPMENT OF CONCEPTUAL STRUCTURES

Conceptual structures about numbers develop early and allow children to experiment with calculations in their preschool years. They quickly master many addition and subtraction strategies, carefully selecting those that are

Our ability to approximate numerical quantities may be embedded in our genes, but dealing with exact symbolic calculation can be an error-prone ordeal.

best suited to a particular problem. As they apply their algorithms, they mentally determine how much time it took them to make the calculation and the likelihood that the result is correct. Siegler and Jenkins (1989) studied children using these strategies and concluded that they compile detailed statistics on their success rate with each algorithm. Gradually, they revise their collection of strategies and retain those that are most appropriate for each numerical problem.

Here is a simple example: Ask a young boy to solve 9–3. You may hear him say, "Nine . . . eight is one . . . seven is two . . . six is three . . . six!" In this instance, he counts backward starting from the larger number. Now ask him to calculate 9–6. Chances are that rather than counting backward as he did in the first problem, he will find a more efficient solution. He might count the number of steps it takes to go from the smaller number to the larger: "Six . . . seven is one . . . eight is two . . . nine is three . . . three!" But how did the child know this? With practice, the child recognizes that if the number to be subtracted is not very close in value to the starting number, then it is more efficient to count backward from the larger number. Conversely, if the number to be subtracted is close in value to the starting number, then it is faster to count up from the smaller number. By spontaneously discovering and applying this strategy, the child realizes that it takes him the same number of steps (three) to calculate 9–3 and 9–6.

Exposure at home to activities involving arithmetic no doubt plays an important role in this process by offering children new algorithms and providing them with a variety of rules for choosing the best strategy. In any case, the dynamic process of creating, refining, and selecting algorithms for basic arithmetic is established in most children before they reach kindergarten.

Exactly how number structures develop in young children is not completely understood. However, in recent years, research in cognitive neuroscience has yielded sufficient clues about brain development, to the point that researchers have devised a timeline of how number structures evolve in the brain in the early years. Sharon Griffin (2002) and her colleagues reviewed the research and developed tests that assessed large groups of children between the ages of 3 and 11 in their knowledge of numbers, units of time, and money denominations. As a result of the students' performance on these tests, they made some generalizations about the development of conceptual structures related to numbers in children within this age range (see also Purpura & Lonigan, 2013). Their work is centered on several core assumptions about how the development of conceptual structures progresses. Three assumptions of particular relevance are as follows:

The dynamic process of creating, refining, and selecting algorithms for basic arithmetic is established in most children before they reach kindergarten.

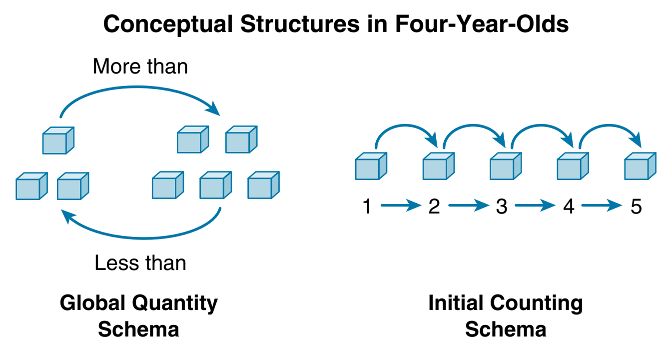
1. Major reorganization in children's thinking occurs around the age of 5, when cognitive structures that were created in earlier years are integrated into a hierarchy.

2. Important changes in cognitive structures occur about every 2 years during the development period. The ages of 4, 6, 8, and 10 are used in this model because they represent the midpoint of the development phases (ages 3–5, 5–7, 7–9, and 9–11).
3. This developmental progression is typical for about 60 percent of children in a modern, developed culture. Thus, about 20 percent of children will develop at a faster rate, while about 20 percent will progress at a slower rate.

Structures in 4-Year-Olds

The innate capabilities of young children to subitize and do some simple finger counting enables them by the age of 4 to create two conceptual structures, one for *global quantity* differences and one for the *initial counting* of objects (Figure 2.1). Looking for global quantity, they can tell which of two stacks of chips is more or less, which of two time units is shorter or longer, and which of two monetary units is worth more or less. On a balance scale, they can tell which side is heavier and/or lighter and which side of the beam will go down. Children at this age are still relying more on subitizing than counting, but they do know that a set of objects will get bigger if one or more objects are added or smaller if one or more objects are removed.

Figure 2.1 At the age of 4, children have developed two major structures: one for global quantity that relies on subitizing and one for counting a small number of objects, mainly through one-to-one correspondence with fingers. (Adapted with permission from Griffin, 2002)

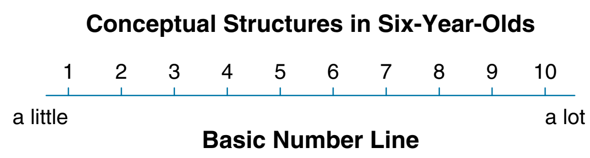


Counting skills are also developing. They know that each number word occurs in a fixed sequence and can be assigned to only one object in a collection. They also know that the last number word said indicates the size of the collection. Most can count to 5, and some can count to 10. Yet, despite these counting capabilities, these children still rely more on subitizing to make quantity determinations. This may be because the global quantity structure is stored in a different part of the brain from the counting structure and because these two regions have not yet made strong neural connections with each other.

Structures in 6-Year-Olds

Children around 6 years of age have integrated their global quantity and initial counting models into a larger structure representing the mental number line we discussed in Chapter 1. Because this advancement gives children a major tool for making sense of quantities in the real world, it is referred to as the *central conceptual structure for whole numbers.* Using this higher-order structure, children recognize that numbers higher up in the counting sequence indicate quantities that are larger than numbers lower down (Figure 2.2). Moreover, they realize that numbers themselves have magnitude—that is, that 7 is bigger than 5. The number line also allows them to do simple addition and subtraction without an actual set of objects just by counting forward or backward along the line. This developmental stage is a major turning point because children come to understand that mathematics is not just something that occurs out in the environment but can also occur inside their own heads.

Figure 2.2 At the age of 6 years, children have developed a mental number line that gives them a central conceptual structure for whole numbers. (Adapted with permission from Griffin, 2002)

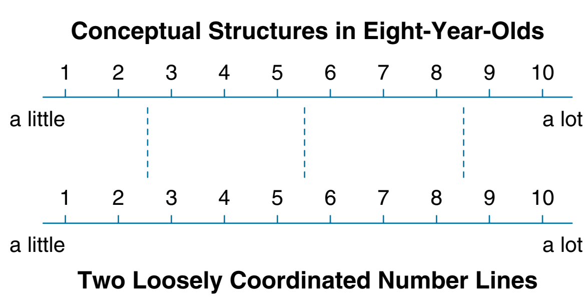


Now children begin using their counting skills in a broad range of new contexts. They realize that counting numbers can help them read the hour hand on a clock, determine which identical-sized money bill is worth the most, and know that a dime is worth more than a nickel even though it is smaller in size. Unlike 4-year-olds, they rely more now on counting than on global quantity in determining the number of objects, such as chips in a stack and weights on a balance.

Structures in 8-Year-Olds

Children at the age of 8 have differentiated their complex conceptual structure into a double mental counting line schema that allows them to represent two quantitative variables in a loosely coordinated fashion (Figure 2.3). Now they understand place value and can mentally solve double-digit addition problems and know which of two double-digit numbers is smaller or larger. The double number line structure also permits them to read the hours *and* minutes on a clock, to solve money problems that involve two monetary dimensions such as dollars *and* cents, and to solve balance-beam problems in which distance from the fulcrum as well as number of weights must be computed.

Figure 2.3 By the age of 8, children can manipulate numbers along two number lines that are loosely coordinated. (Adapted with permission from Griffin, 2002)



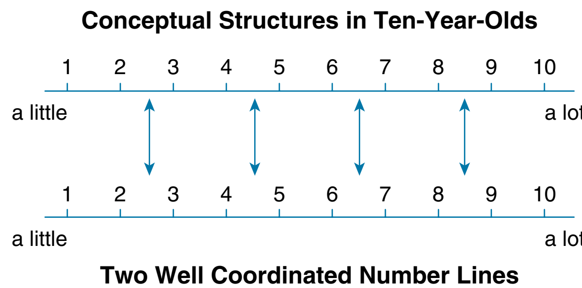
Structures in 10-Year-Olds

By the age of 10, children have expanded the double number line structure to handle two quantities in a well-coordinated fashion or to include a third quantitative variable (Figure 2.4). They now acquire a deeper understanding of the whole number system. Thus, they can perform mental computations with double-digit numbers that involve borrowing and carrying, and can solve problems involving triple-digit numbers. In effect, they can make compensations along one quantitative variable to allow for changes along the other variable. This new structure also allows them to translate from hours to minutes and determine which of two times—say, 3 hours or 150 minutes—is longer. They find it easy to translate from one monetary dimension to another, such as from quarters to nickels and dimes, to determine who has more money, and also to solve balance-beam problems where the distance from the fulcrum and number of weights both vary.

DEALING WITH MULTIPLICATION

Up to this point, we have been exploring how young children manipulate numbers using simple addition and subtraction. In school, they eventually encounter a process called multiplication, sometimes described by teachers as successive addition. However, the mental processes required to perform multiplication are more involved and somewhat different from the innate processes used for addition and subtraction. Imaging studies show that the brain recruits more neural networks during multiplication than during subtraction (Ischebeck et al., 2006; Rosenberg-Lee, Chang, Young, Wu, & Menon, 2011). This should come as no surprise because addition and subtraction were sufficient to allow our ancestors to survive. As a result, humans need to devise learning tools to help them conquer multiplication.

Figure 2.4 By the age of 10, children can manipulate numbers along two mental number lines that are well coordinated and thus can perform mental computations with double-digit numbers. (Adapted with permission from Griffin, 2002)



Why Are Multiplication Tables Difficult to Learn?

Do you remember your first encounters with the multiplication tables as a primary student? Did you have an easy or difficult time memorizing them? How well do you know them today? Despite years of practice, most people have great difficulty with the multiplication tables. Ordinary adults of average intelligence make mistakes about 10 percent of the time. Even some of the single-digit multiplications, such as 8×7 and 9×7, can take up to 2 seconds and have an error rate of 25 percent (Devlin, 2000). Why do we have such difficulty? Several factors contribute to our troubles with numbers. They include associative memory, pattern recognition, and language. Oddly enough, these are three of the most powerful and useful features of the human brain.

Multiplication and Memory

Until the late 1970s, psychologists thought that simple addition and multiplication problems were solved by a counting process carried out primarily by working memory. In 1978, Ashcraft (1995) and his colleagues began a series of experiments to test this notion with young adults. He found that most adults take about the same time to add or multiply two digits. However, it took increasingly longer to do these calculations as the digits got larger, even though the time remained the same for adding or multiplying. It took less than a second to determine the results of 2 + 3 or 2 × 3, but about 1.3 seconds to solve 8 + 7 or 8 × 7. If multiplication is being processed in working memory, shouldn't it take longer to multiply two digits than to add them, since more counting is involved? After many experiments, Ashcraft proposed the only reasonable conclusion that was consistent with the experimental data: solutions to the calculation problems were being retrieved from a memorized table stored in long-term memory. No counting or processing was occurring in working memory.

This effect is not that surprising for three reasons. First, we already noted in Chapter 1 that the accuracy of our mental representation of numerosity drops quickly with increasing number size. Second, the order in which we acquired arithmetic skills plays a role, because we tend to remember best that which comes first in a learning episode. When we began learning our arithmetic facts, we started with simple problems containing small digits, and the difficult problems with large digits came later. Third, because smaller digits appear more frequently in problems than larger ones, we most likely received much less practice with multiplication problems involving larger numbers.

Now, you may be saying: "So what's the big deal? We are using what we memorized in the early grades to solve arithmetic problems today. Isn't that normal?" It may be normal, but it is not natural. Preschool children use their innate but limited notions of numerosity to develop intuitive counting strategies that will help them understand and measure larger quantities. But they never get to continue following this intuitive process. When these children enter the primary grades, they encounter a sudden shift from their intuitive understanding of numerical quantities and counting strategies to the rote learning of arithmetic. Suddenly, progressing with calculations now means acquiring and storing in memory a large database of numerical knowledge, which may or may not have meaning. They also discover that some of the words they use in conversation take on different meanings when doing arithmetic (e.g., "goes into," "difference," and "product"). Many children persevere with this major upheaval in their mental arithmetic and language systems despite the difficulties. Unfortunately, most children also lose their intuition about arithmetic in the process.

Children in the primary grades encounter a sudden shift from their intuitive understanding of numerical quantities and counting strategies to the rote learning of arithmetic facts. Unfortunately, most children lose their intuition about arithmetic in the process.

Is the Way We Teach the Multiplication Tables Intuitive?

Not really. Through hours of practice, young children expend enormous amounts of neural energy laboring over memorizing the multiplication tables, encountering high rates of error and frustration. Yet this is happening at the same time when they can effortlessly acquire the pronunciation, meaning, and spelling of 10 new vocabulary words every day. They certainly do not have to recite vocabulary words and their meanings over and over the way they do their multiplication tables. Furthermore, they remember the names of their friends, addresses, phone numbers, and book titles with hardly any trouble. Obviously, nothing is wrong with their memories, except when it comes to the multiplication tables. Why are they so difficult for children and adults to remember?

One answer is that the way we most often teach the multiplication tables is counterintuitive. Usually, we start with the 1 times table and work our way up to the 10 times table. Teaching step-by-step in this fashion results in 100 (10×10) separate facts to be memorized. But is this really the best way to teach them? Children have little difficulty remembering the 1 and 10 times tables because they are consistent with their intuitive numbering scheme and base-10 finger manipulation strategy. Now that leaves 64 separate facts (each of 2, 3, 4, 5, 6, 7, 8, 9, multiplied by each of 2, 3, 4, 5, 6, 7, 8, 9). But why memorize all 64 separate facts? We noted at the beginning of this chapter that children already recognize the commutativity of addition by age 5. By simply showing them commutativity in multiplication (3×8 is the same as 8×3), we can cut the total number of 64 separate facts nearly in half, to just 36 (the number of four pairs of identical numbers—e.g., 2×2 or 5×5—cannot be reduced). This is a more manageable number, but it still does not solve the problem.

Some critics say that students are just not putting in the effort to memorize their multiplication facts. Others wonder whether this endeavor is even necessary, given the prevalence of electronic calculators. But these ideas raise a question: Why do our ordinarily good memories have such difficulty with this task? There is something to be learned here about the nature of memory and the structure of the multiplication tables.

Patterns and Associations

The human brain is a powerful five-star pattern recognizer. Human memory recall often works by association; that is, one thought triggers another in long-term memory. Someone mentions *mother,* and the associative areas in your brain's temporal lobes generate an image in your mind's eye. Long-term storage sites are activated, and you recall the first time she took you to the zoo. The limbic region in the brain sprinkles your memory with emotions. You were so excited then because you didn't realize that elephants were so wide or giraffes so tall. More connections are made, and you fondly remember the same excitement in your own children on their first zoo visit. The brain's ability to detect patterns and make associations is one of its greatest strengths and is often referred to as *associative memory.* In fact, humans can recognize individuals without even looking at their faces. Through associative memory, they can quickly and accurately identify people they know from a distance by their walk, posture, voice, and body outline.

Answer to Question 2. False: *The brain's ability to detect patterns and make associations is often referred to as* associative memory.

Associative memory is a powerful device that allows us to make connections between fragmented data. It permits us to take advantage of analogies and apply knowledge learned in one situation to a new set of circumstances. Unfortunately, associative memory runs into problems in areas such as the multiplication tables, where various pieces of information must be kept from interfering with one another.

Devlin (2000) points out that when it comes to the multiplication tables, associative memory can cause problems. That's because we remember the tables through language, causing different entries to interfere with one another. A computer has no problem detecting that $6 \times 9 = 54$, $7 \times 8 = 56$, and $8 \times 8 = 64$ are separate and distinct entities. On the other hand, the brain's

Associative memory is a powerful and useful capability. Unfortunately, associative memory runs into problems in areas such as the multiplication tables, where various pieces of information must be kept from interfering with one another.

strong pattern-seeking ability detects the rhythmic similarities of these entities when said aloud, thus making it difficult to keep these three expressions separate. As a result, the pattern 6 × 9 may activate a series of other patterns, including 45, 54, 56, and 58, and load them all into working memory, making it difficult to select the correct answer.

Likewise, Dehaene (1997) stresses the problems that come with memorizing addition and multiplication tables. He notes that arithmetic facts are not arbitrary and independent of one another. Rather, they are closely intertwined linguistically, resulting in misleading rhymes and confusing puns. The following example is similar to one Dehaene uses to illustrate how language can confuse rather than clarify.

Suppose you had to remember the following three names and addresses:

- Carl Dennis lives on Allen Brian Avenue.
- Carl Gary lives on Brian Allen Avenue.
- Gary Edward lives on Carl Edward Avenue.

Learning these twisted combinations would certainly be a challenge. But these expressions are just the multiplication tables in disguise. Let the names Allen, Brian, Carl, Dennis, Edward, Frank, and Gary represent the digits 1, 2, 3, 4, 5, 6, and 7, respectively, and replace the phrase "lives on" with the equal sign. That yields three multiplications:

- $3 \times 4 = 12$
- $3 \times 7 = 21$
- $7 \times 5 = 35$

From this perspective, we can now understand why the multiplication tables present such difficulty when children first encounter them. Patterns interfere with one another and cause problems. Pattern interference also makes it difficult for our memory to keep addition and multiplication facts separate. For example, it takes us longer to realize that $2 \times 3 = 5$ is wrong than to realize that $2 \times 3 = 7$ is false because the first result would be correct under addition. Back in 1990, studies were already revealing that learning multiplication facts interfered with addition (Miller & Paredes, 1990). He discovered that students in third grade took more time to perform addition when they started learning the multiplication tables, and errors such as $2 + 3 = 6$ began to appear. Subsequent studies confirm that the consolidation of addition and multiplication facts correctly into long-term memory continues to be a major challenge for most children.

Over millions of years, our brain has evolved to equip us with necessary survival skills. These skills include recognizing patterns, creating meaningful connections, and making rapid judgments and inferences, even with only a smattering of information. Rudimentary counting is easy because of our abilities to use language and denote a one-to-one correspondence with finger manipulation. But our brains are not equipped to manipulate the arithmetic facts needed to do precise calculations such as multiplication because these operations were not essential to our species' survival.

Studies of the brain using electroencephalographs (EEGs) and other techniques show that simple numerical operations, such as number comparison, are localized in various regions of the brain. But multiplication tasks require the coordination of several widespread neural areas, indicating that a greater number of cognitive operations are in play (Micheloyannis, Sakkalis, Vourkas, Stam, & Simos, 2005; Salillas, Semenza, Basso, Vecchi, & Siegal, 2012). Consequently, to do multiplication and precise calculations, we have to recruit mental circuits that developed for quite different reasons.

Our brains are not equipped to manipulate the arithmetic facts required for precise calculations. To do arithmetic, we need to recruit mental circuits that developed for different reasons.

The Impact of Language on Learning Multiplication

If memorizing arithmetic tables is so difficult, how does our brain eventually manage to do it? One of our strongest innate talents is the ability to acquire spoken language. We have specific brain regions in the frontal and temporal lobes that specialize in handling language. Faced with the challenge of memorizing arithmetic facts, our brain responds by recording them in verbal memory, a sizable and durable part of our language processing system. Most of us can still recall items in our verbal memory, such as poems and songs, that we learned many years ago.

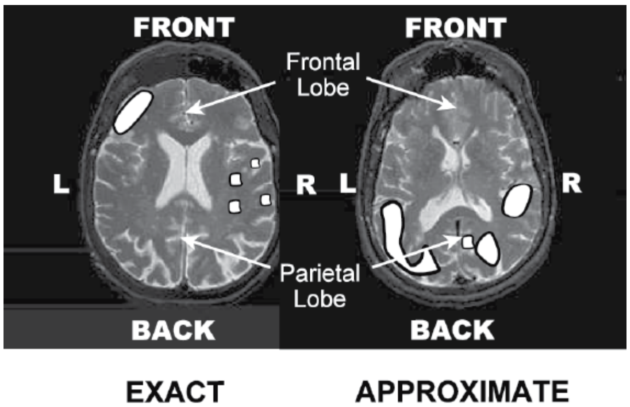
Teachers have long recognized the power of language and verbal memory. They encourage students to memorize items such as rhymes and the multiplication tables by reciting them aloud. As a result, calculation becomes linked to the language in which it is learned. This is such a powerful connection that people who learn a second language generally continue to do arithmetic in their first language. No matter how fluent they are in the second language, switching back to their first language is much easier than relearning arithmetic from scratch in their second language.

Brain imaging studies carried out by Dehaene and his colleagues provided further proof that we use our language capabilities to do arithmetic (Dehaene, Spelke, Pinel, Stanescu, & Tsivkin, 1999). Their hypothesis was that exact arithmetic calculations involved the language regions of the brain because they required the verbal representations of numbers. Estimations requiring approximate answers, however, would not make use of the language facility.

The subjects of the experiments were adult English–Russian bilinguals who were taught two-digit addition facts in one of the two languages and then tested. When both the teaching and the test question were in the same language, the subjects provided an exact answer in 2.5 to 4.5 seconds. If the languages were different, however, the subjects took a full second longer to provide the exact answer. Apparently, the subjects used that extra second to translate the question into the language in which the facts had been learned. When the question asked for an approximate answer, the language of the question did not affect the response time.

During the experiment, the researchers monitored the subjects' brain activity (Figure 2.5). Questions requiring exact answers primarily activated the same part of the left frontal lobe where language processing occurs. When the subjects responded to questions requiring approximate answers, the greatest activity was in the two parietal lobes, the regions that contain number sense and support spatial reasoning. Amazingly, these findings

Figure 2.5 These composite fMRI (functional magnetic resonance imaging) scans show that exact calculations (left image) primarily activate language areas in the left frontal lobe, where verbal representations of numbers are processed. During approximate calculations (right image), the greatest activation was in the two parietal lobes that house number sense and support spatial reasoning (Dehaene et al., 1999).



reveal that we humans are able to extend our intuitive number sense to a capacity to perform exact arithmetic by recruiting the language areas of our brain.

If you need more personal evidence of this connection between language and exact arithmetic, try multiplying a pair of two-digit numbers while reciting the alphabet aloud. You will find that this is quite difficult to do because speaking demands attention from the same language areas required for mental computation and reasoning.

Yet despite this seeming cooperation between the language and mathematical reasoning areas of the brain, it is still important to remember that these two cerebral areas are anatomically separate and distinct. Further proof of this separation comes from case studies showing that one area can function normally even when the other is damaged (Brannon, 2005). Teachers, then, should not assume that students who have difficulty with language processing will necessarily encounter difficulties in arithmetic computation, and vice versa.

Do the Multiplication Tables Help or Hinder?

Answer to Question 3. False: Because the language and number processing areas of the brain are separate, teachers should not assume that students with language problems will necessarily encounter difficulties with computation, and vice versa.

They can do both. Remember that children come to primary school with a fairly developed, if somewhat limited, sense of number. Thanks to their brain's capacity to seek out patterns, they can already subitize, and they also have learned a pocketful of simple counting strategies through trial and error. Too often, as noted above, arithmetic instruction in the primary grades purposefully avoids recognizing these intuitive abilities and resorts immediately to practicing arithmetic facts.

If the children's introduction to arithmetic rests *primarily* on the rote memorization of the addition and multiplication tables and other arithmetic facts (e.g., step-by-step procedures for subtraction), then their intuitive understandings of number relationships are undermined and overwhelmed. In effect, they learn to shift from intuitive processing to performing automatic numerical operations without caring much about their meaning.

On the other hand, if instruction in beginning arithmetic takes advantage of the children's number sense, subitizing, and counting strategies by making connections to new mathematical operations, then the tables become tools leading to a deeper understanding of mathematics, rather than an end unto themselves.

Some students may have already practiced the multiplication tables at home. My suggestion would be to assess how well each student can already multiply single-digit numbers. Then introduce activities using dots or pictures on cards that help students practice successive addition (the underlying concept of multiplication). The idea here is to use the students' innate sense of patterning to build a multiplication network without memorizing the tables themselves. Of course, this may not work for every student, and for some, memorizing the tables may be the only successful option.

WHAT'S COMING?

People are born with a number sense that helps them determine the numerosity of small collections of objects and do rudimentary counting, addition, and subtraction. How can we take advantage of these intuitive skills to help them learn more complex mathematical operations? What is current research in cognitive neuroscience telling us about how the brain focuses, learns, and remembers? How should we use this information when considering effective instruction in mathematics? What are some surprising findings about the impact of technology on attention and memory? These are some of the questions answered in the next chapter.

Chapter 2—Learning to Calculate

QUESTIONS AND REFLECTIONS

Respond to the following questions, and jot down on this page key points, ideas, strategies, and resources you want to consider later. This sheet is your personal journal summary and will help jog your memory.

Why is learning to multiply so difficult, even for adults?

How does language affect learning to multiply?

How do you feel about using the multiplication tables when children are learning to multiply? Why?

3

Reviewing the Elements of Learning

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.

—Godfrey Harold Hardy

In the chapters following this one, we will look at specific ways to approach the teaching of mathematics to young children, preadolescents, and adolescents. Before doing that, however, we should review here some of the basic elements of learning. Effective teachers are continually assessing whether their choices in instructional strategies are consistent with what research is revealing about how the brain learns. This chapter will explore some recent research findings so that teachers can decide how this information compares with what they already know. Suggestions on applying this research when planning mathematics lessons are discussed in Chapter 8.

LEARNING AND REMEMBERING

I have asked teachers all over the world this question: “How long do you want your students to remember what you taught them?” Their answer is always: “Forever!” But is that what really happens? Hardly. Some critics of education have speculated that students in the United States forget more than 80 percent of what they were taught in class within 2 years after they graduate from high school. We have no way of knowing if that figure is accurate, but most of us would agree that many of the facts presented to us in school were never permanently stored.

Because early research studies often used numbers to test the nature of memory, scientists have long known that both short-term and long-term memory can dramatically affect our mathematical capabilities. We will be discussing the effects of memory on calculations here and later in this book. If we want lessons to be remembered, this is a good time to briefly review how the brain's memory components work.

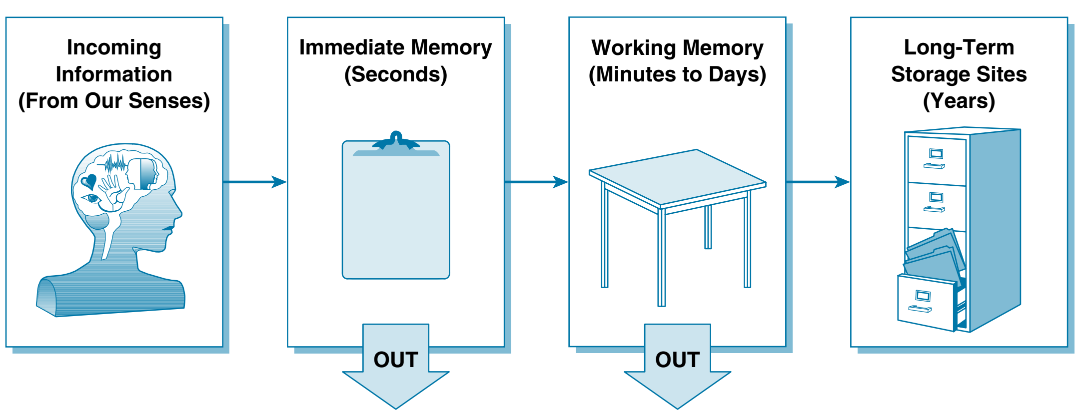
Memory Systems

If you studied memory a few decades ago, you were probably taught that humans have two major memories: a temporary memory called short-term memory and a permanent one called long-term memory. Neuroscientists now believe that we have two temporary memories that perform different tasks. It is a way of explaining how the brain deals briefly with some data but can continue to process other data for extended periods of time, even though that information does not get stored permanently. Short-term memory is the name used by cognitive neuroscientists to include the two stages of temporary memory: *immediate memory* and *working memory* (Cowan, 2009; Squire & Kandel, 1999). Figure 3.1 illustrates the stages of our temporary and permanent memories.

Immediate Memory

Immediate memory is one of the two temporary memories and is represented in Figure 3.1 by a clipboard, a place where we put information briefly until we make a quick decision on how to dispose of it. Immediate memory operates subconsciously or consciously and holds data for up to about 30 seconds. (Note: The numbers used here are averages over time.

Figure 3.1 The diagram illustrates the theory of temporary and permanent memories. Information gathered from our senses lasts only a few seconds in immediate memory. Information in working memory usually endures for minutes or hours, but can be retained for days if necessary. The long-term storage sites (also called permanent memory) store information for years.



There are always exceptions to these values as a result of human variations or pathologies.) The individual's experiences determine the degree of the information's importance. If the information is of little or no importance within this time frame, it drops out of the temporary memory system. For example, when you look up the telephone number of the local pizza parlor, you usually can remember it just long enough to make the call. After that, the number is of no further importance and drops out of immediate memory. The next time you call, you will have to look up the number again.

Working Memory

Suppose, on the other hand, you can't decide whether to call the pizza parlor or the Chinese take-out place, and you discuss these options with someone else in the room. Because this situation requires more of your attention, information is shifted into working memory for full conscious processing. In Figure 3.1, working memory is shown as a worktable, a place of limited capacity where we can build, take apart, or rework ideas for eventual disposal or storage somewhere else. When something is in working memory, it generally captures our focus and demands our attention.

Capacity of Working Memory. Working memory can handle only a few items at one time (see Table 3.1). This functional capacity changes with age. Recent studies suggest that the capacity of working memory in younger individuals is decreasing for reasons we do not yet understand (Cowan, 2009, 2010). One possible explanation is that the widespread use of handheld devices with access to the Internet is inducing the brain to remember *where* to find information, rather than the information itself. Remembering location requires less neural capacity than remembering numerous bits of data. Because of the brain's ability to adapt to its environment (called *neural plasticity*), the theory goes, working memory is less in need of the larger capacity.

The revised capacities suggest that preschool toddlers can deal with one to two items of information at once. Preadolescents can handle three to four items. Through adolescence, further cognitive maturation occurs, and the capacity increases slightly to a range of three to five. For most people, that number will probably remain relatively stable throughout life. (You may recall from Chapter 1, however, that working memory's capacity for *digits* can vary from one culture to another, depending on that culture's linguistic and grammatical system for building number words.)

Table 3.1 Changes in Capacity and Time Limits of Working Memory With Age

Age (in years)	*Average Capacity and Range (in chunks)*	*Average Time Limit (in minutes)*
Younger Than 5	1 to 2	No reliable data
Between 5 and 14	3 to 4	5 to 10
14 and Older	3 to 5	10 to 20

This limited capacity explains why we have to memorize a song or a poem in stages. We start with the first group of lines by repeating them frequently (a process called *rehearsal*). Then we memorize the next lines and repeat them with the first group, and so on. It is possible to increase the number of items within the functional capacity of working memory through this process, called *chunking*. In arithmetic, chunking occurs when the young child's mind quickly recognizes that both 3 + 1 + 1 and 3 + 2 equal 5.

The implication of these findings is that teachers should consider these limits when deciding on the amount of information they plan to present in a lesson. In other words, less is more.

Time Limits of Working Memory. Working memory is temporary memory and can deal with items for only a limited time (see Table 3.1). For preadolescents, that time is likely to be 5 to 10 minutes, and for adolescents and adults, 10 to 20 minutes. These are average times, and it is important to understand what the numbers mean. An adolescent (or adult) normally can process an item in working memory intently for 10 to 20 minutes before fatigue or boredom with that item occurs and the individual's focus drifts. For focus to continue, there must be some *change* in the way the individual is dealing with the item. As an example, the person may switch from listening to an explanation of a concept to physically applying it or talking to someone else about it or making connections to other learnings. If something else is not done with the item, it is likely to fade from working memory.

> ***Answer to Question 4. False:*** *Working memory is short-term memory and can deal with only a few items for a limited time.*

Of course, some items can remain in working memory for hours or even days. Sometimes, we have an item that remains unresolved—a question whose answer we seek or a troublesome family or work decision that must be made. These items can remain in working memory, continually commanding some attention and, if of sufficient importance, interfering with our accurate processing of other information. Eventually, we solve the problem, and it clears out of working memory.

The implication here is that teachers should consider these working memory time limits when deciding on the flow of their lessons. In other words, shorter is better.

> *Working memory has capacity limits and time limits that teachers should keep in mind when planning lessons. Less is more! Shorter is better!*

Impact of Technology on Attention and Memory

Research studies are now revealing that the widespread use of technology is having both positive and negative effects on our students' attention and memory systems. Because young brains are still developing, their frequent exposure to technology is actually wiring their brains differently from the brains of children in previous generations. As these so-called "digital natives" interact with their environment, they are learning how to scan for information efficiently and quickly. Technology allows them to be more creative and to access multiple sources of information, practically simultaneously. But all this comes at a cost.

Learning requires attention. Without it, all other aspects of learning, such as reasoning, memory, problem solving, and creativity, are at risk.

How children develop attention is largely determined by their environment. Modern technology has thrust children into a world where the demands for their attention have increased dramatically. Distraction has replaced consistent attention, and, as we noted earlier, the capacity of working memory appears to be shrinking. Their brains are becoming accustomed to, and are rewarded for, constantly switching tasks, at the expense of sustainable attention. This constant switching from one task to another has a penalty. When students switch their attention, the brain has to reorient itself to the new task, further taxing neural resources. And because of working memory's limited capacity, some of the information from the first task is lost as new information from the second task moves in. Furthermore, the switching causes cognitive overload, a condition where the flow of information exceeds the brain's ability to process and store it. Consequently, the students cannot gain a deep understanding of the new learning or translate it into conceptual knowledge.

Is It Better to Take Notes on Paper or on a Laptop?

High school and college students are often of the belief that taking notes on a laptop enhances their academic performance. After all, laptops allow students to access the Internet, collaborate with other students locally and internationally, engage in demonstrations and other activities, and, of course, take more notes. Because they have grown up with keyboards and technology, many students today can type faster than they can write. Consequently, students who use laptops in the classroom are likely to record more notes on a laptop than they would if they wrote them out in longhand on paper. This would seem to indicate that taking notes on a laptop allows for greater learning and a better review of that learning at a later date—say, during a test. Right? Well, not so fast!

A recent research study that included three different experiments found that college students who took notes on a laptop did not learn as much as those who wrote their notes on paper (Mueller & Oppenheimer, 2014). Students who wrote out their notes had a greater conceptual understanding of the material and were more successful in integrating and applying it than were those who took laptop notes, even though the laptop group took more notes. What happened here? Researchers suggest that because writing by hand is slower than typing, these students' brains had to listen, process, and then jot down a summary of the new learning. These cerebral processes apparently enhanced understanding and retention. Those students who typed their notes essentially recorded a transcription of the teacher's presentation, with little processing of the new material. Ironically, the more verbatim the student's transcript was, the lower that student's retention of the lesson content. Even when a group of laptop students were instructed to think about the lesson's information and type the notes in their own words, they exhibited the same degree of verbatim transcription, and they did no better in summarizing than the laptop students who did not get this instruction. This research reminds us that technology may be faster, but it does not necessarily help students learn the course content better.

Answer to Question 5. False: *Students who take notes in longhand remember more and have a deeper understanding of new material compared with those who take notes on a laptop.*

Reading on Paper Versus Reading on a Screen

Tablets and e-readers are increasingly common in classrooms. Some schools have started phasing out paper textbooks in favor of e-books. What does the brain think about this shift from paper to screens? Apparently, not much. Accumulating research evidence over the past 15 years indicates that we understand better and remember more when we read text on paper rather than on a screen (Jabr, 2013). Screens may be inhibiting comprehension because the digital devices prevent readers from easily and intuitively navigating long texts and making mental maps of the information and concepts presented in the text. In one study, 72 tenth-grade students read one narrative and one expository text, with half the group reading on paper and the other half on screens (Mangen, Walgermo, & Brønnick, 2013). Afterward, the students took comprehension tests while having access to the texts. The students who read on screens performed worse than those who read on paper. This is likely due to having to scroll and click through various screens, increasing demands on the students' cognitive systems and making it harder to remember what they read. Students reading on paper had the entire text in their hands and could easily move among the pages.

Furthermore, e-readers deprive the reader of the tactile experiences of handling paper, which may subconsciously hinder comprehension. The very physicality of paper and the topography of a book, according to researchers, affect reading comprehension in several ways. The reader can easily flip through pages to compare sections of text or scan ahead. Experiments show that when recalling a passage, individuals often picture it on a page, and the four corners of the page act as a physical outline that strengthens these recalls. Reading on paper is less tiring than reading on a screen. Paper reflects ambient light, but screens shine light directly into the reader's face, causing eye strain, blurred vision, and headaches, thereby lowering comprehension. In short, the mental workload required for the *act* of reading on paper is significantly lower than for reading on a screen, leaving more mental capacity available for processing and remembering *what* is read (Wästlund, Norlander, & Archer, 2008).

There are, of course, some positive aspects of technology. Video games and other onscreen media can improve attention, fine motor skills, visual–spatial perceptions, the ability to identify specific objects from clutter, and reaction times. Also, one positive outcome of a decrease in working memory capacity and ultimate storage is that brain regions may be reassigned to engage in more problem solving, critical analysis, reflection, and creativity (Greenfield, 2009). Technology is here to stay, and teachers should not shy away from using it judiciously and with understanding of its effects. Avoid information overload and stick to one mathematics learning objective at a time so students have a chance to do some deep processing. Use the technology sparingly as a tool to help students achieve learning objectives in mathematics, rather than as an end unto itself.

Rehearsal Enhances Memory

Teachers should ensure that they have included instructional strategies *purposefully* designed to increase the probability that students will retain the new learning. Any new learning is more likely to be retained if the learner has adequate time to process and reprocess it. This continuing reprocessing of information is called *rehearsal,* and it is a critical component in the transference of information from working memory to long-term storage.

Types of Rehearsal

Time for Initial and Secondary Rehearsal. Time is a critical component of rehearsal. Initial rehearsal occurs when the information first enters working memory. If the learner cannot attach sense or meaning and if there is no time for further processing, then the new information is likely to be lost. Providing sufficient time to go beyond the initial processing to secondary rehearsal allows the learner to review the information, make sense of it, elaborate on the details, and assign value and relevance, thus significantly increasing the chances of retention.

Brain-imaging studies indicate that the frontal lobe is very much involved during the rehearsal process and, ultimately, in long-term memory formation (Blumenfeld & Ranganath, 2006). This makes sense because working memory is also located in the frontal lobe (Goldberg, 2001). Several studies using fMRI (functional magnetic resonance imaging) scans of human brains and other techniques showed that during longer rehearsals the amount of activity in the frontal lobe determined whether items were stored or forgotten (Buckner, Kelley, & Petersen, 1999; Ofen, 2012; Sharma, Nargang, & Dickson, 2012; Wagner et al., 1998).

Rote and Elaborative Rehearsal. *Rote rehearsal* is used when the learner needs to remember and store information exactly as it is entered into working memory. This is not a complex strategy, but it is necessary to learn information or a cognitive skill in a specific form or an exact sequence. We use rote rehearsal to remember a poem, the lyrics and melody of a song, telephone numbers, steps in a procedure, and, of course, the multiplication tables. *Elaborative rehearsal* is used when it is not necessary to store information exactly as learned but when it is more important to associate the new learnings with prior learnings to detect relationships. This is a more complex thinking process in that the learner reprocesses the information several times to make connections to previous learnings and assign meaning. Students use rote rehearsal to memorize mathematical facts and use elaborative rehearsal to probe the deeper meanings and interrelationships of mathematical concepts.

When students get very little time for, or training in, elaborative rehearsal, they resort more frequently to rote rehearsal for nearly all processing. Consequently, they fail to make the associations or discover the relationships that only elaborative rehearsal can provide.

For example, suppose a teacher presents a lesson on dividing by a fraction this way:

$$15 \div {}^{1}/_{4} = 15 \times {}^{4}/_{1} = 60$$

Instead of exploring and understanding the mathematical rationale used when dividing by a fraction, they simply remember the rote rule: "Ours is not to reason why, just invert and multiply!" Furthermore, they continue to believe that learning mathematics is merely the recalling of information as learned rather than its value for generating new ideas, concepts, and solutions. By simply adding a visual representation of a situation that is relevant to students, greater meaning can be obtained. For instance, in this lesson the teacher could say, "We have 15 pizzas, and we cut (divide) each of them into fourths. How many pieces will we have?" The visual of each pizza cut into four pieces helps students recognize the *meaning* of dividing by a fraction.

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By cutting each of the 15 pizzas into fourths, we have 60 pieces.

Rote rehearsal is valuable for certain limited learning objectives. Nearly all of us learned the alphabet and the addition and multiplication tables through rote rehearsal. But rote rehearsal simply allows us to acquire information in a certain sequence. Too often, students use rote rehearsal to memorize important mathematical terms and facts in a lesson but are unable to use the information to solve problems. They will probably do fine on a true–false or multiple-choice test, mainly because the odds for guessing are not too bad. But they will experience difficulty answering higher-order questions that require them to apply their knowledge to new situations, especially those that have more than one solution. Keep in mind, too, that rehearsal contributes to acquisition of information but does not guarantee that information will transfer into long-term storage. However, there is almost no transfer to long-term memory *without* rehearsal.

The Importance of Meaning

Both experimental and anecdotal evidence reveals that mathematical content often does not have meaning for students. And why is meaning so important? We noted earlier that rehearsal is one way to increase the possibility that new learning will be encoded into long-term memory. Other criteria also play a crucial role. Figure 3.1 shows that information in working memory can be either encoded into long-term memory sites for future recall (from the worktable to the file cabinet) or dropped out of the memory system. Which option will the brain choose? This is an important decision because we cannot later recall what we have not stored.

> *Information is most likely to get stored if it makes sense and has meaning.*

What criteria does the working memory use to make that decision? Information that has survival value is quickly stored, along with strong emotional experiences. But in classrooms, where the survival and emotional elements are minimal or absent, other factors come into play. It seems that the working memory connects with the learner's past experiences and asks just two questions to determine whether an item is saved or rejected:

- **"Does this make sense?"** This question refers to whether the learner can understand the mathematical content on the basis of experience. Does it "fit" into what the learner already knows about numbers and arithmetic operations? When a student says, "I don't understand," it means the student is having a problem making sense of the learning, usually because it doesn't connect to previous learning.

- **"Does it have meaning?"** This question refers to whether the item is *relevant* to the learner. For what purpose should the learner remember it? Meaning is a very personal thing and is greatly influenced by an individual's experiences. The same item can have great meaning for one student and none for another. For instance, a student who has astronomy as a hobby would find activities in geometry more meaningful than would one whose hobby is collecting stamps. Questions such as "Why do I have to know this?" or "When will I ever use this?" indicate that the student has not, for whatever reason, accepted this learning as relevant.

The goal of learning is not just to acquire knowledge but to be able to use that knowledge in a variety of different settings that students see as relevant. To do this, students need a deeper understanding of the concepts involved in the learning. That's one reason mathematics teachers so often hear students asking, "Why do we need to know this?" If teachers cannot answer that question in a way that is meaningful to students, then we need to rethink why we are teaching that item at all.

Whenever the learner's working memory decides that an item does not make sense or have meaning, the probability of its being stored is extremely low (assuming, of course, no survival or emotional component is present). If either sense or meaning is present, the probability of storage increases significantly. If both sense and meaning are present, the likelihood of long-term storage is very high. Brain scans have shown that when new learning is readily comprehensible (sense) and can be connected to past experiences (meaning), there is substantially more cerebral activity followed by dramatically improved retention (Maguire, Frith, & Morris, 1999).

If teachers cannot answer the question, "Why do we need to know this?" in a way that is meaningful to students, then we need to rethink why we are teaching that item at all.

Why Meaning Is So Significant

Of the two criteria, meaning has the greater impact on the probability that information will be stored. Think of all the television programs you have watched that are *not* stored, even though you spent 1 or 2 hours viewing each one. The show's content or story line made sense to you, but if meaning was absent, you just did not save it. Now think of this process when teaching mathematics. Students may diligently follow the teacher's instructions to memorize facts or perform a sequence of tasks repeatedly, and may even get the correct answers. But if they have not found meaning by the end of the learning episode, there is little likelihood of long-term storage. Mathematics teachers are often frustrated by this. They see students using a certain formula to solve problems correctly one day, but they cannot remember how to do it the next day, the next week, or the next month. If the process was not stored, the brain treats the information as brand-new again!

Mathematics teachers get frustrated when they see students using a certain formula to solve problems correctly one day but they cannot remember how to do it the next day. If the process was not stored, the brain treats the information as brand-new again!

Sometimes, when students ask why they need to know something, the teacher's response is, "Because it's going to be on the test." This response may raise the student's anxiety level but adds little meaning to the learning. Students resort to writing the learning in a notebook or typing it into a laptop so it is preserved in writing but not in memory. Then we wonder the next day why they forgot the lesson.

Teachers spend about 90 percent of their planning time devising lessons so students will make sense of the learning objective. But teachers need to spend more time helping students establish meaning, keeping in mind that what was meaningful for students 10 years ago may not necessarily be meaningful for students today (Sousa, 2011a).

Meaning Versus Automatic Response

We have already noted that evolution did not prepare our brains for multiplication tables, complicated algorithms, fractions, or any other formal mathematical operation. So to carry out formal arithmetic, our brain has to make do with whatever networks it has, even if it means following a sequence of steps that its owner does not understand. The result is that as children spend their time memorizing arithmetic tables and facts, they become little calculators who can compute without having any idea of the underlying arithmetic principles involved.

> *Our development as a species did not prepare the brain for multiplication tables, complicated algorithms, or any other formal mathematical operations.*

When students attempt to carry out simple arithmetic computations using memorized facts, they often jump to conclusions without considering the relevant conditions of the problem. They become so skilled at the mechanics of computation that they arrive at answers that do not make sense. Furthermore, the language associated with solving a particular problem may itself interfere with the brain's understanding of what it is being asked to compute. For example, quickly answer the following questions:

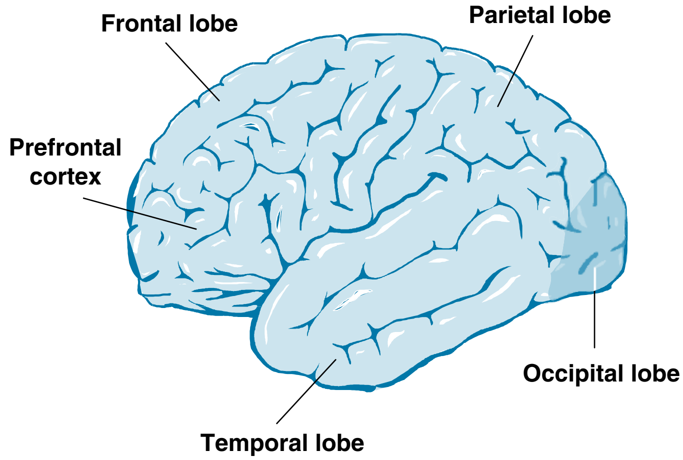
- An aquarium contains 9 fish. All but 6 die. How many fish remain?
- Billy has 6 action figures, which is 3 fewer than Joey. How many action figures does Joey have?

Did you answer 3 to either of the problems? In the first problem, the presence of the numbers 9 and 6 coupled with the words "all but" and the question, "How many remain?" creates a strong temptation to perform the subtraction 9 – 6 = 3, giving the answer as 3. The correct answer is 6, but to get that answer you have to think about what the problem is saying and avoid the blind manipulation of symbols. Similarly, in the second problem, seeing the numbers 6 and 3, along with the words "fewer than," is sufficient to trigger the subtraction mode in your brain: 6 – 3 = 3. When you think about the problem, however, you realize that you should add 3 to 6 to get the correct answer that Joey has 9 action figures.

In both situations, you have to fight the automatic response and actually analyze each problem. This is the job of the front area of the brain's frontal lobe, just behind the forehead, called the *prefrontal cortex* (Figure 3.2). However, the prefrontal cortex develops very slowly and is not fully mature until the age of 22 to 24. Thus, children and adolescents are prone to impulsive decisions while solving problems. Their prefrontal cortex areas have not had much opportunity in school to construct and practice the nonroutine strategies needed to override the automated responses and avoid the arithmetic traps that word problems can harbor. For students to become proficient in mathematical computation, they must resist the automated and meaningless responses and proceed to thoughtfully analyze the situation and select the appropriate calculation algorithm for the problem at hand.

Teachers, then, become the means by which learners can see the links between a mechanical calculation and its meaning. While we recognize the need for learners to remember some basic arithmetic facts, memorization should not be the main component of instruction, and it should not replace exploring the underlying principles of mathematical operations. Depending on memorization erodes the learner's intuitive understanding of approximation and counting, as discussed in Chapter 2. Students then see arithmetic solely as the memorization of mechanical recipes that have no practical applications and no obvious meaning. Such a view can be discouraging, lead to failure, and set the stage for a lifelong distaste for mathematics.

Figure 3.2 The prefrontal cortex is the part of the frontal lobe that, among other things, analyzes problems and implements and controls nonroutine strategies.



How Will the Learning Be Stored?

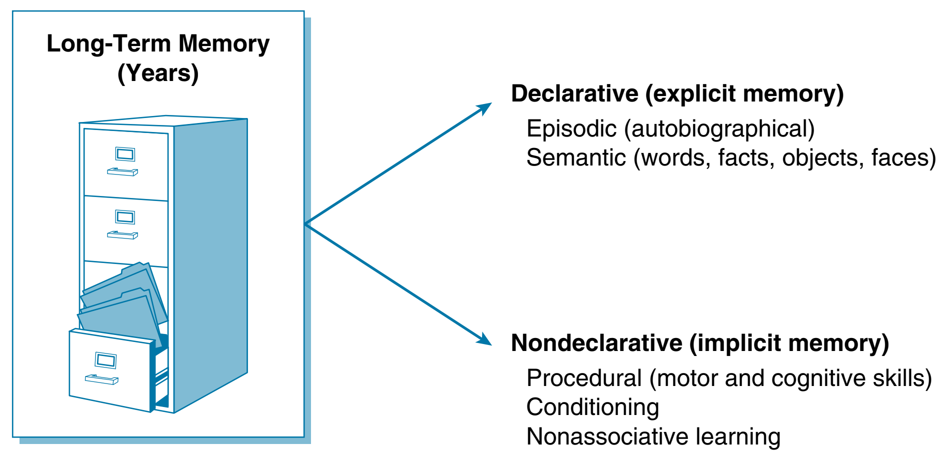
Information can be stored in different ways. Long-term memory can be divided into two major types: *declarative memory* and *nondeclarative memory* (Figure 3.3).

Declarative Memory. *Declarative memory* (also called *conscious* or *explicit* memory) describes the remembering of names, facts, music, and objects. When you think of an important event you attended with someone close to you, such as a concert, wedding, or funeral, note how easily other components of the memory come together. This is declarative memory in its most common form—a conscious and almost effortless recall. Declarative memory can be further divided into *episodic memory* and *semantic memory.*

Episodic memory refers to the conscious memory of events (episodes) in our own life history, such as our 16th birthday party, falling off a bicycle, or what we had for breakfast this morning. It helps us identify the time and place when an event happened and gives us a sense of self. Episodic memory is the memory of personal and autobiographical *remembering.*

Semantic memory is knowledge of facts and data that may not be related to any event. It is knowing that the Eiffel Tower is in Paris, how to tell time, and the quadratic formula. Semantic memory is the memory of factual *knowing.* A student recalling the Pythagorean theorem is using semantic memory; remembering his experiences in the classroom when he learned it is episodic memory.

Figure 3.3 Long-term memory consists of two major types. Declarative memory is our daily recollections of people we know, our vocabulary, and related information. Nondeclarative memory is largely composed of automated procedures, such as driving a car or multiplying a pair of three-digit numbers.



Declarative memory is greatly enhanced by elaborative rehearsal, because our memory of facts, people, and events is preserved best when we can make connections between and among them. This comes through elaborative discussions, new ways of looking at things, analysis of situations, and a deep understanding of why we made specific decisions and behaved in certain ways. The more connections we make through these creative and analytical processes, the stronger and longer lasting the memory is likely to be. Could this have application for how we teach mathematics?

Nondeclarative Memory. *Nondeclarative memory* (also called *implicit* memory) describes all memories that are *not* declarative memories; that is, they are memories that can be used for tasks that cannot be declared or explained in any straightforward manner. Of particular interest to teachers of mathematics is the type of nondeclarative memory called *procedural memory.*

Procedural memory refers to the learning of motor and cognitive skills, and remembering *how* to do something, such as riding a bicycle, driving a car, or tying a shoelace. As practice of the skills continues, these memories become more efficient and can be performed with little conscious thought or recall. The brain process shifts from *reflective* to *reflexive*. Much of what we do during the course of a day—such as breakfast rituals, getting to work, and shaking the hand of a new acquaintance—involves the performance of skills. We do these tasks without being aware that we are using our memory. Although learning a new skill involves conscious attention, skill performance later becomes unconscious and relies essentially on nondeclarative memory.

> *Procedural memory helps us learn things that don't require conscious attention, such as how to perform rote mathematical operations.*

We also learn *cognitive skills,* such as reading, discriminating colors, and figuring out a *procedure* for solving a problem. Cognitive *skills,* such as performing rote mathematical operations, are different from processing cognitive *concepts,* in that cognitive skills are performed automatically and rely on procedural memory rather than declarative memory. Procedural and cognitive skill acquisition involves some different brain processes and memory sites than does cognitive concept learning. If they are learned differently, should they be taught differently?

Procedural memory is enhanced by the repetition of rote rehearsal. In fact, that is the only way we can retain certain information, such as vocabulary words or how to add a column of numbers. Because following a step-by-step procedure usually gives us the desired outcome, we can carry out the steps without much conscious input and without having a clue as to *why* we are doing these steps or how they work.

Brain-imaging studies indicate that procedural and declarative memories are stored in different regions of the brain, and declarative memory can be lost while procedural memory is spared (Rose, 2005; White, 2009). Such division of memory locations makes sense. Declarative memory requires conscious input and processing, so frontal lobe areas are actively involved. Procedural memory, on the other hand, triggers a set of automatic steps that are usually without conscious processing or frontal lobe input. This explains why you can drive your car to work (procedural memory) while your frontal lobe is simultaneously planning your day (declarative memory).

The more arithmetic we can teach through declarative processes involving understanding and meaning, the more likely children will succeed and actually enjoy mathematics.

When Should New Learning Be Presented in a Lesson?

When an individual is processing new information, the amount of information retained depends, among other things, on *when* it is presented during the learning episode. At certain time intervals during the learning, we will remember more than at other intervals.

Primacy–Recency Effect

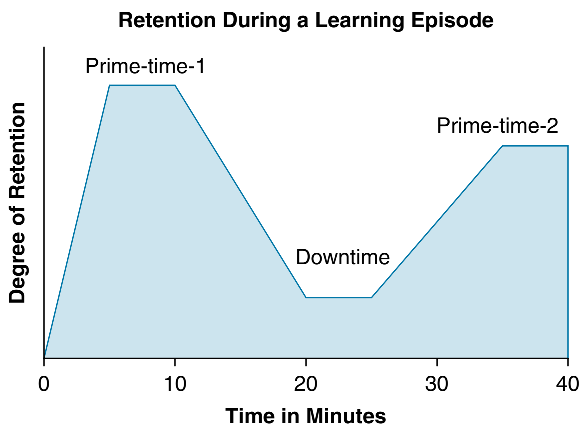
In a learning episode, we tend to remember best that which comes first and remember second best that which comes last. We remember least that which comes just past the middle of the episode. This common phenomenon is referred to as the *primacy–recency effect* (also known as the *serial position effect*). This is not a new discovery. The first studies on this effect were published in the 1880s.

More recent studies help explain why this is so. The first items of new information are within the working memory's capacity limits, so they command our attention and are likely to be retained in semantic memory. The later information, however, exceeds the capacity shown in Table 3.1 and is lost. As the learning episode concludes, items in working memory are sorted or chunked to allow for additional processing of the arriving final items, which are likely held in working memory and will decay unless further rehearsed (Gazzaniga, Ivry, & Mangun, 2002; Stephane et al., 2010; Terry, 2005).

During a learning episode, we remember best that which comes first, second best that which comes last, and least that which comes just past the middle.

Figure 3.4 shows how the primacy–recency effect influences retention during a 40-minute learning episode. The times are averages and approximate. Note that it is a bimodal curve, each mode representing the degree of

Figure 3.4 The degree of retention varies during a learning episode. We remember best that which comes first (prime-time-1) and last (prime-time-2). We remember least that which comes just past the middle.



greatest retention during that time period. In my own work, I refer to the first or primacy mode as *prime-time-1* and the second or recency mode as *prime-time-2.* Between these two modes is the time period in which retention during the lesson is least. I call that area the *downtime.* This is not a time when no retention takes place but a time when it is more difficult for retention to occur.

Does Practice Make Perfect?

Practice refers to learners' repeating a motor or cognitive skill over time. It begins with the rehearsal of the new skill in working memory. Later, the skill memory is recalled, and additional practice follows. Practice is a big part of instruction in mathematics. Therefore, it is important to remember that the quality of the practice and the learner's knowledge base will largely determine the outcome of each practice session.

The old adage that "practice makes perfect" is rarely true. Practice makes permanent! It is very possible to practice the same skill repeatedly with no increase in achievement or accuracy of application. Think of the people you know who have been driving, cooking, or even teaching for many years with no improvement in their skills. Why is this? How is it possible for one to practice a skill continually with no resulting improvement in performance?

Conditions for Successful Practice

For practice to *improve* performance, four conditions must be met (Hunter, 2004):

1. The learner must be sufficiently motivated to *want* to improve performance. If the learner has not attached meaning to the topic, then motivation is low.
2. The learner must have all the knowledge necessary to understand the different ways the new knowledge or skill can be applied.
3. The learner must understand how to apply the new knowledge to deal with a particular situation.
4. The learner must be able to analyze the results of that application and know what needs to be changed to improve performance in the future.

Guided Practice, Independent Practice, and Feedback

Practice may not make perfect, but it does make permanent, thereby aiding in the retention of learning. Consequently, we want to ensure that students practice the new learning correctly from the beginning. This early practice is done in the presence of the teacher (referred to as *guided practice*), who can offer immediate and corrective feedback to help students analyze and improve their practice. We often fail to recognize the power of feedback. A synthesis of more than 900 meta-analyses of 50,000 research studies on student achievement looked at the impact of certain influences on student learning (Hattie, 2012). The effect is measured on a scale called "effect size." It is a useful scale for comparing results on different measures, such as teacher-made tests, student work, and standardized tests. The average effect size for the 50,000 studies was 0.40; however, the effect size for feedback was an impressive 0.75. Feedback, incidentally, is not just about teachers informing students about their learning but also about teachers getting student feedback on their teaching. The message here is that student achievement is likely to improve when student–teacher reciprocal feedback is frequent, providing information to both students and teachers about progress toward successfully accomplishing the learning objectives.

When the practice is correct, the teacher can then assign *independent practice* (usually homework) in which the students rehearse the skill on their own to enhance retention. This strategy leads to perfect practice, and, as coach Vince Lombardi once said, "Perfect practice makes perfect."

Teachers should avoid giving students independent practice before guided practice. Because practice makes permanent, allowing students to rehearse a mathematical operation for the first time while away from the teacher is very risky. If they unknowingly practice the skill or procedure incorrectly, then they will learn the incorrect method well! This will present serious problems for both the teacher and learner later on, because it is very difficult to change a skill that has been practiced and remembered, even if it is not correct. Furthermore, the student will likely get frustrated and annoyed at having spent personal time outside of school practicing a skill incorrectly and may lose the motivation to learn the process correctly. This frequent occurrence contributes to unfavorable attitudes toward mathematics.

Giving students independent practice before guided practice can help them learn an incorrect procedure well.

Unlearning and Relearning a Skill or Process. If a learner practices a mathematical process incorrectly but well, unlearning and relearning that process correctly will be very difficult. The degree to which the unlearning and relearning processes are successful will depend on the

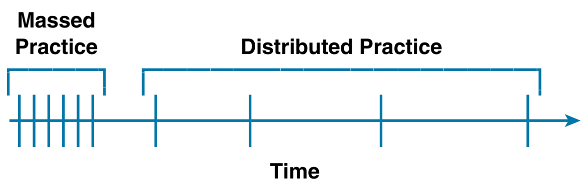
- age of the learner (i.e., the younger, the easier to relearn),
- length of time the skill has been practiced incorrectly (i.e., the longer, the more difficult to change), and
- degree of motivation to relearn (i.e., the greater the desire for change, the more effort will be used to bring about the change).

In any event, both teacher and student have a difficult road ahead to unlearn the incorrect method and relearn it correctly.

Massed and Distributed Practice

Hunter (2004) suggests that teachers use two different types of practice over time. (Hunter uses *practice* to include rehearsal.) Practicing a new learning during time periods that are very close together is called *massed*

Figure 3.5 Practice repeated over a short duration of time is called massed practice. Repeating the practice over increasingly longer periods of time is distributed practice, which is more likely to lead to retention.



practice (Figure 3.5). This produces fast learning, as when one mentally rehearses a multiplication table. Immediate memory is involved here, and the information can fade in seconds if it is not rehearsed quickly.

Teachers of mathematics provide massed practice when they allow students to try different examples of applying a new formula or concept in a short period of time—say, within one classroom period. *Cramming* for an exam is also an example of massed practice. Material can be quickly chunked into working memory but can also be quickly dropped or forgotten if more sustained practice does not follow soon. This happens because the material has no further meaning, and thus the need for long-term retention disappears.

Cramming is an example of massed practice. Material is quickly chunked into working memory for a test and then forgotten unless distributed practice follows soon.

Sustained practice over time, called *distributed practice,* is the key to retention. If you want to remember a multiplication table later on, you will need to use it repeatedly over time. Thus, practice that is distributed over longer periods of time sustains meaning and consolidates the learnings into long-term storage in a form that will ensure accurate recall and applications in the future.

Effective practice, then, starts with massed practice for fast learning and proceeds to distributed practice later for retention. As a result, the student is continually practicing previously learned skills throughout the year(s). Each test should not only address new material but also allow students to practice important older learnings. This method not only helps in retention but also reminds students that the learnings will be useful for the future and not just for the time when they were first learned and tested. That was the rationale behind the idea of the spiral curriculum, whereby critical mathematical facts and skills are reviewed at regular intervals within and over several grade levels. Whatever happened to it?

Assessment as a Form of Practice

No educator would deny that teachers should periodically assess their students' progress toward achieving the intended learning objective. The question that arises is, "What is the best way to do this?" Too often, assessments are in the form of written tests that are given at the end of a unit of study. These tests are usually graded and used to determine a student's final grade for the subject. In addition to the tests created by teachers, many students have to take high-stakes tests, especially in mathematics and language arts. This test-heavy climate engenders in students a fear, distrust, and dislike of tests (Chu, Guo, & Leighton, 2014). They place little value on written tests, mainly because they look at them as "gotcha" experiences.

Assessments become a much more valuable tool for learning when they are used frequently *during* a unit of study to determine each student's progress. These are called *formative assessments* because they allow students and teachers to make adjustments to instructional and learning strategies, if needed. The traditional tests given at the *end* of the unit of study are called *summative assessments.*

Teachers of mathematics in middle schools and high schools have noted that they do few formative assessments because they take time away from all the curriculum material they need to cover. As a result, they have had little interest or practice in designing and implementing the various types of formative assessments in their classrooms (Sutton, 2010). However, when professional development programs provide adequate training for these teachers in this area, student achievement improves (Webb, 2010).

Formative assessments are more of a process than a test. They enhance student learning when they

- allow students to practice what they have learned,
- give teachers information about what each student has learned,
- help teachers analyze how successful they were at teaching their lesson objectives,
- allow teachers to make any needed adjustments to their teaching, and
- enable students to assess themselves and understand how they can improve their learning.

Students are more receptive to formative assessments because they are participants in the process. They have opportunities to see their errors, discuss them, and correct them without penalty. Rather than a "gotcha" activity, they see formative assessments as a "Let's work on this together" activity, thereby improving their motivation and chances of success.

Formative assessments are rarely graded. They come in many forms, such as the following:

- Teacher observations of student performance to determine what they know and do not know
- Graphic organizers where a student can show relationships between concepts
- Exit/admit slips asking students to write down several things they learned today or yesterday
- Questioning to determine the students' depth of understanding
- Think/pair/share conversations allowing students to summarize what they have learned
- Practice presentations several days before giving a final presentation to the class
- Peer/self-assessments to help create a positive learning community in the class
- Kinesthetic assessments where students can incorporate movement to demonstrate their understanding of a concept

Formative assessments should be used frequently because they allow students and teachers to make adjustments to instructional and learning strategies.

With younger students, teachers should consider using written tests mainly for practice, and recording the score of only every third or fourth paper. Oral tests are a good substitute because they are less stressful, and some younger students are better at telling what they know than at writing it.

Include Writing Activities

The Common Core State Standards in Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) include numerous opportunities for writing at all grade levels. Writing is an important component of communication in the classroom, and research studies have highlighted the benefits to students of writing to learn mathematics (Pugalee, 2005; Stonewater, 2002). As the brain's frontal cortex develops during the school years, writing enhances the learner's ability to organize, understand, analyze, and reflect on the new learning. In addition to requiring focus, writing provides another modality for processing information and skills, thereby helping the student find sense and meaning, and increasing the likelihood that the new learning will be remembered.

Benefits of Writing in Mathematics

Through writing activities, teachers help students

- learn a mathematics concept more effectively and develop critical-thinking and problem-solving skills;
- create a permanent record of their thoughts where they can return to reflect on them;
- organize ideas, develop new applications for knowledge, and solve problems involving mathematical operations;
- become active participants in their own learning by engaging in an interaction with the subject or content area;
- maintain a silent dialogue with the content area, in which they internalize knowledge and articulate it in the learning process;
- establish a personal connection to new mathematics concepts;
- get involved in an active intellectual process in which they decide what is important and what is meaningful or relevant to them;
- gain self-understanding and confidence in dealing with their concerns; and
- personalize the subject matter, because it gives them choices for applying their knowledge in areas that interest them.

Besides helping students understand mathematical concepts, writing also enhances their confidence in their writing skills for other curriculum areas. In Chapter 8, you will find specific suggestions for how to incorporate writing into mathematics lessons.

Fixed and Growth Mind-Sets in Mathematics

Psychologists have known for years that our preconceptions about how the world works shape our beliefs and our actions. One psychologist, Carol Dweck (2006), has been looking specifically at preconceptions, called *mind-sets,* about what it means to be smart and successful. Her research has revealed that each of us develops at a young age either of two different mind-sets about our ability and what actions will lead to our success. She called these fixed and growth mind-sets. Below is a comparison of the two types of mind-sets as they apply to learning mathematics.

Students with a fixed mind-set believe the following:

- Success comes from being born smart; some people are smart at mathematics, and some are not.
- Environment can contribute to our success, but the genetic predisposition to be good or poor at mathematics cannot be overcome.
- There is no sense working hard at mathematics, because it is something I know I cannot do.

Students with a growth mind-set believe the following:

- Genetics are just a starting point.
- Determination and persistence are what really predict my success in learning mathematics.
- Mistakes are opportunities to learn and develop.

Because fixed mind-sets are ability centered, they become self-fulfilling prophesies. Students who believe they have limited ability in mathematics (or any other endeavor) are reluctant to expend the effort to do better. As a result, their achievement suffers, and they use that as further evidence that they do not have the ability to improve. They often attribute their lack of success to something beyond their control, making statements such as, "No one in my family is any good at math." Students with a growth mind-set, on the other hand, are effort centered. They put forth the effort, see growth in their progress, and are thus motivated to accept new challenges (Haimovitz, Wormington, & Corpus, 2011).

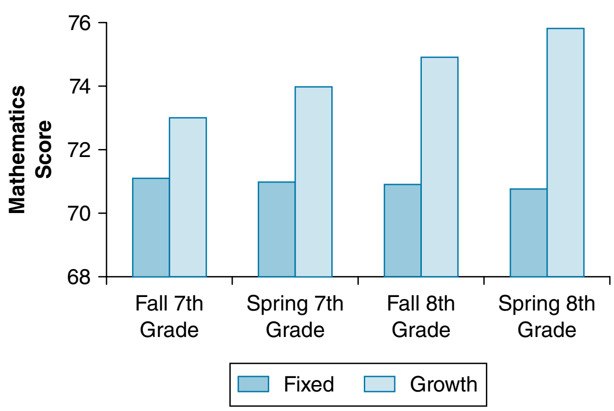
Teachers, of course, also have mind-sets. Those with fixed mind-sets believe that some students will learn mathematics and some will not. Subconsciously, these teachers may separate students by their perceived ability and teach them mathematics accordingly. It makes sense to them to accelerate the smart kids and remediate the others. A fixed mind-set also helps these teachers justify why some students do well in mathematics and others do not.

Teachers with a growth mind-set believe that most students can learn mathematics—or any topic—if they exert the effort to do so. They see their job as eliciting that effort and doing what they can to help their students succeed. These teachers do not accept labeling students but, rather, adjust their instructional strategies to emphasize the *process* a student used to solve a problem and avoid mentioning talents or gifts. They praise students' effort and, when they do not do well, note that everyone learns in a different way and that both teacher and student will keep trying to find a way that works. Some evidence of the power of mind-set in mathematics can be seen in Figure 3.6, which shows the difference in mathematics grades between middle school students with fixed and growth mind-sets over a 2-year period (Blackwell, Trzesniewski, & Dweck, 2007).

Gender Differences in Mathematics

For decades, boys have consistently scored higher than girls on standardized mathematics tests, such as the SAT and National Assessment of Educational Progress. High school and college mathematics classes usually

Figure 3.6 This chart shows the difference in mathematics scores (vertical axis) for students with fixed and growth mind-sets over a 2-year period. The scores of the growth mind-set students increased significantly, while those of the fixed mind-set students fell. (Adapted from Blackwell et al., 2007)



contain more males than females. Those seeking to explain this gender disparity have typically put the blame on outmoded social stereotypes. Recently, however, they have added discoveries in brain science as potential explanations. They cite, for example, that male brains are about 6 to 8 percent larger than female brains. But males are on average about 6 to 8 percent taller than females, which could also explain the similar difference in brain size. And brain imaging studies show that males seem to have an advantage in visual–spatial ability (the ability to rotate objects in their heads); (e.g., Ganley & Vasilyeva, 2011; Lowrie & Diezman, 2011; Syzmanowicz & Furnham, 2011), while females are more adept at language processing (e.g., Kaushanskaya, Gross, & Buac, 2013). In female brains, the bundle of nerves that connects the two cerebral hemispheres, called the *corpus callosum,* is proportionally larger and thicker than in male brains. This suggests that communication between the two hemispheres is more efficient in females than in males. However, in male brains, communication appears to be more efficient within a hemisphere. But whether these differences translate into a genetic advantage for males over females in mathematical processing remains to be seen and proved.

Stereotype Threat

Although the genders have differed on test results in mathematics, researchers believe social context plays an important role in partially explaining these variations. Differences in career choices, for instance, are due not to differing abilities in mathematics but to cultural factors, such as subtle but pervasive gender expectations that emerge in high school. Studies have shown that merely telling females that a mathematics test often shows gender differences is enough to hurt their performance (e.g., Jamieson & Harkins, 2012), but telling females that they have the power to do well in mathematical assessments significantly improves their performance (Van Loo & Rydell, 2013). This phenomenon, called *stereotype threat,* occurs when people believe they will be evaluated based on societal stereotypes about their particular group. In a typical study of stereotype threat, researchers give a mathematics test to males and females. They tell half the females that the test will show gender differences and tell the rest that it will show none. Females who expected gender differences do significantly worse on the test than males. Those females who were told there is no gender disparity perform equally to males on the test (e.g., Franceschini, Galli, Chiesi, & Primi, 2014; Spencer, Steele, & Quinn, 1999).

Another study of stereotype threat was designed to have people think of their strengths rather than their stereotyped weaknesses (McGlone &

Aronson, 2006). Would that serve to improve their performance in areas where they were not supposed to do well, as in mathematics? Ninety college students, half male and half female, completed a questionnaire. One group was asked if they lived in a single-sex or coed dormitory, as this question in previous studies was shown to activate male and female stereotypes. A second group was asked why they chose to attend a private liberal arts college—an attempt, according to the researchers, to activate their "snob schema." The third group, used as a control, was asked to write about their experiences living in the northeastern part of the United States.

After taking a standard test of visual–spatial abilities associated with mathematics performance, the gender gap closed among those who were primed to think about their status as students in an exclusive liberal arts college. The female scores improved, while the male scores were the same as for the control group. There was no significant difference between the male and female scores. Simply manipulating the way female students thought of themselves improved their test performance.

Instructional Approaches Narrow the Gap

Although most neuroscientists will admit to gender differences in how the brain processes information, especially in young children, they are reluctant to support the concept that these differences offer a lifelong learning advantage for one sex over the other in any academic area. Spelke (2005) reviewed 111 studies and papers, and found that most suggest that the male's and female's abilities for mathematics and science have a genetic basis in cognitive systems that emerge in early childhood but give males and females, on the whole, equal aptitude for mathematics and science.

It is important for educators to know about these gender differences and how they change through various stages of human development. The danger here is that people will think that if the differences are innate and unchangeable, then nothing can be done to improve the situation. Such ideas are damaging because they leave the student feeling discouraged, and they ignore the brain's plasticity (the ability to continually change through experience) and exceptional capacity to learn complex information when suitably motivated. A variety of teaching approaches and strategies may indeed make up for these gender differences.

Consider Learning Styles

As adolescents mature, their learning styles also begin to mature and consolidate. Learning style describes the methods and preferences of an individual when in a learning situation and seems to result from a combination of genetic predispositions and environmental influences. These styles comprise a number of variables, including the following:

- Sensory preferences (Do I have a preference for auditory, visual, or kinesthetic–tactile input?)
- Hemispheric preference (Do I usually look at the world more analytically or more globally?)

- Intellectual preferences similar to Howard Gardner's 10 intelligences, mentioned in Chapter 1 (What are my intellectual strengths and weaknesses?)
- Participation preferences (Do I want to do something now with this learning or think about it first?)
- Sensing/intuitive preferences (Do I prefer learning facts and solving problems using established methods, or do I prefer discovering possibilities and relationships on my own?)

Remember, these are preferences and they are not rigid. Everybody can change to another style temporarily if the situation requires it. If you consider just these five learning-style variables and their components, as well as the notion that many of these variables exist along a continuum, you quickly realize that there are thousands of possible permutations. How can a teacher address all these variations in the classroom? It helps if we narrow the field of possibilities.

The concept of learning styles is not accepted by all cognitive researchers. Some argue that there is no solid evidence from neuroscience to suggest that individuals learn in different ways. However, there *is* a growing body of research evidence, mostly related to the gender differences in mathematics and language processing that we discussed earlier in this chapter. Some studies show that during mathematical processing, the brain areas activated in females are different from those activated in males (e.g., Keller & Menon, 2009). Other studies using brain imaging show that the activated brain regions are different among individuals of the same gender when performing the same cognitive task. This would suggest differences in processing styles (e.g., Lai et al., 2012; Miller, Donovan, Bennett, Aminoff, & Mayer, 2012; Okuhata, Okazaki, & Maekawa, 2009).

Despite the controversies, experience tells us that students will benefit from multiple strategies and a range of approaches to the teaching–learning process. Some students will grasp the concept of fractions with one teaching approach, while other students will benefit from different approaches. The variables forming learning style empower teachers with a deeper understanding of individual learning differences so they can devise multiple strategies to ensure student success.

Addressing Multiple Intelligences

In the 30-plus years since Gardner first proposed his theory of multiple intelligences, educators have been developing activities to apply his ideas to classroom practice. You may be surprised to learn that there is little physical evidence from neuroscience to support Gardner's theory. About the best neuroscientists can say is that scanning studies show that different parts of the brain are used to perform certain tasks associated with Gardner's intelligences. For example, language processing is largely devoted to the left frontal lobe, while many visual–spatial operations are generally located in the right parietal lobe. Creating and processing music involve the temporal lobes, and running and dancing are controlled mainly by the motor cortex and cerebellum (see Figure 3.2 earlier in this chapter).

Of course, there is plenty of anecdotal evidence to indicate different degrees and types of intelligence, as anyone who has been a teacher will confirm. We encounter, for example, the star athlete (high bodily/kinesthetic) who can hardly write a complete sentence (low linguistic), or the mathematics whiz (high logical/mathematical) who rarely communicates with classmates (low interpersonal). Classroom observations and studies have shown that more students are likely to be motivated and succeed in classes where teachers use a variety of activities designed to appeal to students whose strengths lie in one or more of the intelligences described by Gardner (Shearer, 2004). Several studies have shown that instructional strategies focusing on the logical/mathematical aspect of this theory improve student motivation and achievement in mathematics (e.g., Karamikabir, 2012; Li, Ma, & Ma, 2012). However, it is important to remember that these intelligences describe the different types of competencies that we all possess in varying degrees and use in our daily lives.

Studies show that more students are motivated and succeed in classes where teachers use activities that address the various intelligences.

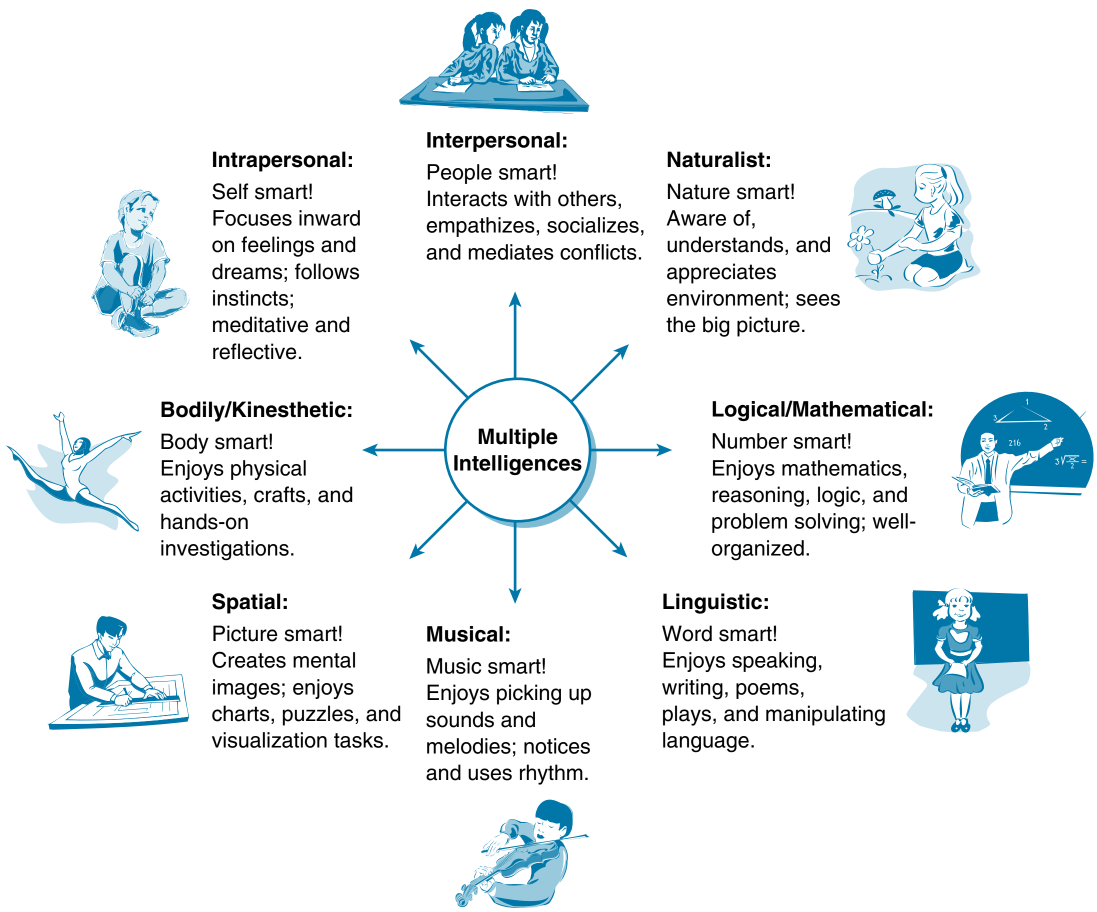
Whether scientists will eventually discover the underlying neurological networks that form different intelligences remains to be seen. In the meantime, Gardner's theory can still be beneficial for two reasons. First, it reminds teachers that students have different strengths and weaknesses and different interests, and that they learn in different ways. By using Gardner's ideas, teachers are likely to address the needs of a wider range of students. Second, teaching the mathematics curriculum through a variety of approaches will probably be more interesting, and this in itself often motivates students to learn.

Figure 3.7 takes a closer look at eight of the intelligences and some of their relevant behaviors as described and revised by Gardner (1993). I am not including in this discussion Gardner's recently proposed intelligences, *spiritualist* and *emotionalist*, because there has not been adequate assessment of their educational implications. Specific suggestions for activities to consider in mathematics lessons that address these intelligences are provided in Chapter 7.

Consider Teaching Styles

Quick, finish this statement: "Teachers tend to teach the way they ______." Did you say "were taught"? That is a common response but not truly accurate. Observational data and research on learning styles show that teachers really tend to teach the way they *learn.* Thus, our learning style drives our teaching style. Teachers who are predominantly auditory learners will do lots of talking in their classes and personally enjoy going to lectures and hearing others recount their vacation trips and other stories. But teachers who are predominantly visual learners will use lots of charts and visual aids in their classes and personally prefer movies, television, museums, and the like for entertainment. Actually, the alternative response that teachers teach the way they were taught is indirectly related to learning style. If a student (prospective teacher) is in a class where the teacher's teaching style closely matches the student's learning style, then the student is more likely to achieve success. Later, the student will feel comfortable emulating that teacher's teaching style because it was so compatible with the student's own learning style. As a result, that student is now teaching as he or she was taught.

Figure 3.7 The eight intelligences describe the different types of competencies that we all possess in varying degrees and use in our daily lives (Gardner, 1993).



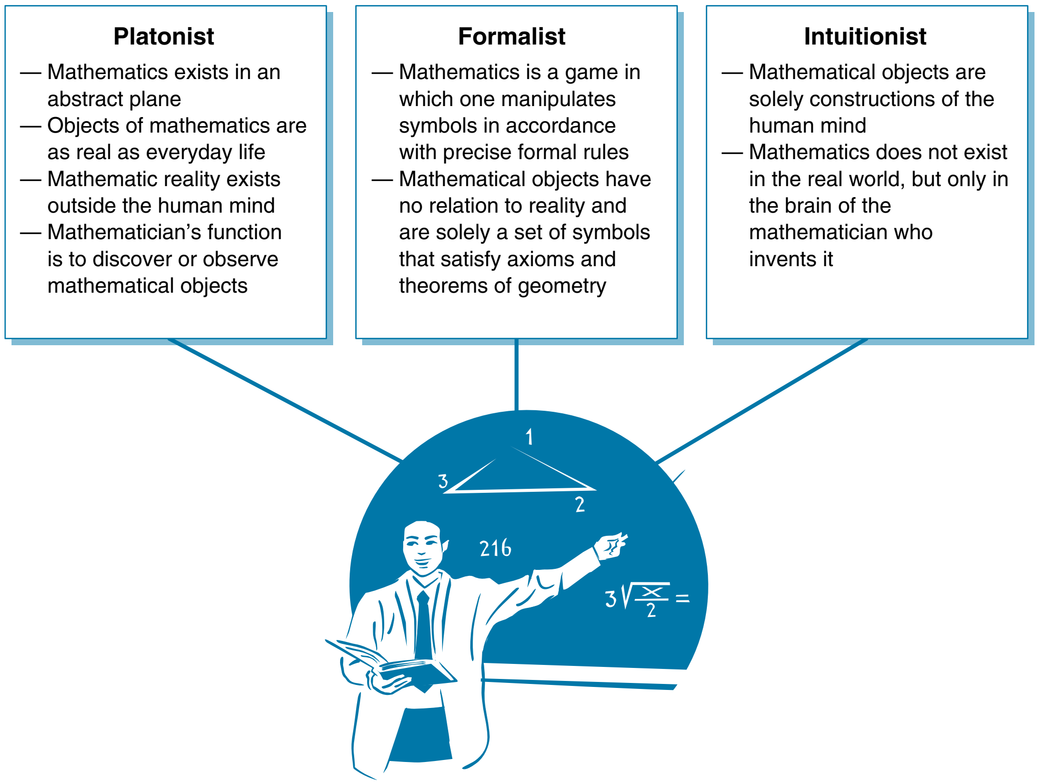
Of course, there are lots of other combinations of teaching styles and learning styles that would take up many pages. That is not our purpose here; so let us turn our discussion to the thinking and learning style of mathematicians. Many high school mathematics teachers majored in mathematics in college and perhaps in graduate school as well. It is fair to say, then, that many of these teachers think and learn like mathematicians.

How Do You Think About Mathematics?

If you are a mathematics educator, how do you think about mathematics? Do you see mathematics as mainly an abstract construct of the human mind and mathematical objects as having no relation to reality? Perhaps you see mathematical objects as real and necessary for our daily experiences. Dehaene (1997), himself a mathematician and researcher in cognitive neuroscience, suggests that mathematicians view their subject from any one of the following three perspectives (see Figure 3.8):

- **Platonist.** For these individuals, mathematics exists in an abstract plane, but the objects of mathematics that they study are as real as everyday life. Mathematic reality exists outside the human mind,

Figure 3.8 Dehaene (1997) suggests that mathematicians view their subject from any one of three different perspectives: Platonist, formalist, or intuitionist. This diagram gives a brief description of each perspective. Whichever perspective a teacher holds will likely affect that individual's approach to presenting mathematics in the classroom.



and the function of the mathematician is to discover or observe mathematical objects.

- **Formalist.** For them, mathematics is only a game in which one manipulates symbols in accordance with precise formal rules. Mathematical objects such as numbers have no relation to reality. Rather, they are defined solely as a set of symbols that satisfy certain axioms and the theorems of geometry.
- **Intuitionist.** These people believe that mathematical objects are merely constructions of the human mind. Mathematics does not exist in the real world but only in the brain of the mathematician who invents it. After all, neither arithmetic nor geometry nor logic existed before human beings appeared on the earth.

After reading the descriptions of these three perspectives, where would you classify yourself? Keep in mind that just as your learning preferences direct your teaching style, so will your perspective on the subject close to your heart affect your approach to designing and presenting lessons in mathematics.

From the information presented in Chapters 1 and 2, it would seem that the intuitionist perspective provides the best account of the relationship between arithmetic and the human brain. We noted in those chapters that

- human beings are born with the innate mechanisms for separating objects and determining the numerosity of small sets of objects;
- number sense is present in animals as well and thus is independent of language and has a long history in the development of our species;
- in children, the capability to do numerical estimation, comparison, finger counting, simple addition, and subtraction arises spontaneously without much direct instruction; and
- mental manipulation of numerical quantities is carried out by neural networks located in the parietal areas of both brain hemispheres.

Thus, intuition about numbers is deeply rooted in our brains. It is one of the ways we search for structure in our environment. Just as specialized brain circuits allow us to locate objects in space, so do circuits in our parietal lobes allow us to effortlessly determine numerical quantities.

MOTIVATING STUDENTS IN MATHEMATICS

Students in the elementary grades are still interested in their studies, and the arithmetic they need to learn is achievable without too much effort. They generally maintain a positive attitude toward mathematics in these grades. But attitudes change when students get into middle school and high school, where mathematics is taught by subject-area teachers and the coursework is more demanding. Teachers now hear many students saying, "I don't like math!" There are several reasons for this emotional response to secondary-level mathematics. One reason is due to what is generally termed "math anxiety." We will discuss math anxiety in greater detail in Chapter 7. For now, we will note that anxiety triggers an increase in cortisol—the stress hormone—in the blood, and that produces an unwelcome and uncomfortable mental state.

A second reason is the increased weight that states have attached to the scores of high-stakes testing in mathematics, another stressful situation. A third reason is that students often do not see any usefulness to the mathematics they are learning, now or in the future. The lament, "When will I ever use this?" is a clear sign that what they are learning does not appear relevant to them. And let us not forget that another important reason for the distaste for mathematics may be that the student has not done well in this subject in the past. In the student's brain, the equation is simple: mathematics = failure. We do not like to repeat things that we fail at.

Regardless of the source of the negative feelings about mathematics, they can be overcome if students become sufficiently *motivated* to see that mathematics can be engaging, it has practical applications in the real world, and they can learn it successfully. Not surprisingly, studies show that motivation improves student performance in mathematics (e.g., Steinmayr, Wirthwein, & Schöne, 2014). Some studies found that identifying the motivation of students in mathematics classes provides teachers with important information regarding instructional approaches (Cleary & Chen, 2009).

Motivation Surveys

It could be very helpful for mathematics teachers to know how their students feel about mathematics at the beginning of the school year (Whitin, 2007). A simple survey would give the teacher a quick summary of the extent of any negative feelings the students might harbor about mathematics. Such a survey should be anonymous and include statements that ask students to respond using a scale such as Strongly Agree (SA), Agree (A), Neither Agree nor Disagree (N), Disagree (D), and Strongly Disagree (SD). Here is a sample survey. Replace or reword the statements as appropriate for the particular school and grade level.

Student Survey in Mathematics

What grade are you in? _____

Please respond to the following statements circling the appropriate letters, indicating whether you *Strongly Agree (SA)*, *Agree (A)*, *Neither Agree nor Disagree (N)*, *Disagree (D)*, or *Strongly Disagree (SD)* with the statement.

1.	I enjoy studying math	SA	A	N	D	SD
2.	I find math boring	SA	A	N	D	SD
3.	I am good at math	SA	A	N	D	SD
4.	I usually do my math homework	SA	A	N	D	SD
5.	I keep trying in math even when the problem is difficult	SA	A	N	D	SD
6.	I solve problems on my own without the help of others	SA	A	N	D	SD
7.	I ask for help when I need it	SA	A	N	D	SD
8.	I enjoy helping someone else in math	SA	A	N	D	SD
9.	I enjoy using manipulatives when learning math	SA	A	N	D	SD
10.	I use technology when learning math	SA	A	N	D	SD
11.	I like working in small groups with my classmates to solve problems	SA	A	N	D	SD
12.	Math is important throughout life	SA	A	N	D	SD
13.	If I work hard, I can be successful in math	SA	A	N	D	SD
14.	I know how to assess my understanding in math	SA	A	N	D	SD
15.	Homework is important to my learning in math	SA	A	N	D	SD

Strategies for Motivating Students in Mathematics

Teachers sometimes spend more time with the students who are motivated in mathematics than with those who are not. Yet it is possible to get disinterested students motivated with a few strategies, such as the following (Posamentier, 2013):

- ✓ **Highlight a void in students' knowledge.** This strategy involves teasing the students with a familiar situation followed by an unfamiliar situation. Can we solve the unfamiliar situation? For example, the students may know that we can use radar to calculate the distance to the moon. But how would we calculate the distance to a star?
- ✓ **Show passion for your subject.** Students are motivated when they see teachers who are passionate about the subject they teach. Their natural curiosity urges them to discover what the teachers see in this subject matter that they do not. Passion is contagious.
- ✓ **Look for an unexpected pattern.** Set up a situation that leads students to discover a pattern they have not seen before. For example, ask the students how they would find the sum of the numbers from 1 to 100. They could, of course, start by adding them one at a time. After they do that for a bit, point out that they can add the first and last numbers (1 + 100 = 101), and then add the second and next-to-last numbers (2 + 99 = 101). See a pattern? So how could they find the sum of all the numbers faster? Simply multiply 50 (the number of similar pairs) by 101 (the sum of each pair): 50 x 101 = 5,050.
- ✓ **Pose a challenge.** Many students play video games because they are challenging and engaging. They are motivated to keep playing. Challenging them in the mathematics class can be motivating, but the challenge should relate to the lesson objective and be within the students' abilities.
- ✓ **Show the practical applications of the topic.** Students often do not see how a particular topic in mathematics has any practical application. Showing the practical application whenever possible will raise student interest and motivation.
- ✓ **Entice the class with a surprising mathematical result.** The laws of probability can often surprise people. One classic example is the birthday phenomenon, which shows an unexpectedly high probability that two people in a relatively small group will have the same birthday. This one is always surprising to students when they encounter it for the first time.
- ✓ **Tell a historical story about the use of mathematics.** A historical story can be a motivating event. Stories can be about famous mathematicians or the mathematics involved in building the pyramids of Egypt or the Verrazano Bridge in New York City, one of the longest suspension bridges in the world. To be really interesting, the story should be recited slowly and be rich in detail.
- ✓ **Use games, puzzles, and other recreational activities in mathematics.** Games, puzzles, paradoxes, and the like can be really motivating for students, especially in this technology-rich environment. The activity should not be too complicated or time-consuming, and should have some relationship to the learning objective at hand.
- ✓ **Stimulate interest in mathematical curiosities.** Students can be motivated by trying to explain sets of numbers that have curious properties. Some of these are the Fibonacci series, the golden ratio (phi), Pythagorean triples (whole numbers that satisfy the Pythagorean theorem, such as 3, 4, 5; 5, 12, 13; etc.).
- ✓ **Mathematics in the movies.** Mathematics can be found in numerous movies. Some funny scenes come from old black-and-white

films (a novelty in themselves for some students). Examples include *In the Navy,* with Bud Abbott and Lou Costello, in which Costello tries to prove that 7 times 13 equals 28, or *Ma and Pa Kettle Back on the Farm,* which shows how 5 times 14 equals 25. Having students discuss these clever routines gives them a deeper understanding of mathematical processes.

Another activity is to determine if some of the stunts performed in movies are really possible. For example, in *Speed,* a bus full of passengers has a bomb onboard that will explode if the speed of the bus drops below 50 mph. The bus is traveling on an unfinished highway that has a 50-foot gap. Ask students to calculate at what speed the bus would need to travel to successfully jump this gap? Many mathematics-related film clips are available on the Internet. (Note: Be careful not to violate copyright laws. It is best to find the clip on the Internet and then purchase the DVD for classroom use.)

Any of these strategies can motivate students in mathematics. They may even prompt students to come up with other suggestions that teachers can use in the classroom. Bring mathematics to life, and help the students recognize that it is a human endeavor that enriches us all.

WHAT'S COMING?

Now that we have looked at the major components of learning, the next step is to decide how these components need to be adapted to serve the needs of preschool, preadolescent, and adolescent learners. Are there limits to what mathematics concepts we can teach at each developmental level? How will the Common Core State Standards for Mathematics affect how and what we teach? Are preschoolers even ready for mathematics? Is the preadolescent brain really mature enough to learn algebra? Are we sufficiently challenging adolescents in mathematics classes, or just scaring them off? The answers to these questions will emerge in the next three chapters. First, we start with preschoolers and kindergartners.

Chapter 3—Reviewing the Elements of Learning

QUESTIONS AND REFLECTIONS

Respond to the following questions, and jot down on this page key points, ideas, strategies, and resources you want to consider later. This sheet is your personal journal summary and will help jog your memory.

What seems to be happening to students' working-memory capacity?

What are the kinds of rehearsal, and why are they important to learning mathematics?

What effects does technology seem to be having on the developing brain?

4

Teaching Mathematics to the Preschool and Kindergarten Brain

I recommend you to question all your beliefs, except that two and two make four.

—Voltaire

Chapters 1 and 2 provided detailed discussions about the development of a child's innate capabilities for subitizing, estimating, counting, and doing simple arithmetic processing. By the time most children reach the age of 4, their interactions with the environment have offered them opportunities to practice these basic numerical operations. The purpose of preschool is to provide children a wide variety of learning experiences that they might not otherwise get. But should these experiences include mathematics? Is the brain of a 4-year-old sufficiently prepared to tackle numerical operations beyond a young child's limited inborn talents? Let's see what research is telling us.

SHOULD PRESCHOOLERS LEARN MATHEMATICS AT ALL?

Should there even be mathematics in preschool? Some people think that the preschool brain is not sufficiently mature to deal with number manipulation. But as we have already explained in earlier chapters, humans are born with intuitive capabilities for handling simple numerical quantities. Mathematics at the preschool level, then, should take advantage of a young child's innate number sense. This means that instructional activities should be deeper and broader than mere practice in counting and adding.

Researchers suggest the following reasons why basic mathematical concepts should be taught in preschool (Claessens, Engel, & Curran, 2014; Clements, 2001; Golbeck, 2005; Klibanoff, Levine, Huttenlocher, Vasilyeva, & Hedges, 2006; Purpura & Lonigan, 2013):

- Preschoolers already encounter curricular areas that include only a small amount of mathematics. Supplemental instruction would help make these areas more understandable.
- Many preschoolers, especially those from low-income and minority groups, have often experienced difficulties with mathematics in their later years. This potential gap can be narrowed by including more mathematics at the preschool level.
- Preschoolers already possess numbering and geometry abilities ranging from counting objects to making shapes. Children use mathematical ideas in their everyday life and can develop surprisingly sophisticated mathematical knowledge. Preschool activities should extend these abilities so these students can be successful with mathematics in kindergarten.
- Preschoolers can develop their mathematics-related vocabulary. When preschool teachers talk to young children about mathematics, the children will be better able to understand word problems, communicate their learning strategies, and discuss their solutions.
- Recent brain research affirms that preschoolers' brains undergo significant development, that their learning and experiences affect the structure and organization of their brains, and that their brains develop most when challenged with complex activities and not with rote skill learning.

Because the human brain is such a powerful pattern seeker, preschoolers are self-motivated to investigate shapes, measurements, the meaning of numbers, and how numbers work. Activities in preschool mathematics, therefore, should be designed to raise their intuitive number sense and pattern-recognition abilities to an explicit level of awareness. Teachers should not assume that preschoolers perceive situations, problems, or solutions the same way adults do. Clements (2001) reports how one researcher asked a student to count six marbles. Then the researcher covered the marbles, showed the student one more, and asked how many marbles there were in total. The student responded that there was just one marble. When the researcher pointed out that he had six marbles hidden, the preschooler replied adamantly that she didn't see six. For her, no number could exist unless there were objects to count.

Mathematics activities in preschool should raise children's intuitive number sense and pattern-recognition abilities to an explicit level of awareness.

Preschool teachers need to interpret what the student is thinking and doing, and use these interpretations to assess what concepts the student is learning and how they can be linked to the student's own experiences. Young students do not see the world as separate subject areas. They try to link everything together. Their play is usually their first encounter with mathematics, be it counting objects or drawing geometric designs.

Assessing Students' Number Sense

One of the first tasks of a preschool and kindergarten teacher is to determine the level of number sense that each student has already reached. Designing a number knowledge test is no easy task. Researchers Sharon Griffin and Robbie Case tackled this problem. Starting in the 1980s, they refined their assessments over the years on the basis of their research to ensure that the items reflected the capabilities possessed by a majority of students at ages 4, 6, 8, and 10 (Griffin, 2002; Griffin & Case, 1997).

By administering this test, a teacher can determine how far an individual student's number sense has progressed. The teacher can then use differentiated activities to develop the number sense for students of the same age who may be at different levels of competence. Table 4.1 shows the current version of the test for 4-year-olds, reprinted here with permission (Griffin, 2002). Tests for the other grade levels are provided in Chapter 5.

Table 4.1 Number Knowledge Test for 4-Year-Olds

Level 0 (4-year-old level): Go to Level 1 if 3 or more correct (see Chapter 5).	
1	Can you count these chips and tell me how many there are? (Place 3 counting chips in front of the child in a row.)
2a	(Show stacks of chips, 5 versus 2, same color.) Which pile has more?
2b	(Show stacks of chips, 3 versus 7, same color.) Which pile has more?
3a	This time I'm going to ask you which pile has less. (Show stacks of chips, 2 versus 6, same color.) Which pile has less?
3b	(Show stacks of chips, 8 versus 3, same color.) Which pile has less?
4	I'm going to show you some counting chips. (Show a line of 3 red and 4 yellow chips in a row, as follows: R Y R Y R Y Y). Count just the yellow chips and tell me how many there are.
5	Pick up all chips from the previous question. Then say: Here are some more counting chips (Show mixed array, not in a row, of 7 yellow and 8 red chips). Count just the red chips and tell me how many there are.

SOURCE: Griffin (2002). Reprinted with permission of the author.

Preschoolers' Social and Emotional Behavior

A young child's social and emotional functioning will have an impact on practically any content the child studies, including the development of mathematical competence. Not surprisingly, recent studies done with preschoolers show that those students who had initiative, self-control, and attachment, regardless of gender, were better at acquiring mathematics skills than were students with behavior, social, and attention problems. Moreover, students who received interventions designed to address their social and emotional problems improved their mathematics skills, as compared with the students who did not receive the interventions (Dobbs, Doctoroff, Fisher, & Arnold, 2006; Dobbs-Oates & Robinson, 2012; Garner & Waajid, 2012). The obvious implication here is that young students who consistently display social and emotional problems will need to have those problems addressed before we can expect them to successfully acquire and develop their mathematics skills.

> ***Answer to Question 6. False**: A young child's social and emotional functioning will have an impact on the development of mathematical competence and any content the child studies.*
>
> *Preschoolers with social and emotional problems will need to have those problems addressed before they can successfully develop their mathematical skills.*

WHAT MATHEMATICS SHOULD PRESCHOOLERS AND KINDERGARTNERS LEARN?

Preschool teachers do not always agree among themselves as to what is appropriate curriculum for this age group. A study of prekindergarten teachers revealed that the teachers' beliefs about the appropriateness of early mathematics instruction differed with the socioeconomic status (SES) of the children (Lee & Ginsburg, 2007).

Teachers of low-SES children at publicly funded preschools believed that

- mathematics education should be a priority or the children will have a difficult time catching up later,
- parents should be encouraged to work with their children at home,
- mathematics instruction should be organized around a theme or topic,
- teachers should use ready-made mathematics instructional materials,
- teachers should encourage even those children who show little interest to engage in mathematics activities, and
- computers are effective instructional tools.

Teachers of middle-SES children, on the other hand, believed that

- social development should be more of a priority than mathematics education and the children will be able to catch up later in kindergarten,
- parents demand a rigorous academic education in prekindergarten,
- curriculum should be based on the children's interests and adapted to their pace of learning,
- teachers should not use ready-made curriculum materials,
- teachers should postpone the introduction of mathematics activities when the children do not seem interested, and
- computers are inappropriate instructional tools.

Professional development programs will need to take into account these dramatically different views of what constitutes appropriate preschool instruction in mathematics. The quality of instruction should not be determined by children's SES, any more than it should be determined by their skin color. The National Council of Teachers of Mathematics' (2006) focal points are a good starting point for establishing a common instructional approach for preschool mathematics instruction, regardless of the children's SES.

The Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) do not address standards for preschool. However, *Curriculum Focal Points* (National Council of Teachers of Mathematics, 2006) suggests that prekindergarten children should be

- developing an understanding of whole numbers, including concepts of correspondence, counting, cardinality, and comparison;
- identifying shapes and describing spatial relationships; and
- identifying measurable attributes and comparing objects by using these attributes.

The Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) suggest the following standards for kindergarten children:

Counting and Cardinality

- Know number names and the count sequence.
- Count to tell the number of objects.
- Compare numbers.

Operations and Algebraic Thinking

- Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

Number and Operations in Base Ten

- Work with numbers 11–19 to gain foundations for place value.

Measurement and Data

- Describe and compare measurable attributes.
- Classify objects and count the number of objects in each category.

Geometry

- Identify and describe shapes.
- Analyze, compare, create, and compose shapes.

Mathematical Practices

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.

3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

More specifically, early childhood researchers and mathematics educators agree that preschoolers and kindergartners should be exposed to the following areas and skills:

- **Numbers.** Children learn about numbers by counting objects and talking about their results. "You gave Billy five cards. How many does Mary need?" Children count spaces on board games. "You are now on Space 3. How many more spaces do you need to go to get to Space 7?" They count the days until their birthdays. The teacher might say, "Yesterday there were 9 days until your birthday. How many days are there now?" Children read counting books and recite nursery rhymes with numbers.
- **Geometry and spatial relations.** Children practice constructing various shapes and discussing their properties. They can see thin triangles, fat triangles, and upside-down triangles, and gradually realize that they are all still triangles.
- **Measurement.** Children compare the height of a block tower with the height of a chair or table. They measure each other's heights and the distance from the desk to a wall. They learn that a block is too short or too long to complete a project.
- **Patterns/geometry.** Children become aware of patterns in their environment. They learn to recognize patterns of different colors and sizes in beads, blocks, and their clothes. They practice reproducing simple patterns by stringing beads and copying designs with colored blocks.
- **Analyzing data.** Children sort objects by color, size, and shape; count them; and record the data on graphs and charts. These charts might reflect how many bean plants have sprouted, the class pet's growth, the number of rainy days in March, or the number of children with a birthday in January.

PRESCHOOL AND KINDERGARTEN INSTRUCTIONAL SUGGESTIONS

General Guidelines

Here are some suggestions that should guide preschool and kindergarten instruction:

✓ Plan a learning environment conducive to mathematical explorations, including unit blocks and manipulatives. This includes building on the students' languages and cultural backgrounds, as well as their current mathematical ideas and counting strategies.

- ✓ Recognize whether a student's mathematical thinking is developing or stalled. When thinking is developing, you can observe and take notes, leave the students alone, and later talk with the students or the entire class to explain the mathematics involved. If the thinking is stalled, then intervene to clarify and discuss the ideas. For example, two students might be arguing about whose block set is bigger. It might be that one child is talking about height, while the other is looking at width or volume. Use this opportunity to discuss how size can be measured in different ways.
- ✓ Introduce activities that specifically rely on mathematics. For example, card games that use numbered cards and board games with number cubes offer students experiences with counting and comparison. Many students' books have mathematical themes that develop classifying and ordering, and strengthen students' number and geometric knowledge.
- ✓ Use a variety of instructional strategies that create meaningful age-appropriate contexts and require students to be active participants in their learning.
- ✓ Continue to enhance the students' thinking about mathematics by posing higher-order questions, such as "Have you tried this way?" or "What do you think would happen if . . . ?" and "Is there another way we could . . . ?"

Suggestions for Teaching Subitizing

In Chapter 1 we discussed the innate skill of subitizing, which is the ability to know the number of objects in a small collection without counting. If, as it seems, conceptual subitizing is the prerequisite skill for learning counting, then strengthening this skill should make learning to count easier for young students.

It may seem odd to suggest that it is possible to strengthen an innate ability, but we do this continually as we grow. Humans are born with the innate abilities to move and to speak, abilities that are strengthened through developmental learning experiences. The ability to subitize can be developed as well, and studies show that doing so significantly improves children's counting skills (e.g., Kroesbergen, Van Luit, Van Lieshout, Van Loosbroek, & Van de Rijt, 2009). Thus, preschool and kindergarten teachers should consider incorporating activities that strengthen this capability in their young students.

In searching for activities that strengthen subitizing, teachers should use cards or objects with dot patterns, and avoid using manipulatives. Why? The brain is a superb pattern seeker, and we want to take advantage of this capability by getting students to form mental images of number patterns. If students use manipulatives, they are more likely to rely on counting by ones rather than on mental imagery.

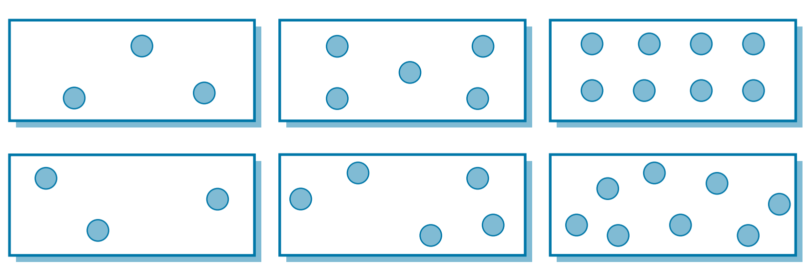
Subitizing is best practiced with dot patterns, rather than with manipulatives, to enhance imagery and eliminate counting by ones.

Clements (1999) suggests four guidelines that should be followed when designing activities that encourage conceptual subitizing in young students. The groups to be subitized should (1) stand alone and not be embedded in pictures; (2) be simple forms, such as groups of circles or squares rather than pictures of animals (which could be distracting); (3) emphasize regular arrangements that include symmetry; and (4) have strong contrast with the background. Here are some examples of activities for preschool and kindergarten students:

✓ **Dot patterns on cards.** Draw circles on cards (or punch holes in cards for use with an overhead projector). The circles (or holes) should be arranged in geometric patterns on some cards and randomly on others. Some examples are shown in Figure 4.1 (Clements, 1999).

- One activity uses cards with randomly placed dots and asks the students to say how many dots are on a card without counting them.
- Another activity is to play a matching game. Display several cards that have the same number of dots, except one. Ask the students to say which card does not belong with the others without counting the dots.
- Select decks of cards that have 0 to 10 dots arranged randomly and in patterns. Give a deck to each student. Ask the students to spread the cards out in front of them. Say a number and ask students to find the matching card as fast as they can and to hold it up. On other days, use different sets of cards with different arrangements.
- Display a card and ask students to say the number that is one more than the number of dots on the card. You can have them respond aloud, by writing down the numeral, or by holding up a numeral card. Remind them to try to avoid counting the dots.
- Place dots on a large sheet of paper or poster board in various arrangements. Point to an arrangement and ask the students to say its number as fast as possible. Each time you play this game, rotate the paper or board so it is in a different orientation.
- Another variation is to flash one particular pattern on the whiteboard or overhead projector for just 3 seconds. The goal here is to encourage the students to think about the parts of the image. Ask them to tell how many dots were shown and to describe what they saw. You may want to flash it a second time for 3 seconds to give them a chance to organize their images. That second look will be unnecessary once the students get better at recognizing patterns instantaneously. Timing is important. If you show the pattern for too long, the students will work from the picture rather than from their mental image. Showing it too briefly will not give them sufficient time to form the mental image (Kline, 1998).

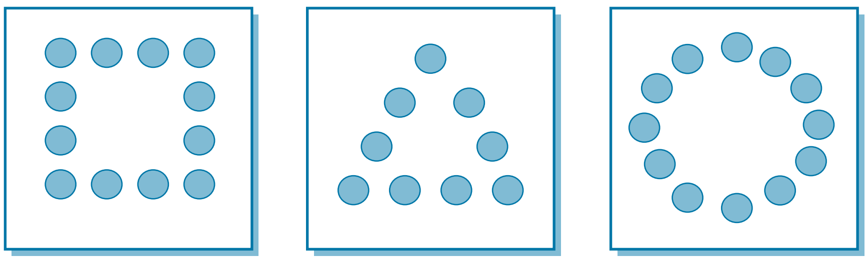
Figure 4.1 Cards like these that have dots placed randomly and in patterns help young children enhance their ability to subitize, which is determining the number of a collection without counting.



✓ **Visualization.** Subitizing relies heavily on visualization because the goal is to determine the number of a small group of objects with a quick visual glance and without counting. Visualization abilities develop rapidly in young children. Thus, activities that rely on visual cues enhance this development and allow students to make mental connections between patterns of objects and their numerosity.

- For instance, cards displaying dot patterns in specific geometric shapes (Figure 4.2) help students associate number and geometry by purposefully combining the two.

Figure 4.2 These types of cards combine numbers with geometry and are useful in developing conceptual subitizing in young students.

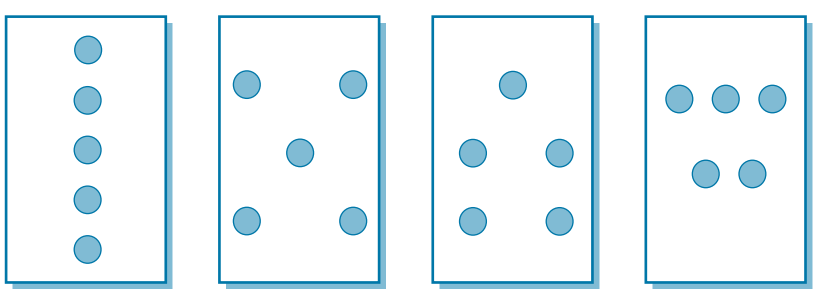


- Visuals also help young students see that different patterns can show the various ways a number can be partitioned, or decomposed (Figure 4.3). Through partitioning, students come to understand the idea that numbers can be broken down into other numbers. They also begin to recognize the relationship of parts to the whole. When students interpret numbers in terms of part–whole relationships, they think about numbers as being made up of other numbers, and this way of thinking is the major conceptual achievement of the early school years.

Subitizing and Understanding Part–Whole Relationships

Enhancing subitizing helps students as young as 4 understand part–whole relationships. Thus, preschool and kindergarten students should

Figure 4.3 Showing different arrangements of the same number of objects helps students recognize different decompositions, or partitioning, of a number (Clements, 1999).



be investigating different ways of splitting up numbers in a variety of contexts. If the quantities are small and the activities meaningful, the idea of the parts and the whole can be introduced successfully in the early childhood years. As students become better able to take numbers apart and put them back together without even thinking, they develop a fluency with small numbers that will help them later when working with larger numbers.

Two other ideas that are central to the notion of quantifying without counting (i.e., subitizing) are *covariation* (the idea that the whole quantity increases/decreases if one of the parts is increased/decreased) and *compensation* (the idea that removing some items from one part and adding them to the other part leaves the whole quantity unchanged). Studies have shown that children as young as 4 were able to give the correct answer on problems involving covariation and compensation. Furthermore, they were able to justify their answers by giving appropriate reasons for their responses. These results support the idea that children need experiences that draw their attention to the dynamic relationship between the parts and the whole, and the effect on the whole when there are changes to one of the parts. Having an understanding of part–whole relationships is important for learning numeracy. Counting is a valuable tool, but it need not be the first step toward the development of part–whole understanding. Early childhood teachers have an important role in helping students appreciate the ways numbers are composed of other numbers and how these part–whole relationships can be used to solve arithmetic problems (Young-Loveridge, 2002).

Using visualization to enhance conceptual subitizing eventually will help young students develop ideas about addition and subtraction. Visuals provide a basis for students to see addends and the sum by recognizing that two apples and two apples make four apples. Some students may advance to counting on 1 or 2, solving 3 + 2 by saying "3, 4, 5." Counting on 1 or 2 gives students an idea of how counting on works. Later they can learn to count on using larger numbers by developing their conceptual subitizing.

Studies indicate that the maximum size for objects to be subitized is about five, even for adults (Dehaene, 1997). Eventually, we have to deal with quantities that exceed our ability to subitize, and counting becomes necessary. But remember that persistent practice with activities that enhance subitizing will make it much easier for young students to count and to manipulate numbers for basic arithmetic.

Subitizing With Audio Input

Would adding audio input improve a child's ability to perform simple arithmetic by subitizing? The answer to this question is yes, and it came from a study conducted by Hilary Barth and her colleagues at Harvard University (Barth, La Mont, Lipton, & Spelke, 2005). In these experiments, 5-year-olds who had no real experience using number symbols were able to add two arrays of dots and compare them to a third array. When researchers replaced the third array of dots with beeps, the children integrated the sight and sound quantities easily. The children performed all these tasks successfully, without actually counting or having any knowledge of number symbols.

To study this ability further, the Harvard researchers investigated the responses of preschoolers to both visual and audio inputs. On the first test, the children were shown some blue dots. After these were covered, they saw red dots. They were asked if there were more red or blue dots. The preschoolers had no trouble answering correctly even when the difference was only a matter of a few dots. On another test, the children had to visually add two arrays of blue dots and compare them with the number of red dots in a third array. This they did without problems. Then sound was added. First, they compared numbers of dots with numbers of beeps. After that, they added two arrays of dots and compared them with a sequence of beeps. Surprisingly, the children added and compared dots and beeps as easily as they had dots alone.

At the conclusion of this study, the researchers reflected on how their findings could be used to help the many young children who experience trouble learning basic arithmetic. Devising new teaching strategies in elementary education that harness children's preexisting arithmetic intuitions can help them acquire knowledge about symbolic numbers and operations. They came up with two suggestions:

- ✓ First, youngsters who struggle with symbols for numbers might be encouraged and reassured if they discover that they can successfully play the kinds of games mastered by the children in the Harvard experiments. This play could show them that they already have the abilities they need to do the operations their mathematics teachers are presenting in the classroom.
- ✓ Second, joining nonsymbolic play with symbolic arithmetic problems could help children master the symbol system. Numerical symbols and operations may be less confusing to children if they are coupled with examples of sets of dots that are added, subtracted, multiplied, or divided—events they may already intuitively understand.

Learning to Count

There are several activities that can be used to help young children learn the principles of counting.

- ✓ Reinforce the cardinal principle with number line activities, using, for instance, chips that have different-colored sides. Lay the chips out with the same color facing up. Turn each chip as it is counted (Figure 4.4). Cover up a group that has just been counted and ask the students, "How many did you count?" Keep extending the wait time before asking the question. You can also ask the students to count the items in their hands and to put their hands behind their backs. Then ask them the "how many" question. Vary the activity by having students do counting without the number line and in various pattern arrangements (Solomon, 2006).
- ✓ Students also need to realize that number words describe "how many" objects and not their arrangement or size. This concept is reinforced when students have practice in counting objects that are arranged in different patterns (Figure 4.5) or a collection of objects of different sizes.

Figure 4.4 Using chips with different-colored sides can enhance students' understanding of the cardinal principle. Turn each chip over as it is counted, cover up the counted group, and then ask the students to say how many have been counted. The idea is to get students to recognize that the last number said indicates the number of total items in a group.

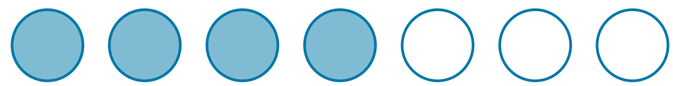
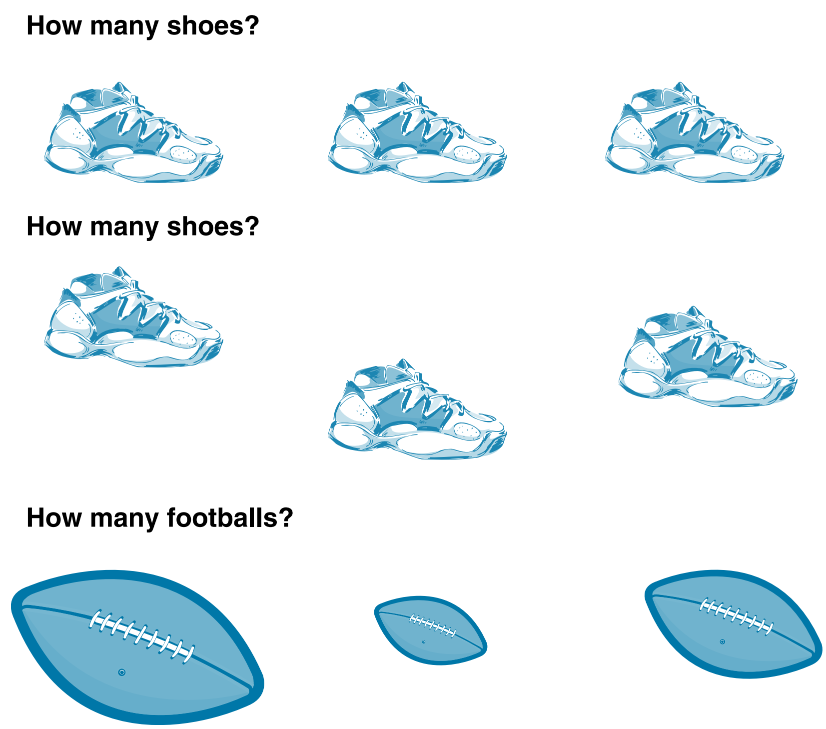


Figure 4.5 Activities that allow young students to practice counting objects in different arrangements and sizes help reinforce the notion that number is independent of other physical qualities.



By the age of 4, most children have mastered basic counting and can apply counting to new situations. Exactly how this competence emerges is not fully understood. It most likely starts with the genetic predisposition to understand the numerosity of a small group of objects in the environment. Reciting number words in a fixed sequence is a natural outcome of our facility with language. And the principle of one-to-one correspondence is actually widespread in the animal kingdom.

Once children learn to count using objects, the next challenge is learning to count mentally without objects. Children who have been practicing activities that enhance visualization are likely to learn mental counting more easily because it is so heavily dependent on imagery. The next step is the realization that when two quantities are joined, counting can begin from the last number of one quantity rather than starting all over from 1. This is called *counting on* and is an advanced strategy used by children to solve problems in addition.

An Easier Counting System

We discussed in Chapter 1 that young Asian children have a much easier time learning how to count because their language logically describes the counting sequence. Some early childhood researchers suggest trying out this approach with English-speaking children, using English counting words in a pattern similar to that used in some Asian languages. This method requires just 10 different words to count from 1 to 100, instead of the 28 English words needed for the traditional counting method.

Teaching how to count might be easier using a method modeled after some Asian languages, such as Japanese and Chinese. The system is easier and more logical, and gives students a deeper understanding of our base-10 system.

The numbers in this approach are not shorter or faster to say, but they make a lot more sense and help children get a deeper understanding of our base-10 system. Some people fear that adding this approach may be confusing, but schools in North America where this has been used have not reported any significant confusion. Children will learn the traditional counting words without any instruction through their exposure to adults, television, and other media. No one believes it is practical to suggest that this method replace the traditional one, but its use may help those students who struggle to understand our numbering system. Table 4.2 shows what the easier counting system from 1 to 100 would look and sound like.

Teacher Talk Improves Number Knowledge

A research study showed that preschool teachers who use numbers in their everyday speech aid the growth of their students' conventional mathematical knowledge over the school year. Klibanoff and her colleagues (2006) recorded the speech of 26 preschool teachers during a randomly selected hour of class instruction, including the times when the teacher gathered the class for a story or games. Although the teachers did not present planned mathematics activities during the recorded hour, many incorporated counting and even calculation into their speech.

The researchers assessed the students' skills in mathematics at the beginning and the end of the school year. Students who were in classrooms where the teachers talked a lot about numbers tended to improve more over the course of the school year than students who were less exposed to mathematics vocabulary. Furthermore, the improvements were unrelated to general teacher quality, the complexity of the teachers' sentence structure, or the students' socioeconomic status (Klibanoff et al., 2006).

Questioning

Questioning about numbers can be a very helpful tool for teachers as they assess the development of their students' mathematical thinking. It is

Table 4.2 A More Logical Counting System for Numbers 1 to 100

1 one	**2** two	**3** three	**4** four	**5** five	**6** six	**7** seven	**8** eight	**9** nine	**10** ten
11 ten-one	**12** ten-two	**13** ten-three	**14** ten-four	**15** ten-five	**16** ten-six	**17** ten-seven	**18** ten-eight	**19** ten-nine	**20** two-ten
21 two-ten one	**22** two-ten two	**23** two-ten three	**24** two-ten four	**25** two-ten five	**26** two-ten six	**27** two-ten seven	**28** two-ten eight	**29** two-ten nine	**30** three-ten
31 three-ten one	**32** three-ten two	**33** three-ten three	**34** three-ten four	**35** three-ten five	**36** three-ten six	**37** three-ten seven	**38** three-ten eight	**39** three-ten nine	**40** four-ten
41 four-ten one	**42** four-ten two	**43** four-ten three	**44** four-ten four	**45** four-ten five	**46** four-ten six	**47** four-ten seven	**48** four-ten eight	**49** four-ten nine	**50** five-ten
51 five-ten one	**52** five-ten two	**53** five-ten three	**54** five-ten four	**55** five-ten five	**56** five-ten six	**57** five-ten seven	**58** five-ten eight	**59** five-ten nine	**60** six-ten
61 six-ten one	**62** six-ten two	**63** six-ten three	**64** six-ten four	**65** six-ten five	**66** six-ten six	**67** six-ten seven	**68** six-ten eight	**69** six-ten nine	**70** seven-ten
71 seven-ten one	**72** seven-ten two	**73** seven-ten three	**74** seven-ten four	**75** seven-ten five	**76** seven-ten six	**77** seven-ten seven	**78** seven-ten eight	**79** seven-ten nine	**80** eight-ten
81 eight-ten one	**82** eight-ten two	**83** eight-ten three	**84** eight-ten four	**85** eight-ten five	**86** eight-ten six	**87** eight-ten seven	**88** eight-ten eight	**89** eight-ten nine	**90** nine-ten
91 nine-ten one	**92** nine-ten two	**93** nine-ten three	**94** nine-ten four	**95** nine-ten five	**96** nine-ten six	**97** nine-ten seven	**98** nine-ten eight	**99** nine-ten nine	**100** ten-ten

important that teachers follow up on a student's initial response if it will encourage additional explanation. This approach takes time, but studies show that when teachers take the time to examine a student's explanation through targeted questions, they can make more informed decisions about ensuing instructional strategies (e.g., Franke et al., 2009).

✓ Help young students think mathematically by asking questions about numbers and following up with an appropriate activity. Some examples follow (Burns, 1998):

- **"How many are there?"** Young students love to count but need practice learning the correct sequence of numbers. To develop an understanding of the *meaning* of numbers, they must learn about one-to-one correspondence—counting objects one by one, pointing to them as they say the numbers in the sequence. Students must also grasp the concept of cardinality—that the last number in the sequence tells how many objects there are. **Activity idea:** Play "How Many Buttons?" Ask the students to come to the front of the classroom one by one, and count how many buttons are on each student's shirt. This is a good way to introduce the idea that zero means none at all.

- **"How many of each kind?"** Students develop classification and counting skills as they think about this question. "How many?" asks them to count the number of items, but they must first sort the items to determine "each kind." Students learn that different types of things belong to different groups. **Activity idea:** Provide collections of almost anything in your classroom, such as buttons, crayons, blocks, markers, and beads, for students to sort and count.
- **"How are these items the same or different?"** To answer this question, the students look at two items and identify how they are alike or different. Answering the question requires students to observe, compare, analyze, and then reach a conclusion—the basic skills of mathematical and scientific exploration. **Activity idea:** Gather students in a circle and ask each child to remove one shoe. Pick up two shoes and ask, "How are these shoes the same?" Allow the students to share their ideas. Then ask, "How are they different?"
- **"Which has more or fewer?"** Comparing quantities is key to setting the stage for students' later thinking about subtraction. **Activity idea:** Play "Coin Toss." Give a student an odd number of pennies to toss, such as five or seven, to start. Ask, "Which are there more of, heads or tails?" This activity also helps students become familiar with coins.
- **"Which is taller, longer, or shorter?"** Young students are most comfortable comparing lengths by using direct comparison and matching up objects to see which is taller, longer, or shorter. If this is not possible, such as figuring out which table is longer when they are on different sides of the room, students can use a variety of nonstandard measures (paper clips, pencils, baby steps). Making direct comparisons and using nonstandard measures help prepare students for learning standard units such as inches, centimeters, and feet. **Activity idea:** Choose one length of ribbon from a basket of ribbons, and ask students to sort the rest according to whether they're longer or shorter.

Developing Sorting and Classifying Skills

Young children use sorting and classifying skills to help them organize the world around them. Both of these skills, which emerge around the age of 3, are essential in developing a child's understanding of the real world. As these skills develop, children begin to recognize the differences between plants and animals, day and night, and different geometric shapes. They enhance their number sense and their intuitive understandings about how to manipulate numbers during mathematical operations. As a result, they begin to apply logical thinking to the mathematical concepts they encounter. Platz (2004) offers the following suggestions on how to teach young children sorting and classifying skills.

Teaching young students sorting and classifying skills enhances their number sense and their intuitive understandings about how to manipulate numbers.

Sorting Versus Classifying. Sorting and classifying are terms often used synonymously, but they really represent two separate levels of logical thinking. Sorting is a beginning type of grouping task in which the student is told how the objects will be sorted: "Give me all the green blocks." Classifying, however, requires students to discover how a given set of objects might be grouped: "Look at these different blocks and show me how you could put them into groups." Unlike sorting tasks, the students are not told to put objects into groups based on a particular attribute. With classifying tasks, students must decide how the objects in each group might be alike.

✓ Keep the following key developmental factors in mind when selecting activities for young students on sorting and classification tasks.

- **Age.** Tasks that challenge a 3-year-old student may not be challenging to a 4-year-old student. Younger students may be assigned simple sorting tasks in which real objects, such as fruit, are shown and their task is to find all the objects in a group that are like the one shown. Older students may work with a set of attribute blocks and be asked to place the attribute blocks into different groups so the blocks in each group are alike in some way.
- **Perceptions.** How things look to students becomes the foundation for understanding their environment. As they engage in sorting and classifying, things that look more similar may be considered the same. Thus, for younger students, it is more helpful at first to use objects that are more dissimilar in their appearance.
- **Constructing information.** Because of their limited experience, young students construct information differently than adults. Adults may expect that a young student will sort or classify a group of objects by triangles and circles when the student actually groups the objects by things that roll or do not roll. Students often see categories or groupings that adults do not anticipate.
- **Tactile and kinesthetic tasks.** Even before young students learn numbers, using tactile or kinesthetic tasks with real objects permits signals about the numerosity of items to be sent directly to the brain. This has great value for learning mathematical concepts that include sorting and classifying.
- **Quantity of objects.** Three-year-old students are less likely to attend to sorting and classifying tasks if there are too many objects involved. Starting with four to five objects and increasing to six to eight objects should be sufficient when starting with sorting and classifying tasks.
- **Mathematical talking.** As students sort and classify objects, they should communicate their thinking aloud as to how they sorted or classified them. We know that task-related talking is important for learning the vocabulary of mathematics. Providing students the opportunity to communicate their actions can clarify mathematical terms and phrases.

- **Make it fun and offer choices.** Providing students with various opportunities to sort and classify in fun ways through individual play and group-time activities will promote healthy learning as they engage in these activities.

Levels of Sorting. Sorting tasks are excellent beginning activities for promoting understandings related to grouping. The teacher's responsibility here is to provide a set of objects to students and identify how the set is to be grouped. For example, the teacher may show students five different fruits and ask them to pick out all the red apples. In sorting, the tasks can become more challenging by increasing the number of objects to be sorted, by having students consider more attributes, and by giving verbal instead of visual clues. Platz (2004) suggests the following four levels for moving from simple to complex sorting tasks (Figure 4.6).

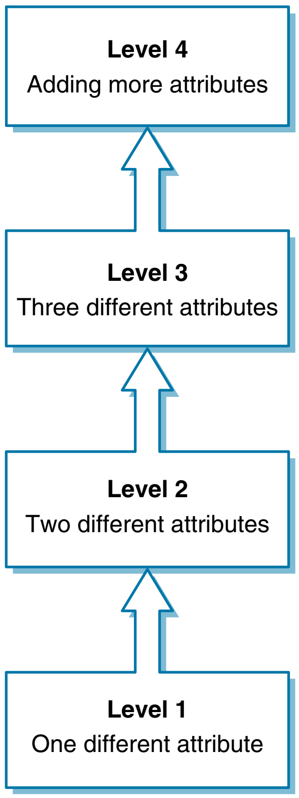
✓ **Level 1: One different attribute.** To start, the students complete a number of tasks with four or five objects that contain only one different attribute. For example, the student may be given all different shapes of the same color or size and be asked to indicate all the shapes that are like the one shown, perhaps a square. Students need to sort by shape only. At this level, show the object and ask the students to place all like objects in a container (or similar space). After the students learn shapes, ask them to select the squares—a verbal cue instead of a visual cue. The students should also explain why they picked out the objects they did. This communication component helps teachers gain insight into the students' rationale for picking the objects they did while helping students clarify their own understanding of the task. Increase the number of objects to five to eight as they work through Level 1.

✓ **Level 2: Two different attributes.** Provide students with six to eight objects that have different colors as well as different shapes. Show the students an object with two attributes, such as a red circle, and ask them to give you all the objects like the one shown. Include several different shapes and colors in the pile of objects used. Students who select all the red circles are classifying based on two different attributes. Again, provide students with the opportunity to talk about how they are sorting into groups. You can add more objects as the student becomes successful.

✓ **Level 3: Three different attributes.** The next step is to sort objects with three different attributes: color, size, and shape. Show a large green square and ask the students to give you all the objects like this one. Make sure that the pile contains a variety of shapes that represent the three different attributes.

✓ **Level 4: Adding more attributes.** Another attribute, such as thickness, is added as the students become more successful in their sorting. A different task might include sorting based on a function, such as with a spoon, fork, and knife. Have a group of objects containing some used for eating and some not used for eating, and ask students to give you things they would use to eat with. When sorting by function, ask the students to explain why they selected the objects for each grouping.

Figure 4.6 These are four levels of sorting tasks for children in preschool (Platz, 2004).



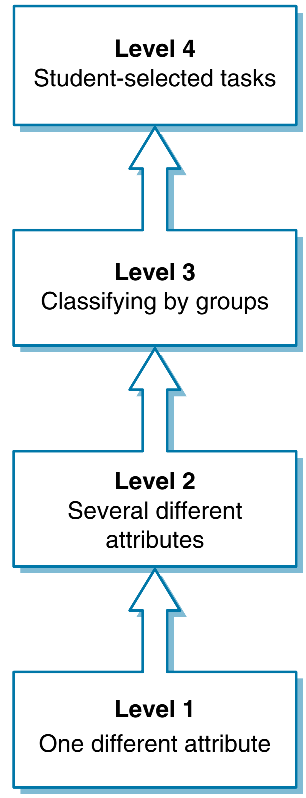
Levels of Classifying. As students become proficient in sorting tasks, teachers can introduce classification tasks. When using classification tasks, the students are not told how to classify. When asking students to classify sets of objects based on their thinking, teachers should also ask them to explain their reasoning behind the classifications they have made. Basic attribute blocks of different colors, sizes, and shapes are useful for the four levels of strategies for developing classification tasks (Figure 4.7; Platz, 2004).

- ✓ **Level 1: One different attribute.** Initial classification tasks follow similar strategies to those used in sorting tasks. Start with four to five objects that have only one different attribute. For example, the objects may have the same color and size but have different shapes. Ask the student to put objects into different piles that are alike in some way. If the students do not classify them by shapes, the teacher puts the objects back into a pile and asks them to show another way they can be classified. Any reasonable explanation by students as to how objects were classified is acceptable. As students work through Level 1, add more objects.
- ✓ **Level 2: Several different attributes.** The next stage is to give students a group of objects that have several different attributes and ask them to show several ways they could be classified into groups. For example, if given attribute blocks with different colors, sizes, and shapes, the students could first classify them by color, and then the objects could be put into groups of the same size or shape. Normally, young students will look first for the attributes of color and shape before the attribute of size. Here again, the students should explain why particular objects were placed into certain groups, giving you an insight into their thinking and an opportunity to ask for clarification, if needed.
- ✓ **Level 3: Classifying by groups.** This level challenges students to classify objects in such a way that the objects fit into a specific number of groups. For example, a collection may include objects with three different colors, two different shapes, and two different sizes. Ask the students to put the objects into three different groups so the objects in each group are alike in some way. Some students of 4 and 5 years of age who can complete classification tasks for Level 2 may struggle at first with Level 3 tasks. The intent of this level is to have students think logically about the possible ways a set of objects could be classified and deduce the one way that best fits the specific number of groups being sought.
- ✓ **Level 4: Student-selected tasks.** Beyond the teacher-directed tasks in Levels 1 through 3, provide opportunities for student-to-teacher and student-to-student classification tasks. A student can select a

group of objects for you or other students to classify based on a system the student has in mind. This reversal of roles provides students the opportunity to develop another level of understanding with regard to classification. By setting up tasks for others to solve, students perform classification thinking in new ways that add to, and clarify, their understandings.

By using sorting and classification tasks, teachers help develop students' thinking in terms of grouping and regrouping, which is important to learning mathematical operations. Selecting and developing tasks organized in some sequential manner will give students the opportunity to expand the ways they think about new situations and will assist them in organizing new information.

Figure 4.7 These are the four levels of classifying tasks for children in preschool (Platz, 2004).



WHAT'S COMING?

We have looked here at some basic activities in mathematics for students in preschool and kindergarten. The focus has been on assessing number sense, helping students learn to count, and developing sorting and classifying skills. With brain growth and development proceeding at a breakneck pace, students moving into the elementary and middle school grades are ready for more difficult and complex challenges. The next chapter examines the components and strategies that teachers of elementary and middle school students should consider when constructing lessons in mathematics.

Chapter 4—Teaching Mathematics to the Preschool and Kindergarten Brain

QUESTIONS AND REFLECTIONS

Respond to the following questions, and jot down on this page key points, ideas, strategies, and resources you want to consider later. This sheet is your personal journal summary and will help jog your memory.

What are some basic concepts to teach preschoolers in mathematics?

What are the advantages and disadvantages of teaching young children to count based on the Asian number system?

What are some sorting and classifying skills these children should learn?

5

Teaching Mathematics to the Preadolescent Brain

The different branches of arithmetic: Ambition, Distraction, Uglification, and Derision.

—Lewis Carroll
Alice in Wonderland

WHAT IS THE PREADOLESCENT BRAIN?

Exactly what is the preadolescent brain? Teachers and parents have been struggling with the answer to this question for a long time. Many ideas have emerged about the internal and external forces acting on the young brain as it develops through the preadolescent years. Now neuroscience weighs in with evidence from brain-imaging technologies that may shed more light on the mysteries of the preteen brain. Let's take a look at how nature (genetic contribution) and nurture (environmental influence) interact to sculpt the preadolescent mind. For our purposes, we will define preadolescence as the growth period between the ages of 6 and 12.

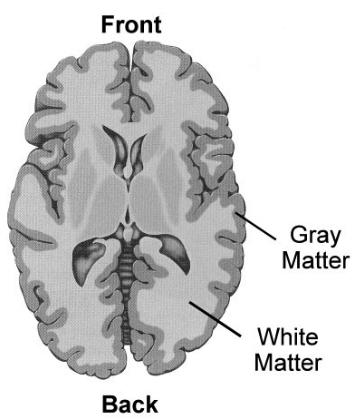
How Nature Influences the Growing Brain

Humans are born with certain innate abilities to help them survive during their long trek from infancy to adulthood. Our genetic predispositions for spoken language, numerosity, and social bonding, among others, allow us to develop the skills needed to remain alive and become a contributing part of a family unit. The brain guides these actions, and while it grows and develops, new capabilities emerge. But brain growth is not linear; it is sporadic, and different regions of the brain develop at different rates. At any given moment, some parts of the brain are already sufficiently developed to carry out their functions while others are just beginning to get their neural networks organized. Early learning, therefore, depends largely on the maturity level of those brain regions responsible for acquiring cognitive, motor, and emotional skills.

Gray Matter and White Matter

No doubt you have come across references to the brain's gray and white matter. But what are they, exactly? *Gray matter* (Figure 5.1) is the one-tenth-of-an-inch-thick, six-layered covering of the brain (also called the *cerebral cortex*) that contains mainly the cell bodies of neurons and their support cells. It is so named because it appears gray in preserved brains. The gray matter includes areas of the brain responsible for sensory perception, such as seeing and hearing, muscle control, speech, numerosity, and emotions. This cortex is where most conscious thinking, creating, and problem solving occur. Some recent research evidence suggests a positive correlation between the thickness of the cortex layer and certain aspects of intelligence (e.g., Burgaleta, Johnson, Waber, Colom, & Karama, 2014) and degree of attention (e.g., Ducharme et al., 2012).

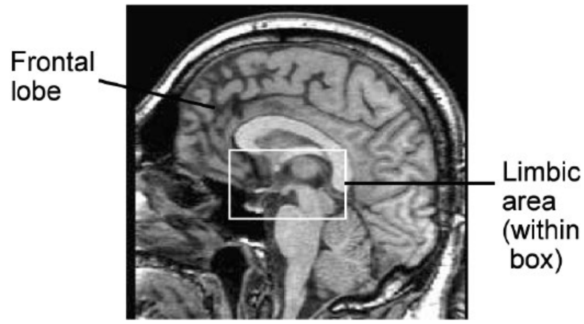
Figure 5.1 A horizontal cross section of the brain. The gray matter is composed primarily of neurons, and the white matter below is composed of myelinated axons.



The *white matter* is below the gray matter. It looks lighter than the gray matter in preserved tissues, due to *myelin,* the milky and fatty substance that surrounds the transmitting arm of each neuron, called the *axon.* Nerve axons are connected in a complex array of neural networks that relay information back and forth between the rest of the body and the cerebral cortex. These networks also interact with systems that regulate the body's autonomic (unconscious) functions, such as blood pressure, heart rate, and body temperature. Certain nuclei in the limbic system (Figure 5.2) are responsible for the expression of emotions, the release of hormones, and the regulation of food and water intake.

What Brain Scans Show. Longitudinal brain-imaging studies of individuals between the ages of 4 and 21 have revealed some interesting clues about how parts of the young brain develop. There may not be many surprises here for people who work with preadolescents, but it is interesting to see how examining brain development can explain the learning and behavior of these children. Table 5.1 shows the major imaging findings related to preadolescents, along with my own thoughts on some possible implications for learning and teaching (Gogtay et al., 2004). Findings related to the adolescent brain are discussed in the next chapter.

Figure 5.2 Emotional responses are generated in the limbic area. Moderating the intensity of those responses is one function of the frontal lobe.



Emotional and Rational Behavior

Deep within the brain lies the limbic area, which is largely involved in generating emotional responses. Because these emotions can get out of hand, one of the functions of the frontal lobe is to assess and control the types and intensity of emotions emanating from the limbic area (Figure 5.2). But this is

Table 5.1 Preadolescent Brain Development and Some Implications

Research Finding	*Possible Implications for Learning Mathematics*
The volume of gray matter and white matter continues to increase from childhood until puberty as the brain grows in size.	Children can tackle problems of increasing difficulty as they move through the intermediate grades. There is no "learning pause" in the intermediate grades as some people still believe.
At puberty, when the brain is nearly at its full adult size, gray-matter volume begins to decrease because unneeded and unhealthy neurons are pruned away.	By sixth grade, creative problem solving should start becoming easier, include more options, and show greater sophistication of thought.
Parts of the brain associated with basic functions mature early. Motor and sensory functions (taste, smell, and vision) mature first, followed by areas involved in spatial orientation, speech and language development, and attention (upper and lower parietal lobes).	Primary-grade children may have some difficulty solving complex visual–spatial problems. Boys may do better than girls at these types of challenges in the early grades, but the gap narrows in the intermediate grades. A multimodality approach is likely to be successful.
Later to mature are those areas involved in executive functions (creativity, problem solving, reflection, analysis), attention, and motor coordination (frontal lobes).	These skills are just emerging in the intermediate grades, so problems with multiple approaches and answers are a challenge but doable.
Most areas of the temporal lobes mature early. These areas are involved mainly in auditory processing. Maturing last is a small section of the temporal lobe involved in the integration of memory, audiovisual association, and recognition of objects.	Because the auditory areas are rapidly maturing, reading problems aloud is helpful. Three-dimensional object rotation and manipulation would be difficult for intermediate-grade students.

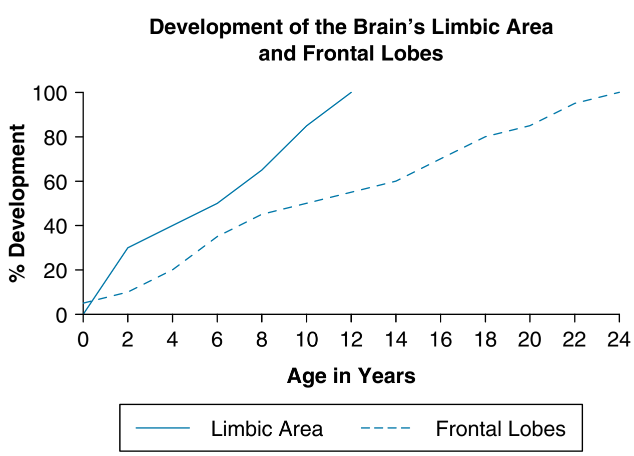
SOURCE: Gogtay et al. (2004).

hardly a balanced system in preadolescents. Among other things, human survival depends on the family unit, where emotional bonds increase the chances of producing children and raising them to be productive adults. The human brain has learned over thousands of years that survival and emotional messages must be given high priority as it filters through all the incoming signals from the body's senses. So it is no surprise that studies of human brain growth show that the emotional (and biologically older) system develops faster and matures much earlier than do the frontal lobes (Paus, 2005; Smith, Chein, & Steinberg, 2013; Steinberg, 2005). Figure 5.3 shows the percentage development of the brain's limbic area and frontal lobes from birth through the age of 24 years. The limbic area is fully mature around the age of 10 to 12, but the frontal lobes mature closer to 22 to 24 years of age. Consequently, the emotional system is more likely to win the tug-of-war for control of behavior during the preadolescent years.

Answer to Question 7. False: Emotional attention comes before cognitive recognition.

What does this mean in a classroom of preadolescents? Emotional messages guide the individual's behavior, including directing his or her attention to a learning situation. Specifically, emotion drives attention and attention drives learning. But even more important to understand is that emotional attention comes *before* cognitive recognition. For instance, you see a snake in the garden, and within a few seconds your palms are sweating, your breathing is labored, and your blood pressure is rising—all this before you know whether the snake is even alive. That's your limbic system acting without input from the cognitive parts of your brain (frontal lobe). Thus, your brain is responding emotionally to a situation that could be potentially life threatening, without the benefit of cognitive functions, such as thinking, reasoning, and consciousness.

Figure 5.3 Based on recent studies, this chart suggests the possible degree of development of the brain's limbic area and frontal lobes (Paus, 2005; Smith et al., 2013; Steinberg, 2005).



Preadolescents are likely to respond emotionally to a learning situation much faster than rationally. Getting their attention for a lesson in mathematics means trying to find an emotional link to the day's learning objective. Starting a lesson with, "Today we are going to study fractions," will not capture their focus anywhere near as fast as asking whether they would rather have one third, one fourth, or one sixth of a pie or pizza. Whenever a teacher attaches a positive emotion to the mathematics lesson, it not only gets attention but also helps the students see mathematics as having real-life applications.

Environmental Influences on the Young Brain

Part of our success as a species can be attributed to the brain's persistent interest in novelty—that is, changes occurring in the environment. The brain is constantly scanning its environment for stimuli. When an unexpected stimulus arises—such as a loud noise from an empty room—a rush of adrenaline closes down all unnecessary activity and focuses the brain's attention so it can spring into action. Conversely, an environment that contains mainly predictable or repeated stimuli (like some classrooms?) lowers the brain's interest in the day's lesson and tempts it to search elsewhere, such as in whatever technology the student is carrying, for novel sensations.

We often hear teachers remark that students are more different today than ever before in the way they learn. They seem to have shorter attention spans and grow bored easily. Why is that? Is there something happening in the environment of learners that alters the way they approach the learning process? Let's look at Table 5.2 and review the kind of environment the brain of today's preadolescent is facing, along with some of my thoughts on implications for learning. Note how students today are immersed in multimedia experiences and are acclimated to shifting among different tasks. They want to participate in their learning experiences. When students appear uninterested in a topic, it may be because the lesson is almost entirely teacher directed and there is little opportunity for active student participation. Furthermore, they may not see meaning in what is being presented.

Students today are immersed in multimedia, are acclimated to shifting among different tasks, and want to participate in their learning experiences.

In Chapters 1 and 2, we saw evidence that humans are born with an innate number sense that enables them to estimate small numbers of objects without counting (subitizing) and to understand some basic rules of numbers in base 10, using one-to-one correspondence with finger manipulation. Their brains are excellent pattern recognizers that seek meaning in whatever they encounter. Will the learning environment support or hinder the development of these innate capabilities? As children begin their formal schooling, they will find out there is a lot more to learn about manipulating numbers. How teachers introduce numbers in the primary grades will affect how children view mathematics later. Will we insist that they focus on memorizing symbols without understanding the numbers those symbols represent? Will we show them how to perform symbolic procedures, such as how to add fractions, in an essentially mindless fashion?

Because the human brain has superb adaptive capabilities, it can be coaxed to learn the procedures for manipulating symbols during the addition of fractions. After all, lots of school children get good grades in mathematics because they mastered a sequence of actions without any understanding of what they were doing. Many intelligent adults cannot add fractions, even though they could do so as children in school. What happened? If only they had really understood what was going on in their lessons on the addition process, they would never forget how to do it. Instead, they carried the rote procedures in working memory long enough to take the test. After that, the information was dropped out of working memory because it had no meaning.

Teaching for Meaning

We already explained in Chapter 3 the importance of teaching mathematics in a way that makes it meaningful for students. Recognizing that

Table 5.2 The Preadolescent Environment and Some Implications

Environmental Factor	*Possible Implications for Learning*
Family units are not as stable as they once were. Single-parent families are more common, and children have fewer opportunities to talk with the adults who care for them. Their dietary habits are changing as home cooking is becoming a lost art.	More preadolescents come to school looking to have their emotional needs met before they can focus on course content. Also, many have low blood sugar because they do not eat breakfast. Make sure they are fed, and make them feel that you really care about their success.
They are surrounded by media: cell phones, movies, computers, video games, e-mail, and the Internet, where they spend 17 hours a week, on top of the 14 hours a week they spend watching television.	Media are a part of their learning experience. Because most media are so interactive, students today want to participate in their learning experiences. Use all the technology and active participation you can in your lessons.
They get information from many different sources besides school.	Students are exposed to so much information outside of school that they come with many preconceived notions about numbers, geometry, and problem solving—not all of them accurate. Find out what they know and what their interests are, and use that information for motivation.
Children have become accustomed to their information-rich environment and its rapidly changing messages. It divides their attention, so they try to pay attention to several things at once, but they seldom go into any one thing in depth.	Modern technology has not shortened children's attention spans, but it has made it more difficult to get them to focus on one concept long enough to probe it in depth. That is one reason they are in a hurry to get the answer. By offering problems that can be solved in different ways, we force them to spend more time analyzing the situation as they look for various solutions.
They spend much more time indoors with their technology, thereby missing outdoor opportunities to develop the gross motor skills and socialization skills necessary to communicate and interact personally with others.	Look for opportunities to present and solve problems outdoors or in a large indoor area such as the gymnasium. Movement and greater social interaction stimulate long-term memory and create interest in the lesson.
Young brains have responded to technology by changing their functioning and organization to accommodate the large amount of stimulation occurring in the environment. In acclimating themselves to these changes, brains respond more than ever to the unique and different—or *novelty.*	Doing the unexpected is a form of novelty. Every day, students have a fairly accurate expectation of how their teachers will present lessons. Anytime you disrupt that expectation, you create novelty. How many ways can you think of to introduce different types of polygons?

meaning is a key criterion for long-term storage, teachers at all grade levels should *purposefully* plan for meaning in their lessons. We also noted that closure was a valuable strategy for helping students attach meaning to their new learning. Here are two basic ideas, with examples, about how to teach arithmetic for meaning using models (Dehaene, 1997) and closure (Sousa, 2011a).

Using Models

- ✓ **Use multiple models.** Arithmetic and mathematical knowledge should be based first on concrete situations rather than abstract concepts. Numerical representations help students develop mental models of arithmetic that connect to their intuitive number sense. For instance, a simple subtraction problem such as $8 - 3 = 5$ can be presented in different ways using concrete situations. It can be shown using a set-of-objects model: a box has eight toys. Take away three, and there are five toys left. It can also be applied to a temperature model: If it is only 8 degrees outside and the temperature drops 3 degrees, then it will be 5 degrees. A distance model is another option: In a board game, a chip moving from space 3 to space 8 requires five moves. While these examples may seem the same to an adult, they are new for a young student who must discover that subtraction is the arithmetic process applied to them all.

  The use of various models is important because relying on just one model may not be sufficient. Suppose you introduce negative numbers, for example, and you ask the class to compute $3 - 8$. A student who relies solely on the set-of-objects model will say that this operation is illogical and impossible because you cannot take eight toys away from three. But this problem would be logical using the temperature model, because most young students can comprehend the concept of negative degrees.
- ✓ **Select the correct model**. Children encounter fractions in real life long before they meet them in school. They have a few concrete examples, such as portions of pie or cake. When first confronted with the problem of adding the fractions 1/2 and 1/3, they can relate these numbers to their intuitive notions of sections of a pie. They may soon realize that these two portions will add up to just less than 1. However, children who have no intuitive understanding of fractions are very likely to simply add the numerators and denominators and get the incorrect result: $1/2 + 1/3 = 2/5$.

  This result is not as far-fetched as it seems, because it does have a concrete representation in the real world. If a baseball player gets one hit out of two times at bat, his average is 1/2. In his next game, if he gets one hit out of three times at bat, his average is 1/3. For both games, his total performance is two hits for five times at bat, or 2/5. Here is a situation where 1/2 "plus" 1/3 equals 2/5. How do you explain this seeming conflict? When teaching fractions, it is important to make clear to students that they should have the "portion-of-pie" model in their head, not the "scoring-average" model.

Using Cognitive Closure to Remember Meaning

Closure in a lesson does not mean to pack up and move on; rather, it is a cognitive activity that helps students focus on what was learned and whether it made sense and had meaning. Attaching meaning greatly increases the probability that the learning will be remembered. Remembering the meaning increases the likelihood that the learning will be used again in a new, future situation.

✓ One way to assist students in remembering the meaning of what was learned is to have them record key information in a paper or computer journal after each lesson in which something new was presented. It is important that they record the answers to these three questions:

- **What did we learn today?** This question ensures that the students have made sense of what they learned.
- **How does what we learned today connect or add to something we already have learned?** This question increases the likelihood that the new learning will be associated in memory with similar or related concepts, making future recall easier.
- **How can what we learned today help us in the future?** This question goes to the heart of meaning. The human brain is apt to save information that can be useful to its owner in the future.

Figure 5.4 Sample journal entry sheet to help students remember meaning in a lesson

Today's Lesson: *Commutative property in multiplication*

1. **Today I learned:** *That I can multiply numbers in any order and get the same answer, so 3 × 6 × 7 is the same as 7 × 3 × 6. This is known as the commutative property.*
2. **This connects/adds to what I know about:** *The same rule worked when learned in addition that 3 + 6 + 7 was the same as 7 + 3 + 6.*
3. **What I learned today can help me later when:** *I will be able to rearrange longer lists of numbers to add or multiply faster.*

This task should not take more than a few minutes (depending on the age of the students). Preprinted journal pages or onscreen templates can make this activity go faster with younger students. Figure 5.4 suggests one way this preprinted page or template could be organized and shows a sample student entry.

Mathematics becomes much easier when students understand what it is about and when the symbols have meaning for them and thus become a means to an arithmetic end rather than an end unto themselves. People who do not see meaning in arithmetic computations are often the ones who say they are not good at mathematics.

■ WHAT CONTENT SHOULD WE BE TEACHING?

Elementary and middle school teachers have expressed to me their concern that there are too many elements in the mathematics curriculum at each grade level and not enough time to cover them. In some cases, important concepts are not given enough time while less important topics are repeated in different grade levels. We have all heard of the "mile-wide and inch-deep curriculum." There was also the question of whether the topics at each grade level were consistent with the students' cognitive growth. This lack of focus led to a widely disparate mix of topics that varied among grade levels and school districts. Topics that were taught in Grade 5 in District A were presented in Grade 4 in District B and in Grade 6 in District C. Consequently, student results on national mathematics achievement tests in the elementary grades were often lower than expected, not because the students had low ability in mathematics but because they had not yet been taught some of the topics that were tested.

Common Core State Standards for Mathematics

Because of this lack of consistency and our students' disappointing performance in mathematics and reading on national assessments, state governors and school chiefs who are members of the National Governors Association and the Council of Chief State School Officers led an effort in 2009 to develop a consistent set of standards in K–12 mathematics (and English/language arts). These standards are known as the Common Core State Standards (CCSS; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) and were developed in collaboration with teachers, school administrators, and subject-area experts. The standards are research and evidence based, and are meant to reflect the skills needed for students to succeed in college, career, and life. In addition, they compare favorably with standards in other countries, particularly in those that consistently outscore U.S. students in international assessments. The standards focus on what students need to *learn,* but they do not tell teachers how they should teach; schools and teachers will decide how best to teach the standards. The idea is to develop higher-order thinking skills and to increase the rigor of instruction, especially given the complaints from professionals in higher education that an increasing number of students need remediation in mathematics (and reading) at the beginning of their college studies.

Nearly all the states have adopted these standards, and some are already implementing them in their respective school districts. Continued implementation is currently facing resistance from some political groups who see the standards as an inappropriate overreach of the federal government to gain further control over what is taught in the nation's schools. Supporters say that CCSS builds on the best of the already existing standards, with a few key shifts to improve rigor and critical-thinking skills. Furthermore, the standards may greatly benefit students with high mobility: students in Tennessee should be learning the same curriculum as those in Oregon or Minnesota. Regardless of how these different viewpoints get resolved, CCSS for Mathematics (CCSS-M) currently offers a reasonable, research-based set of learning objectives for K–12 students that are consistent with what we currently know about brain growth and development.

Key Shifts in CCSS-M

CCSS-M (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) includes three key shifts from previous standards to ensure that students are prepared in mathematics for college and career. These shifts provide a greater focus on fewer topics, link topics and thinking across grade levels, and increase academic rigor.

Greater Focus on Fewer Topics. The Common Core calls for greater focus in mathematics by asking teachers of mathematics to cut down significantly on the number of topics covered and spend more time with those that matter most. This means focusing deeply on the major work of each grade as follows:

Grades K–2: Concepts, skills, and problem solving related to addition and subtraction

Grades 3–5: Concepts, skills, and problem solving related to multiplication and division of whole numbers and fractions

Grade 6: Ratios and proportional relationships, and early algebraic expressions and equations

Grade 7: Ratios and proportional relationships, and arithmetic of rational numbers

Grade 8: Linear algebra and linear functions

This focus helps students gain a solid understanding of concepts, a high degree of procedural skill and fluency, and the ability to apply the mathematics they know to solve problems inside and outside the classroom.

Coherence: Linking Topics and Thinking Across Grade Levels. Too often, students see mathematics as a disjointed collection of concepts that have no relationship to one another. Mathematics is a coherent body of knowledge made up of interconnected concepts; therefore, the standards for mathematics are designed around coherent progressions from one grade level to the next. Topics are carefully connected across grades so students build new understandings on the foundations they built in previous years. For example, in fourth grade, students must "apply and extend previous understandings of multiplication to multiply a fraction by a whole number" (Standard 4.NF.B.4). This extends to fifth grade, when students are expected to build on that skill to "apply and extend previous understandings of multiplication to multiply a fraction or whole number by a fraction" (Standard 5.NF.B.4). Each standard is not a new event but, rather, an extension of previous learning. Coherence is also built into the standards in how they reinforce a major topic by using supporting, complementary topics. For example, instead of presenting the topic of data displays as an end in itself, the topic is used to support grade-level word problems in which students apply their mathematical skills to solve problems.

Rigor: Pursuing Conceptual Understanding, Procedural Skills and Fluency, and Application With Equal Intensity. Rigor does *not* mean making mathematics harder by introducing certain topics in earlier grades. Rather, it refers to deep, authentic command of mathematical concepts. To help students meet the standards, educators should pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, procedural skills and fluency, and application.

- **Conceptual understanding:** The standards call for conceptual understanding of key concepts, such as place value and ratios. Students must be able to understand concepts from a number of perspectives to see mathematics as more than just a set of mnemonics or discrete, rote procedures.
- **Procedural skills and fluency:** The standards call for speed and accuracy in calculation. Students must practice core functions, such as single-digit multiplication, to gain access to more complex concepts and procedures. Fluency must be addressed in the classroom or through supporting materials, as some students might require more practice than others.
- **Application:** The standards call for students to use math in situations that require mathematical knowledge. Correctly applying mathematical knowledge depends on students' having a solid conceptual understanding and procedural fluency.

TEACHING PROCESS SKILLS

Understanding how to manipulate numbers, detect and analyze patterns, solve problems, and apply mathematical knowledge to the real world requires the acquisition of certain process skills. CCSS describes five important processes that have long been part of mathematics education. These processes, which come from the National Council of Teachers of Mathematics (2000) *Principles and Standards for Mathematics,* are (1) problem solving, (2) reasoning and proof, (3) communication, (4) connections, and (5) representation.

The Common Core State Standards for Mathematics (CCSS-M) shift the focus of instruction to conceptual understanding, procedural fluency, and the ability to apply mathematical concepts and processes to solve problems.

Three areas related to the process standards that cognitive neuroscience has explored to some extent are number sense, estimation (a by-product of subitizing), and reasoning. When planning and presenting lessons for preadolescents, here are some questions teachers should ask about these skills.

Does the Lesson Enhance Number Sense?

Chapter 1 provides an in-depth look at that innate quantifying capability that we call number sense, particularly as it applies to cognitive development in a child's early years. We have already noted that cognitive neuroscientists view number sense as a biologically based innate quality that is limited to simple intuitions about quantity, including the rapid and accurate perception of small numerosities (subitizing) and the ability to count, compare numerical magnitudes, and comprehend simple arithmetic operations.

As we discussed in Chapter 2, mathematics educators have a much broader view of number sense than do cognitive neuroscientists. Because the development of number sense is not limited to the primary grades, teachers of mathematics at all grade levels should be determining which of the number-sense abilities are being addressed in each lesson.

Assessing Students' Number Sense

As teachers prepare lessons to develop number sense in their students, it is important for them to know the level of number sense that each student has already reached. We noted in Chapter 4 that designing a number knowledge test is no easy task but that researchers Sharon Griffin and Robbie Case started addressing this problem in the 1980s. While using their assessments for more than 20 years, they made refinements based on their research to ensure that the items reflected the capabilities possessed by a majority of students at each age level (Griffin, 2002; Griffin & Case, 1997). Other assessments that have subsequently appeared use a similar approach.

As teachers prepare lessons to develop number sense, they should determine what level of number sense each student has already reached.

By administering these tests, a teacher can determine how far an individual student's number sense has progressed at ages 4, 6, 8, and 10. The teacher can then use differentiated activities to develop number sense for students of the same age who may be at different levels of competence. Table 5.3 shows the current version of the test for ages 6, 8, and 10, reprinted here with permission (Griffin, 2002). The test for age 4 is provided in Chapter 4.

Table 5.3 Number Knowledge Test for 6-, 8-, and 10-Year-Olds

Level 1 (6-year-old level): Go to Level 3 if 5 or more correct.	
1	If you had 4 chocolates and someone gave you 3 more, how many chocolates would you have altogether?
2	What number comes right after 7?
3	What number comes two numbers after 7?
4a	Which is bigger: 5 or 4?
4b	Which is bigger: 7 or 9?
5a	This time, I'm going to ask you about smaller numbers. Which is smaller: 8 or 6?
5b	Which is smaller: 5 or 7?
6a	Which number is closer to 5: 6 or 2? (Show visual array after asking question)
6b	Which number is closer to 7: 4 or 9? (Show visual array after asking question)
7	How much is 2 + 4? (OK to use fingers for counting)
8	How much is 8 take away 6? (OK to use fingers for counting)
9a	(Show visual array of 8 5 2 6 and ask child to point to and name each numeral.) When you are counting, which of these numbers do you say first?
9b	When you are counting, which of these numbers do you say last?
Level 2 (8-year-old level): Go to Level 3 if 5 or more correct.	
1	What number comes 5 numbers after 49?
2	What number comes 4 numbers before 60?
3a	Which is bigger: 69 or 71?
3b	Which is bigger: 32 or 28?
4a	This time I'm going to ask you about smaller numbers. Which is smaller: 27 or 32?
4b	Which is smaller: 51 or 39?
5a	Which number is closer to 21: 25 or 18? (Show visual array after asking question)
5b	Which number is closer to 28: 31 or 24? (Show visual array after asking question)
6	How many numbers are there in between 2 and 6? (Accept either 3 or 4)
7	How many numbers are there in between 7 and 9? (Accept either 1 or 2)
8	(Show card: 12 54) How much is 12 + 54?
9	(Show card: 47 21) How much is 47 take away 21?
Level 3 (10-year-old level):	
1	What number comes 10 numbers after 99?
2	What number comes 9 numbers after 999?
3a	Which difference is bigger: the difference between 9 and 6 or the difference between 8 and 3?
3b	Which difference is bigger: the difference between 6 and 2 or the difference between 8 and 5?
4a	Which difference is smaller: the difference between 99 and 92 or the difference between 25 and 11?
4b	Which difference is smaller: the difference between 48 and 36 or the difference between 84 and 73?
5	(Show card: 13 39) How much is 13 + 39?
6	(Show card: 36 18) How much is 36 – 18?
7	How much is 301 take away 7?

SOURCE: Griffin (2002). Reprinted with permission of the author.

Developing Multidigit Number Sense

Students in primary grades have developed a notion of counting but have a difficult time studying subject matter that contains large numbers, such as the population of a country, distances to the planets and stars, and the cost of running a space mission. Although they are fascinated by large quantities, they have a limited understanding of them and often express exaggerated amounts in their conversation—as in, "There were thousands of people at my birthday party." When students lack an understanding of large numbers, they cannot reason effectively with the information they are given. In this situation, teachers need to develop the students' ability to process large numbers—that is, develop their multidigit number sense.

The concept of multidigit number sense refers to the students' understanding of, and flexibility in, using numbers of more than one digit. It includes intuitive feelings for large numbers and their uses, as well as the ability to make judgments about the reasonableness of multidigit numbers in different problem situations (Jones, Thornton, & Putt, 1994). As can be expected, research studies show that the progressing brain development of Grade 4 students allows them to deal with three-digit and larger numbers more easily than can students in Grade 2. This seems to be because manipulating numbers greater than two digits requires both sequential and parallel processing, a more sophisticated cerebral strategy (Mann, Moeller, Pixner, Kaufmann, & Nuerk, 2012; Meyerhoff, Moeller, Debus, & Nuerk, 2012). Because of the complexity of this topic, teachers should select meaningful activities that help students make sense of how large numbers are used in context.

> *Multidigit number sense allows students to acquire an understanding of large numbers and to make judgments about their reasonableness in different problem situations.*

Diezmann and English (2001) have found success working with students in the primary grades by selecting activities that help the students read large numbers, develop meaningful examples for large numbers, and understand large numbers that represent quantity, distance, and money.

- ✓ **Reading large numbers.** In this activity, students are introduced to the pattern for reading large numbers. Numbers of increasing magnitude are displayed for the students, starting with the ones column, progressing to the thousands column, and ending with the millions column. The name of each column is added to facilitate students' reading.
- ✓ **Developing physical examples of large numbers.** Concrete examples help students understand the nature of ever-increasing numbers. One activity to show visually the quantities 1, 10, 100, and 1,000 is to use colored sprinkles (confectionery decoration) on buttered bread that is cut into four pieces. The students add 1 sprinkle on the first piece of bread, 10 sprinkles on the second piece, about 100 sprinkles on the third piece (by estimating groups of 10), and about 1,000 sprinkles on the final piece (by estimating groups of 100).

The sprinkles activity provides a meaningful example for the students' understanding of the relative magnitude of numbers to a thousand. Some students may extrapolate beyond the physical examples and observe that you probably cannot fit 1 million sprinkles on one piece of bread.

✓ **Appreciating large numbers in money.** What size of container would be needed to carry a million dollars? Before solving this problem, the students should complete two tasks. The first involves making posters that are labeled with the amounts $1, $10, $100, $1,000, $10,000, $100,000, and $1,000,000. The students identify items in magazine and newspaper advertisements that cost roughly each of these amounts and then glue the pictures of items under the corresponding amounts. This activity raises students' awareness of the monetary value of expensive items. In the second task, the students calculate how much money is in a Monopoly game.

After completing these tasks, the students tackle the main problem of determining the container size needed to hold a million dollars. The students should use the Monopoly money to help them solve this problem. No containers are provided, as the students are encouraged to model different container sizes with their hands. Through discussion, the students should realize that there is more than one answer to the problem. For instance, the size of the container is dependent on the denomination of the notes that are used to make a million dollars. Some students may observe that a larger-sized container would be required if notes of low value were used and vice versa.

✓ **Appreciating large numbers in distance.** How far away are the brightest stars? The purpose of this activity is to develop the students' understanding of large distances within the context of space travel. One approach is to have the students make 10 paper stars and label them with the names of the 10 brightest stars in the sky, their brightness, and their distance from Earth. The stars can be fastened onto upturned paper cups for ease of mobility. The students initially arrange the stars by order of brightness, beginning with the brightest star.

Next, consideration is given to the stars' distances from Earth. After the students discuss the notion of measuring stellar distances in light-years, they rearrange the stars in order from the closest to Earth to the most distant. Extend the activity by asking the students to discuss whether there is a relationship between the brightness of a star and its distance from Earth.

To represent the stars' relative distances from Earth in light-years, draw a timeline and mark it in 100s, from 0 to 1,000. Ask the students to position each star at the correct number of light-years from Earth. Then they can discuss the fact that when we see a star today, the light from that star was actually emitted many years ago. Older students may be able to connect the year when light was emitted from particular stars to significant historic events on Earth. In this way, students make links between their mathematical understanding and their scientific knowledge.

Does the Lesson Deal With Estimation?

A close correlate to number sense is estimation.

Estimation is an extension of the brain's innate ability to subitize. Estimating how many animals to hunt or how many crops to plant to feed the village was a survival skill. Our ancestors were good at it. Are we?

Mathematics educators often comment on the poor estimation skills of students. A frustrated teacher once told me that a middle school student felt very pleased with himself after calculating the size of a molecule to be just over 1 meter in length. The unreasonableness of this measurement never occurred to him. Yet, ironically, youngsters often successfully use estimation skills *outside* of school. For example, they can quickly make the computations needed to cross a street with traffic, decide if a sibling is sharing equally, or accurately throw, catch, or hit a ball in sports. Poor estimation skills, it seems, are more likely to appear *inside* school when dealing with arithmetic estimation, and they can result from at least three factors.

- First, students at an early age are programmed to get the exact answer to a problem; so they have few experiences with estimation. Furthermore, activities that ask students for both an estimated and exact answer undermine the value of estimation. Why should students estimate if they are going to find the exact answer, too?
- Second, when students use a calculator in their work, they assume the calculator's answer must be right, with no thought that they could have inadvertently entered an incorrect number or a misplaced decimal. Consequently, they rarely reflect on the reasonableness of their answers.
- Third, students want to get the answer quickly and avoid estimation because it often takes more time.

Activities involving estimation should begin as early as possible in the primary grades. However, they should not be isolated as a single unit of instruction but, rather, should be taught in the context of other mathematics skills throughout all grade levels. If we want to emphasize the value of estimation, then students should be given assignments that require them only to estimate.

Estimation is an extension of the brain's innate ability to subitize. Estimating how many animals to hunt or how many crops to plant to feed the village was a survival skill. Our ancestors were good at it. Are we?

Methods of Estimation

The common methods of estimation include (1) rounding, which involves finding a number to the nearest 10, 100, or 1,000, or the nearest one, tenth, hundredth, or thousandth; (2) front-end estimation, which entails computing the higher place values or leftmost digits, then adjusting the rounded sum using the lower place values or digits to the right; and (3) clustering, which involves grouping numbers and is useful whenever a group of numbers cluster around a common value. These methods of estimation are most helpful when students are doing computational tasks. They can check whether their answers come close to the estimated answer and use it to determine if their answer makes sense.

Students need to be aware that methods of estimation may not work in the real world. If you want to buy a shirt for $17.45, rounding down to the nearest dollar will not give you enough money to buy it. This is also true for estimations related to measurement. If you need exactly 3 1/4 yards of fabric to make a dress, you will not succeed if you round down to just 3 yards. Thus, rounding down for estimations of quantity in real-life situations is often impractical. So what other types of estimation are available?

Types of Estimation

Taylor-Cox (2001) suggests four distinct types of estimation:

1. **True approximations** are used when an estimate is acceptable, especially when dealing with very large numbers. Is it really important to know that the average distance from Earth to the sun is 92,955,630 miles, or will 93,000,000 miles suffice? True approximations are more applicable to problems in the intermediate and upper grades. Unfortunately, the numbers that youngsters work with in the primary grades are typically smaller and less complex than the numbers that lend themselves to true approximations. Making true approximations has little advantage with simpler numbers because we can easily calculate to ensure that we know the exact amount.
2. **Overestimating** is used when rounding up might be beneficial, such as overestimating the amount of food to buy for a child's birthday party. The major drawback for this option is that it may be wasteful if you get way too much. But if you underestimate, some kids may not get enough food.
3. **Underestimating** is used when rounding down is applicable. This can be helpful in certain situations. For example, better to underestimate the amount of profit a business will make so as to avoid overspending.
4. **Range-based estimations** broaden the applicability and understanding of estimation. Some situations call for underestimating and some for overestimating. Range-based estimation involves thinking about quantity in terms of the upper end and the lower end that encompass an estimate: "What are the minimum and maximum quantities I need for this?" In the primary grades, teachers can design mathematical tasks that are productive and worthwhile by using range-based estimation, thereby encouraging students to become better estimators.

Meaningful Estimation Activities

For estimation activities to be meaningful rather than futile, Taylor-Cox (2001) suggests that the activities include the following five components:

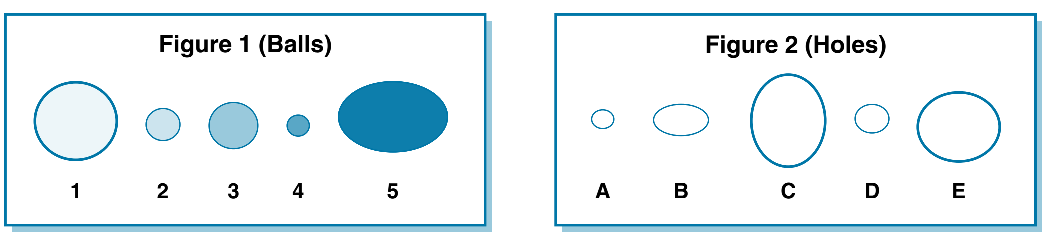
- ✓ **Purpose.** Whenever you ask students to estimate a number, give them a reason for doing so. These contexts offer a purpose and give students a reason to engage in real-life mathematical problem solving. Otherwise, students may ask, "Who cares?" Making the task relevant, interesting, and significant invites students to care and, consequently, to engage in mathematics. Offering a purpose does not ensure that the "Who cares?" response will disappear. But by listening to students and reflecting on their perspectives and feelings, you can manage the continuing challenge of providing meaningful mathematics.
- ✓ **Referents (benchmarks).** To help students succeed, give them a referent or benchmark they can use when making estimations. For instance, if you ask students to estimate the number of marbles in a

jar, it would be helpful to provide a smaller container with a known number of marbles of the same size. This container gives students a point of reference on which to base their estimates for the larger jar.

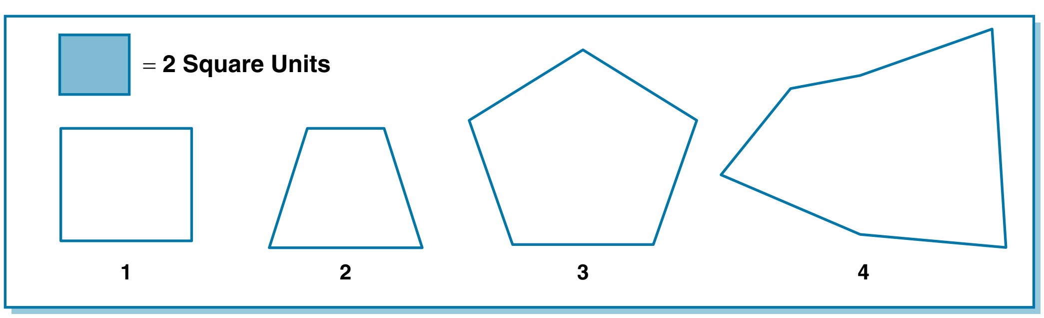
- ✓ **Pertinent information.** Clarify the actual mathematical problem to be solved so the students can decide what type of estimation is most appropriate. For example, students do not need to estimate the number of marbles in a jar if they are going to open the jar and count the actual number of marbles. As explained earlier, doing so counteracts the purpose of, and time spent on, estimating. Rather, you should ask whether estimating or counting to find out the actual number is more appropriate. Which methods will be used to check for accuracy? What kind of information is pertinent to the given mathematical situation?
- ✓ **Diverse experiences.** Students need numerous and diverse experiences with estimation in the context of other content areas, such as in time and measurement. Primary-grade students often have difficulty estimating time. Teachers are no help when they attach inaccurate time constraints to their statements. When we say, "I will be there in just a minute" or "Wait one second," we really mean, "I will be there when I can" or "Wait indefinitely." Perhaps asking students to time the teacher would encourage them to check estimates of time while enhancing their experiences and improving their precision with estimating time.

Young students work on measurement skills by comparing lengths, weights, and capacities. For estimating size they use comparative language, such as *larger, smaller, heavier,* and *lighter.* Figure 5.5 shows two examples of

Figure 5.5 These are sample activities that develop estimation skills in different grade levels.



Primary grades: Which balls in Figure 1 can fit through the holes in Figure 2?



Intermediate grades: Estimate how many square units are in each of the four shapes.

activities for different age groups that require estimating size. In these types of tasks, students engage in estimation that is related to size rather than quantity and recognize that estimation is an important tool for dealing with real-life mathematical experiences.

- ✓ **Range-based techniques.** Estimation should involve using mathematical skill to predict information within a reasonable range. If, for example, the actual number of a quantity or measurement is in the 70s, an appropriate estimation may be in a range of 10 or less. But if the actual number is in the 700s, an appropriate range may include up to 60 or 70 numbers. Although suitable ranges vary with the problem situation, the aim is to estimate within an appropriate range. However, many students still want to estimate the precise amount. To combat this need for the right answer, it may help to use terminology such as the *actual answer.* A range of about 10 to 20 per hundred is reasonable. This type of ranged-based estimation is particularly helpful in situations that call for approximating a quantity that may need to be overestimated.

Estimation experiences improve the students' estimation skills, increase their confidence in their level of mathematical expertise, enhance their perception of the value of mathematics, and improve their mathematics achievement test scores (Booth & Siegler, 2006). Each estimation activity is an opportunity for teachers to connect mathematics with the everyday lives of students.

From Memorization to Understanding

We discussed earlier in this chapter the importance of teaching children the meaning of what they are doing when they manipulate numbers during arithmetic computations. Meaning not only increases the chances that information will be stored in long-term memory but also gives the learner the opportunity to change procedures as the nature of the problem changes. Without meaning, students memorize procedures without understanding how and why they work. As a result, they end up confused about when to use which procedure. Teachers who use primarily a declarative approach emphasize not only arithmetic facts but how they are related to one another and connected to other concepts the students have already learned. They use elaborative rehearsal and provide for cognitive closure.

Are We Teaching Elementary-Grade Arithmetic for Understanding?

In some schools, we teach too much arithmetic through procedural approaches and very little with declarative methods. Could it be because that is how most teachers learned arithmetic themselves? Could this explain why arithmetic instruction in the primary grades has not changed very much over the years? We teach students a procedure for solving computation problems, which they then repeatedly practice (procedural memory). But the practice does not result in computational fluency because we rarely talk about how and why the procedure works. Consequently, when we give the students a problem to solve, they reflexively draw on their knowledge of the practiced procedure and apply that procedure quickly and efficiently, but with little understanding of the mathematical concepts involved.

To develop understanding and meaning, teachers should show students (at the earliest possible age) why they are performing certain arithmetic operations.

Of course, students need to learn some basic procedural activities, such as memorizing a short version of the multiplication tables mentioned in Chapter 2, along with a few number facts. But the emphasis should be on showing students (at the earliest possible age) *why* they are performing certain arithmetic operations. The more arithmetic we can teach through declarative processes involving understanding and meaning, the more likely students will be to succeed and actually enjoy mathematics.

Example of a Declarative-Based Approach

A declarative-based approach focuses on capitalizing on the students' innate number sense, intuitive notions of counting by finger manipulation, and an understanding of a base-10 model for expressing quantities. It includes allowing students to create their own procedures for arithmetic computations so they truly understand the algorithms involved. Researchers have long recognized, and recent studies confirm, that students in the primary grades are capable of constructing their own methods of computation (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Fuson et al., 1997; Guerrero & Palomaa, 2012; National Science Teachers Association, 2014). In doing so, primary-grade students pass through three predictable developmental levels:

- At the first level, students deal with all the quantities in a problem. To add a group of objects, they count out separate groups of objects, combine the groups, and then recount the total. To subtract, students count out and separate a group, and then recount what is left.
- At the next level, students consider all parts of the problem before solving it. They demonstrate this ability by counting on from, or back to, a quantity to determine an answer.
- At the most advanced level, students use abstract knowledge and consider quantities in flexible ways. They make use of what they already know to solve new problems. For example, students might use their prior knowledge to realize that $6 + 7$ is equal to $6 + 6 + 1$, or that $7 + 9$ is equal to $6 + 10$, by decomposing and recombining tens and ones.

Understanding the development of mathematical thinking in young students allows teachers to anticipate procedures that students are apt to invent, and find ways to support students as they progress through the different levels. When teachers encourage students to invent alternative problem-solving strategies, the learning objectives are different from those that result from instruction using standard memorization procedures. The emphasis is on making sense and finding meaning in the methods that students create and successfully use (Scharton, 2004).

Mathematics educator Susan Scharton (2004) has been a strong advocate for giving primary-grade students opportunities to solve computational problems, create their own procedures for solving them, and explain their methods to others. She found that this approach improved the students' accuracy, as well as their understanding of the methods they had created. When she asked students to explain their methods, their understanding of their own procedures deepened as a result of this elaborative rehearsal. Listening to the methods that others had used prompted some students to experiment with other students' methods of computing.

One example that Scharton (2004) has used to demonstrate how students can resort to different solution strategies involves the following problem: "Paul has 28 markers. He got 34 more. How many does he have?" One second-grade student wrote 34 under 28 and attempted classic addition of two-digit numbers in a column. Here is what he wrote down and then stopped:

$$\begin{array}{r} 1 \\ 28 \\ +\ 34 \\ \hline 2 \end{array}$$

He tried to follow a standard procedure that typically begins by combining the numbers in the ones column. However, he did not understand how this step in the standard procedure works when the resulting sum is a two-digit number: He was unsure about where to place which digit. He treated each digit separately but did not associate each digit with a value. He was trying to use an efficient procedure that he did not understand and lacked confidence in his method because he could not recall the steps he thought he had learned. If he had devised his own solution strategy, he would have been forced to rely on his own understanding of the numbers and operations he was using, rather than on someone else's.

Another second-grade student, who came from a class that encouraged sharing problem-solving approaches, used a different method. She was able to accurately and efficiently solve the problem using the method she described in Figure 5.6. Her process demonstrated essential aspects of number sense and place–value relationships. She decomposed numbers into tens and ones, and then recombined these parts. She understood that each digit in a two-digit number has a different and separate value. She recognized the commutative and associative properties of the problem, and she created her own sequence of steps to follow that were comfortable and meaningful to her. Not only could she use a method that made sense, but she could clearly explain in writing why and how her method worked.

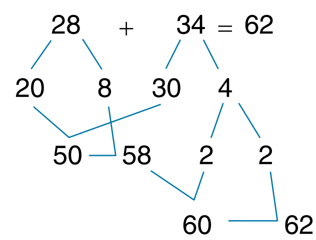
An Instructional Model for First and Second Grade. Hoping to encourage young students to develop their own procedures for solving arithmetic problems, Scharton (2004) devised an instructional model in which students in first and second grade alternate between small-group sessions and whole-class discussions.

- ✓ The model focuses on the students' invention of meaningful computation methods and how they can effectively communicate these methods to one another through discussions and written work. Here's how it works:

Figure 5.6 A second-grade student describes her method for solving the addition of a pair of two-digit numbers. (Adapted from Scharton, 2004)

What I did was I put 20 from 28 together with 30 from 34 and got 50. I put 50 and 8 together and 58. Then I broke up 4 into 2 and 2. Then I put 58 plus one of the twos and got 60. I put 60 plus the other two and then I got 62.

$20 + 30 = 50$
$50 + 8 = 58$
$58 + 2 = 60$
$60 + 2 = 62$



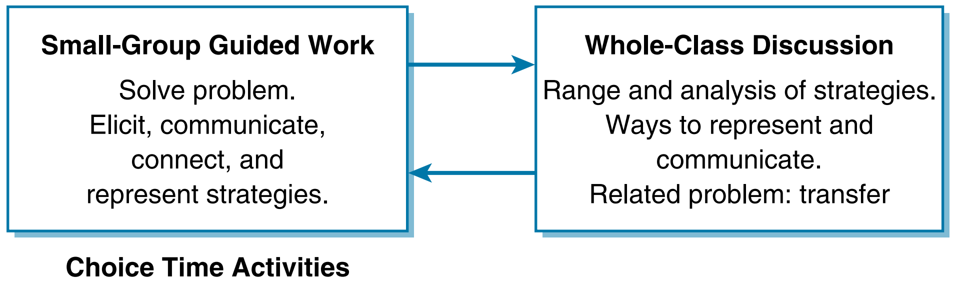
- **Small-group work.** Students first participate in a small, heterogeneous group of four to six students every week, usually during choice time. Give the group an arithmetic problem to solve. Meanwhile, the rest of the class works on activities that expose students to different types of problems and promote the development of various invented strategies for solution. In the small group, students independently solve the problem using methods they have invented or have learned from other students. They explain their methods to other group members (elaborative rehearsal) and discuss how their methods are similar to and different from one another, as well as the ways the methods are related. Students write down the ways they solved the problem.
- **Whole-class discussion.** Following the small-group session, the whole-class discussion allows students to explain a range of methods to the larger group. Class discussions should focus on the efficiency of each method, the relationships between the methods, ways to effectively represent and communicate them, and how and why each method is successful. After the whole-class discussion, give students another similar problem to solve to determine the degree of transfer—that is, which strategies the students will apply to solve the new problem. The students repeat this cycle weekly. Figure 5.7 illustrates this instructional model.

Too often, the goal of arithmetic instruction in most primary classrooms is the accurate and rapid use of a teacher-demonstrated algorithm. Scharton's model, however, helps young students build fluency by inventing their own computation procedures, explaining these procedures clearly to their peers, and analyzing their procedures for relevancy, efficiency, and effectiveness.

Other Strategies to Enhance Understanding

- **Connecting new learning to past learning.** The brain is more likely to understand new learning if it can link it to something already in

Figure 5.7 This instructional model by Scharton (2004) for first and second grade encourages students to develop strategies in small-group settings and to share them in whole-class discussions. (Used by permission of the author)



its long-term memory. This is known as *transfer* and is one of the most powerful learning principles. Transfer also refers to how students can use their new learning in new situations. Whenever possible, teachers should look for ways to activate the students' relevant background knowledge before introducing a new mathematical concept. Using a K-W-L chart can often facilitate this process. Students become aware of what they already know (K), determine what they want to know (W), and discover ways of applying what they learned to new settings (L). The K-W-L chart will also inform the teacher about what the students already know, which may allow skipping past certain topics and make more time to delve deeper into the new and essential learnings.

- **Predicting patterns.** Remember that the brain is programmed to detect patterns. Mathematics is fundamentally a study of patterns, from Pascal's triangle to the Fibonacci sequence. Help students use this innate pattern-seeking ability by giving them some information about a new topic and asking them to identify any patterns. For example, in the lower grades, ask the children to shout out the missing number in a sequence—say, the times table: 4, 8, 12, 16, 20, 28, 32, . . . The first student to call out "24" wins. For upper grades, use more complicated number patterns, such as 1, 2, 4, 7, 11, 16, 22, . . . (adding one more each time: 1 + **1** = 2, 2 + **2** = 4, 4 + **3** = 7, 7 + **4** = 11, 11 + **5** = 16, 16 + **6** = 22, etc.), or 32, 16, 8, 4, 2, 1, 1/2, 1/4, 1/8, . . . (halving each time).
- **Questioning.** We discussed in Chapter 4 the value of using questioning to encourage students to reflect more on the concepts they are learning. In addition to the questions teachers ask, students should get into the habit of asking themselves these important questions when moving into a new area: What do I already know (K)? What do I need to find out (W)? Are there any special conditions to consider (C)? Modify the K-W-L chart to a K-W-C chart for this activity so students can keep a written or computer record of their progress toward the learning objective.
- **Visualization.** Students can often get a better understanding of the complexities of a problem if they can visualize it. This makes it easier to represent the mental image on paper or on a screen. Furthermore, this strategy helps teachers move from the concrete to

the abstract when dealing with higher mathematics. Studies show that regularly using visual images increases student performance in mathematical problem solving (e.g., van Garderen, 2006). The visualization strategy seems to be particularly helpful in improving the problem-solving accuracy of students with mild learning disabilities (Krawec, 2014).

Multiplication With Understanding

An elementary school principal once told me about conversations she had with parents regarding the third-grade mathematics curriculum. The parents felt that there should be heavy emphasis on memorizing multiplication facts. To them, third-grade mathematics should include memorizing facts through drill and practice, worksheets, flash cards, and other memorization aids. But this school principal was promoting an approach that encouraged problem solving and understanding. She explained to the parents that this approach would help children remember the processes of multiplication for a much longer time. She recounted from her own experiences that students who had mastered their multiplication tables during third grade were barely able to remember them the following year. Apparently, memorizing multiplication facts during third grade had accomplished little because it did not build understanding of multiplication concepts. Despite having experienced a "back-to-basics" curriculum, they still did not know what multiplication was.

Students typically develop the ability to add quite naturally, but multiplication is much more complex than addition and requires guidance to understand the actions that are important elements of the process. By memorizing facts before developing an understanding of multiplication, students get a mistaken impression about the need to understand what it means to multiply and the situations in which multiplying is the appropriate thing to do.

So what does it mean to understand multiplication? The mathematics education literature suggests that a basic understanding of multiplication requires four interconnected concepts: (1) quantity, (2) problem situations requiring multiplication, (3) equal groups, and (4) units relevant to multiplication. Most of these understandings can develop from experiences using counting and grouping strategies to solve meaningful problems in the early grades (Smith & Smith, 2006).

- **Understanding quantity.** The meaning of quantity often gets overlooked in addition, but it provides an important foundation for understanding multiplication. A quantity is a characteristic of objects that can be counted or measured, and it consists of a *number* and a *unit*. Seven dollars is an example of a quantity because it includes both the number 7 and the unit *dollars.* Number words (e.g., *seven*) are often used to describe the number portion of a quantity, but other representations, such as pictures (e.g., seven bills representing seven dollars), can be used. In addition to the number, a unit must be specified to provide the complete quantity.

  A count is a particular type of number that is part of the quantity characteristic of collections of objects. It answers the question, "How many?" Counting begins with counting by ones and progresses to skip counting using larger, equal-sized units. Students need sufficient experience in counting collections of objects to clearly understand these two aspects of quantities and the various ways of representing them. A measure (e.g., length) is a particular type of quantity that is a continuous characteristic of individual objects. Measuring includes selecting an appropriate unit of measure (e.g., an inch) and determining the number of these units in the continuous characteristic of the object. Thus, to fully understand quantity, students need to understand the differences between discrete and continuous quantities, recognizing that they use both different units and different processes (counting vs. measuring) to determine the number portion of the quantity.

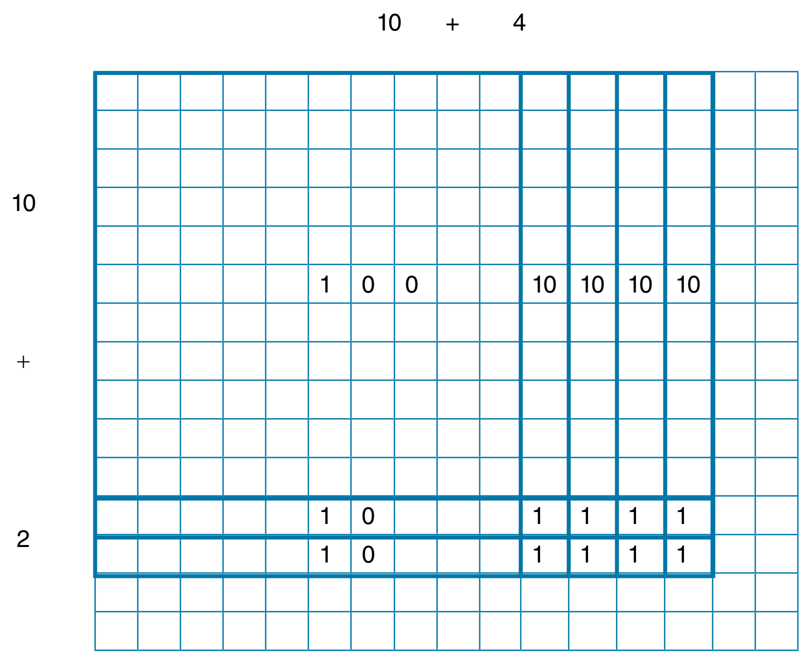
- **Understanding problem situations requiring multiplication.** Students need experience with interpreting word problems that require multiplication and distinguishing them from other situations requiring addition, subtraction, or division. Students also need to understand the relationships between multiplication and division and be able to find each of the three possible unknown quantities when provided with any two of these three pieces of information (e.g., $3 \times 7 = ?$ or $3 \times ? = 21$).
- **Understanding equal groups.** Students need experience with arranging objects into groups to understand the role of equal groups in multiplication and the efficiency of multiplying equal groups instead of counting all the objects in the problem. Number sense includes the ability to compose and decompose numbers. Reasoning in multiplication includes using factors and multiples as equal groups when composing and decomposing numbers, instead of using adding. For example, eight objects can be arranged into groups representing *multiplication* (one group of eight, two groups of four, four groups of two, or eight groups of one) rather than groups representing *addition* (one and seven, two and six, four and four, and eight and zero). Visual images are particularly helpful in understanding grouping (e.g., the difference between a disorganized collection of 60 items and the same 60 items organized into five groups of 12 items or an array of 6 rows and 10 columns).
- **Understanding units relevant to multiplication.** Students need experience with counting and arranging objects into groups to understand the differences between various kinds of units that are relevant to multiplication. Addition most often involves the joining of unequal quantities of the same unit (e.g., adding 35 cents and 24 cents). However, the two factors in multiplication most often refer to different units (e.g., multiplying 12 dogs by four legs for each dog). Students also need to understand how units are sometimes transformed in multiplication. For example, adding 7 oranges to 7 oranges makes 14 oranges, but multiplying 7 inches by 3 inches equals 21 *square inches.*

Area Model With Base-10 Blocks

Another way to increase the students' deeper understanding of the process of multiplication is to show different ways multiplication can be carried out by hand. One method uses the area model along with base-10 blocks as an engaging, hands-on tool for exploring single and multidigit multiplication. The process can be shown in various ways. Figure 5.8 offers some examples using graph paper and the base-10 blocks.

Figure 5.8 The box on the left shows the traditional algorithms used to multiply multidigit numbers. The graph on the right uses a model that represents the numbers by area, giving the students a deeper understanding of the process of multiplication.

14	14
12	12
28	8 = 2 x 4
140	20 = 2 x 10
	40 = 4 x 10
168	100 = 10 x 10
	168



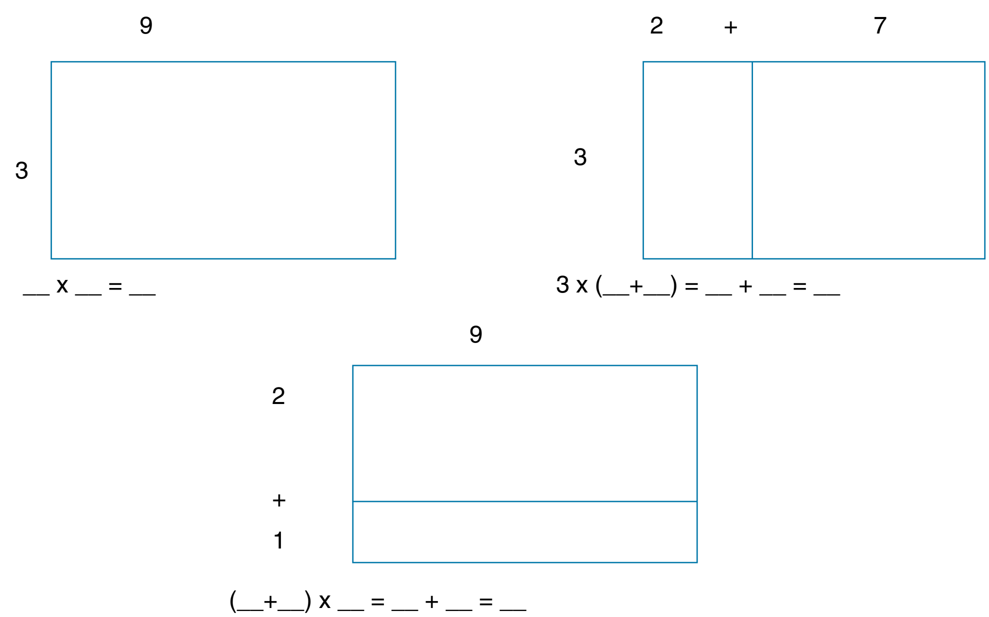
The task is to multiply 14 by 12. Ask students to do so on paper, using the traditional algorithm they learned. This usually produces the two results shown on the left in Figure 5.8, depending on which algorithm the student has learned. Then ask the students to decompose the two numbers by place value and to mark out the rectangular area represented by the partial numbers, either by drawing the base-10 area on graph paper or creating the area with base-10 blocks. Adding up the areas, 100 + 40 + 20 + 8 = 168; so 14 x 12 = 168.

Area models can also be used to show the distributive property of mathematics. For example, Figure 5.9 illustrates the distributive property of 3 x 9 using different area combinations.

Does the Lesson Develop Mathematical Reasoning?

CCSS (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) calls for increased attention to developing mathematical reasoning as early as first grade, where students should be able to solve two-digit addition and subtraction problems with strategies they understand and can explain. Mathematical reasoning also includes spatial and quantitative concepts as well as metacognition, which

Figure 5.9 These three rectangles illustrate the various ways to multiply 3 x 9 using the distributive property.



is thinking about what you are doing and why you are doing it, and making adjustments as needed.

Mathematical competence involves a blend of skills, knowledge, procedures, understanding, reasoning, and application. But too often, instruction focuses on skills, knowledge, and performance—that is, *what* students know and are able to do. Thus, students learn to use routine methods, leading to superficial understanding. We do not spend enough time on reasoning and deep understanding—that is, *why* and *how* mathematics works as it does. Knowledge and performance are not reliable indicators of either reasoning or understanding. For deep understanding, the what, the why, and the how must be well connected. Then students can attach importance to different patterns and engage in mathematical reasoning.

Too often, mathematics instruction focuses on skills, knowledge, and performance but spends little time on reasoning and deep understanding.

Can Preadolescents Do Mathematical Reasoning?

Have the brains of young children developed sufficiently to carry out reasoning skills? The answer: yes, but it depends on which reasoning skill. By the age of 6, most students can demonstrate deductive reasoning using concrete objects. Abstract reasoning, on the other hand, is possible but more difficult and becomes easier in the early teen years and over the course of adolescence as the brain's frontal lobes mature. Teachers of preadolescents can find many activities in books and on the Web designed to enhance reasoning skills. Using activities that move students from concrete to abstract reasoning in effect also moves them from arithmetic thinking to algebraic thinking (Carpenter & Levi, 2000; Hewitt, 2012).

✓ Here is one example: Show students in the first and second grade a series of number sentences. Ask them to discuss the series and to make a general statement (conjecture) about that series.

Number Sentences	Conjecture by Students
7 – 9 = –2	When you subtract a number bigger than your starting number,
10 – 14 = –4	you will always get a negative number for your answer.
15 – 20 = –5	Then: 7 – 16 = ?
8 + 0 = 8	Zero plus a number equals that number.
11 + 0 = 11	Then: 19 + 0 = ?
0 + 15 = 15	
6 – 6 = 0	If you subtract the same number from the same number, you
12 – 12 = 0	will get zero.
14 – 14 = 0	Then: 21 – 21 = ?
4 – 0 = 4	If you subtract zero from a number, you will end up with the
13 – 0 = 13	same number.
21 – 0 = 21	Then: 18 – 0 = ?
3 + 3 = 6	If you add two identical whole numbers that are higher than
6 + 6 = 12	zero, you will get an even number.
11 + 11 = 22	Then: 14 + 14 = ?
5 + 3 = 8	If you add two odd whole numbers, you will get an even
11 + 7 = 18	number.
15 + 9 = 24	Then: 13 + 17 = ?
8 + 3 = 11	If you add an even and an odd whole number, you will always
14 + 7 = 21	get an odd whole number.
18 + 9 = 27	Then: 6 + 11 = ?

Interpreting the Equal Sign

Although most students in the primary grades can come up with these conjectures, they have difficulty interpreting the equal sign as expressing a relationship. Even into the upper elementary grades, most students interpret the equal sign as an operational symbol meaning "find the total" or "put the answer here." When asked to define the equal sign, students not only provide operational interpretations but also believe that interpretations such as "the total" and "the answer" are more accurate than interpretations such as "equal to" or "two amounts are the same" (McNeil & Alibali, 2005; Stephens et al., 2013). One study provided evidence that age alone cannot account for students' operational—rather than relational—interpretation of the equal sign. When researchers asked first- through sixth-grade students what number should be placed on the line to make the number sentence 8 + 4 = __ + 5 true, they found that fewer than 10 percent in any grade gave the correct answer. Further, that performance did not improve with age (Carpenter, Franke, & Levi, 2003). The obvious question here is whether the preadolescents' difficulties in interpreting the equal sign are due to immature cognitive structures or to their earlier experiences with arithmetic.

Several studies were conducted to find the answer to this question. A case study of a second-grade classroom provided a systematic examination of the contexts in which students actually see the equal sign. The researchers analyzed two mathematics textbooks used by students in the classroom. They found that the equal sign was nearly always presented in the operations-equal-answer context (e.g., 4 + 5 = __). This finding is in line with the belief that students' understanding of the equal sign can be explained by their experiences (Seo & Ginsburg, 2003) and to some extent by their culture (Jones, Inglis, Gilmore, & Dowens, 2012).

Most middle school students do not interpret the equal sign as a relational symbol, setting the stage for difficulties with algebraic operations. Their textbooks don't help.

In an examination of four popular middle school mathematics textbooks, researchers found that the textbooks frequently present the equal sign in an operations-equal-answer context, and rarely present the equal sign in an operations-on-both-sides context. This practice likely reinforces the students' interpretation of the equal sign as an operational symbol. Thus, middle school mathematics textbooks may not be designed to help students acquire a relational understanding of the equal sign. Although the proportion of equal signs presented in an operations-equal-answer context declined across the middle grades, even in eighth grade many students continue to interpret the equal sign as an operational symbol (McNeil et al., 2006).

Researchers found that students in middle school did not exhibit a relational understanding of the equal sign unless they had contextual support. Seventh-grade students were randomly assigned to view the equal sign in one of three contexts: alone (=), in an operations-equal-answer equation (4 + 5 + 6 + 4 = __), and in an operations-on-both-sides equation (4 + 5 + 6 = 4 + __). Only 11 percent of the students in the alone and only 25 percent of the students in the operations-equal-answer contexts had a relational understanding of the equal sign. By contrast, 88 percent of the students in the operations-on-both-sides context exhibited a relational understanding of the equal sign. Apparently, students in seventh grade did not interpret the equal sign as a relational symbol of equivalence in general, but they were able to interpret the equal sign as a relational symbol in the context of an equation with operations on both sides of the equal sign (McNeil & Alibali, 2005). This finding is important because middle school (or the upper elementary grades) is where students make the transition from arithmetic to algebra, and where a relational understanding of the equal sign is necessary for success.

Clearly, middle school students benefit from seeing more equal signs in an operations-on-both-sides context. The notion of *equal* is complex and difficult for students to comprehend, yet it is a central mathematical idea within algebra. However, improving students' understanding of the equal sign and their preparation for algebra may require changes in teachers' instructional practices, as well as changes in elementary and middle school mathematics curricula and textbooks. Teachers should present students with statements of equality in different ways to further develop their ideas of equivalence.

Using Practice Effectively With Young Students

We noted in Chapter 3 that practice allows the learner to use the newly learned skill in a new situation with sufficient accuracy so that it will be correctly remembered. Before students begin practice, the teacher should model the thinking process involved and guide the class through each step of the new learning's application.

Since practice makes permanent, the teacher should monitor the students' early practice to ensure that it is accurate and to provide timely feedback and correction if it is not. This guided practice helps eliminate initial errors and alerts students to the critical steps in applying new skills. Here are some suggestions by Hunter (2004) for guiding initial practice, especially as it applies to young students:

- ✓ **Limit the amount of material to practice.** Practice should be limited to the smallest amount of material or the skill that has the most relevancy for students. This allows for sense and meaning to be consolidated as the learner uses the new learning. Remember that most preadolescents can deal with only about five items in working memory at one time.
- ✓ **Limit the amount of time to practice.** Practice should take place in short, intense periods of time when the student's working memory is running on prime time. When the practice period is short, students are more likely to be intent on learning what they are practicing. Keep in mind the 5- to 10-minute time limits of working memory for preadolescents, discussed in Chapter 3.
- ✓ **Determine the frequency of practice.** New learning should be practiced frequently at first so it is quickly organized (massed practice). Vary the contexts in which the practice is carried out to maintain interest. Young students tire easily of repetitive work that lacks interest. To retain the information in long-term memory and to remember how to use it accurately, students should continue the practice over increasingly longer time intervals (distributed practice), which is the key to accurate retention and application of information and mastery of skills over time.
- ✓ **Assess the accuracy of practice.** As students perform guided practice, give prompt and specific feedback on whether the practice is correct or incorrect, and why. Ask the students to summarize your feedback comments in their own words. This process gives you valuable information about the degree of student understanding and whether it makes sense to move on or reteach portions that may be difficult for some students.

Graphic Organizers

Today's students have grown up in a visual world. They are surrounded by television, computer screens, movies, portable DVD players, tablets, and cell phones with screen images. Using visual tools in the mathematics classroom, then, makes a lot of sense. A graphic organizer is one type of visual tool that not only gets students' attention but is also a valuable device for improving understanding, meaning, and retention.

Many different types of graphic organizers are available in books and on the Internet (see websites in the **Resources** section of this book). The following are just two examples created by Dale Graham and Linda Meyer (2007) for use in middle school mathematics classes, used here with their permission.

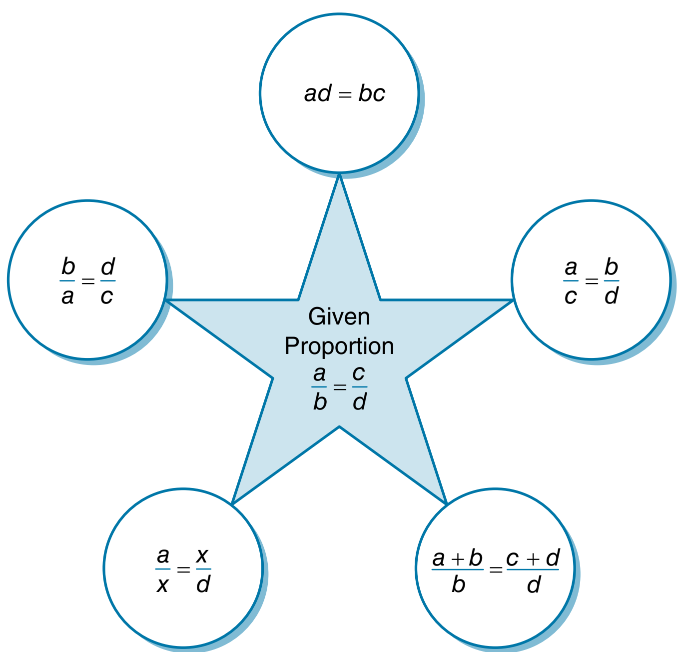
What Are the Properties of the Real Number System?

Property	*Addition*	*Multiplication*
Closure		
Commutative		
Associative		
Identity		
Inverse		
Distributive		

SOURCE: Graham and Meyer (2007). Adapted with permission of the authors.

What Are the Properties of Proportions?

What Are the Properties of Proportions?



SOURCE: Graham and Meyer (2007). Adapted with permission of the authors.

Taking Advantage of Technology

As we discussed in Chapter 3, students today have grown up with technology. Using new technologies involves time, effort, and a rethinking of instructional approaches. It also has its benefits and risks.

Teachers are sometimes torn between the enthusiasm for using technology in mathematical investigations and the cautions about undermining students' computational skills. Research studies show that—particularly in middle-grade mathematics—technology, including online peer tutoring, can have positive effects on students' attitudes toward learning, on their confidence in their abilities to do mathematics, and on their motivation and time on task. Furthermore, technology use can help students make significant gains in mathematical achievement and conceptual understanding (Tsuei, 2012).

Research studies also suggest that using technology for *nonroutine* (i.e., *novel*) applications, such as exploring number concepts and solving complex problems, leads students to greater conceptual understanding and higher achievement, whereas using technology for *routine* calculations does not. Students often perceive calculators as simply computational tools. But when they engage in mathematical exploration and problem solving with calculators and other technologies, they broaden their perspective and see these instruments as tools that can enhance their learning and understanding of mathematics (Gibson et al., 2014).

Answer to Question 8. False: *Using technology for nonroutine mathematics applications leads students to greater conceptual understanding and higher achievement, whereas using technology for routine calculations does not.*

Technology cannot replace the instructional strategies teachers use to deepen students' understanding of mathematical concepts or their problem-solving skills. It can, however, extend their ability to make sense of mathematics, gain access to additional content not available in the classroom, enhance their mathematical reasoning, and improve their computational fluency. It is equally important for professional development programs to continually update teachers' knowledge of various technologies and their practical application in the teaching–learning process.

WHAT'S COMING?

As the brain's frontal lobes continue to mature during adolescence, students should be able to successfully engage in solving more complex and abstract problems. But many adolescents are not successful in high school mathematics courses. Why is that, and what can teachers do to improve mathematics achievement for this age group? How can we help these students see mathematics as a meaningful, practical, and enjoyable endeavor? The answers to these and other questions about adolescent performance in mathematics are provided in the next chapter.

Chapter 5—Teaching Mathematics to the Preadolescent Brain

QUESTIONS AND REFLECTIONS

Respond to the following questions, and jot down on this page key points, ideas, strategies, and resources you want to consider later. This sheet is your personal journal summary and will help jog your memory.

What are some characteristics of the developing preadolescent brain?

What are some process skills preadolescents should learn in mathematics?

How can we develop mathematical reasoning?

6

Teaching Mathematics to the Adolescent Brain

No employment can be managed without arithmetic, no mechanical invention without geometry.

—Benjamin Franklin

WHAT IS THE ADOLESCENT BRAIN?

Only in recent years have cognitive science researchers begun to focus on understanding the capabilities and limitations of the adolescent brain. You will recall from Chapter 5 that the brain's frontal lobe matures much more slowly than the limbic area. So the processes that control voluntary behavior are not yet fully operational. Adolescents may look and sometimes even act like adults. But as recent studies indicate, differences in the development of the frontal lobe may be one of the most important distinctions between adolescents and adults. Although the strategies suggested in this chapter focus on adolescents, some of them can be easily adapted for use with preadolescents.

Overworking the Frontal Lobes

Much of the new information about brain growth and development has come from imaging studies using fMRI (functional magnetic resonance imaging). One of the first seminal studies involved scanning 8- to 30-year-old subjects performing visual–motor tasks. The researchers found that adolescents used more of their prefrontal cortex than did adults. Actually, the amount of prefrontal cortex used was similar to what adult brains use when performing much more complex tasks. The teens' excessive reliance on a brain region that is not mature can lead to problems. A mature prefrontal cortex makes it easier for an individual to use reason to override reflexive or emotional behavior. But for adolescent brains, deliberately overriding the automatic response is more difficult than for adult brains. Adults, on the other hand, recruit other parts of the brain to collaborate and better distribute the workload (Luna et al., 2001).

Figure 6.1 These representative fMRI scans show that adolescent brains draw heavily on the frontal lobe areas to accomplish a visual–motor task, while adults distribute the workload over other brain regions (Luna et al., 2001).

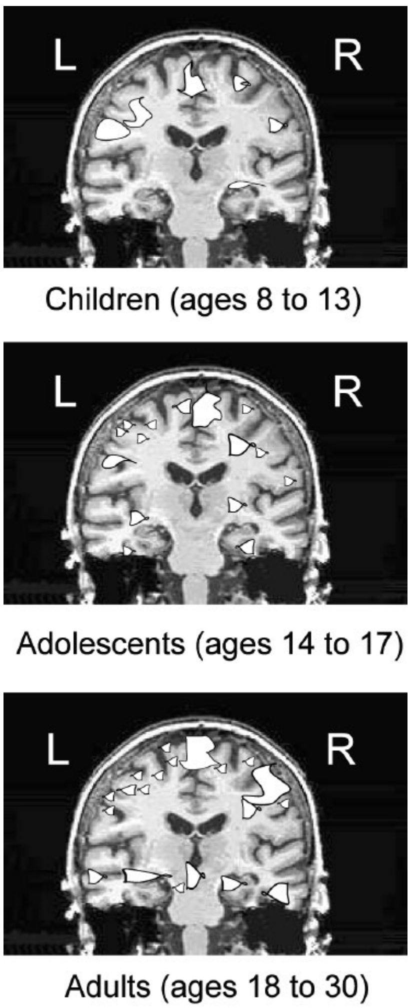


Figure 6.1 shows how the workload to handle a visual–motor task is drawing heavily on frontal lobe resources in the adolescent brain. The adult brain recruits more resources from other parts of the brain to distribute the workload and collaborate to handle the task. Thus, if something unexpected occurs in an already stressful situation, an adolescent may exhaust his or her prefrontal cortex resources. That explains why adolescents often exhibit impulsive or thoughtless behavior. Subsequent studies have found similar results (e.g., Blakemore, 2012; Smith, Cobb, Farran, Cordray, & Munter, 2013).

Working Memory Still Developing

One of the functions of working memory is to control and guide voluntary behavior. Working memory is still developing in adolescents. Thus, fMRI scans reveal that adolescents are not as efficient as adults in recruiting brain areas that support working memory. Investigations of spatial working memory showed that early adolescents performed well on spatial working-memory tests, but they needed to engage more neural circuits than did older adolescents. Further, they also became much less efficient if they were stressed when asked to perform an additional task. This is likely due to cortisol, a hormone released into the blood stream when the body is under stress. Cortisol prepares the body to deal with the stress and reduces working memory's ability to focus on unrelated and less important learning tasks.

Older adolescents seem to recruit fewer neurons and use different strategies to perform the same job, compared with younger adolescents. Researchers found that the older teens solved the task through a verbal strategy rather than by rote spatial rehearsal. As adolescents mature, the brain uses more areas in general and distributes certain tasks to specialized regions. This process reduces the total neural effort necessary to achieve the same level of performance (Nagel, Herting, Maxwell, Bruno, & Fair, 2013).

We discussed in Chapter 5 the results of longitudinal brain-imaging studies of individuals between the ages of 4 and 21. Table 6.1 summarizes the results of those studies, along with additional scanning studies that have focused mainly on adolescents.

Can Adultlike Challenges Accelerate Maturity?

Not all neuroscientists and psychologists accept the notion that the rate of maturation of the frontal lobes in adolescents is so closely linked to

Table 6.1 Adolescent Brain Development and Some Implications

Research Finding	*Possible Implications for Learning Mathematics*
After puberty, gray-matter volume continues to decrease until about the age of 20 to 22 as unneeded and unhealthy neurons are destroyed. Meanwhile, the white matter is thickening as more myelin surrounds neurons to increase protection and transmission of signals.	As neural networks begin to consolidate in the frontal lobes, learners can tackle more complex problem solving requiring inductive and deductive reasoning.
Some regions of the temporal lobes (located just above the ears) are the last to mature, even though some of the areas with which they are associated (such as the visual and language-processing areas) mature earlier.	The temporal lobes are responsible mainly for auditory processing, but they also contribute to visual object identification and the association of vocabulary with objects. Older adolescents will be better able to name and discriminate plane and solid objects visually and auditorily compared with younger adolescents.
During problem solving, more frontal-lobe areas are used in the adolescent brain than in the adult brain.	Overworking of the frontal lobes leads to impulsive and more emotional (rather than rational) responses during problem solving.
Working memory (located mainly in the frontal lobes) matures slowly.	Adolescents can have difficulty working with problems that have more variables and/or components than working memory's limited capacity can handle.

SOURCES: Gogtay et al. (2004); Luna et al. (2001); Nagel et al. (2013).

genetic influences. They point out that teens in other countries spend much more time with adults than with their peers and, consequently, do not exhibit the immature behavior of teens in North America (Sabbagh, 2006). If the environment can provide the adolescent brain with more adultlike experiences, then perhaps adolescents can avoid the stress and antisocial behavior associated with this period of growth. Furthermore, they contend that allowing adolescents to make adult decisions (e.g., serving in the military) will accelerate frontal-lobe maturation.

The extent to which genetic forces and environmental demands affect the developing brain will most likely be unique to each adolescent. Nonetheless, the implication of this approach is that schools should not view adolescents as being so biologically immature that they cannot take on challenging tasks, including those in mathematics (Moshman, 2013).

The Search for Novelty

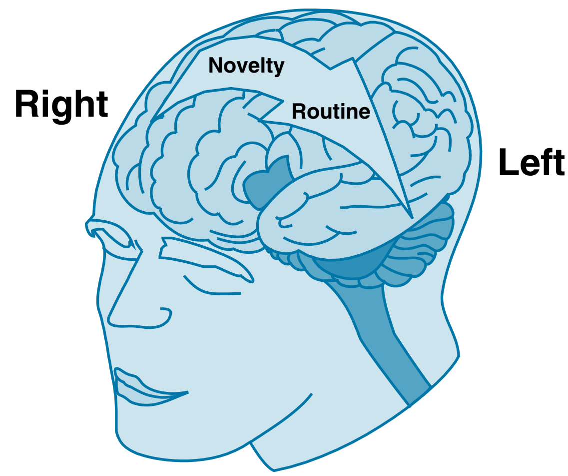
In Chapter 5, we discussed how the developing preadolescent brain responds to novelty. With the onset of puberty and during the adolescent years, the search for novelty becomes more intense. When curious adolescents try a new challenge, such as a video game, they keep at it until they master it. Then the novelty wears off, they get bored with it, and they move on to a new challenge. Cognitive neuroscientists attribute this phenomenon to the specialized functions of each cerebral hemisphere and how these functions affect new learning. Research seems to indicate that hemispheric specialization may center on the differences between novelty and routine. Closer examination of brain-damaged patients shows that those with severe right-hemisphere problems experience difficulty in facing new learning situations but can perform routine, practiced tasks, such as language, normally. Conversely, patients with severe left-hemisphere damage can create new drawings and think abstractly but have difficulty with routine operations and language (Fridriksson, Richardson, Fillmore, & Cai, 2012; Goldberg, 2001, 2005).

These findings give us a different way of looking at how the brain learns. They suggest that upon encountering a novel situation for which the individual has developed no coping strategy, the right hemisphere is primarily involved and attempts to deal with the situation. In mathematics, for instance, that novel situation could be the student's first encounter with solving quadratic equations. With repeated exposure to similar situations, coping strategies eventually emerge and learning occurs because this process results in a change of behavior. In time, and after sufficient repetition, the responses become routine and shift to the left hemisphere (Figure 6.2). The amount of time and the number of situational exposures needed to accomplish this right-to-left–hemisphere transition vary widely from one person to the next. But it may be that one component of mathematical aptitude is the ability of a student's brain to make right-to-left transitions involving mathematical operations in less time and with fewer exposures than average.

> *One component of mathematical aptitude may be the ability of a student's brain to make right-to-left–hemisphere transitions involving mathematical operations in less time and with fewer exposures than average.*

For more than a decade, studies using neuroimaging have provided evidence to support this right-to-left transition. In one early study, researchers used positron emission tomography (PET) scans to measure the changes in brain-flow patterns when subjects were asked to learn various types of information. Changes in blood-flow levels indicate the degree of neural activation. When the information was novel, regions in the right temporal lobe were highly activated. After the information had been presented several times to the subjects, activity in the right temporal lobe decreased dramatically (Figure 6.3). In both instances, however, the level of activation in the left temporal lobe remained constant (Martin, Wiggs, & Weisberg, 1997).

Figure 6.2 With repeated exposure, novel experiences become routine, and their cortical processing areas shift from the right hemisphere to the left hemisphere.

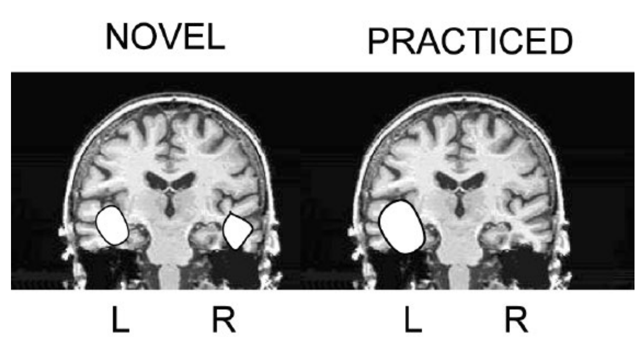


Similar results were reported from other studies involving a variety of learning tasks, such as recognizing faces and symbols (Schwartz et al., 2003), learning a complex motor skill (Bassett et al., 2011; Krakauer & Shadmehr, 2006), and learning and relearning different words or systems of rules (Berns, Cohen, & Mintun, 1997; Doron, Bassett, & Gazzaniga, 2012; Habib, McIntosh, Wheeler, & Tulving, 2003). The same shifts were detected no matter what type of information was presented to the subjects. In other words, the association of the right hemisphere with novelty and the left hemisphere with routine appears to be independent of the nature of the information being learned.

Novelty and Mathematics

Teachers ultimately decide whether mathematics is full or devoid of novelty. If adolescents have already mastered a mathematical operation but we continue to give them more of the same assignments, they will see no purpose in completing repetitive practice. They will lose interest, they will see mathematics as boring and hum-drum work, their motivation will drop, and their grades will slump. The key here is for the teacher to find different and meaningful applications of the mathematical operation or concept to maintain interest and attention, key components of motivation. And even more so, the teacher should recognize how much practice each student needs to show mastery, and stop at that.

Figure 6.3 In this representation of PET scans, the white areas show the changes in regional blood flow for novel and practiced tasks. The images reveal areas of high activation in the left and right temporal lobes for novel tasks, but only in the left temporal lobe for practiced tasks.



Novelty and motivation are also undermined by a mathematics curriculum that focuses mainly on a strict formal approach, heavy in memorizing abstract axioms and theorems. This model, which emerged in the 1970s, was based on the brain–computer metaphor—the notion that cognitive processing in the human brain is similar to that in a computer. As neuroscience reveals more about the underlying mechanisms that power the brain's cognitive processes, it is clear that the differences between the brain's operations and those of a computer are far greater than their similarities. An adolescent's brain, unlike a computer, is a structured entity that requires facts only insofar as they can be integrated into prior knowledge to elucidate new situations. It is adapted to represent continuous quantities and mentally manipulate them in analogical form. Conversely, it is not innately prepared to handle vast arrays of axioms or symbolic algorithms. For most people, to do so requires heavy doses of motivation, interest, and novelty. The Common Core State Standards for Mathematics shift away from a heavy emphasis on memorization to deeper understanding of fewer key concepts and applying that understanding to solve problems in real-world situations.

The Adolescent Brain and Algebra

The adolescent brain's heavy reliance on the frontal lobe for cognitive processing may have an upside. One curious finding from fMRI studies is that adolescents could have an advantage over adults when learning algebra. The studies indicate that after several days of practice, adolescents, like adults, rely on the prefrontal cortex regions for retrieving algebraic rules to solve equations. However, unlike adults, after practice, adolescents decrease their reliance on the brain's parietal region that is holding an image of the equation. To the researchers, this suggests that, compared with the brains of adults, the developing prefrontal regions of the adolescent brain are more plastic and thus change more with practice, resulting in an enhanced ability for learning algebra (Qin et al., 2004).

It seems that the adolescent brain may actually have an enhanced aptitude for learning algebra more easily than the adult brain.

Another study found that Algebra I students who began the course with a good prior understanding of fractions and fraction magnitude achieved better in algebra than did those students with poor knowledge of fractions (Booth, Newton, & Twiss-Garrity, 2014). The researchers suggest that Algebra I teachers give incoming students a pretest to determine their prior knowledge of fractions and fraction magnitude. A brief reteaching of these topics could improve the achievement levels of all students in the class.

When used properly, showing students *incorrect* examples and explaining why they are incorrect can improve students' conceptual understanding and procedural skills in algebra (Booth, Lange, Koedinger, & Newton, 2013). However, it is important to make certain that the students have a good understanding of the correct method before introducing the incorrect examples. Also, the incorrect examples should be presented as guided practice, with the teacher working through the explanation of the errors with students.

LEARNING STYLES AND MATHEMATICS CURRICULUM

Qualitative Versus Quantitative Learning Styles

Cognitive researchers suggest that adolescent students approach the study of mathematics with different learning styles that run along a continuum from primarily quantitative to primarily qualitative (Augustyniak, Murphy, & Phillips, 2005; Farkas, 2003; Liston, 2009; Sharma, 2006). Students with a quantitative style approach mathematics in a linear, routine fashion. They prefer working with numbers over concrete models and may run into difficulty with solutions requiring multistep procedures. On the other hand, students with a qualitative style prefer concepts over routine steps and models over numbers. The implication of this research is that students are more likely to be successful in learning mathematics if teachers use instructional strategies that are compatible with their students' cognitive styles, but exposure to both kinds of strategies can strengthen students' weak areas.

✓ Tables 6.2 and 6.3 illustrate teaching strategies appropriate for the mathematical behaviors exhibited by quantitative and qualitative learners, respectively. The strategies are meant to help teachers address specific mathematical behaviors they identify in individual students. Such strategies target specific needs and, with practice, can strengthen a student's weak areas. It is unrealistic, however, to expect teachers to identify and select individual strategies for problems encountered by all their students during a single learning episode. By

Table 6.2 Teaching Strategies for Learners With Quantitative Style

Mathematical Behaviors	*Teaching Strategies to Consider*
Approaches situations using recipes	Emphasize the meaning of each concept or procedure in verbal terms.
Approaches mathematics in a mechanical, routine fashion	Highlight the concept and overall goal of the learning.
Emphasizes component parts rather than larger mathematical constructs	Encourage explicit description of the overall conceptual framework. Look for ways to link parts to the whole.
Prefers numerical approach rather than concrete models	Use a step-by-step approach to connect the model to the numerical procedure.
Prefers the linear approach to arithmetic concept	Start with the larger framework and use different approaches to reach the same concept.
Has difficulty in situations requiring multistep tasks	Separate multiple tasks into smaller units, and explain the connections between the units.

Table 6.3 Teaching Strategies for Learners With Qualitative Style

Mathematical Behaviors	*Teaching Strategies to Consider*
Prefers concepts to algorithms (procedures for problem solving)	Connect models first to the concept and then to procedures before introducing algorithms.
Perceives overall shape of geometric structures but misses the individual components	Emphasize how individual components contribute to the overall design of the geometric figure.
Has difficulties with precise calculations and explaining procedure for finding the correct solution	Encourage explicit description of each step used.
Can offer a variety of approaches or answers to a single problem	Use simulations and real-world problems to show application of concept to different situations.
Prefers to set up problems but cannot always follow through to a solution	Provide opportunities for the student to work in multistyle cooperative learning groups. To ensure full participation, give one grade for problem approach and setup, and one grade for exact solution.
Benefits from manipulatives and enjoys topics related to geometry	Provide a variety of manipulatives and models (e.g., Cuisenaire rods, tokens, or blocks) to support numerical operations. Look for geometric links to new concepts.

understanding the different approaches to the learning of mathematics, teachers are more likely to select instructional strategies that will result in successful learning for all students.

Developing Mathematical Reasoning

As teenage brains mature over the course of adolescence, teachers should present challenging mathematical problems involving increasingly complex reasoning. Inductive and deductive reasoning are among the most common types of reasoning used in mathematics. Inductive reasoning, sometimes called the bottom-up approach, moves from parts to a whole or from the specific to the general. In inductive reasoning, we begin with specific observations and measures, look for patterns and regularities, formulate some tentative hypotheses we can explore, and develop general conclusions or theories. "The sun rose today, yesterday, the day before, and so on. I conclude the sun will rise tomorrow."

In deductive reasoning, sometimes called the top-down approach, one draws a conclusion from principles (or premises) that are already known or hypothesized. "Triangle A has three 60-degree angles. Triangles with three 60-degree angles are called equilateral triangles. Therefore, Triangle A must be an equilateral triangle." Inductive reasoning is often used to make a guess at a property, and deductive reasoning is then used to prove that the property must hold for all cases or for some set of cases.

Table 6.4 suggests a sequence for using inductive and deductive approaches when introducing a new mathematical concept. The order first accommodates qualitative learners and then moves to techniques for quantitative learners.

Instructional Choices in Mathematics

We have already discussed in earlier chapters the irony that although people are born with number sense, many feel that they are not able to learn or remember basic mathematical operations. This feeling of incompetence is particularly evident in high school classrooms and poses a significant obstacle for both students and teachers to overcome. Motivation, of course, has a lot to do with this attitude, and plenty of research studies show that low motivation leads to low achievement in mathematics, as in other subjects.

Educators for years have explored strategies and models to help motivate students to higher achievement levels. Kathie Nunley (2004, 2006, 2011) has developed a student-centered teaching method based on research

Table 6.4 Inductive to Deductive Approach for Introducing a New Concept in Mathematics

Steps for the Inductive Approach for Qualitative Learners	✓ Explain the linguistic aspects of the concept.
	✓ Introduce the general principle or law that supports the concept.
	✓ Provide students opportunities to use concrete materials to investigate and discover proof of the connection between the principle and the concept.
	✓ Give many specific examples of the concept's validity using concrete materials.
	✓ Allow students to discuss with each other what they discovered about how the concept works.
	✓ Demonstrate how these individual experiences can be integrated into a general principle or rule that applies equally to each example.
Steps for the Deductive Approach for Quantitative Learners	✓ Reemphasize the general principle or law that the concept relates to.
	✓ Demonstrate how several specific examples obey the general principle or law.
	✓ Allow students to state the principle and suggest specific examples that follow it.
	✓ Ask students to explain the linguistic elements of the concept.

Layered Curriculum consists of three layers of differentiated instruction that enhance student motivation and encourage complex teaching.

in cognitive neuroscience. Her model is called Layered Curriculum and consists of three layers of differentiated instruction that enhance student motivation and encourage complex thinking.

Nunley notes that mathematics teachers who strive for brain-compatible classrooms share three basic goals. First, they want to increase student motivation by engaging students emotionally in their learning. Second, they want to enable students to master mathematics skills to a level of proficiency that allows practical use of the skill, thus creating meaning. And third, they look for ways to encourage higher-level thinking and connect new learning to prior knowledge in a complex manner. Engaging students is first and foremost, because without engagement and motivation, teachers cannot begin to address the other two goals. Improving motivation and engagement in students requires only that teachers add one simple thing to their classroom—choice.

Lack of motivation, Nunley suggests, remains one of the major reasons students do not succeed in mathematics, or any other subject (Legault, Pelletier, & Green-Demers, 2006; Pintrich, 2003). In mathematics class, students may feel that they lack the ability to be successful, or they may feel that they cannot sustain the effort long enough for success, or they may simply be bored and unable to concentrate on the task. Some students experience lack of motivation due to learned helplessness—the feeling that no amount of effort will ever lead to success, so there is no point in trying. Other unmotivated students simply have placed no personal value on the learning task. Whatever the cause, they all share one common thread: students without self-determined motivation are generally not successful in school.

To be motivated, Nunley continues, students must see a relationship between their behavior and the outcome. This requires that they perceive they have some sense of control in their environment. With a sense of control comes a sense of responsibility. Unfortunately, traditional teacher-centered, autocratic classrooms do little to encourage responsibility in students. If the teacher makes all the decisions regarding rules and instruction, the student is immune from all responsibility.

Thus, we see the shift in education to student-centered classrooms. In student-centered classrooms, students are allowed some choice and decision making through differentiated instruction. Studies reveal that student-centered classrooms have higher-achieving students, higher standardized test scores, fewer classroom-management problems, more on-task behavior, and fewer dropouts (Pekrun, Maier, & Elliot, 2006). So mathematics teachers want to create classrooms of motivated learners because motivated learners actively process information, have better conceptual understanding of material, and show greater problem-solving skills. Such an approach is also consistent with what we know about how today's students want to participate actively in their own learning (Sousa, 2011a).

Three Steps to Layering the Curriculum

Layering the curriculum is a simple way to differentiate instruction, encourage higher-level thinking, prepare students for adult-world decision making, and hold them accountable for learning. Any lesson plan can be converted into a layered unit with three easy steps (Nunley, 2004, 2006, 2011).

✓ **Step 1: Add some choice.** Choice transforms a classroom instantly. Choice suddenly turns unmotivated students into motivated ones, ensures student attention, and gives students the perception of control. Choice is the centerpiece to student-centered, differentiated classrooms. Traditionally, mathematics teachers have seen their subject as so regimented and sequential that it leaves little room for student choice. But within even the tightest-structured curriculum, some student choice is possible.

- Take your teaching objectives and offer two or three assignment choices as to how students can learn those objectives. Not all objectives need to be taught through choices, but offer as many as you can. These assignment options could include teacher lecture, small-group peer instruction, hands-on tactile projects, or independent study.
- For example, if your objective is that students be able to determine the area of a triangle, you may offer a quick chalk/white board lesson on that topic. Then allow the students to do some practice problems themselves, work in small groups, play a computer game that practices that concept, or complete a task using manipulatives.
- One suggestion worth considering is to make your lectures optional and award points for attending them. Tell the students that they can either listen to your lecture (direct instruction) or work on another assignment from the unit instead. What you will discover is that all students will probably listen to the lecture. But the fact that it is now their decision, rather than the teacher's mandate, changes the whole perception of the task and increases attention. Recording lectures or lessons has another benefit. In addition to allowing you to offer them as an elected assignment, it also gives you the freedom to move the placement of your lectures.
- Because technology now allows teachers to make better use of the time spent in the school day, many are moving to *flipped* or *inverted classrooms.* These terms refer to the idea of taking the traditional classroom of the past century—where we lectured to students during class time and then assigned homework to be done at home—and flip the whole thing. In a flipped classroom, instruction is recorded and uploaded online. Students watch these recordings at home outside the school day. Then the actual class time is spent on interaction between students and teachers—asking questions, providing answers, and doing assignment drills.

  However, an objection to the flipped classroom is that having every student watch recorded instruction at night for class discussion the next day is one giant step backward in differentiating for the needs of our diverse learners. These mediated lessons should not be relegated and mandated as homework for all. This ignores too many of our students who have different learning modalities, presents problems for students without access to technology outside of school, and creates too much distance between receiving the lesson and student questions and feedback.

In a Layered Curriculum classroom, teachers have the option to go to a more practical plan in what Nunley (2013) calls a sideways classroom. Rather than flipping the classroom completely on its head (which, let's face it, gives us the same lesson, just upside down), we instead tip it sideways a bit and let things flow broadly from side to side. So the teacher records the instruction but uses the recorded lessons not for homework but, rather, as one of the classroom day assignment options. These can easily be added into the C Layer of the Layered Curriculum unit (explained further in Step 3).

There are many advantages of using this sideways Layered Curriculum classroom, especially for mathematics. Students who might benefit from listening to a lecture or watching a demonstration can make use of the opportunity. Teacher class time is then freed up to work with students individually. Students are near a teacher during and directly after the instruction for questions and clarification. Another advantage is that students can catch the lesson missed from a day of absence, or just watch it more than once for a relearning opportunity or for study. And best of all, the classroom learning environment remains open and accessible to a wide variety of learners.

✓ **Step 2: Hold students accountable for learning.** One of the unfortunate developments in our traditional grading system is the wide variation in how grading points are awarded in our classrooms. Some teachers award points simply for practicing a skill, some for just doing assignments, and of course some points are eventually awarded for demonstrating mastery in the test. Because grading schemes are nearly as numerous and varied as the number of teachers, a heavy weight is frequently put on the points awarded for doing assignments. This means that students can earn enough points to pass a course without actually learning much at all. In fact, so many points have been awarded for doing classwork and homework that many students never understand that the purpose of doing an assignment is actually to *learn* something from it. They say, "I did it. Doesn't that count?"

- A key to layering the curriculum is to award grade points for the actual learning of the objective rather than for the assignment that was chosen for the learning. For example, if our objective is that students learn how to determine the area of a triangle, then points are awarded for the assignment based on whether or not the student can do that. Whether they chose to do the bookwork, a manipulative exercise, or a computer game is immaterial. What is important is that they learned the objective. This can be done through oral defense, small-group discussions, or unannounced quizzes. Have sample problems on index cards that you or their classmates can pull at random. Two or three sample problems can easily check for the skill. Award points for acquiring the skill rather than for the journey chosen to get there.

✓ **Step 3: Encourage higher-level thinking.** One of the main components of brain-compatible learning is helping students make complex connections with new information—finding relationships, hooking new learning to previous knowledge, and cross-connecting between memory networks. These are the keys to real learning. Layering the curriculum encourages more complex learning by dividing the instructional unit into three layers: (1) basic rote information, (2) application and manipulation of that information, and (3) critical analysis of a real-world issue. Rather than just calling them Layer 1, Layer 2, and Layer 3, the complexity of the learning is tied into the actual grade a student will earn, so the layers are called C Layer, B Layer, and A Layer.

- The C Layer consists of all the objectives that have to do with the lower levels of Bloom's taxonomy. This layer consists of rote learning and concrete facts. All students begin in this layer. Even the highest-ability students can add to their current bank of knowledge; so the entire class starts here.
- After students complete the C Layer, they move to the B Layer, which asks them to connect the new information gained in the C Layer to prior knowledge. This layer includes assignments that require problem solving, application, demonstration of mastery, or unique creations. The purpose of this layer is to attach new knowledge to prior knowledge to make a more complex picture or network in the student's brain. Interdisciplinary assignments work beautifully in this layer. A student who satisfactorily completes the C and B Layers will earn the grade of B on this unit.
- Finally, the A Layer asks students to mix the facts and basic information they have learned with more sophisticated brain concepts such as values, morality, and personal reflection to form an opinion on a real-world issue or current event. This layer asks for critical thinking and prepares students for their roles as voters and decision makers in the adult world. Many educators may refer to this area as the essential question. A student who successfully completes this layer will earn the grade of A on this unit.

All students are expected to complete the three layers. Many students may not be able to show sufficient mastery of a skill or handle an A-Layer issue with the sophistication needed to gain enough points for a letter grade of A or B. Nonetheless, they all must still tackle the three layers. We are preparing these students for an adult world that will ask them to gather and manipulate information, and make community decisions based on that information. Thus, all students need to practice these types of thinking. At the outset, teachers help students walk through all the layers so they experience success and understand the process. As the year progresses, units may be left more open in their structure so students are free to move among the layers as they are ready.

Examples of Layered Curriculum Units

✓ **Eighth-Grade Layered Curriculum Unit Content: Understanding Graphs and Data-Analysis Objectives**

- Collect, organize, analyze, and display data (including scatter plots) to solve problems.
- Approximate a line of best fit for a given scatter plot; explain the meaning of the line as it relates to the problem, and make predictions.
- Identify misuses of statistical and numerical data.

C Layer:

1. Listen to the teacher "chalk talk" lesson each day. (5 points/day)
2. Book practice problems: Choose one from each section. (10 points each)
   a. Page 235, numbers 1 to 20, any seven problems
   b. Page 238, numbers 1 to 21, any seven problems
   c. Page 240, numbers 1 to 17, any six problems
3. Choose one of the lab analysis projects. With one or two classmates, calculate mean, median, mode, and range, and draw a linear regression line. Answer the prediction questions. (10 points each)

B Layer: Choose one (20 points)

1. What will be the price of gasoline in the year 2018? Research a 10-year history of gas prices, plot the data, and use them for your prediction.
2. What will be the price of school lunches in the year 2018? Research a 10-year history as in Question 1 above.
3. What's the value of a scatter plot? Surf the Internet. Find 30 Internet sites that use scatter plots to make predictions or explain situations. Compile an annotated bibliography of your findings.

A Layer: Choose one (20 points)

1. Do you feel that politicians misuse statistical data? If yes, find three to five pieces of evidence to support your argument. If no, choose another question.
2. The media often misuse graphs and data. Find three to five examples, and make an argument for being an educated consumer of media.
3. How many people will die on our highways next year? Find the research to support your answer. Are our laws helping reduce highway deaths? What else could be done?

✓ Sample of a Layered Curriculum Unit in Algebra: Polynomials in the Real World

C Layer: Evaluating, adding, subtracting polynomials (50 points)

1. I have an understanding of the following terms. (Pass the quiz, 10 points)

   _____ monomial _____ binomial _____ trinomial _____ polynomial

Suggestions of how to learn these terms (try two):

a. Listen to the lecture on Day 1.
b. Read about them in your textbook.
c. Divide a white sheet into four parts, and fill each part with samples of these.

2. I know and can identify these parts of a polynomial. (Pass quiz, 10 points)

_____ leading term _____ constant term _____ coefficients _____ identify degrees

Suggestions on how to learn these parts (try two):

a. Listen to the lecture on Day 1.
b. Read about them in a textbook.
c. Look them up on Purplemath.com, and explain them to a family member.

3. ____ I can evaluate polynomials! (15 points)

How can I learn to evaluate polynomials? Suggestions (try two):

a. Listen to the lecture on Day 2.
b. Build two versions of a cube. Build two versions of a square. Build a row for each. Write it "mathematically" on a piece of paper, or dictate it to a friend to write.
c. Do some practice paper problems (in sheet bin).
d. Watch Mr. Keegan evaluate them using Sketchup.com.

4. ____ I can add polynomials! (20 points)

a. Listen to the lecture on Day 3 or read about it on Purplemath.com.
b. Add together the two problems you built for Question 3b. Write it out.
c. Do some practice paper problems (in sheet bin).

B Layer: (30 points)
Model a section of a warehouse where iPods are stacked in cubes of 64 ($x = 4$). You start on Monday with 6 cubes ($6x^3$). On Friday you have 8 iPods left. You sell them in flats (x^2), quads (x), and single units. Show how this could have happened (choose one).

- Use Sketchup.com.
- Use our plastic cube station.
- Draw it on a poster (showing three dimensions).
- Write a story.

A Layer: (20 points)
I need a barn for my two milk cows. I want a place to store hay, store feed, milk the cows, and park a tractor. Materials come in ($30) cubes, ($20) squares, ($15) rows, and ($10) single, 2-ft blocks. The cubes are the cheapest cost per foot, and it goes up from there.

1. With a partner, design my barn using Sketchup.com. What's it going to cost me? Defend your design.
2. Using the materials in the shoe boxes, with a partner or alone, build the above barn. Defend your design.

✓ Sample of a Layered Curriculum Unit in Calculus Content: Introducing Derivatives

C Layer:

Day 1—Topic: Define derivatives and rules for finding derivatives

1. Lecture (5 points)
2. Practice: Choose one (5-point quiz)
   a. Problem Set 3.1, numbers 1 to 9
   b. Work derivatives unit on our *Journey Through Calculus* software.

Day 2—Topic: Derivatives of trigonometric functions

1. Lecture (5 points)
2. Practice: Choose one (5-point quiz)
   a. Problem Set 3.2, numbers 1 to 10
   b. Find a website that teaches this topic. Create a mini-lesson, and teach it to two classmates.

Day 3—Topic: The chain rule

1. Lecture (5 points)
2. Practice: Choose one (5-point quiz)
   a. Problem Set 3.3, numbers 1 to 12
   b. Make a poster that teaches the chain rule. Give a mini-lesson to two classmates.

B Layer: Choose one (10 points)

1. Write a one-page library report on tides, and explain how derivatives are used to predict high tide and low tide.
2. Write a one-page library report on the actual use of derivatives in business transactions and corporate risk management.
3. Solve these three problems:
   a. A rectangular piece of paper measures 20 cm by 28 cm. Equal-sized squares are to be cut out from each corner of the paper, and the remaining flaps are to be folded up to make an open-topped

box. Find the dimensions of the square that should be cut out to maximize the volume of the box.

b. A metal cylindrical soda can is to be constructed to have a known volume, V. What is the ratio of the diameter of the can to the height of the can if the amount of metal used in the construction of the can is to be a minimum? Assume that the metal used is of uniform thickness.

c. A Norman window, formed by placing a semicircle on top of a rectangle, still remains a popular architectural feature. If the perimeter of the window is 300 cm, find the radius of the semicircle that will maximize the window's area and let in the most light.

A Layer: Choose one. Find and summarize three pieces of research on your topic. Write a two-paragraph opinion using the research as a basis. (15 points)

1. Asteroid A2004 MN4 is heading toward Earth and projected to impact our planet in 2029. Impact dates continue to be revised. How do they calculate this event? Should we be worried?
2. When Hurricane Katrina hit New Orleans in 2005, how did the tide position at the time the hurricane made landfall impact the devastation of New Orleans? Would a change in tidal position have made the impact worse or better? How much importance should be placed on tidal position during coastal storms for making evacuation decisions?

You will also have a 50-point quiz over this unit.

For more information on Nunley's Layered Curriculum, see the **Resources** section at the end of this book.

Graphic Organizers

We mentioned in Chapter 5 that today's adolescents have grown up in a visual world. They are surrounded by television, computer screens, movies, portable DVD players, and cell phones with screen images. Using visual tools in all mathematics classrooms, then, makes a lot of sense. A graphic organizer is one type of visual tool that not only gets students' attention but is also a valuable device for improving understanding, meaning, and retention.

Many different types of graphic organizers are available in books and on the Internet (see websites provided in the **Resources** section of this book). The following are just a few examples created by Dale Graham and Linda Meyer (2007) for use in high school mathematics classes, used here with their permission.

How to Determine if a Function Is Continuous

$f(x)$ is continuous
at $x = a$ if three conditions are met:

↓

1st: $f(a)$ exists
if the value a is substituted for x, the function has a value

↓

2nd: $\lim_{x \to a} f(x)$ exists
The limit of the function as x approaches a can be found

↓

3rd: $\lim_{x \to a} f(x) = f(a)$
The value of the limit is the same as the value of the functions when x equals a

SOURCE: Graham and Meyer (2007). Adapted with permission of the authors.

How to Graph Quadratic Functions

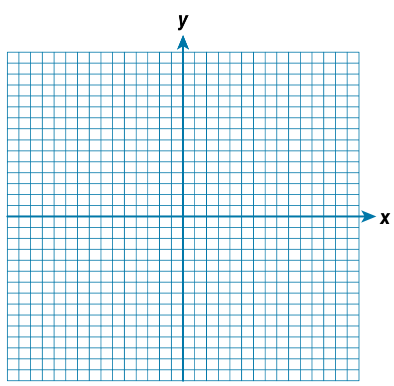
Intercept Form

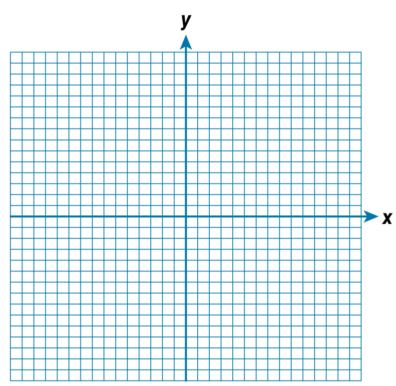
$f(x) = a(x - b)(x - c)$

	Example $f(x) = 0.5(x + 5)(x - 4)$	**Your Turn** $f(x) = -\frac{1}{2}(x - 3)(x + 1)$
Find and plot the x-intercepts and the vertex. $x = \dfrac{b + c}{2}$		
Find and sketch the axis of symmetry.		
Make a table of values using two values of x higher than the vertex. Plot these points.		

Using symmetry, find two other points on the curve and plot them.

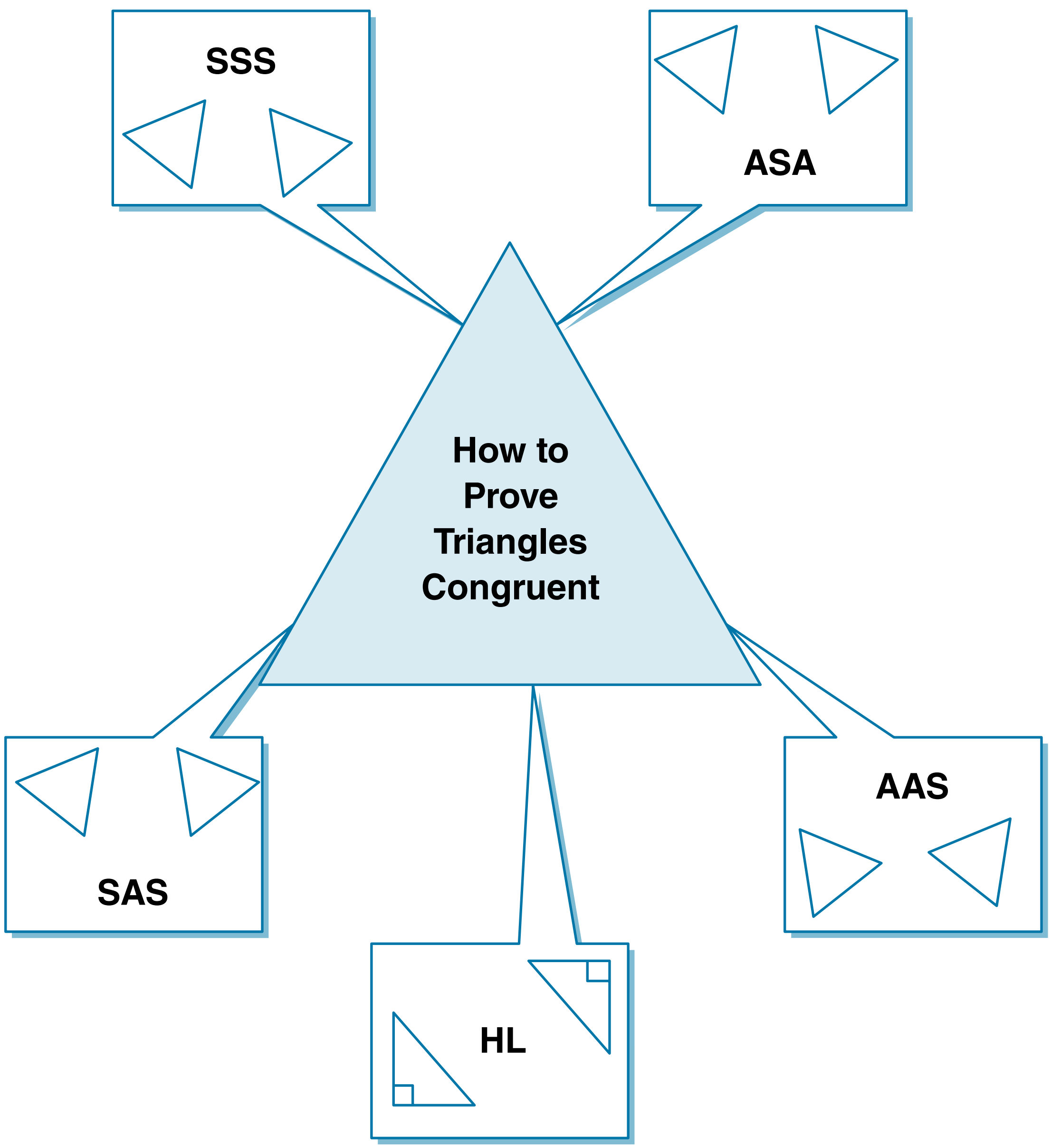
Connect the points with a smooth curve.





SOURCE: Graham and Meyer (2007). Adapted with permission of the authors.

How to Prove Triangles Congruent



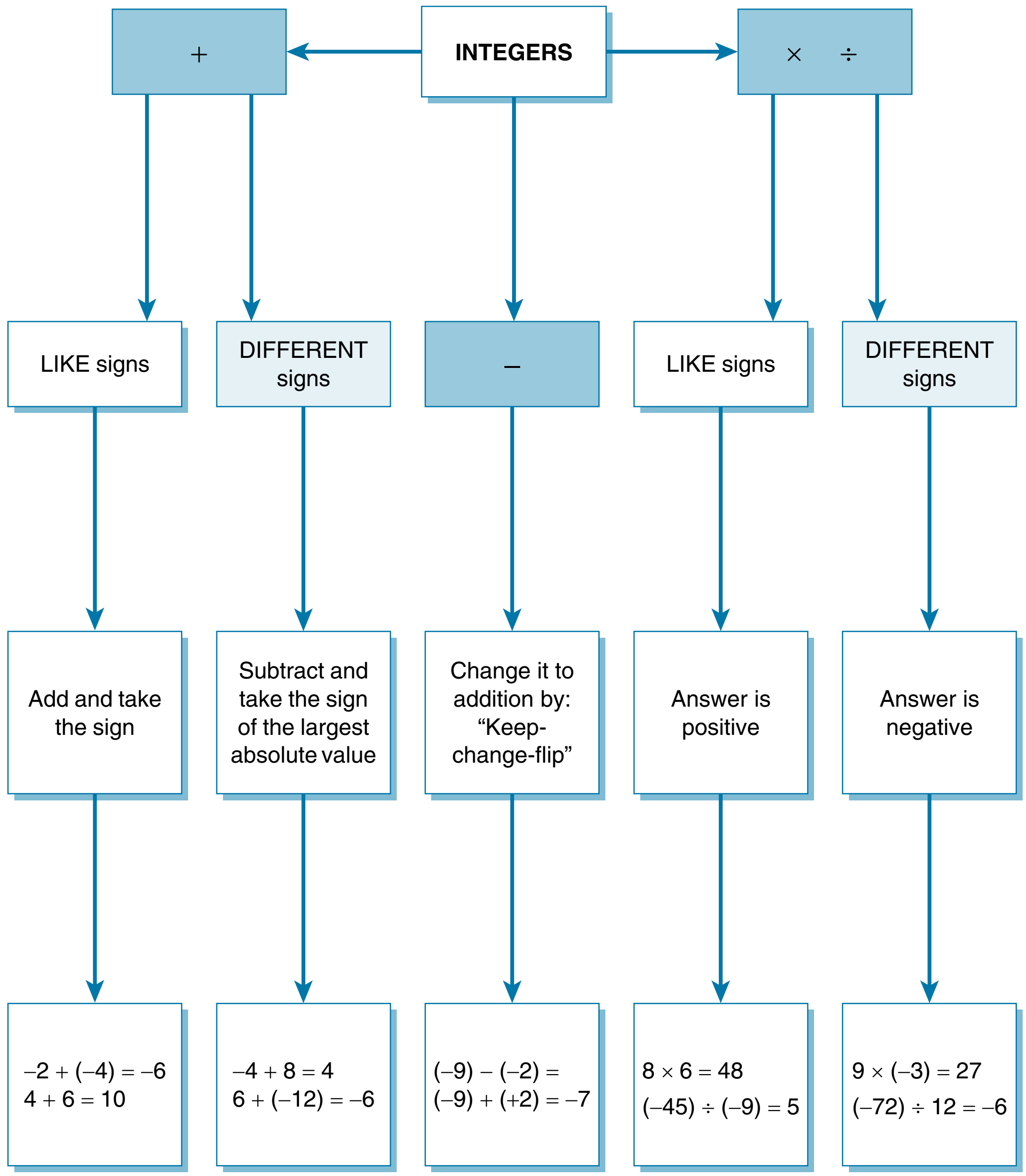
SOURCE: Graham and Meyer (2007). Adapted with permission of the authors.

How to Write the Equation of a Line

Given	*Example*	*Your Turn*
Slope and *y*-intercept Use the slope intercept form of the equation: $y = mx + b$ Substitute for m and b	Slope = 3/4, *y*-intercept = −2	Slope = −1/2, *y*-intercept = 5
Slope and a Point Use the point-slope form of the equation: $y - y_1 = m(x - x_1)$ Substitute for m, x_1, and y_1. Solve for y. Solve for y.	Slope = −3, Point = (−2, 4)	Slope = 4, Point = (−6, −4)
Two Points Use the slope formula and the coordinates of the two points to find the slope. $m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$ Use this slope and one of the given points to write the equation of the line following the **Slope and a Point** method above.	Points are (−6, 3) and (2, −7)	Points are (−9, −2) and (1, −8)

SOURCE: Graham and Meyer (2007). Adapted with permission of the authors.

Integer Rules



SOURCE: Graham and Meyer (2007). Adapted with permission of the authors.

Interpreting Word Problems

Even students who are proficient at solving mathematical expressions can have difficulty interpreting the meaning of word problems. Barton and Heidema (2002) suggest that authors of mathematics texts do not always follow the principles of writing that students have learned in their language arts classes. For example, students learn that an author's main idea usually appears in the passage's opening sentences. In mathematics problems, however, the main idea often appears in the last sentence. Here's a typical example:

> Billy is sorting out blue, green, and yellow marbles into single color groups. He has 58 marbles altogether. There are twice as many blue marbles as green marbles, and three more yellow marbles than blue marbles. How many marbles of each color does Billy have?

Students must wade through numerous details before they get to the point of the problem: "How many marbles of each color does Billy have?" Teachers can help students interpret word problems by using strategies designed to focus on what the problem is asking and then select a solution.

The SQRQCQ Process

✓ One way to help students get to important information in a word problem is through a six-step process called SQRQCQ (Barton & Heidema, 2002; Heidema, 2009). This strategy is designed to help students think through what the problem is asking and to determine the method for solving it. The six steps are as follows.

- Survey: Read the problem quickly to get a general understanding of it.
- Question: Ask what information the problem requires.
- Read: Reread the problem to identify relevant information, facts, and the details needed to solve it.
- Question: Ask what must be done to solve the problem. "What operations must be performed and in what order?"
- Compute: Do the computations or construct a solution.
- Question: Ask whether the solution process seems correct and the answer reasonable.

Here is a simple example:

Problem: *Billy has 26 apples. His friend Patrick gives him 9 more apples. Billy now has 13 more apples than Michael. How many apples does Michael have?*

Following the SQRQCQ process, the student's thinking should go like this:

Survey: *Billy had 26 apples, but Patrick gave him 9 more, so he has 13 more than Michael.*

Question: *I need to find out how many apples Michael has.*

Reread: *I can see that Michael will have fewer apples than Billy.*

Question: *If I know that Billy started out with 26 apples and that Patrick gave him 9 more, then I have to add 9 to the 26 to find out Billy's new total number of apples. Then, to find out how many apples Michael has, I need to subtract 13 from Billy's total number of apples.*

Compute: *26 + 9 = 35—This is Billy's new total.*

35 – 13 = 22—This is how many apples Michael has.

Question: *Is it true that 26 plus 9 minus 13 is 22? Yes, so the answer must be correct.*

Note that in this strategy, the student is specifically told to reread the problem. Students often rely on their working memory to retain all the relevant information after one quick reading. Rereading the problem (a form of rehearsal) increases the likelihood that important details will be found, processed, and retained in working memory, while unimportant items will be set aside. This methodical process has been particularly effective for improving recording abilities, study habits, and thinking skills among students who have learning difficulties and also with English language learners.

This process can be used in cooperative learning groups so students talk each other through the six steps. Each step builds on the previous one and gives students the opportunity to look deeper into the meaning of each problem.

Word Problem Roulette

Another successful strategy for working with word problems, suggested by Heidema (2009), is known as Word Problem Roulette. In this process, the students work in small groups and collaborate on solving a word problem. Then they report out to the whole group orally and in writing the processes they went through to solve the problem. The students benefit from communicating their own thinking to others and from hearing how other students think about the problem they are solving. Here is how it works:

1. Students get into groups of three or four. Each student has a copy of the word problem for the group. The teacher explains that they are to solve this problem as a group.
2. The students discuss ways to solve the word problem. They talk about what the problem is asking and orally (no writing at this time) suggest ideas for solving the problem. Then the students agree on a solution method and the steps they will use to solve the problem.
3. After the students agree on a solution, they take turns writing the steps to the solution in words, not in mathematical symbols. Each student writes one step or sentence and passes the paper to the next student, who then adds the following step or sentence (this is the

roulette part). Although the students can confer on what individuals write, the solution paper should have contributions from everyone in the group.

4. When all the groups have finished writing down their solutions, each group in turn presents its solution to the class. While one member of a group reads the solution steps, another group member writes the symbolic representation of the solution on the board.
5. If all the groups had the same problem, then the teacher compares the methods and results of the different groups. If the groups have different problems, volunteers from other groups can be asked to review and comment on another group's solution.

Here is a sample problem to try with Word Problem Roulette:

A family of three adults and four children goes to an amusement park where an adult's admission is twice as much as a child's admission. The total cost of admissions for the family is $260. How much is an adult's admission? How much is a child's admission?

Making Mathematics Meaningful to Teenagers

As mentioned in Chapter 3, it is important for students to find meaning in what they are learning because meaning is one of the criteria the brain uses to identify information for long-term storage. One way to help learners find meaning is to connect what they are learning to their daily lives. Yet, too often, students in secondary-school mathematics classes have difficulty seeing the practical and concrete applications of mathematics to everyday living. Here are just a few suggestions for how mathematical concepts can be meaningfully related to common experiences.

Probability

✓ **Determining odds.** Millions of people visit casinos, buy lottery tickets, play the stock market, join in the office football pool, and meet with friends for a game of poker. They invest their money in chance, believing they can beat the odds. The mathematical principle of probability can tell us how often we are likely to win, helping us decide whether to risk the odds and our money.

How do we determine probability? Let's say there are 12 apples in a fruit basket. Five are red and seven are green. If you close your eyes, reach into the basket, and grab one apple, what is the probability that it will be a red apple? Five of the 12 apples are red, so your chances of picking a red apple are 5 out of 12 or, as a fraction, 5/12, which is about 42 percent. Or let's say you are choosing between two colleges, one in Texas and one in Connecticut. You decide to flip a coin. The chances are one out of two, or 1/2, of getting heads or tails. The odds are 50 percent for each.

What are your odds for winning the state lottery if you buy only one ticket?

✓ **Does gambling pay off? Odds in roulette.** Is roulette a good bet at a casino? Actually, the casino will win more often than the player. Here's why: The roulette wheel is divided into 38 numbered slots. Two of these slots are green, 18 are red, and 18 are black. To begin the round, the wheel is spun, and a ball is dropped onto its outside edge. When the wheel stops, the ball drops into one of the 38 slots. Players bet on which slot they believe the ball will land in. If you bet your money that the ball will land in any of the 18 red slots, your chances of winning are 18 out of 38, or about 47 percent. If you bet your money on a certain number, such as the red slot numbered with a 10, your chances of winning fall to 1 in 38, or 2.6 percent.

The mathematics of probability guarantees that the roulette wheel will make money even if the casino doesn't win every time. Remember, there are 18 each of the red and black slots. There are also 2 green slots. Whenever the ball lands in one of those green slots, the house wins everything that was bet on that round. So again, let's say you bet that the ball will land in a red or black slot. This is the safest possible bet in roulette, since the odds are 18 out of 38 (47 percent) that you will win. But there are 20 out of 38 chances (53 percent) that you will lose.

Calculating Interest on Buying a Car

✓ **How much are you actually paying when you finance a car purchase?** Understanding interest can help you manage your money and help you determine how much it will cost you to borrow money to pay for your car purchase. Interest is expressed as a rate, such as 3 percent or 18 percent. The dollar amount of the interest you pay on a loan is figured by multiplying the money you borrow (called the principal) by the rate of interest.

Suppose you want to buy a used car for $10,000. The car salesman says that the dealership will finance your car at a rate of 8.4 percent and estimates your monthly payments at about $200 over a period of 5 years. How much money are you actually paying back to the dealer over the term of the loan? Is this a good deal, or should you shop around? What if a bank offered to loan you the $10,000 at a rate of 9.0 percent for 4 years? Which offer is better?

Exponential Changes/Progressions

✓ **Population growth.** In 2014, the world population was estimated at 7.2 billion people. The number of people living on the earth has grown dramatically in the past few centuries. There are now 10 times more people on our planet than there were 300 years ago. How can population grow so fast? Think of a family tree. At the top are two parents, and beneath them are the children they had. Listed beneath those children are the children *they* had, and so on down through many generations. As long as the family continues to reproduce, the tree will increase in size, getting larger with each passing generation. This same idea applies to the world's population.

New members of the population eventually produce other new members so that the population increases exponentially as time passes. Population increases cannot continue forever. Living creatures are constrained by the availability of food, water, land, and other vital resources. Once those resources are depleted, population growth will plateau, or even decline, as a result of disease or malnutrition.

How fast will the population grow? Arriving at a reasonable estimate of how the world's population will grow in the next 50 years requires a look at the rates at which people are born and die in any given period. If birth and death rates stayed the same across the years in all parts of the world, population growth could be determined with a fairly simple formula. But birth and death rates are not constant across countries and through time because disease or disaster can cause death rates to increase for a certain period. A booming economy might mean higher birth rates for a given period.

The rate of the earth's population growth is slowing down. Throughout the 1960s, the world's population was growing at a rate of about 2 percent per year. By 2010, that rate was down to 1.14 percent and is estimated to drop to less than 1 percent by the year 2020. Family planning initiatives, an aging population, and the effects of diseases such as AIDS are some of the factors behind this rate decrease. Even at these very low rates of population growth, the numbers are staggering. Can you estimate how many people will be living on the earth in 2020? By 2050? Can the planet support this population? When will we reach the limit of our resources? How could this affect the lifestyle of your children or grandchildren?

✓ **Is this job offer a good deal?** Looking to make a million dollars? Let us examine a plan for earning a million dollars based on a contract between an employee and an employer. First, let's agree on a contract.

Contract for Employment

Employee ____________ (Your name)

Employer ____________ (A company agreeing with these terms)

Points of Agreement

1. The employee will work a 5-day workweek.
2. The employee will be paid for the week's wages each Friday.
3. The employee will be hired for a minimum of 30 workdays.
4. The salary schedule is as follows:
   - The base pay for Day 1 is one penny.
   - Each subsequent day, the salary is double that of the previous day.

   Signed __________________________ (Employee)

   Signed __________________________ (Employer)

   Date: __________________________

Is this a good deal? Take a guess how much money this employee will have earned in the 30 working days. My guess: $__________________. Calculate the amount one would earn working 6 weeks (40 hours a week) at minimum wage. Minimum-wage salary (before taxes and other deductions): $____________. Now let's calculate the earnings for this contract and see whether the employer or the employee has made the better deal. In Week 1, the wages would be as follows: Monday, 1 cent; Tuesday, 2 cents; Wednesday, 4 cents; Thursday, 8 cents; and Friday, 16 cents, for total weekly earnings of 31 cents. Doesn't seem like much does it? Now continue calculating the daily wages for the next 5 weeks.

There is a formula that allows one to calculate a particular day's wages without having to calculate every step. This is an example of a geometric progression, a sequence of numbers in which the ratio of any number to the number before it is a constant amount, called the common ratio. For example, the sequence of numbers 1, 2, 4, 8, 16, . . . has a common ratio of 2. A geometric progression may be described by calling the first term in the progression X (in this example, $X = 1$ cent), the common ratio R (in this example, $R = 2$), and, in a finite progression, the number of terms n. Then the nth term of a geometric progression is given by the expression, $X_n = X_1R^{n-1}$.

Questions about this job:

1. How does the total amount of money earned compare with your original guess?
2. Suppose you wanted to buy a car. On which day could you purchase your car and pay in cash?
3. Can you develop a formula for the daily salary? (Answer: Daily Salary = $2^{n-1}X$, where n = the number of days you've been working and X = your base pay on Day 1.)

This counting principle can also be applied to social causes. Efforts to address social issues are often started by just a scant few individuals who are committed to a cause. Suppose you tell one person a day about your issue. A one-on-one plea will be much more effective in convincing the listener. On the second day, there will be two of you who can approach two more people. On the third day, there are four of you to approach four more people. On the fifth day, the eight of you convince eight more people, and so on. By the twelfth day, more than 2,000 people know about your cause, and by the thirtieth day, more than 1 billion people are talking about the issue that is so close to your heart! Yet you personally talked to only 30 people. By the way, now you know how unfounded rumors spread so quickly.

Ratio/Proportion

- ✓ **The challenges of cooking: Altering recipes.** Recipes involve mixing together ingredients that have relationships to one another. In mathematics, this relationship between two quantities is called a ratio. If a recipe calls for 1 egg and 2 cups of flour, the relationship of eggs to cups of flour is 1 to 2. In mathematical language, that relationship can be written in two ways: 1/2 or 1:2.

All recipes are written to serve a certain number of people or yield a certain amount of food. For example, suppose you have a cookie recipe that makes 2 dozen cookies. What if you want 1 dozen or 4 dozen cookies? Understanding how to increase or decrease the yield without spoiling the ratio of ingredients is a valuable skill for any cook.

Let's look at the cookie recipe:

1 cup flour	1/2 tsp. baking soda
1/2 tsp. salt	1/2 cup butter
1/3 cup brown sugar	1/3 cup sugar
1 egg	1/2 tsp. vanilla
1 cup chocolate chips	

This recipe yields 3 dozen cookies. If you want 9 dozen cookies, you will have to increase the amount of each ingredient in the recipe while ensuring that the relationship between the ingredients stays the same. To do this, you will need to understand proportion. A proportion exists when you have two equal ratios, such as 2:4 and 4:8. Two unequal ratios, such as 3:16 and 1:3, do not result in a proportion. The ratios must be equal.

In the cookie recipe, you will need to set up a proportion to make sure you get the correct ratios to make 9 dozen. Start by figuring out how much flour you will need to make 9 dozen cookies by setting up this proportion:

$$1 \text{ (cup)} / 3 \text{ (dozen)} = X \text{ (cups)} / 9 \text{ (dozen)}$$

To find X (number of cups of flour needed in the new recipe), multiply the numbers like this: X times 3 = 1 times 9, or $3X = 9$. Now find the value of X by dividing both sides of the equation by 3. The result is $X = 3$. To extend the recipe to make 9 dozen cookies, you will need 3 cups of flour. Follow the same process to determine how much of each ingredient is needed for 9 dozen cookies.

Other Meaningful Activities

- ✓ **Cell phone plans.** Advertisers often use mathematics to confuse rather than educate the public, and cell phone network companies are no exception. Ask students to collect advertisements for different cell phone network providers. Their task is to compare the various plans offered by these companies and determine which plan is the most economical. They should also note what variables are important, such as number of lines, roll-over minutes, optional texting, and so on, and decide which would be the most appropriate for their individual situation. If they currently have a cell phone plan, how does it compare to what they have discovered in this activity?
- ✓ **Choosing an environmentally friendly car.** The news is full of stories about how much car exhaust emissions contribute to polluting the atmosphere. Environmental and governmental groups are calling on car buyers to choose "green" cars, those that produce low emissions.

If students want to be eco-friendly when they buy a car, what information will they need to decide which make/model is the greenest? What statistics should they gather? (Note: www.fueleconomy.gov is a valuable source for this information.)

✓ **Discovering a famous mathematician.** Contrary to some students' beliefs, mathematics is a human endeavor. There are famous mathematicians just as there are famous rap, movie, and sports stars. Ask each student to conduct research on a famous mathematician. Websites can be valuable sources for finding these individuals. They should prepare a presentation for the class that includes at least the person's name, place of birth, dates of birth and death, education, at least two interesting facts, why the person is famous, and an example of this person's work.

These are just a few examples of activities that can help make seemingly abstract mathematical operations more interesting and practical to students. For suggestions on where to find more examples, see some of the websites in the **Resources** section of this book.

WHAT'S COMING?

Although many students have occasional difficulties learning mathematics, they often find ways to overcome them. Math anxiety is an example of a stumbling block that can be overcome with simple interventions. But some students have persistent difficulties with even simple arithmetic operations. The next chapter discusses how teachers can recognize those students who have persistent problems and what can be done to help them learn mathematics concepts.

Chapter 6—Teaching Mathematics to the Adolescent Brain

QUESTIONS AND REFLECTIONS

Respond to the following questions, and jot down on this page key points, ideas, strategies, and resources you want to consider later. This sheet is your personal journal summary and will help jog your memory.

Explain how the brain's search for novelty can enhance instruction in mathematics. ______________________________

____

____

____

____

____

____

How do the qualitative and quantitative learning styles differ?

____

____

____

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____

How can graphic organizers help in learning mathematics? ______________

____

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7

Recognizing and Addressing Mathematics Difficulties

Do not be troubled by your difficulties with Mathematics. I can assure you mine are much greater.

—Albert Einstein

Some children are adept at mathematical calculations, while others struggle despite much effort and motivation. In the past three decades, the percentage of school-age children who experience difficulties in learning mathematics has been growing steadily. Why is that? Is the brain's ability to perform arithmetic calculations declining? If so, why? Does the brain get less arithmetic practice because technology has shifted computation from brain cells to inexpensive electronic calculators? What makes a child do poorly in mathematics? The answer to this question is complicated by at least two considerations:

1. We need to distinguish whether the poor achievement is due to inadequate instruction or some other environmental factor, or whether it is due to an actual cognitive disability.
2. Exactly *how* is mathematics being taught? Instructional approaches can determine whether a cognitive deficit is really a disability at all. For example, one instructional approach emphasizes conceptual

understanding while deemphasizing the learning of procedures and mathematical facts. Another approach places heavy emphasis on procedures and facts. A student with a deficit in retrieving arithmetic facts might not be considered as having a learning disability in the first approach because of the deemphasis on memory-based information. However, that deficit would be a serious disability in the second approach. Recognizing this dichotomy, the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) have focused more on conceptual understanding than on memorization.

The growing number of students who are having trouble learning mathematics has spurred research interest in how the brain does calculations and the possible causes of mathematical difficulties. In this chapter, the term *mathematics difficulties* includes those students performing in the low average range, regardless of whether their difficulties are due to environmental factors or cognitive deficits. It is important to remember that because mathematics achievement tests include many types of items, it is possible that students may demonstrate average performance in some areas of mathematics and show deficits in other areas.

DETECTING MATHEMATICS DIFFICULTIES

As with any learning difficulty, the earlier mathematics difficulties are detected, the better. Studies have shown that using intense tutoring with first graders who display problems with calculations significantly improved their end-of-year achievement in mathematics (Smith, Cobb, Farran, Corday, & Munter, 2013). The key, of course, is early detection so interventions can begin as soon as practicable.

Determining the Nature of the Problem

The first task facing educators who deal with students with mathematics difficulties is to determine the nature of the problem. Obviously, environmental causes require different interventions than developmental causes. Low performance on a mathematics test *may* indicate that a problem exists, but tests do not provide information on the exact source of the poor performance. Standardized tests, such as the *Brigance Comprehensive Inventory of Basic Skills—Revised*, are available and provide more precise information on whether the problems stem from deficits in counting, number facts, or procedures.

Educators should examine the degree to which students with mathematics difficulties possess the prerequisite skills for learning mathematical operations. What skills are weak, and what can we do about that? They also should look at the mathematics curriculum to determine how much mathematics is being taught and the types of instructional strategies teachers are using. Are we trying to cover too much? Are we using enough visual and manipulative aids? Are we developing student strengths and not just focusing on their weaknesses?

Prerequisite Skills

Examining the nature of mathematics curriculum and instruction may reveal clues about how the school system approaches teaching these topics. A good frame of reference is the recognition that students need to have mastered a certain number of skills before they can understand and apply the principles of more complex mathematical operations. Mathematics educators have suggested that the following seven skills are prerequisites to successfully learning mathematics (Sharma, 2006):

1. Following sequential directions
2. Recognizing patterns
3. Estimating by forming a reasonable guess about quantity, size, magnitude, and amount
4. Visualizing pictures in one's mind and manipulating them
5. Having a good sense of spatial orientation and space organization, including telling left from right, compass directions, horizontal and vertical directions
6. Doing deductive reasoning—that is, reasoning from a general principle to a particular instance or from a stated premise to a logical conclusion
7. Doing inductive reasoning—that is, coming to a natural understanding that is not the result of conscious attention or reasoning, easily detecting the patterns in different situations and the interrelationships between procedures and concepts

Students who are unable to follow sequential directions, for example, will have great difficulty understanding the concept of long division, which requires retention of several different processes performed in a particular sequence. First, one estimates, then multiplies, then compares, then subtracts, then brings down a number, and the cycle repeats. Those with directional difficulties will be unsure which number goes inside the division sign or on top of the fraction. Moving through the division problem also presents other directional difficulties: One reads to the right, then records a number up, then multiplies the numbers diagonally, then records the product down below while watching for place value, then brings a number down, and so on.

Diagnostic Tools

Primary-Grades Assessments

Teachers in the primary grades, of course, often rely on their own observations of students' performance to determine when a particular child is having problems with arithmetic computations. Although teacher observations are valuable, other measures should be considered as well. Research studies have shown that several measures are reliable in detecting and predicting how well young students are mastering number manipulation and basic arithmetic operations. Table 7.1 summarizes the

Table 7.1 Description of Selected Screening Measures in Early Mathematics

Measure	*Description*
Digit span	Student repeats a string of numbers either forward or backward
Fact retrieval	Student solves simple problems in addition and subtraction
Magnitude comparison	Student chooses the largest of four visually or verbally presented numbers
Missing number	Student names a missing number from a sequence of numbers between 0 and 20
Number knowledge test	Basic measure of number sense (see Chapter 5)
Numbers from dictation	Student writes numbers from oral dictation
Number identification	Student identifies numbers between 0 and 20 from printed numbers
Quantity array	Student names the number of dots arrayed in a box
Quantity discrimination	Student identifies the larger of two printed numbers

SOURCES: Gersten et al. (2005, 2011); Griffin (2002); Lembke and Foegen (2005).

screening measures that can be used with kindergarten and first-grade students to determine whether mathematics difficulties exist (Gersten, Clarke, Haymond, & Jordan, 2011; Gersten, Jordan, & Flojo, 2005; Griffin, 2002; Lembke & Foegen, 2005).

Postprimary-Grades Assessments

Past the primary grades, research studies suggest that five critical factors affect the learning of mathematics. Each factor can serve as a diagnostic tool for assessing the nature of any learning difficulties students may experience with mathematical processing (Augustyniak, Murphy, & Phillips, 2005; Bohlmann & Weinstein, 2013; Sharma, 2006). Here are the factors to consider:

✓ **Level of cognitive awareness.** Students come to a learning situation with varying levels of cognitive awareness about that learning. The levels can range from no cognitive awareness to high levels. Your first task is to determine the students' levels of cognitive awareness and the strategies each student brings to the mathematics task. This is not easy, but it can be accomplished by doing the following:

- Interview the students individually and observe how each one approaches a mathematical problem that needs to be solved.
- Ask, "What is the student thinking?" and "What formal and informal strategies is the student using?"

- Determine what prerequisite skills are in place and which are poor or missing.
- Determine if a mathematics answer is correct or incorrect, and ask students to explain how they arrived at the answer.

Knowing the levels of the students' cognitive awareness and prerequisite skills will give you valuable information for selecting and introducing new concepts and skills.

✓ **Mathematics learning profile.** As discussed in Chapter 6, researchers agree that each person processes mathematics differently and that these differences run along a continuum from primarily quantitative to primarily qualitative. You will recall that quantitative learners prefer entities with definite values, use procedural approaches to problem solving, and focus on deductive reasoning. Qualitative learners, on the other hand, prefer holistic and intuitive approaches, look for relationships between concepts and procedures, are social learners, and focus on visual–spatial aspects of mathematical information.

Because both types of learning profiles are present in mathematics classes, teachers need to incorporate multiple instructional strategies. Teaching to one style alone leaves out students with the other style, many of whom may do poorly in mathematics as a result. In fact, some may even exhibit the symptoms of mathematics difficulties.

✓ **Language of mathematics.** Mathematical difficulties often arise when students fail to understand the language of mathematics, which has its own symbolic representations, syntax, and terminology. Solving word problems requires the ability to translate the language of English into the language of mathematics. The translation is likely to be successful if the student recognizes English language equivalents for each mathematical statement. For example, if the teacher asks the class to solve the problem "76 take away 8," the students will correctly write the expression in the exact order stated: "76 – 8." But if the teacher says, "Subtract 8 from 76," a student following the language order could mistakenly write "8 – 76." Learning to identify and correctly translate mathematical syntax becomes critical to student success in problem solving.

Language can be an obstacle in other ways. Students may learn a limited vocabulary for performing basic arithmetic operations, such as *add* and *multiply*, only to run into difficulties when they encounter expressions asking for the *sum* or *product* of numbers. You can avoid this problem by introducing synonyms for every function: "Let us *multiply* 6 and 5. We are finding the *product* of 6 and 5. The product of 6 *times* 5 is 30."

✓ **Prerequisite skills.** As noted earlier, the seven prerequisite skills for learning mathematics successfully are nonmathematical in nature. However, they must be mastered before even the most basic understandings of number concepts and arithmetic operations can be learned. You should assess the extent to which these seven skills are present in each student.

Consider using this simple profile diagram (see example on the next page) to assist in assessment of the seven skills. After assessing the student's level on each skill, analyze the results and decide on a plan of action that will address any areas needing improvement.

Students with four or more scores in the 1 to 2 range will have significant problems learning the basic concepts of mathematics. They will need instruction and practice in mastering these skills before they can be expected to tackle mathematical content.

Prerequisite Skills Profile for Mathematics

Student's Name: ______________________________ Date: __________

Directions: On a scale of 1 (lowest) to 5 (highest), circle the number that indicates the degree to which the student displays mastery of each skill. Connect the circles to see the profile.

Skill					
Follows sequential directions	5	4	3	2	1
Recognizes patterns	5	4	3	2	1
Can estimate quantities	5	4	3	2	1
Can visualize and manipulate mental pictures	5	4	3	2	1
Has sense of spatial orientation and organization	5	4	3	2	1
Is able to do deductive reasoning	5	4	3	2	1
Is able to do inductive reasoning	5	4	3	2	1

Action Plan: As a result of this profile, we will work together to

____

____

____

by doing

____.

✓ **Levels of learning mastery.** How do you decide when a student has mastered a mathematical concept? Certainly, written tests of problem solving are one of the major devices for evaluating learning. However, they are useful tools only to the extent that they measure actual mastery rather than rote memory of formulas and procedures. Cognitive research suggests that a person must move through the following six levels of mastery to truly learn and retain mathematical concepts:

- Level 1: Connects new knowledge to existing knowledge and experiences (Example: Recognizes that three multiplied by four is the same as three plus three plus three plus three, and the same as four plus four plus four)
- Level 2: Searches for concrete material to construct a model or show a manifestation of the concept (Example: Uses manipulatives such as blocks or coins to lay out four groups of three objects or three groups of four objects)
- Level 3: Illustrates the concept by drawing a diagram to connect the concrete example to a symbolic picture or representation (Example: Draws four groups of three objects or three groups of four objects, such as animals or stars)
- Level 4: Translates the concept into mathematical notation using number symbols, operational signs, formulas, and equations (Example: Writes $3 \times 4 = 3 + 3 + 3 + 3 = 12$ or $3 \times 4 = 4 + 4 + 4 = 12$)
- Level 5: Applies the concept correctly to real-world situations, projects, and story problems (Example: Solves prewritten or student-created story problems)
- Level 6: Can teach the concept successfully to others, or can communicate it on a test (Example: Explains the concept orally to a peer or the class)

Too often, paper-and-pencil tests assess only Level 6. Thus, when the student's results are poor, the teacher may not know where learning difficulties lie. By designing separate assessments for each level, teachers will be in a much better position to determine what kind of remedial work will help each student.

ENVIRONMENTAL FACTORS

Students without cognitive deficits may still display difficulties with arithmetic and mathematical operations. Environmental factors, such as emotional responses to mathematics and instructional quality, can play a vital role in determining how well young students and adolescents will achieve in their mathematics classes.

Answer to Question 9. False: *Students without cognitive deficits may still display difficulties with arithmetic and mathematical operations.*

Student Attitudes About Mathematics

In modern American society, reading and writing have become the main measures of a good student. Mathematics ability has been regarded more as a specialized function than as a general indicator of intelligence. Consequently, the stigma of not being able to do mathematics has been reduced and has become socially acceptable. Just hearing their parents say, "I wasn't very good at math," allows children to embrace the social attitudes that regard mathematics failure as acceptable and routine.

In recent years, schools have placed a heavy emphasis on raising standards in all curriculum areas. At the same time, the No Child Left Behind Act requirements included high-stakes assessments in reading and mathematics. Despite these initiatives, student attitudes about mathematics have not improved much. Surveys show that most students (including those who like mathematics) find making nonmathematical mistakes much more embarrassing than making mathematical mistakes. Furthermore, regardless of the efforts toward gender equity, female high school students still rate themselves as less confident in mathematics than do their male peers (Martinez & Guzman, 2013).

These findings are unsettling, especially because other research studies have shown that attitudes are formed by social forces and predict academic performance. Not surprisingly, students with positive attitudes about what they are learning are more motivated and therefore achieve more than students with poor attitudes (Bramlett & Herron, 2009; Glynn, Taasoobshirazi, & Brickman, 2007). Apparently, higher standards, the STEM (science, technology, engineering, and mathematics) initiative, and increased testing are not sufficient as yet to improve how students feel about learning mathematics. They do not yet view competency in mathematics as a basic life skill. Until this view changes, students will have little incentive to master mathematics and teachers will continue to have their work cut out for them.

Fear of Mathematics (Math Anxiety)

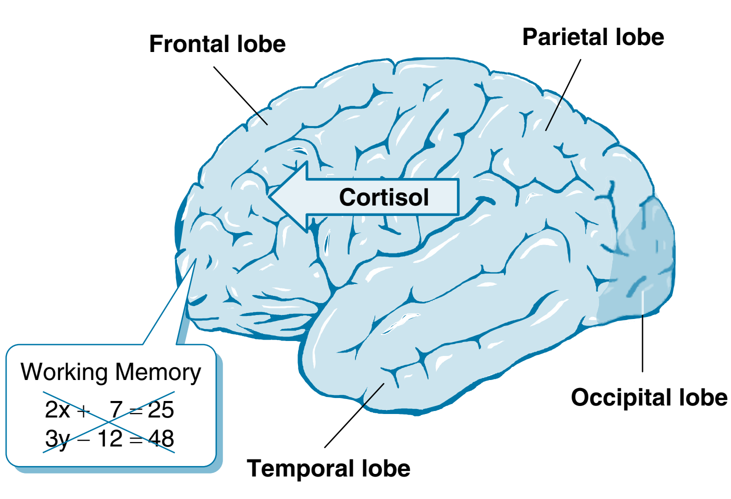
Anxiety about learning and doing mathematics (commonly referred to as math anxiety) has been around a long time. It can be described as a feeling of tension that interferes with the manipulation of numbers and the solving of mathematical problems in academic and ordinary-life situations. It occurs in many individuals regardless of age, race, or gender, and can prevail in the home, classroom, or society. Some studies suggest that more than 60 percent of secondary students have a fear of mathematics (National Mathematics Advisory Panel, 2008).

Students at all grade levels often develop a fear (or phobia) of mathematics because of negative experiences in their past or current mathematics class, or they have a simple lack of self-confidence with numbers. Math anxiety appears to be more prevalent in girls than in boys (Devine, Fawcett, Szucs, & Dowker, 2012). It conjures up fear of some type; perhaps it is the fear that one won't be able to do the calculations or that it's too difficult, or the fear of failure that often stems from having a lack of confidence. In people with math anxiety, the fear of failure often causes their minds to draw a blank, leading to more frustration and more blanks. The added pressure of time limits on mathematics tests also raises the levels of anxiety for many students (Popham, 2008; Tsui & Mazzocco, 2007). Ironically, when we ask a student to spell a new vocabulary word, we do not say to the student, "Spell that word as fast as you can." Rather, we encourage the student to take the time needed to spell it correctly. Yet, so often, we insist that students solve mathematics problems as fast as possible. As a result, we produce students who rely more on rote memorization than on understanding concepts.

Typically, students with this phobia have a limited understanding of mathematical concepts. They may rely mainly on memorizing procedures, rules, and routines, without much conceptual understanding; so panic soon sets in. Mathematics phobia can be as challenging as any learning disability, but it is important to remember that these students have normal neurological systems for computation. They need help primarily in replacing the memory or fear of failure with the possibility of success. As we shall see later, students with mathematical disorders, on the other hand, have a neurological deficit that results in persistent difficulty in processing numbers.

Regardless of the source, the most prevalent consequences of this anxiety are poor achievement and poor grades in mathematics. One reason for the poor performance is biological. Anxiety of any type causes the body to release the hormone cortisol into the bloodstream. Recall that cortisol's main function is to refocus the brain on the source of the anxiety and determine what action to take to relieve the stress (Figure 7.1). Heart rate increases, and other physical indicators of worry appear. Meanwhile, the frontal lobe is no longer interested in learning or processing mathematical operations because it has to deal with what may be a threat to the individual's safety. As a result, the student cannot focus on the learning task at hand and has to cope with the frustration of inattention. Furthermore, the anxious feelings disrupt working memory's ability to manipulate and retain numbers and numerical expressions (Trezise & Reeve, 2014; Vukovic, Kieffer, Bailey, & Harari, 2013).

Figure 7.1 Math anxiety causes the release of cortisol into the bloodstream. Cortisol refocuses the frontal lobe to deal with the anxiety. Meanwhile, any unrelated learning in working memory is disrupted and lost.



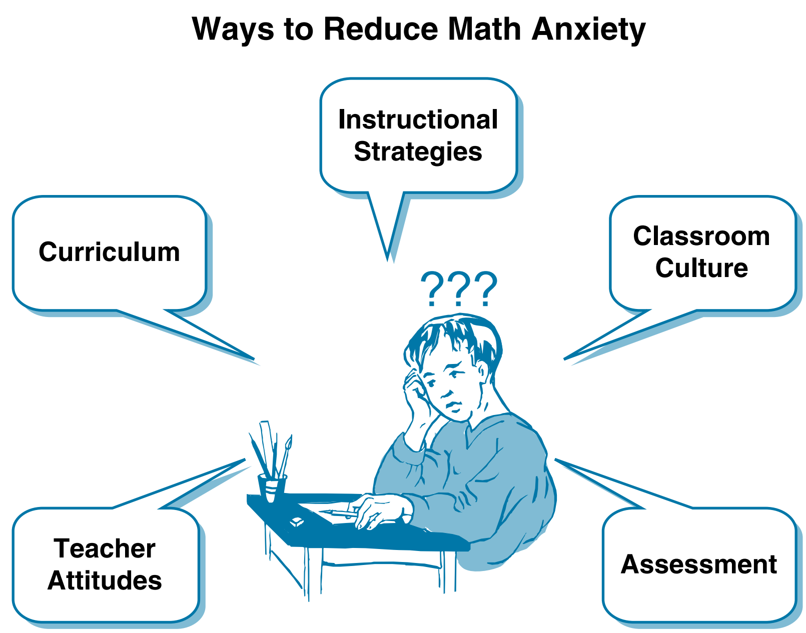
Alleviating Math Anxiety in the Classroom

Researchers suggest that five areas contribute in one way or another to math anxiety: teachers' attitudes, curriculum, instructional strategies, the classroom culture, and assessment (Figure 7.2). Let's take a look at what research studies say about each of these five areas, as well as what can be done to lessen anxiety and improve student achievement in mathematics (Finlayson, 2014; Geist, 2010; Lyons & Beilock, 2012; Shields, 2005).

✓ **Teacher attitudes.** Research studies confirm that teacher attitudes greatly influence math anxiety and represent the most dominating factor in molding student attitudes about mathematics (Beilock, Gunderson, Ramirez, & Levine, 2010; Brady & Bowd, 2005). Here are some things teachers can do to maintain a positive attitude in themselves as well as their students:

- Present an agreeable disposition that shows mathematics to be a great human invention.
- Show the value of mathematics by how it contributes to other disciplines as well as society.

Figure 7.2 Math anxiety is a common problem with both students and adults. In schools, math anxiety can be lowered by making modifications in the five areas shown in this figure.



- Promote student confidence and curiosity by assigning appropriate, interesting, and relevant tasks.
- Focus on the goals and process of learning rather than just searching for the correct answer.
- Create opportunities for success. Teachers need to build in a significant success rate for students to remain engaged with work that is challenging enough to demand effort but easy enough to expect success.
- Resist the temptation to believe that males have a greater innate ability than females to do mathematics or that females have to put forth more effort than males to succeed.
- Display confidence in your teaching. Teachers, especially at the elementary grade levels, who are math anxious themselves or who lack confidence with the subject often inadvertently transmit this fear to students.

✓ **Curriculum.** Studies of mathematics curricula in kindergarten through Grade 8 reveal much repetition of subject matter. One comprehensive study of 183 mathematics topics taught in kindergarten through Grade 8 by more than 7,000 teachers in 27 states showed a considerable amount of redundancy (Polikoff, 2012). From 70 to 80 percent of instructional time in Grade 3 through middle school repeated material taught in the previous grade. Thus, only 20 to 30 percent of time was devoted to new topics. The study also noted that the Common Core State Standards for Mathematics have even more redundancy in the early grades but much less repetition in middle school.

Despite this repetition, students in the primary grades usually rate mathematics as one of the subjects they like most. Through positive learning experiences, the students believe that they have the competence to do mathematics and that hard work will bring success. But by fourth grade, math anxiety often surfaces because the curriculum shifts from using manipulatives and concrete applications to more rote memorization and abstract thinking. By middle school, the abstract nature of the curriculum content causes students to believe that success in mathematics is due to innate ability and that effort matters little. The material gets even more abstract in high school, and students realize that memorization is not sufficient to succeed. Teachers help students make this transition to abstract thinking when they

- devote more time in the elementary and middle school grades to new material, discovery, and application;
- provide activities that constantly train students to apply their knowledge to new ideas and to use mathematics as a tool for discovery;
- prune the mathematics curriculum to eliminate the less important items so it focuses on a deeper understanding of major topics and enhances skills; and
- avoid repeating the same topics annually unless they are critical to learning, applying, and discovering new opportunities in mathematics.

✓ **Instructional strategies.** One critical factor in how well students learn mathematics is the quality of the teaching. Studies show that student achievement in mathematics is strongly linked to the teacher's expertise in mathematics. Students of a teacher expert in mathematics perform better on achievement tests than do students of a teacher with limited training in mathematics (National Science Foundation, 2004). Teaching techniques that center on "explain–practice–memorize" are among the main sources of math anxiety because the focus is on memorization rather than on understanding the concepts and reasoning involved. Students taught with this approach do not have the skills to deal successfully with material that goes beyond memorization. Students are more successful in mathematics classes where teachers

- possess a mathematics skill level that goes beyond basic understanding;
- show an awareness and understanding of student confusion and frustration;
- pose questions in an effort to help students continuously learn;
- limit the frequency of memorizing, doing rote practice, searching for one right answer, and making calculations that can be performed by computers and calculators;
- develop meaning by showing practical applications that are related to students' lives;
- incorporate projects that allow students to explore solutions to problems individually and in groups;
- encourage students to investigate and formulate questions involving mathematical relationships; and
- provide opportunities for students to represent everyday situations verbally, numerically, graphically, and symbolically.

✓ **Classroom culture.** Classroom culture can be defined as the norms and behaviors that regularly guide classroom interactions. Structured, rigid classes where there is little opportunity for debate can become a source of math anxiety. If the culture includes that inevitable search for the one right answer, then students feel that there is no recognition or reward for the cognitive processes involved. The culture may also place a strong emphasis on answering quickly and on timed tests. Emphasizing speed does not encourage students to reflect on their thinking processes or to analyze their results. Math anxiety is likely to be lower in classrooms where teachers

- create a culture where students can ask questions, discover learning, and explore ideas;
- foster an environment where students can feel secure in taking risks and not feel embarrassed for giving wrong answers;
- provide for a calming period so the emotional aspect of the anxiety can abate and allow the rational thought processes to emerge;
- discourage valuing speed over time for reflection; and
- encourage students to make sense of what they are learning rather than just memorize steps or procedures. Remember that making sense is one of the criteria (along with meaning) that the brain uses to determine whether information is worth tagging for long-term storage.

Several researchers suggest that students can overcome math anxiety and find learning mathematics to be a rewarding and successful experience when teachers establish a classroom culture oriented toward making sense, rather than a more traditional culture oriented toward memorizing, being correct, recalling quickly, and listening (Flewelling & Higginson, 2001; Martin, 2009; Rudduck & McIntyre, 2007). More specifically, their comparison of some of the characteristics of the sense-making classroom to the traditional classroom is shown in Table 7.2.

These researchers further suggest that changing the typical classroom culture to a sense-making culture in mathematics can be achieved by having

Table 7.2 Comparison of Sense-Making and Traditional Classroom Cultures

Traditional Classroom	*Sense-Making Classroom*
Mathematics is a collection of procedures	Mathematics is a way of thinking
Working with the inexplicable	Working with things that make sense
Significance of material lost on learner	Material significant to learner
Student is passive	Student is active
Validated by teacher	Validated by student
Truth is as presented	Truth is as constructed
Teacher owned	Student owned

Traditional Classroom	*Sense-Making Classroom*
Described/explained in teacher language	Described/explained in student language
Often forgotten, not retrievable	Remembered, retrievable
Pops into existence	Grows into being
Ignores student readiness	Considers student readiness
Nonexperiential	Experiential
Presented at beginning of lesson	Developed at end of lesson
Reliance on memory aids	Minimal reliance on memory aids
Isolated and superficial	Connected and thorough
Follow procedures	Develop procedures
Anxious about mathematics	Sense of personal efficacy and confidence
Deadens the mind and spirit	Enlivens the mind and spirit

SOURCES: Flewelling and Higginson (2001); Martin (2009); Rudduck and McIntyre (2007).

students and their teacher focus on, engage in, and experience rich learning tasks. They need to see what learning looks and feels like and the kind of interaction that is involved when they are truly engaged. The researchers define rich learning tasks as those that give learners the opportunity to (1) use their knowledge in an integrated, creative, and purposeful fashion to conduct investigations, inquiries, and experiments, and to solve problems; (2) acquire knowledge with understanding; and (3) develop the attitudes and habits of a lifelong sense maker. Rich learning tasks help students recognize the role mathematics plays in their lives, and how mathematical reasoning can be an important tool for their decision making as students and adults. Table 7.3 offers a comparison of traditional and rich tasks that can be conducted in the mathematics classroom.

Table 7.3 Comparison of Traditional and Rich Tasks

Traditional Tasks	*Rich Tasks*
Prepare for success in school	Prepare for success outside of school
Address learning outcomes in mathematics	Address learning outcomes in mathematics and other subject areas
Focus on the use of relatively few skills	Provide an opportunity to use broad range of skills in an integrated and creative fashion
Are more artificial and out of context	Are authentic and in context
Encourage recollection and practice	Encourage thinking, reflection, and imagination

(Continued)

Table 7.3 (Continued)

Traditional Tasks	*Rich Tasks*
Allow for demonstration of a narrow range of performance	Allow for demonstration of a wide range of performance
Usually require enrichment to be added after the task	Provide enrichment within the task
Permit the use of fewer teaching and learning strategies	Encourage the use of a wide variety of teaching and learning strategies
Keep students and teachers distanced from the task	Encourage greater engagement of students and teachers in the task

SOURCES: Flewelling and Higginson (2001); Martin (2009); Rudduck and McIntyre (2007).

✓ **Assessment.** Tests can be the primary source of students' anxiety in any subject. But the anxiety may be greater in those subjects, such as mathematics, that are the basis for the high-stakes tests that have emerged since the adoption of curriculum standards and minimum competencies. Tests often diminish the students' confidence because they have no flexibility in the testing process and, as a result, the tests do not stir their curiosity or inventiveness. Furthermore, tests are often used to determine which students will enter classes of advanced mathematics. One can question whether poor assessment techniques should be used to determine how students advance in the mathematics curriculum, especially since these decisions can affect their post–high school choices. Teachers can alleviate the math anxiety caused by testing when they do the following:

- Limit class tests and do not time them. Timed tests increase the pressure on students, which disrupts processing in both working and long-term memory (Tsui & Mazzocco, 2007). This issue is controversial, and there may be occasions where this accommodation is not possible. However, it should be seriously considered whenever it can be implemented.
- Reduce the weight given to tests in determining grades, ranking students, or measuring isolated skills.
- Assess students on how they think about mathematics.
- Include multiple methods of assessment, such as oral, written, or demonstration formats.
- Provide feedback that focuses on a lack of effort rather than a lack of ability so students remain confident in their ability to improve (Dweck, 2006).
- Use the six National Council of Teachers of Mathematics (1995) assessment standards, which still make sense in today's educational climate, as a guide for testing practices. In brief, these standards state that assessment should (1) include real-life activities, (2) enhance mathematics learning, (3) promote equity, (4) be an open process, (5) promote valid inferences about mathematics learning, and (6) be a coherent process.

We have already discussed how student performance in mathematics improves when anxiety is alleviated. Teachers ease that anxiety when they demonstrate excitement and confidence in the subject, develop a relevant mathematics curriculum, use effective instructional strategies, create classrooms centered on discovery and inquiry, and assess students in a meaningful and fair manner.

NEUROLOGICAL AND OTHER FACTORS

Apart from environmental factors that can cause poor performance in mathematics, researchers look also at potential neurological causes. Just a few years ago, neuroscientists studied reading disabilities and were finally able to separate the factors that *cause* reading problems from those that are the *consequences* of these factors. Most researchers now agree that the major causal deficits in reading difficulties result from impairments in the brain regions responsible for phonological processing (Sousa, 2014).

The problem facing researchers in the field of mathematics disabilities is similar: distinguishing those factors that are causal from those that are consequential. Because students with moderate mathematical difficulties are often of average or higher intelligence and possess good reading skills, the brain regions involved in mathematics difficulties are likely localized or modular. In other words, the neurological causes of mathematics difficulties can be limited and not affect other cognitive areas. As we have noted in previous chapters, there is already a substantial body of brain imaging and case studies research supporting the existence of number modules.

Dyscalculia

About 3 to 6 percent of school-age students have serious difficulty processing mathematics (Butterworth, 2010). This is about the same number of students who have serious reading problems. However, because of the strong emphasis our society places on the need to learn reading, many more research studies have focused on problems in this area than on problems in mathematics. But that situation is slowly changing as neuroscientists get a deeper understanding of the various neural networks responsible for mathematical processing.

The condition that causes persistent problems with processing numerical calculations is referred to as *dyscalculia* (pronounced dis-kal-KOOL-ee-ah), from the Greek meaning "counting badly." Dyscalculia is a difficulty in conceptualizing numbers, number relationships, outcomes of numerical operations, and estimation—that is, what to expect as an outcome of an operation. If the condition is present from birth, it is called *developmental dyscalculia*. Genetic studies of twins reveal that developmental dyscalculia is moderately inheritable (Tosto, Malykh, Voronin, Plomin, & Kovas, 2013). If the condition results from an injury to the brain after birth, it is called *acquired dyscalculia*. Whether developmental or acquired, for most individuals, this disorder is the result of specific disabilities in basic numerical processing and not necessarily the consequence of deficits in other cognitive abilities (Landerl, Göbel, & Moll, 2013). People with dyscalculia have difficulty

- mastering arithmetic facts by the traditional methods of teaching, particularly those involving counting;
- learning abstract concepts of time and direction, telling and keeping track of time, and ordering the sequence of past and future events;
- acquiring spatial orientation and space organization, including poor left/right orientation, trouble reading maps, and grappling with mechanical processes;
- following directions in sports that demand sequencing or rules, and keeping track of scores and players during games such as cards and board games; and
- following sequential directions and sequencing (including reading numbers out of sequence, substitutions, reversals, omissions, and doing operations backward), organizing detailed information, and remembering specific facts and formulas for completing their mathematical calculations.

The neurological basis of developmental dyscalculia is an impairment in the child's innate ability to subitize. Because they cannot see the "twoness" or "threeness" of a group of objects, they learn to count differently than other students, relying heavily on sequencing and memorizing. They typically have no difficulty remembering the sequence of number words, and they can place those words in a one-to-one correspondence with objects in an array. But even when they say that four objects are present, they do not have an innate sense of the "fourness." They simply have confidence that their counting process has led them to the correct answer.

The neurological basis of developmental dyscalculia is an impairment in the child's innate ability to subitize.

We explained in Chapter 2 that after children learn to count, their brains will quickly associate a digit with a quantity; that is, the digit 5 automatically produces a mental image of five items. But in individuals with developmental dyscalculia, seeing the digit does not generate this mental representation of quantity, making it very difficult to perform mental arithmetic operations involving symbols. On the other hand, these individuals can still differentiate the number of objects contained in concrete (nonsymbolic) collections of objects. Apparently, it is the symbol (digit) that causes the problem (Mejias, Grégoire, & Noël, 2012). Dyscalculia can be (1) quantitative, which is a difficulty in counting and calculating; (2) qualitative, which is a difficulty in the conceptualizing of mathematics processes and spatial sense; or (3) mixed, which is the inability to integrate quantity and space.

Some simple tests are available that could indicate the presence of dyscalculia. A common one is a reaction time test in which subjects are asked which is the larger of two numbers. You will recall from Chapter 1 that as the distance between two numbers increases, most people find it easier to say which is larger. It is easier to recognize that 8 is larger than 3 than to recognize that 4 is larger than 3. But the responses from people with dyscalculia are exactly the opposite. Because they cannot subitize, people with dyscalculia must rely on counting and sequencing. Counting from 3 to 8 takes longer than counting from 3 to 4.

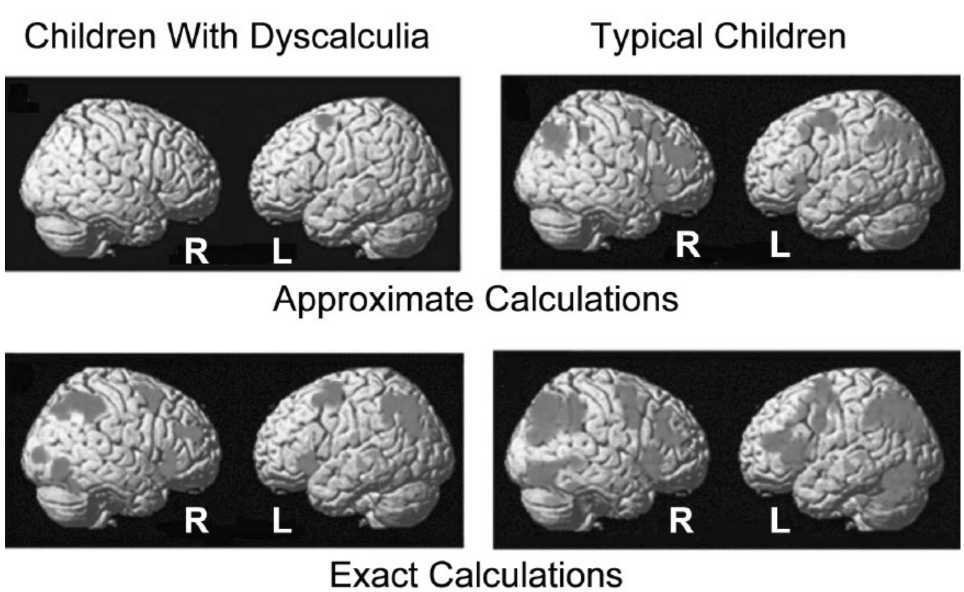
Possible Causes

The difficulty that individuals with developmental dyscalculia have in subitizing may be due to deficits in the number-processing regions of the brain. Several fMRI (functional magnetic resonance imaging) studies have found that the parts of the brain responsible for making the approximations necessary to subitize are much less activated in children with developmental dyscalculia than in typical children. However, brain activation during exact calculations was similar for both groups (Cappelletti & Price, 2014; Castelli, Glaser, & Butterworth, 2006; Kucian et al., 2006). Figure 7.3 shows only a small activated area in the brains of children with dyscalculia during approximate calculations, compared with the brains of typical children, but a similar amount of activation during exact calculations.

Because the parietal lobe is heavily involved with number operations, damage to this area can result in mathematics difficulties. Studies of individuals with Gerstmann's syndrome—the result of damage to the parietal lobe—showed that they had serious problems with mathematical calculations as well as right–left disorientation, but no problems with oral language skills (Chen, Xu, Shang, Peng, & Luo, 2014; Lemer, Dehaene, Spelke, & Cohen, 2003; Roitman, Brannon, & Platt, 2012).

Individuals with visual-processing weaknesses almost always display difficulties with mathematics. This is probably because success in mathematics requires one to visualize numbers and mathematical situations, especially in algebra and geometry. Students with sequencing difficulties also may have dyscalculia because they cannot remember the order of mathematical operations or the specific formulas needed to complete a set of computations.

Figure 7.3 These representative fMRI scans show that during approximate calculations, the right (R) and left (L) hemispheres of children with dyscalculia are much less activated than those of typical children. During exact calculations, however, the activation is very similar in both groups (Kucian et al., 2006).



Genetic factors also seem to play a significant role. Studies of identical twins reveal close mathematics scores. Children from families with a history of mathematical giftedness or learning disorders show common aptitudes with other family members. Girls born with Turner syndrome (a condition caused by the partial or complete absence of one of the two X chromosomes normally found in females) usually display dyscalculia, among other learning problems (Mazzocco & Hanich, 2010).

Types of Mathematical Disorders

The complexity of mathematics makes the study of mathematical disorders particularly challenging for researchers. Learning deficits can include difficulties in mastering basic number concepts, counting skills, and processing arithmetic operations, as well as procedural, retrieval, and visual–spatial deficits. As with any learning disability, each of these deficits can range from mild to severe.

Number Concept Difficulties. As discussed in Chapter 1, an understanding of small numbers and quantity appears to be present at birth. The understanding of larger numbers and place value, however, develops during the preschool and early elementary years. A poor understanding of the concepts involved in a mathematical procedure will delay the adoption of more sophisticated procedures and limit the child's ability to detect procedural errors. Studies show that most children with mathematical disorders nevertheless have their basic number competencies intact. However, they often are unable to use their number concept skills to solve arithmetic problems (Mussolin, Mejias, & Noël, 2010).

Counting Skill Deficits. Studies of children with mathematical disorders show that they have deficits in counting knowledge and counting accuracy. Some may also have problems keeping numerical information in working memory while counting, resulting in counting errors (Moeller, Neuburger, Kaufmann, Landerl, & Nuerk, 2009).

Difficulties With Arithmetic Skills. Children with mathematical disorders have trouble solving simple and complex arithmetic problems, and they rely heavily on finger counting. Their difficulties stem mainly from deficits in both numerical procedures (solving 6 + 5 or 4 × 4) and working memory. They tend to use developmentally immature procedures, such as counting all rather than counting on.

At the same time, they do not show the shift from procedure-based problem solving to memory-based problem solving that is found in typically achieving children, most likely because of difficulties in storing arithmetic facts or retrieving them from long-term memory. Moreover, deficits in visual–spatial skills can lead to problems with arithmetic because of misalignment of numerals in multicolumn addition. Although *procedural, memory,* and *visual–spatial* deficits can occur separately, they are often interconnected.

Procedural Disorders. Students displaying this type of disorder

- use arithmetic procedures (algorithms) that are developmentally immature;
- have problems sequencing multistep procedures, such as 52 × 13 and 317 + 298;

- have difficulty understanding the concepts associated with procedures; and
- make frequent mistakes when using procedures.

The exact cause of this disorder is unknown, but research studies have yielded some intriguing findings. Children with developmental or acquired dyscalculia can still count arrays of objects, say the correct sequence of number words while counting, and understand basic counting concepts, such as cardinality. However, they have difficulties in solving complex arithmetic problems. Researchers suspect one possible cause may be a dysfunction in the brain's left hemisphere, which specializes in procedural tasks and working-memory deficits (Roşca, 2009).

Memory Disorders. Students displaying this type of disorder

- have difficulty retrieving arithmetic facts,
- have a high error rate when they do retrieve arithmetic facts,
- retrieve incorrect facts associated with the correct facts, and
- rely on finger counting because it reduces the demands on working memory.

This disorder likely involves the manipulation of information in the language system. Here again, a dysfunction of the left hemisphere is suspected, mainly because these individuals frequently have reading disorders as well (Ashkenazi, Black, Abrams, Hoeft, & Menon, 2013; D'Amico & Guarnera, 2005). This association further suggests that memory deficits may be inheritable.

Memory disorders can be caused by two separate problems. One involves disruptions in the ability to retrieve basic facts from long-term memory, resulting in many more errors than for typically achieving children. Research findings indicate that this form of memory disorder is closely linked to the language-processing system and may indicate developmental or acquired deficits in the left hemisphere.

The second possibility involves disruption in the retrieval process caused by difficulties in inhibiting the retrieval of irrelevant associations. Thus, the student seems impulsive. For example, when asked what is 7 + 3, a student might quickly blurt out 8 or 4 because those numbers come next in counting (Passolunghi & Siegel, 2004). Solving arithmetic problems becomes much easier when irrelevant information is prevented from entering working memory. When irrelevant information is retrieved, it lowers working memory's capacity and competes with correct information for the individual's attention. This type of retrieval deficit may be caused by deficits in the brain's executive areas of the prefrontal cortex responsible for inhibiting working-memory operations (Geary, Hoard, & Bailey, 2012; Peng, Congying, Beilei, & Sha, 2012).

Visual–Spatial Disorders. Students with this type of disorder

- have difficulties in the spatial arrangement of their work, such as aligning the columns in multicolumn addition;
- often misread numerical signs, rotate and transpose numbers, or both;

- misinterpret spatial placement of numerals, resulting in place value errors; and
- have difficulty with problems involving space in areas, as required in algebra and geometry.

Studies indicate that this disorder is closely associated with deficits in the right parietal area, which specializes in visual–spatial tasks. Individuals with injuries to this area often show a deficit in spatial-orientation tasks and in the ability to generate and use a mental number line. Some studies suggest that the left parietal lobe also may be implicated (Ashkenazi, Rosenberg-Lee, Metcalfe, Swigart, & Menon, 2013; Szucs, Devine, Soltesz, Nobes, & Gabriel, 2013).

Many students eventually overcome procedural disorders as they mature and learn to rely on sequence diagrams and other tools to remember the steps of mathematical procedures. Those with visual–spatial disorders also improve when they discover the benefits of graph paper and learn to solve certain algebra and geometry problems with logic rather than through spatial analysis alone. However, memory deficits do not seem to improve with maturity. Studies indicate that individuals with this problem will continue to have difficulties retrieving basic arithmetic facts throughout life. This finding may suggest that the memory problem not only exists for mathematical operations but also signals a more general deficit in retrieving information from memory.

Children often outgrow procedural and visual–spatial difficulties, but memory problems may continue throughout life.

Associating Dyscalculia With Other Disorders

Reading Disorders. Students with dyscalculia can also have developmental reading difficulties, or dyslexia. Although these disorders do not appear to be genetically linked (Fletcher, 2005), nearly 50 percent of children with mathematics difficulties also have reading difficulties (Ashkenazi, Black, et al., 2013). No one knows for sure why these conditions appear simultaneously in so many children. Some research suggests that this comorbidity may be because both reading and mathematics share cerebral regions responsible for working memory, processing speed, and verbal comprehension (Willcutt et al., 2013). It is also possible that students with both disorders are less successful in solving mathematics problems than are those who have only dyscalculia because students in the former group have difficulty translating word problems into mathematical expressions.

Attention-Deficit Hyperactivity Disorder (ADHD). Because many children with ADHD have difficulty with mathematics, some researchers wondered if these two conditions had related genetic components, increasing the possibility that they would be inherited together. But studies show that these two disorders are transmitted independently and are connected to distinctly different genetic regions (Hart et al., 2010; Monuteaux, Faraone, Herzig, Navsaria, & Biederman, 2005). These findings underscore the need for separate identification and treatment strategies for children with both conditions.

Nonverbal Learning Disability (NLD). This disability is thought to be caused by deficits in the brain's right hemisphere, especially in the occipital lobe (Semrud-Clikeman & Fine, 2011). Individuals with NLD have difficulty processing nonverbal information but are very good at processing verbal information. They tend to be excessively verbal and expressive, and show weaknesses in visual and spatial tasks and in tactile perception. Although there is little evidence that NLD is directly associated with dyscalculia, NLD affects one's ability to manage and understand nonverbal learning assignments (Volden, 2013). Thus, students with NLD will have problems with handwriting, perceiving spatial relationships, drawing and copying geometric forms and designs, and grasping mathematics concepts and skills. We will discuss later in this chapter some of the strategies that can help these individuals.

ADDRESSING MATHEMATICS DIFFICULTIES

Research Findings

Numerous research studies have looked at the effectiveness of instructional strategies in improving achievement by students with mathematics difficulties. As expected, some strategies work better than others, and a particular strategy's effectiveness can depend on the nature of the learning difficulties found in the individuals being studied.

Because of this wide range of effectiveness for instructional strategies, it is usually helpful to rely on a meta-analysis of studies to determine which strategies have a significant impact on student achievement. This impact is often measured by effect size. As a general rule, an effect size of 0.20 is considered a small effect, 0.40 a moderate effect, and 0.60 or above a large effect. The most recent meta-analysis of instructional interventions in mathematics for students with learning difficulties was conducted by Russell Gersten and his colleagues (2009). They looked at 42 different interventions and found that the following two had significantly important effect sizes: the use of explicit instruction and the use of heuristics (experience-based techniques for problem solving and discovery). Table 7.4 shows those strategies that yielded effect sizes of 0.20 or larger.

Among these studies, the use of heuristics had the largest effect size. A heuristic was any strategy that represented a generic approach to solving a problem. It could be as simple as asking the student to read the problem, underline key words, solve it, and then check the work. Heuristics are not problem specific. Instead, they show the student multiple ways to solve a problem, encourage reflection to assess alternate solutions, and direct them to select one to solve the problem. Explicit instruction also yielded a large effect size, a result found in earlier meta-analyses of similar studies. To be included in this meta-analysis, explicit instruction meant that the teacher demonstrated a step-by-step strategy for solving the problem that was specific for a set of similar problems and asked the students to use the same procedure to solve a given problem.

That student verbalization scored high as a strategy is not surprising. Verbally explaining thinking processes is a form of rehearsal that helps students find sense and meaning in what they are doing. It also gives teachers the opportunity to answer questions and correct student thinking, thereby leading them on a path to successfully solving a problem.

Table 7.4 Mean Effect Sizes of Strategies in Mathematics for Students With Learning Difficulties

Instructional Strategy	*Mean Effect Size*
Use of heuristics (general techniques for problem solving)	1.56
Explicit instruction	1.22
Student verbalizations	1.04
Cross-age tutoring	1.02
Sequence and/or range of examples	0.82
Visuals combined	0.47
Visuals for teacher and student	0.46
Visuals for teacher only	0.41
Teacher feedback plus options for addressing instructional needs	0.34
Teacher feedback combined	0.23
Student feedback	0.23
Teacher feedback	0.21

SOURCE: Gersten et al. (2009).

Cross-age tutoring also yielded an impressive effect size. In these studies, the tutors were well-trained upper-elementary students who tutored students in lower grades. Their training focused on teaching lessons to the students as well as on what type of specific feedback they should give to students who had difficulty or made mistakes.

The large effect size for sequence and/or range of examples indicates how important it is for teachers to present mathematics examples to these students in a logical pattern. That pattern could be showing the distinctive features of a particular type of problem set, going from easy to progressively more complex and difficult problems, or finding common features of outwardly dissimilar word problems. Visual representations during problem solving have long been shown to be effective with students who have difficulties in learning mathematics, mainly because these representations become tools for thought, calculation, and communication in mathematics. For these students, abstract concepts, such as fractions and proportions, are easier to understand with visual representations.

Feedback through formative assessments is a valuable tool for teachers to monitor student progress and make any needed adjustments to their instructional strategies. The teacher feedback could include progress plus options for addressing any additional instructional activities the students might need to improve their achievement. Providing feedback to the students is also helpful because it keeps them engaged by letting them know where they have been successful or unsuccessful in their learning of mathematics.

Meta-analyses such as this one show us that we do have instructional strategies that, when appropriately implemented, can significantly help students who have difficulties in learning mathematics.

Basic Guidelines

From the research findings we have discussed in previous chapters and those included in Table 7.4, we can develop some basic guidelines for strategies that are likely to be effective with students who have difficulties in learning mathematics.

- ✓ Recalling that these students often have limitations with working memory, strategies should avoid including too much information or too many skills at one time. Allow the students to practice and achieve mastery in the current learnings before moving on to new material.
- ✓ Use worksheets that are simple and clear, and avoid including too much visual information.
- ✓ Use tactile, visual, and auditory examples as much as possible. Concrete manipulatives are an excellent way to help young students gain confidence when facing equations.
- ✓ Many students benefit from drawing to visualize a mathematics problem.
- ✓ Remember to use guided practice *before* independent practice so students do not practice incorrect procedures out of school.
- ✓ Whenever possible, use games involving a mathematical concept, such as probability, to maintain student interest and engagement.
- ✓ Use real-world examples so students see the relevance and practicality of mathematics, and why learning mathematical skills is so important.
- ✓ Include technology when appropriate to encourage and maintain student motivation.
- ✓ Use feedback and formative assessments so students can keep track of their progress.
- ✓ Enhance retention of learning by revisiting important concepts as part of distributed practice.

The Concrete–Representational–Abstract Approach

Students who have difficulties with mathematics can benefit significantly from lessons that include multiple models that approach a concept at different cognitive levels. Mathematics educators have recognized a substantial body of research showing that the optimal presentation sequence for new mathematical content is *concrete–representational–abstract*, or the CRA approach. This approach has also been referred to as concrete–pictorial–abstract or concrete–semiconcrete–abstract. Regardless of the name, the instructional approach is similar and originally based on the work of Jerome Bruner (1960). Concrete components include manipulatives (e.g., Cuisenaire rods, foam-rubber pie sections, and markers), measuring tools, or other objects the students can handle during the lesson. Pictorial representations include drawings, diagrams, charts, or graphs created by the students or provided for the students to read and interpret. The abstract aspect refers to symbolic representations, such as numbers or letters that the student writes or interprets to demonstrate understanding of a task. This approach is recommended by numerous researchers because of its success with students who have difficulties learning mathematics (e.g., Allsopp et al., 2008; Fahsl, 2007; Mancl, Miller, & Kennedy, 2012).

When using the CRA approach, the sequencing of activities is critical. Activities with concrete materials should come first to impress on students that mathematical operations can be used to solve real-world problems. Pictorial examples show representations of the concrete manipulatives and help students visualize mathematical operations during problem solving. It is important here that the teacher explain how the pictorial examples relate to the concrete examples. Finally, formal work with symbols is used to demonstrate how they provide a shorter and more efficient way to represent numerical operations. Ultimately, students need to reach that final abstract level by using symbols proficiently with many of the mathematical skills they master; however, the meanings of those symbols must be firmly rooted in experiences with real objects. Otherwise, their performance of the symbolic operations will simply be rote repetitions of meaningless memorized procedures.

The CRA approach benefits all students but has been shown to be particularly effective with students who have mathematics difficulties, mainly because it moves gradually from actual objects to pictures and then to symbols. These students often get frustrated when teachers present mathematics problems only in the abstract. Mathematics teachers need to organize content into concepts and provide instruction that allows students to process the new learning in meaningful and efficient ways.

Research studies support the effectiveness of this approach. Witzel and his colleagues have conducted several studies of students identified as having difficulties in learning algebra. Students who learned how to solve algebra transformation equations through the CRA approach scored higher on postinstruction and follow-up tests than did the control peers receiving traditional instruction. Furthermore, students who used the CRA sequence of instruction performed fewer procedural errors when solving for algebraic variables (Witzel, 2005; Witzel, Mercer, & Miller, 2003).

Teachers of mathematics in elementary schools have recognized the importance of using concrete and pictorial activities when introducing new concepts. Yet despite newer research in cognitive neuroscience lending support to the CRA method, it is not in widespread use in middle and high school mathematics classrooms and is seldom mentioned in textbooks. Perhaps teachers at the secondary level feel that concrete objects may be perceived by students as too elementary, or it may be that the content demands of the curriculum push teachers directly to the abstract level to save time. But students with difficulties in learning mathematics are encouraged when concrete and pictorial representations eventually lead them to interpret abstract concepts accurately (Walkowiak, 2014).

Those teachers who want to try the CRA approach are left on their own with minimal or no guidance from textbooks. To help teachers with implementing CRA, Witzel has developed a seven-step process that uses the mnemonic CRAMATH as a guide (Witzel, Riccomini, & Schneider, 2008).

1. **Choose a topic in mathematics.** The teacher looks for a broad idea that may join apparently dissimilar items together. A middle school topic might involve working with algebraic equations, such as $24 = 10 - 4x$.

2. **Review procedures for solving the problem.** For example, when solving equations, the teacher lists the steps, such as identifying the variable and coefficient, balancing across the equal sign, calculating according to the order of operations, continuing until the coefficient is 1, determining the answer, and, finally, checking the answer.

3. **Adjust the steps to eliminate notation or calculation tricks**. After presenting the steps, the teacher eliminates any tricks or shortcuts so students do not spend time learning tricks or shortcuts that work only in specific situations and not in others. For example, take the equation, $x - 9 = 14$. Students might assume that this can be easily solved by just adding nine to both sides. However, the first step should be to make the coefficient of the unknown equal to 1. Thus, the student should first divide each side of the equation by the coefficient of 1. Otherwise, when presented with the equation, $3x - 4 = 17$, the teacher has to tell the students that if there is a coefficient greater than 1, they must divide by that coefficient after adding 4 to both sides.

4. **Match the abstract steps to the appropriate concrete manipulative.** The teacher should choose objects that apply to multiple skills to help students recognize the generalizability of the rules, procedures, and concepts in mathematics. For example, algebra blocks are helpful for simple equations involving whole-number coefficients, such as $5x = 30$, but not helpful for equations with fractional coefficients, such as $3/5\ x = 18$. Here, it is more useful to use concrete objects representing groupings, such as containers. (See the **Resources** section at the end of this book for a website that suggests concrete/virtual representations for mathematics topics at different grade levels.)

5. **Arrange concrete and representational lessons**. After completing the work with concrete objects, the teacher creates pictorial representations that mimic the concrete ones. For example, subtraction using toothpicks involves removing the toothpicks from the desktop. Pictorially, this could be represented by drawing an X over the toothpicks that were subtracted.

6. **Teach each concrete, representational, and abstract lesson to mastery.** Teachers use frequent and accurate assessments to evaluate their students' progress as they transition from one stage to the next toward mastering the learning objectives. Transitioning from the representational to the abstract stage is the most difficult for students with learning difficulties in mathematics. Therefore, teachers should use clear and appropriate terminology and language related to the mathematical principle. For example, a coefficient may be called a *cup* in the concrete stage, a *group* in the representational stage, but a *coefficient* in the abstract stage.

7. **Help students apply what they learned through word problems**. Students with learning difficulties in mathematics do not usually

apply concepts or skills to new settings without appropriate guidance and explicit instruction. Teachers should present instructional activities that improve students' ability to generalize across different scenarios. Using real-world examples in word problems can help students recognize the practical applications of mathematics to new settings and understand the value of learning mathematics skills.

Concrete and pictorial representations should be used at all grade levels. By using cognitive strategies such as CRA, teachers provide students with a technique for tackling mathematics problems rather than just searching for an answer. Here is a simple example of presenting an algebraic word problem at the three cognitive levels.

✓ Example: Algebraic Word Problem

High school students Bob and John both work part-time on weekends at the local fast-food restaurant and are paid at the end of the day on Sunday. When they receive their pay, Bob gets $10 more than John. Together they have $130. How much money does each person have?

- **Concrete**: Count out $130 in play money. Give Student A (Bob) $10. Then divide the rest of the money ($120) between Student A (Bob) and Student B (John). Find out how much money each student has. Bob has $70 and John has $60.
- **Representational/pictorial:** Represent the $130 as $10 drawings on an overhead or a board.

  $10 $10 $10 $10 $10 $10 $10 $10 $10 $10 $10 $10 $10

  Identify the $10 for Bob (shown in bold italic).

  ($10) $10 $10 $10 $10 $10 $10 $10 $10 $10 $10 $10 $10

  Count how much money is left ($120).

  Divide the remaining money equally between Bob and John.

  Bob: ***($10)*** + $10 $10 $10 $10 $10 $10

  John: $10 $10 $10 $10 $10 $10

  Count how much money Bob has: $70.

  Count how much money John has: $60.

- **Abstract:**

Bob = x	John = (x – $10)
x + (x – $10)	= $130
2x – $10	= $130
2x	= $130 + $10
2x	= $140
x	= $70 (Bob)
x – 10	= $60 (John)

Figure 7.4 is an example of a simple planning worksheet that reminds teachers to select instructional strategies that address all three cognitive levels.

Numeracy Intervention Process

Many students with dyscalculia have difficulties with basic numerical knowledge and conceptual knowledge. Interventions designed to address these deficits can be effective in improving student achievement in basic arithmetic. Kaufmann, Handl, and Thöny (2003) reported the results of a pilot study that used an intervention program involving third-grade students with dyscalculia. The interventions were conducted three times weekly for a period of about 6 months. Figure 7.5 lists the components of the intervention program, which were conducted in semi-hierarchical order from the bottom up. Results from subtests administered before and after the interventions showed that the students with dyscalculia exhibited positive and partly significant improvement in basic numerical knowledge, in conceptual knowledge, and in arithmetic fact and procedural knowledge. Studies that have been carried out with kindergartners have had similar positive findings (Kaufmann, Delazer, Pohl, Semenza, & Dowker, 2005; Toll & Van Luit, 2013).

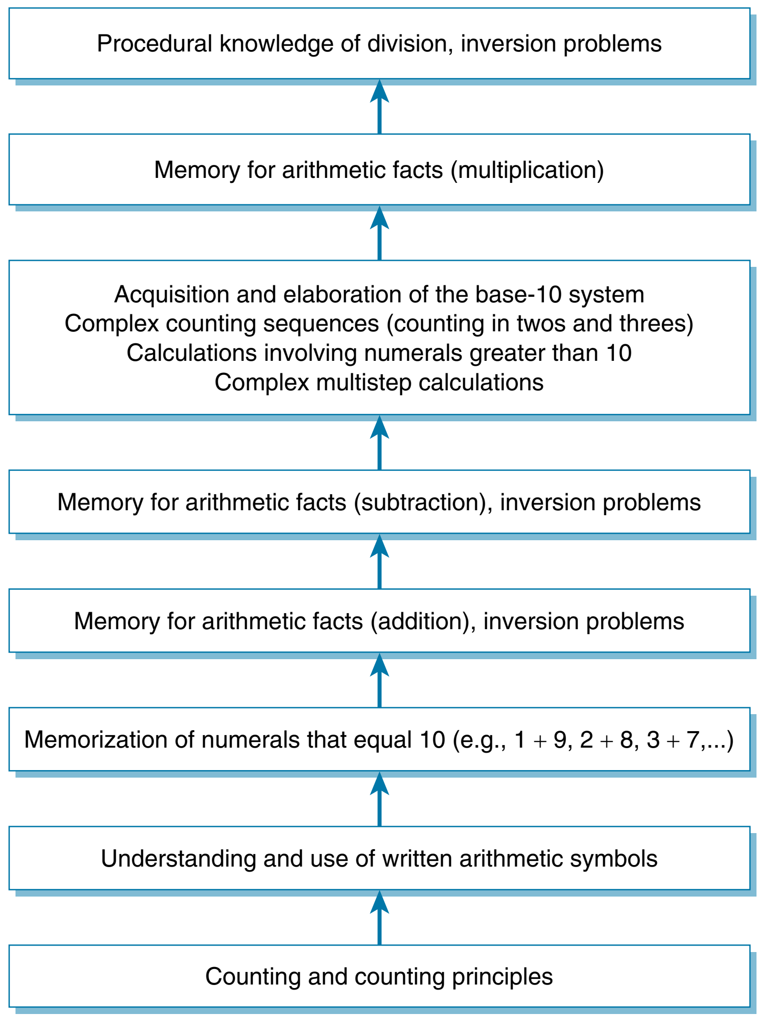
Students With Nonverbal Learning Disability

Students with nonverbal learning disability (NLD) have good verbal processing skills but will have problems comprehending the visual and spatial components of mathematics skills and concepts, especially when dealing with geometric shapes and designs. Although it may be difficult for students with NLD to understand mathematics concepts and solve problems, they may have no trouble applying a mathematical formula that has been explicitly taught. They generally learn verbal information quickly.

Figure 7.4 A planning grid for selecting instructional strategies at three representational levels.

Planning Sheet for CRA Representational Levels	
Mathematics Concept:	
Level	Instructional Strategy
Concrete	
Representational	
Abstract	

Figure 7.5 The modules in the intervention program are shown here. They were conducted in semi-hierarchical order from the bottom up (Kaufmann et al., 2003, 2005).



But when they look at a diagram for the first time, they look at a detailed piece of it. When they look a second time, they see a different piece, and another piece when they look for the third time. Because there is no visual overview, the diagram may not make sense. Additionally, due to their poor spatial-organization ability, they may have difficulty aligning problems on a page to solve them correctly (Gillis, Sevlever, & Roth, 2012).

Teachers of arithmetic and mathematics who work with students with NLD should consider the following strategies (Serlier-van den Bergh, 2006):

- ✓ Rely heavily on the student's verbal and analytic strengths. These students begin to work when speech is used, so use speech as the starting point. For example, have the student read the mathematics problem aloud before attempting to solve it.
- ✓ Gain a commitment from the student to collaborate to improve visual and spatial weaknesses. Drawing diagrams and graphic organizers

that are related to mathematics concepts and problems may help considerably.

- ✓ Use words to describe visual and spatial information. Ask the student to do the same while pointing to the corresponding places on the diagram or concrete model.
- ✓ Use language as the bridge to connect new learnings in mathematics to past learnings.
- ✓ Teach the student mnemonic devices to help remember mathematical procedures.
- ✓ Provide sequential verbal instructions for nonverbal tasks.
- ✓ Break down tasks and procedures into manageable segments.
- ✓ Young students with NLD may feel awkward handling manipulatives because their tactile sense is not developed. However, manipulatives can help students develop mental images of geometric shapes and visualize spatial relationships, as well as improve their visual memory skills. Ask them to touch objects first with their dominant hand, then with their nondominant hand, and finally with both hands at once.
- ✓ Minimize paper-and-pencil tasks, and substitute the computer where possible.
- ✓ Encourage the student to slowly integrate sensory information: Read it, say it, hear it, see it, write it, do it.

Students With Both Mathematics and Reading Difficulties

Students who have both reading and mathematics difficulties are obviously at a double disadvantage. However, even though the reading and mathematical processing areas of the brain are separate from each other, these two cerebral regions interact whenever the learner must translate word problems into symbolic representations (Dehaene, 2010). Here are some strategies that are effective with these students.

- ✓ **Cue words in word problems.** Help these students decode language into mathematical operations by alerting them to common phrases or cue words found in word problems that identify which operation to use. For example:

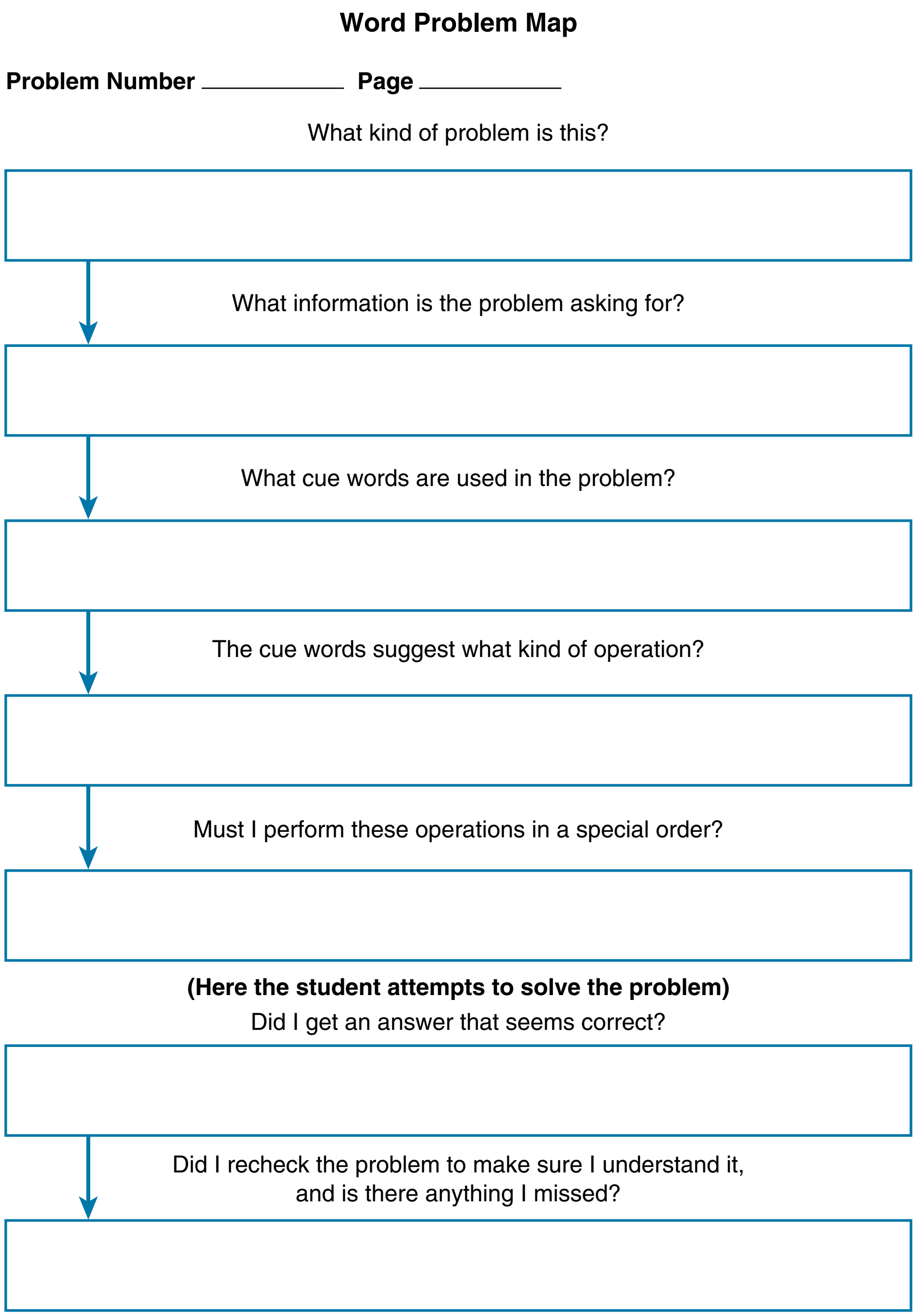
Common Phrases/Cue Words	*Example*	*Operation*
Add how many, altogether, in all, put together	When the apples were put together, how many were there?	Addition
Took away, take, left, give away	How many apples did they give away?	Subtraction
Problems that start with one and then ask for a total	Each rock weighed 6 pounds. How much did five rocks weigh?	Multiplication
Problems that start with many and then ask about one	Four boxes of the same cereal cost 12 dollars. How much did each box cost?	Division

✓ **Word problem maps.** Students with reading problems are often given a story map to help them highlight certain important aspects of the story, such as introduction, plot line, characters, timeline, and story climax. Gagnon and Maccini (2001) have developed a similar learning aid, called a word problem map, to help students with mathematics difficulties organize their thoughts as they tackle word problems. The map asks a series of questions to guide the student through the problem-solving process. It essentially becomes a scaffold on which to build a better understanding of what the problem is asking them to solve. The map can be completed by an individual student or by students working together in groups of two or three. It also serves as an excellent study guide for future assessments on the mathematical material involved. Figure 7.6 shows one variation of the word problem map (Gagnon & Maccini, 2001).

✓ **The RIDD strategy.** The RIDD strategy was developed by Fay Balch Jackson (2002) for students with learning disabilities. In practice, it has shown to be particularly helpful to students who have difficulties in both reading and mathematics. RIDD stands for *read*, *imagine*, *decide*, and *do*. The following is a description of these four steps.

- **Step 1: Read the problem.** Read the passage from beginning to end. This helps students focus on the entire task rather than just one line at a time. Good readers often skip words within a text, or they substitute another word and continue reading. In this step, students decide ahead of time what they will call a word they do not recognize. In mathematics word problems, substitutions can be made for long numbers rather than saying the entire number on the first reading. Teachers should model this substitution when they read the problem aloud to the class.
- **Step 2: Imagine the problem.** In this step, the students create a mental picture of what they have read. Using imagery when learning new material activates more brain regions and transforms the learning into meaningful visual, auditory, or kinesthetic representations of information. This makes it easier for the new information to be stored in the students' own knowledge base. Imagery helps students focus on the concept being presented and provides a way of monitoring their performance.
- **Step 3: Decide what to do.** To generate a mental picture of the situation, this step encourages students to read the entire mathematics problem without stopping. They then decide what to do and in what order to solve the problem. For example, in a word problem requiring addition and then subtraction, students would read the problem, create a mental picture, and then decide whether to add or subtract first. For young students, teachers can guide them through this step with appropriate questioning so the students can decide what procedures to use. Note how this step combines reading, visualization, and problem solving.
- **Step 4: Do the work.** During this step the students actually complete the task. Often, students start reading a mathematics problem, stop partway through it, and begin writing numerical expressions. This process can produce errors because the students do not have

Figure 7.6 This is one example of a word problem map that can help students with mathematics difficulties organize their approach to solving the problem.



all the information. By making this a separate step, students realize that there are things to do between reading the problem and writing it down. Jackson (2002) observed that when students used RIDD to solve mathematics problems, they liked this strategy because they perceived the last step as the only time they did work. Apparently, the students did not realize that what they did in the first three steps was all part of the process for solving problems.

The RIDD Strategy

R Read the problem
I Imagine the problem
D Decide what to do
D Do the work

RIDD is effective because it combines several strategies, such as visualization, reflection, and thoughtful questioning, all of which serve to increase understanding and retention of learning. Furthermore, RIDD allows the student to recognize that there are multiple cognitive processes involved between reading a problem and rushing to do the mathematics involved. This learning strategy, like most, is effective if used regularly, increasing the likelihood that it will become a permanent part of the student's study skills toolkit.

English Language Learners

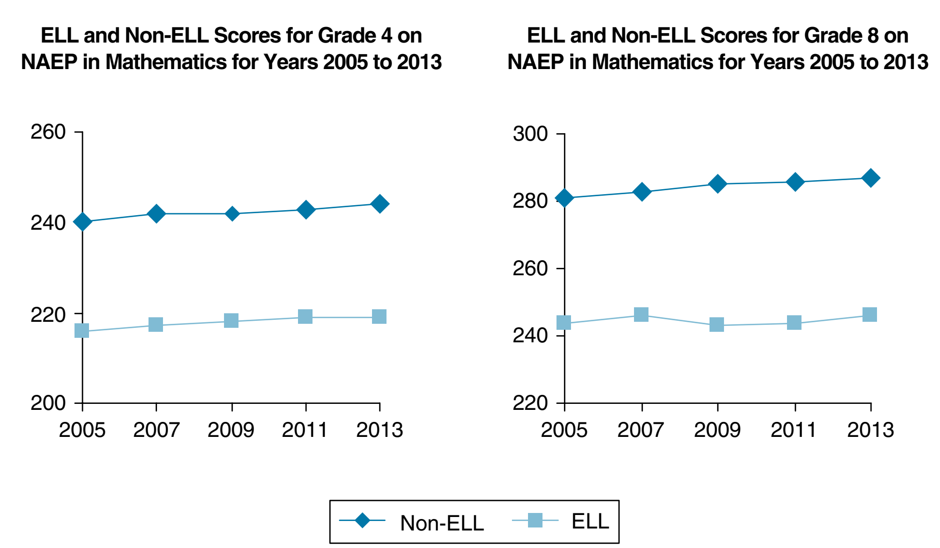
English language learners (ELLs) are the fastest-growing population in U.S. public schools, the largest portion of them being native Spanish speakers. ELL students have difficulty learning mathematics, as evidenced by their scores in Grades 4 and 8 on the National Assessment of Educational Progress (NAEP, 2013) over recent years: They are consistently lower than those of non-ELL students (Figure 7.7). Clearly, there is an achievement gap in mathematics that educators need to address.

The chart for Grade 4 indicates that, although the scores of ELL students are lower than those of non-ELL students, the ELL scores have risen about the same amount as the non-ELL scores from 2005 to 2013. For Grade 8, however, the ELL students have made little progress over the same time period, while the non-ELL scores have risen. What are some of the issues involved, and what can be done to improve the achievement of ELL students in mathematics?

Language Issues

We already discussed in Chapter 1 how the brain relies on three cerebral systems—visual processing, symbolic processing, and language

Figure 7.7 These charts compare the mathematics scores of ELL and Non-ELL students on the NAEP from 2005 to 2013 (NAEP, 2013).



processing—when dealing with quantities. Still, many people, including some educators, believe that mathematics is a nonverbal discipline. That mathematics processing relies heavily on language systems is reason enough to allow ELL students, whenever possible, to master basic mathematics in their native language before trying to learn mathematics in English. Language is a major concern in mathematics teaching because most of the content is conveyed through oral language, as teachers tend to do the majority of the talking in mathematics classes. ELL students do not derive a significant portion of their learning from reading mathematics textbooks.

The language issue is becoming more significant now because the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) curriculum shifts instruction in mathematics from more emphasis on numbers to more emphasis on word problems. Consequently, to understand and be successful in mathematics, students need to be able to read, solve problems, and communicate using technical language in a specialized context—and to properly discuss and explain mathematics content, teachers must use technical language. Students lacking proficiency in the English language and in the specialized language of mathematics understandably frustrate teachers who are faced with an increasing number of ELL students in their classrooms.

Two Planning Objectives. Teachers also face challenges when working with ELL students. ELL teachers may not be well trained in mathematics, and mathematics teachers are typically not well trained in working with ELL students. Regardless, when planning a lesson, teachers need to decide on a language objective in addition to the mathematics content objective. While the language objective includes those English mathematical terms and expressions that describe the problem and the operations, the content objective demonstrates the steps involved in solving it. Understandably, the language of mathematics presents an array of challenges to ELL students.

Grammar and Vocabulary Issues. Features such as symbolic notation, graphs, technical vocabulary, and complex grammatical phrases all pose substantial barriers to understanding. For example, the phrase "7 multiplied by 12" is very different from "7 increased by 12." Similarly, "divided by" and "divided into" will produce very different results. Even number notation can pose problems. For instance, some countries use a comma to separate whole units from decimals, instead of the period commonly used in North America, and use a period to separate thousands (e.g., 1 million dollars in Europe is written as $1.000.000,00). The difficulty of learning the already foreign language of mathematics is compounded when the instruction is also in a nonnative language.

The multiple meanings of words and the rules of English syntax allow us to interchange terms or expressions to identify the same mathematical concept. Teachers of mathematics are so accustomed to the content vocabulary that they are often not aware of the multiple terms used to describe the same operation. Addition, for example, uses *plus, total, add, combine, sum, put together, altogether, increase by, more than,* and *in all* to indicate its operation. Subtraction has its own list: *less, take away, difference, subtract, decrease by, minus, fewer than, are left, take from,* and *remain.* As a result, an

ELL student who has learned only the words *add* and *subtract* will be confused most of the time whenever a teacher uses other words to describe those operations.

Some words used in mathematics and in everyday life can overlap, but others are not found in daily usage. Words such as *regroup, hypotenuse, coefficient*, and *exponent*, or more complex terms such as *least common denominator, greatest common factor*, and *rational function* are common in mathematics class but not in the ELL students' social environs. However, many mathematical terms, such as *product, foot, chance, table, plane, round, scale*, and *value*, are deceptively familiar; thus, ELL students may believe that they comprehend the concepts these terms represent long before they really do. Undoubtedly, one of the major challenges facing ELL students in learning mathematical language is that most of it has to be acquired in the classroom because it does not occur in casual conversation.

Adolescent ELL Students. Complications from language issues are more pronounced for ELL adolescents. Carol Beal and her colleagues conducted a study looking at the relationship of English language proficiency to mathematics problem solving and mathematics motivation in adolescent ELL students (Beal, Adams, & Cohen, 2010). The sample was nearly 450 ninth-grade students in Algebra I, about half of whom were English learners and half of whom spoke English as their primary language. The researchers used multiple measures of mathematics problem solving, including state achievement test scores, computer software pre- and posttest scores, and correct solutions to word problems recorded as students worked with an online tutorial software for prealgebra review. Also, the students completed a survey of their mathematics self-concept and the perceived value of mathematics in their lives.

Many of these students were struggling with basic mathematics, and the teachers rated almost half their students as failing or at risk of failing algebra. Although overall mathematics performance was poor, there were significant variations related to English language proficiency. As expected, the ELL students scored lower in mathematics than did the native English-speaking students. The ELL students' reading skills in English were significantly related to mathematics performance, whereas their English conversational proficiency (speaking and listening) was not. More surprisingly, reading proficiency also predicted the ELL students' self-concept in mathematics. Thus, the ability to read English seems critical for success in mathematics for adolescent ELL students.

Cognitive Issues

Because cognitive processing is so closely tied to language processing, mathematics teachers of ELL students are faced with trying to determine whether the students have mastered a concept even if they have difficulty expressing their understanding in English. One effective method for dealing with this issue and assisting students in solving mathematics problems is called *reciprocal teaching*. In reciprocal teaching, students read in small groups, using cognitive strategies to comprehend the text. One study taught ELL students how to use four cognitive strategies to help them deal with the language challenge. The strategies were (1) clarifying the meaning of words and phrases so the students knew the basic components of the problem, (2) questioning extensively to identify the key elements of the problem, (3) summarizing to each other the purpose of the problem, and (4) devising a plan to solve the problem (van Garderen, 2004). Student performance improved when this strategy was used regularly.

Research Findings

Findings from research studies suggest that teachers of mathematics who have ELL students in their classes should consider the following instructional strategies (Burchinal, Field, López, Howes, & Pianta, 2012; Freeman & Crawford, 2008; Keengwe & Hussein, 2013; Kersaint, Thompson, & Petkova, 2013; Kim, Chang, & Kim, 2011; Orosco, 2014; Orosco, Swanson, O'Connor, & Lussier, 2011; Robinson, 2010; Sousa, 2011b; Stansfield, 2011):

- ✓ **Draw pictures and symbols.** These students need considerable help in representing word problems through symbols and pictures. This includes providing the necessary pictures or having the students draw their own diagrams, which helps translate the word problem into a visual representation and facilitates a better understanding of vocabulary.
- ✓ **Create vocabulary lists.** When introducing a new topic, prepare lists of mathematics vocabulary words and phrases ELL students will encounter, and be sure to provide simple, clear explanations of what they mean.
- ✓ **Help students select the correct operation.** These students often are able to solve the arithmetic algorithm. Their difficulty arises in deciding which arithmetic operation to use, based on interpreting the language in the word problem.
- ✓ **Reinforce basic concepts.** Do not assume that these students have a sound understanding of number relationships. Instruction should include activities that continuously reinforce basic concepts, such as manipulating the number line, estimating, evaluating answers, and representing number relationships.
- ✓ **Use all the information.** Encourage these students to use all the information provided, including diagrams, to solve the problem.
- ✓ **Use manipulatives.** Besides helping students construct physical models of abstract ideas, manipulatives also build understanding of mathematics vocabulary. They can be used to confirm a student's reasoning before proceeding to solve the problem.
- ✓ **Rewrite problems.** The language used in word problems is often more complex than it needs to be. Consider rewriting problems using simpler language. Also consider putting the question—that is, what the student is to solve—first. This helps the student's attention system focus on only the information needed to solve the problem.
- ✓ **Design oral assessments.** Use oral assessment whenever possible to evaluate the students' knowledge of mathematical concepts. This approach helps mediate for any lack of English proficiency the students may have and lowers their test-related anxiety levels.
- ✓ **Take advantage of technology.** Consider using computer programs, such as the HELP Math program, that are specifically designed to assist ELL students with mathematics and have met with success.

- ✓ **Use students' native language.** When possible, allow students in small groups to discuss a mathematics topic in their native language to encourage their participation in a safe environment. Avoid assuming that a primary-grade student's poor performance on a mathematics assessment in English is due to a mathematics deficit. It may be because of the language. Studies show that kindergarten and first-grade Hispanic students performed better on mathematics assessments when tested in Spanish instead of English.
- ✓ **Seek professional development opportunities.** Professional development programs should offer teachers of mathematics opportunities to learn about the needs of ELL students and the challenges they face when learning mathematics. Similarly, ELL specialists should have opportunities to enhance their knowledge about mathematical concepts and effective strategies for teaching mathematics.

The success ELL students will have in learning mathematics comes down, as always, to individual teachers. Teachers should have the information and strategies they need to be effective in conveying mathematics concepts to these students, should understand how to implement them, and should have the necessary professional development and administrative support to carry all this out.

■ WHAT'S COMING?

In Chapter 3, we discussed what the findings from research in cognitive neuroscience are telling us about how lessons should be planned and delivered so the new learning is likely to be remembered. How do we incorporate that information into lesson plans? To what degree do we include writing? How do we differentiate instruction to meet the needs of a diverse student population? How can integrating the arts into mathematics lessons improve student motivation and achievement? These are some of the questions we will tackle in the next, and final, chapter.

Chapter 7—Recognizing and Addressing Mathematics Difficulties

QUESTIONS AND REFLECTIONS

Respond to the following questions, and jot down on this page key points, ideas, strategies, and resources you want to consider later. This sheet is your personal journal summary and will help jog your memory.

What is math anxiety, and what can teachers do to reduce it?

Briefly describe the concrete–representational–abstract approach as an instructional strategy.

What are some other strategies research studies have shown to be effective with students who have learning difficulties in mathematics?

8

Putting It All Together

Planning Lessons in PreK–12 Mathematics

It is hard to convince a high-school student that he will encounter a lot of problems more difficult than those of Algebra and Geometry.

—Edgar W. Howe

At first glance, the title of this chapter could give the reader the impression that planning lessons in mathematics might be different from planning lessons in other subject areas. To some extent, it is. Studying mathematics requires not only the mastery of content but also the acquisition and enhancement of certain process skills needed for that study to be successful and meaningful. Consequently, teachers planning lessons in mathematics at *any* grade level should consider whether those lessons will provide learners with both the content and requisite process skills.

QUESTIONS TO ASK WHEN PLANNING LESSONS

After deciding a lesson's content objective, one of the next steps is to design the learning episode. When teachers keep in mind what is now known about how the brain learns, they are more likely to develop lessons wherein students learn and remember the content objective while they enhance their process skills. Here are some questions to keep in mind while planning for effective instruction.

Is the Lesson Memory Compatible?

Do you remember the capacity and time limitations we discussed in Chapter 3? These limitations become very important when deciding *how much* and *how long* to present new learning to students.

Keep the number of items in a lesson objective within the capacity limits of working memory, and students are likely to remember more of what you presented. Less is more!

✓ **Capacity limitations** of working memory must be considered when planning lessons. The elementary-grade teacher who tells the class, "We have seven mathematical facts to learn today," is already in trouble. So is the high school teacher who plans to cover eight different ideas in one lesson. By keeping the number of items in a lesson objective within the capacity limits of working memory, you increase the likelihood that students will remember more of what you presented. Less is more!

✓ **Time limitations** mean that lessons in the elementary grades should be taught in 12- to 15-minute segments, and those in secondary grades in 15- to 20-minute segments. So, for example, one segment could be direct instruction and the next could be practice alone or in small groups, or computer work or research. These time restrictions are particularly critical in high schools that have block scheduling. In this format, periods are from 80 to 90 minutes long. Teachers have already learned from experience that doing direct instruction (mainly through teacher talk) for this entire period of time is not generally effective. Breaking the 90-minute block into four segments of 20 minutes or so is much more productive because each new segment starts the time-limit clock all over again. Sometimes, incorporating a brief off-task activity between segments also serves to refresh the working-memory clock. Shorter is better!

Teaching within the time limits of working memory will enable students to stay focused and remember more of what you presented. Shorter is better!

Does the Lesson Include Cognitive Closure?

Cognitive closure describes the covert process whereby the learner's working memory summarizes for itself its perception of what has been learned. It is during closure that a student often completes the rehearsal process and attaches sense and meaning to the new learning, thereby increasing the chances that the learning will be retained in long-term memory.

✓ **Initiating closure.** The teacher gives directions that focus the students on the new learning—for example, "I'm going to give you two minutes to think of what we learned today about how to multiply two double-digit numbers. Be prepared to talk about it with a partner." In this statement, the teacher tells the students how much quiet time they have to summarize mentally and identifies the overt activity (discussion) that will be used for student accountability. During the discussion, the teacher listens carefully, assesses the quality and accuracy of what occurred during closure, and makes any necessary adjustments in teaching.

✓ **Closure is different from review.** In review, the teacher does most of the work, repeating key mathematics concepts introduced during the lesson and rechecking for student understanding. In closure, the student does most of the work by mentally rehearsing and summarizing those concepts and deciding whether they make sense and have meaning. Closure also gives the students an opportunity to think of questions that can clarify any misunderstandings.

✓ **When to use closure.** Closure can occur at various times in a lesson.

- It can start a lesson: "Think of the steps we learned yesterday about adding fractions, and be prepared to discuss them."
- It can occur during the lesson (called *procedural closure)* when the teacher moves from one sublearning to the next: "Review those two geometry rules in your mind before we learn the third rule."
- It should almost always take place at the end of the lesson (called *terminal closure)* to tie all the sublearnings together. Remember, this may be the last opportunity the learner has to make sense and attach meaning to the new learning.

Cognitive closure is a small investment in time that can pay off dramatically in increased retention of learning.

Cognitive closure is a small investment in time that can pay off dramatically in increased retention of learning.

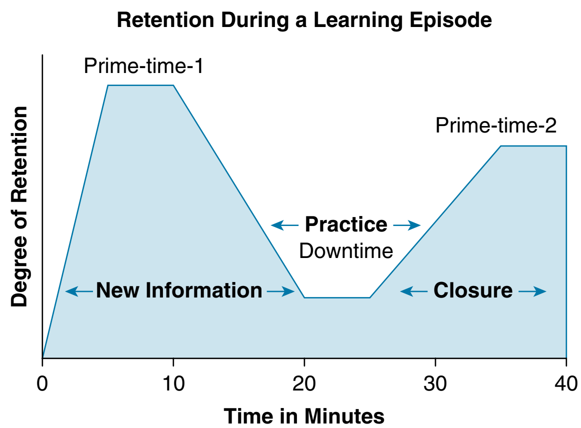
Will the Primacy–Recency Effect Be Taken Into Account?

In Chapter 3, we discussed the impact of the primacy–recency effect on retention. This impact should be taken into account when planning and teaching a lesson. The learning episode begins when the learner focuses on the teacher with intent to learn (this is indicated by 0 in Figure 8.1). New information or a new skill should be taught first, during prime-time-1, because it is most likely to be remembered. Keep in mind that the students will remember almost any information coming forth at this time. It is important, then, that only *correct* information be presented. This is not the time to be searching for what students *might* know about something. I remember watching a teacher of mathematics start a class with, "Today, we are going to learn the differences between the *mean,* the *median,* and the *mode.* Does anyone have any idea what the differences are?" After several wrong guesses, the teacher finally explained the differences. Regrettably, those same wrong guesses appeared as answers in the follow-up test. And why not? They were mentioned during the most powerful retention position, prime-time-1.

When you have the students' focus, teach the new information. Don't let prime time get contaminated with incorrect information.

Presenting new material right at the beginning of the class without student input might seem to contradict other views of teaching that encourage students to construct their own model of what they are learning. This "constructivist" approach is valid in some situations but not very useful if the students have little or no knowledge of the concept the teacher is introducing. We cannot construct a concept accurately with ignorance. Keep in mind, too, that nature did not prepare our brains for the variety of symbols, ratios, and abstract correlations that are found in advanced

Figure 8.1 New information should be presented in prime-time-1 and closure in prime-time-2. Practice is appropriate during downtime.



algebra and calculus. Consequently, there are no intuitive constructs that the student can use to develop a model.

The new material being taught should be followed by practice or review during the downtime. At this point, the information is no longer new, and the practice helps the learner organize it for further processing. Cognitive closure should take place during prime-time-2, since this is the second most powerful learning position and an important opportunity for the learner to determine sense and meaning. Adding these activities to the graph in Figure 3.4 shows how we can take advantage of research on retention to design a more effective lesson (see Figure 8.1).

Should a Lesson Start With Mathematics Homework?

A standard practice with many teachers of mathematics is to start a lesson by reviewing the students' homework from the previous day. This can be an effective strategy, but there are some cautions to be observed. Because this review is being carried out during prime-time-1, the teacher should emphasize the *correct* way to solve the homework problems. Spending too much time focusing on student errors during this powerful memory segment might cause students to remember the errors inadvertently. A discussion of common errors and how to avoid them should occur only after the students have accurately and fully learned the correct method. Teachers should not use up valuable prime time if the homework review is merely casual (about the mechanics of *doing* it) rather than substantive (about its *content*). Get on with today's learning objective, and collect the homework during downtime.

Here is a summary of how teachers can take advantage of the primacy–recency effect in the classroom:

- ✓ Teach the new material first (after getting the students' focus), during prime-time-1. This is the time of greatest retention. Alternatively, this would also be a good time to reteach any concept that students may be having difficulty understanding.
- ✓ Avoid asking students at the beginning of the lesson if they know anything about a *new* topic being introduced. If it is a new topic, the assumption is that most students do not know it (this can be determined using a pretest). However, there are always some students eager to make a guess—no matter how unrelated. Because this is the time of greatest retention, almost anything that is said, including incorrect information, is likely to be remembered. The teacher should provide the information and examples to ensure that they are correct.
- ✓ Avoid using precious prime-time periods for classroom management tasks, such as collecting absence notes or taking attendance.

Do these before getting focus, or during the downtime when students are engaged in practice.

- ✓ Use the downtime portion to have students practice the new learning or to discuss it by relating it to past learnings. "How does what we learned today about calculating the area of a polygon tie into what we already learned about calculating the area of a triangle?" Remember that retention of learning can occur during the downtime, but it takes more effort and concentration.
- ✓ Do cognitive closure during prime-time-2. This is the learner's last opportunity to attach sense and meaning to the new learning, to make decisions about it, and to determine where and how it will be transferred to long-term memory. It is important, then, that the students' brains do the work at this time as in prime-time-1. If a review is desired, then it should be done *before* closure to increase the chances that the closure experience is accurate. But doing review *instead* of closure is of little value to retention.
- ✓ When packaging lesson objectives (or sublearnings) into teaching episodes of about 20 minutes, link the sublearnings with procedural closure. This approach helps students recognize why these sublearnings should be integrated into the same memory network.

What About Practice?

We noted in Chapter 3 that practice makes permanent, not perfect. Practice is more likely to be effective when teachers do the following:

- ✓ Start by selecting the smallest amount of material that will have maximum *meaning* for the learner. Stay within the capacity limits of working memory for the students' age group. Excessive homework erodes motivation, builds frustration, and often leads to poor attitudes about studying mathematics.
- ✓ Model the application of the concept step-by-step. Use concrete manipulatives whenever possible. This helps students develop visual and spatial representations of the concept or skill being taught. Studies show that the brain uses observation as a means for determining the spatial learning needed to master a skill (Gatti et al., 2013).
- ✓ Insist that the practice occur in your presence (guided practice) over a short period of time while the student is focused on the learning.
- ✓ Watch the practice and provide the students with prompt and specific feedback on what variables need to be altered to correct and enhance the performance. When the guided practice is correct, then assign limited independent practice.

What Writing Will Be Involved?

Chapter 3 discussed how important writing is in communicating mathematical concepts. Adding this kinesthetic activity engages more brain areas and helps students organize their thoughts about the concept.

Strategies for Using Writing in Mathematics

Here are some strategies for incorporating writing in the mathematics classroom that I think can be effective, along with some research recommendations (Burns, 2004; Ediger, 2006):

- ✓ **Clarify the purpose.** Students, especially in secondary schools, may not be thrilled about writing in mathematics class or see its purpose. Make clear that writing activities are a tool to help them gain understanding of the mathematical concepts involved in their lessons. Treating writing as a learning and memory tool and not as an assignment to be graded helps students feel more comfortable with using writing in the mathematics classroom.
- ✓ **Review vocabulary.** Before beginning the writing activity, review and explain any new vocabulary terms encountered in the lesson. Do they know the meaning of *proper* and *improper* fractions, or of *complementary* and *supplementary* angles? Posting a chart containing all the new words is also helpful.
- ✓ **Discuss before writing.** By talking about their ideas in class before writing, students are able to formulate and clarify their thinking, select appropriate vocabulary, and decide on the main points to include in their writing sample. They can write about any mathematical idea they heard during the discussion, as long as they can explain it.
- ✓ **Work individually or in groups.** Although writing is often done alone, some students may prefer to work in groups so they can discuss with others what they are writing. Working in groups also allows students to share their written ideas and to hear different points of view. Allow for both opportunities.
- ✓ **Add interest.** Maintaining student interest is an important component of motivation. Writing activities can be made interesting by including historical information, such as how the Roman and Arabic number systems developed, the invention of the zero placeholder, or the introduction of negative numbers. People have been using geometry for thousands of years. How did it develop, and what was its impact on the growth of ancient societies?
- ✓ **Prompt when necessary.** Sometimes younger students need prompts to get them started. Write some prompts on the board, such as, *I think the answer is _____ because _______*, or *Today I learned __________. It is important to know this because _____________.*
- ✓ **Avoid rewriting the textbook.** Students are expected to use their own thoughts, phrasing, and vocabulary to write their sample, and not simply copy what is in their textbook. The point here is to get the brain to do elaborative rehearsal of the new concepts learned so as to increase the likelihood of retaining the learning.
- ✓ **Provide individual assistance.** Some students, especially those in the elementary grades or English language learners, will no doubt have difficulty getting their thoughts down. Talk with these students individually to ensure that they understand what they have learned and what they are expected to do in the writing activity. Get them to talk to you by asking, "What have you learned about this idea? What do you think about other people's ideas?" For younger

students, just the physical act of writing can be such an effort that they have difficulty keeping their thoughts in working memory. Suggest they recite silently to themselves what they intend to write before and while they are writing it down.

✓ **Use students' ideas.** Student writings can often provide useful ideas for clarifying or extending a mathematics concept. Sharing student notions in this manner places value on their work and provides motivation for future writing activities.

Writing as an Assessment Tool

✓ Writing can be an effective assessment tool for both students and teachers. Because writing provides a permanent record of students' thoughts, it documents the students' progress in learning. Students reflect on their own learning when they return to their writing. Once the writing is complete, students have a permanent record of their learning that can help them revise information or expand their application of knowledge in the future. The writing can help teachers by

- diagnosing error patterns,
- giving insights about where instruction should begin or what topics need to be retaught,
- providing evidence of where and why a student has failed to make connections, and
- showing the beliefs and attitudes students hold about mathematics.

✓ Consider keeping student writing samples in individual folders. They provide a chronological collection of each student's thoughts and progress, and can be very helpful in parent conferences as well.

Are Multiple Intelligences Being Addressed?

Dozens of books are available that suggest specific activities in all subject areas for applying Howard Gardner's theory of multiple intelligences in the classroom. My purpose here is to offer some general activities that mathematics teachers can use to apply and strengthen the eight intelligences through instruction.

In Table 8.1, simply replace "MC" with the mathematical concept you are teaching. The variety of activities helps you differentiate instruction. Some activities serve as enrichment for students who have already mastered the concept, while others serve to provide additional information to help students whose understanding of the concept may be shaky. Of course, it might not be appropriate to require all students to practice all the activities, because they may not be suitable for students with a weakness in a particular area or for those who, for example, are not comfortable doing performance-style activities in some situations. Students should have some choice in selecting which of these activities to do.

Does the Lesson Provide for Differentiation?

Today's teachers work in classrooms filled with a broad diversity of students. Besides their different learning capabilities, students come from

Table 8.1 Activities for Multiple Intelligences

Intelligence	*Activity for the Mathematical Concept (MC)*
Linguistic "The word player"	• Use storytelling to explain the MC. • Write a poem, myth, legend, or news article about the MC. • Give a presentation on the MC. • Lead a class discussion on the MC. • Create a talk-show radio program about the MC.
Logical/ mathematical "The questioner"	• Translate the MC into a mathematical formula. • Design a proof for the MC. • Make a strategy game that includes the MC. • Collect and interpret data related to the MC. • Write a computer program for the MC.
Spatial "The visualizer"	• Chart, map, or graph the MC. • Design a poster, bulletin board, or mural about the MC. • Create a piece of art that demonstrates the MC. • Make a film or advertisement of the MC.
Musical "The music lover"	• Write a song that explains the MC. • Give a presentation on the MC with appropriate musical accompaniment. • Explain how the music of a song relates to the MC. • Create a musical game that relates to the MC.
Bodily/kinesthetic "The mover"	• Rehearse and perform a play that explains the MC. • Choreograph a dance that shows the MC. • Build a model that explains the MC. • Plan and attend a field trip that will show or explain the MC.
Interpersonal "The socializer"	• Conduct a class meeting that discusses the MC. • Organize or participate in a group that will deal with the MC. • Suggest ways to accommodate learning differences and the MC. • Participate in a service project that uses the MC.
Intrapersonal "The individual"	• Create a personal analogy for the MC. • Set a goal to accomplish the MC. • Describe how you feel about the MC. • Use some form of emotional processing to understand the MC.
Naturalist "The nature lover"	• Describe any patterns you detect in the MC. • Explain how the MC can be found in the environment. • Show how the MC could be applied in nature. • Demonstrate how this MC can be linked to other MCs we have learned.

different cultures, speak many languages, and possess varying learning styles. Direct instruction based mainly on a one-size-fits-all approach does not work effectively with such a diverse group. What *can* work is an approach whereby teachers differentiate their instruction by using a variety of techniques and strategies that address the varying needs of all their students. In differentiated instruction, teachers enhance learning by matching their students' characteristics to instruction and assessment. Teachers can differentiate content, process, and/or product for students (Sousa & Tomlinson, 2011).

- **Differentiation of content** refers to a change in the material being learned by an individual student. For example, if the classroom objective is for all students to determine whether two triangles are congruent, some students may learn this by working with diagrams of triangles, while others may learn it through solving word problems.
- **Differentiation of process** describes the method by which a student accesses material. One student may explore a learning center, another may conduct an interview, and a third may collect information from the Internet.
- **Differentiation of product** refers to the way students demonstrate what they have learned. For instance, to demonstrate understanding of a geometric concept, one student may solve word problems while another builds a model.

When teachers commit themselves to using differentiated instruction, they switch their goal from teaching a collective class to teaching individual students. This is a major paradigm shift, and to do this successfully, teachers respond to an individual student's readiness, interest, and learning profile. A teacher may differentiate based on any one of these factors or any combination of factors (Sousa & Tomlinson, 2011).

- **Readiness** describes the student's skill level and background knowledge in mathematics. Some of this information can be gathered at the beginning of the year by reviewing student records, standardized test scores, and previous mathematics grades. Diagnostic assessments can also be used to determine student readiness. These assessments can be formal or informal. Teachers can give pretests or question students about their background knowledge of a particular topic.
- **Interest** refers to topics related to mathematics that the student may want to explore or that will motivate the student. Student interests can be determined by using interest inventories throughout the year. Teachers may discover, for example, that some students are interested in sports statistics or architecture. Including students in the lesson planning process is another way to explore interests. Teachers ask students what specific interests they have in a topic and then try to incorporate them into their lessons.
- **Learning profile** includes the students' learning style preferences (i.e., visual, auditory, tactile, or kinesthetic input), grouping preferences (i.e., individual, small group, or large group), and environmental preferences (i.e., lots of space or a quiet area to work). Learning styles

can be measured using learning style inventories. Teachers can also get information about student learning styles by asking students how they learn best and by observing student activities.

Some Guidelines for Differentiating Instruction

Dozens of books and Internet sites suggest specific ways to differentiate instruction in mathematics at every grade level (see the **Resources** section at the end of this book). Here are some general guidelines:

- ✓ If you are new to differentiation, start small. Try it first with one short unit, and then use your regular method for the next unit. Reflect on what was successful and what needs improvement. Students also need to adjust to this method of learning; so do not get discouraged if at first it does not go as well as planned. Like any new strategy, learning to differentiate instruction is a process. To succeed, implement it gradually, and constantly revise what does not work.
- ✓ Provide a variety of materials and opportunities for student projects. These can include reference books, manipulatives, construction and drawing materials, computers with Internet access, and other audiovisual materials. Arrange the classroom so workstations can be quickly set up when needed.
- ✓ Let students choose among several projects that cover the lesson objective. Options should reflect various learning styles. For example, visually preferred learners may want the option of showing a poster or brochure to present what they have learned, rather than just talking about it.
- ✓ The assignment options should represent various levels of difficulty and complexity and involve different types of thinking skills. Bloom's revised taxonomy remains an effective model for designing activities at different levels of cognitive thought (Sousa, 2011a).
- ✓ Vary your lesson delivery style during each class period to appeal to different learning styles. For example, use discussions for auditorily preferred learners, provide handouts outlining the topic for visually preferred learners, and incorporate hands-on activities for kinesthetically preferred learners. Remember that all students benefit when they use a variety of modalities while learning.
- ✓ For each project and activity, consider grouping students according to ability or interest. During units on percentages or graphing, for instance, several students may be interested in examining population growth.
- ✓ Consider using a variety of assessment tools, and offer students several assessment options. Design assessments with various skill levels, learning styles, and thinking skills in mind. Show sample work, and share with students the rubric or scoring criteria you will use to evaluate open-ended assessments and projects.

■ INTEGRATING THE ARTS

Teachers of mathematics are no doubt familiar with the STEM initiative. STEM is an acronym for science, technology, engineering, and mathematics. Spurred by a report from the National Science Foundation and other agencies regarding the declining state of education in the STEM subjects, the U.S. Congress in 2007 authorized funding for STEM initiatives from kindergarten through graduate school. Numerous school districts have obtained funding to support increased instruction in the STEM areas. Yet, despite the funding and increased instruction, student progress in achievement in these areas has been slower than expected. Mathematics scores for high school seniors on the 2013 National Assessment of Educational Progress (NAEP) were unchanged from the 2009 scores.

One frequently heard explanation for the slow progress is that these subjects, especially science and mathematics, are still being taught in many schools with a heavy emphasis on rote memorization rather than the creative problem solving and analytical skills the NAEP and international tests measure. In other words, these courses still focus too much on following traditional curricula with a vast amount of *content* and not enough on the *skills* that students will need to successfully apply what they learn in STEM courses to real-life situations. These skills include creative problem solving, linear and spatial analysis, improved memory systems, persistence, attention, and motor coordination. Neither the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) nor the Next Generation Science Standards (National Research Council, 2012) will do much to improve student achievement unless educators involved in the STEM subjects make a concerted effort to introduce activities that promote creativity and develop these important skills.

From STEM to STEAM

Mathematics and the arts are closely related. One interesting brain-imaging study found that when mathematicians looked at certain equations, the same region of the brain was activated as when people listen to beautiful music or look at beautiful artwork (Zeki, Romaya, Benincasa, & Atiyah, 2014). Practicing mathematicians, scientists, and engineers recognize that the arts are vital to their work, and they frequently use skills associated with the arts as mathematical and scientific tools. These skills include the following (Sousa & Pilecki, 2013):

- Capitalizing on curiosity
- Observing accurately
- Working effectively with others
- Constructing meaning
- Expressing one's observations accurately
- Perceiving an object in a different form
- Thinking spatially by rotating an object in one's head
- Perceiving kinesthetically how an object moves

These skills are not expressly taught in the STEM subjects but are an integral part of music, visual art, dance, drama, and writing. Because of the potential for the arts to infuse the STEM subjects with their creative skills, a new initiative has emerged called STEAM, an expanded acronym that adds the *A* for arts. In practice, there are no boundaries between the arts and the STEM subjects. Music, in fact, can be thought of as the artistic expression

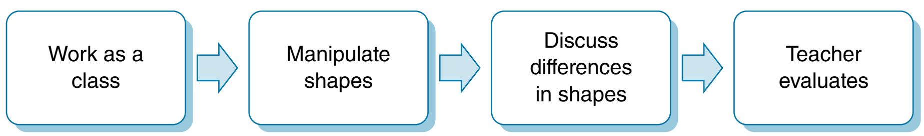
Answer to Question 10. False: *Mathematics and the arts are closely related. The same region of the brain is activated when an individual is listening to music or examining a mathematical equation.*

of mathematics. Mathematics is everywhere in music: Frequency intervals in the notes of the scale, harmonic octaves, notation, tempo, and beat are all examples. With appropriate planning, activities that incorporate the skills of the arts can be integrated into STEM courses without a lot of additional work for the teacher. Very often, volunteer teacher artists are available to help guide this process. The STEAM initiative is gaining momentum. Several universities, such as the State University of New York at Potsdam, have developed STEAM curricula, and a number of school districts have included STEAM activities in their courses. Here are a few examples of successful teacher-tested lessons that integrate the arts in mathematics (Sousa & Pilecki, 2013).

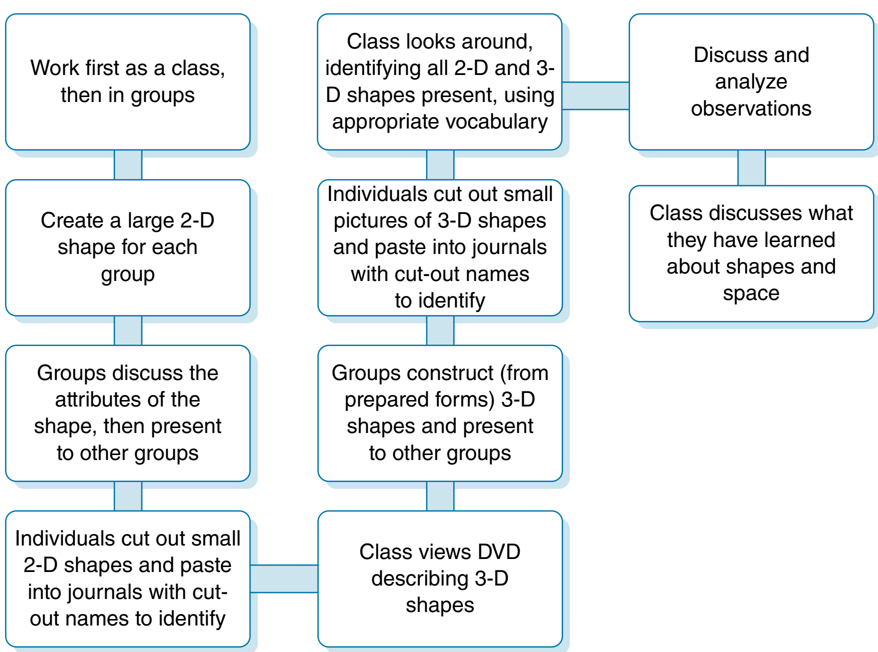
Examples of How to Integrate the Arts in Mathematics Lessons

Kindergarten Lesson in Mathematics: Describe Shapes and Sizes

Traditional approach: The teacher would probably have the students work as a class to manipulate the shapes, lead a class discussion of the shapes, and then evaluate how well the students understood the different shapes and sizes.

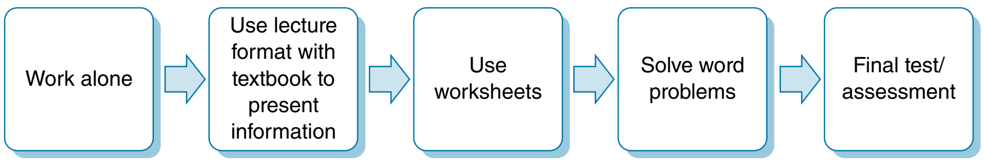


Arts-integrated approach: This approach involves much more student engagement, including artistic activities that help students gain a deeper understanding of the attributes of the different shapes and sizes. It focuses more on showing how two-dimensional objects differ from three-dimensional objects, thereby developing these young students' spatial reasoning and analysis skills.

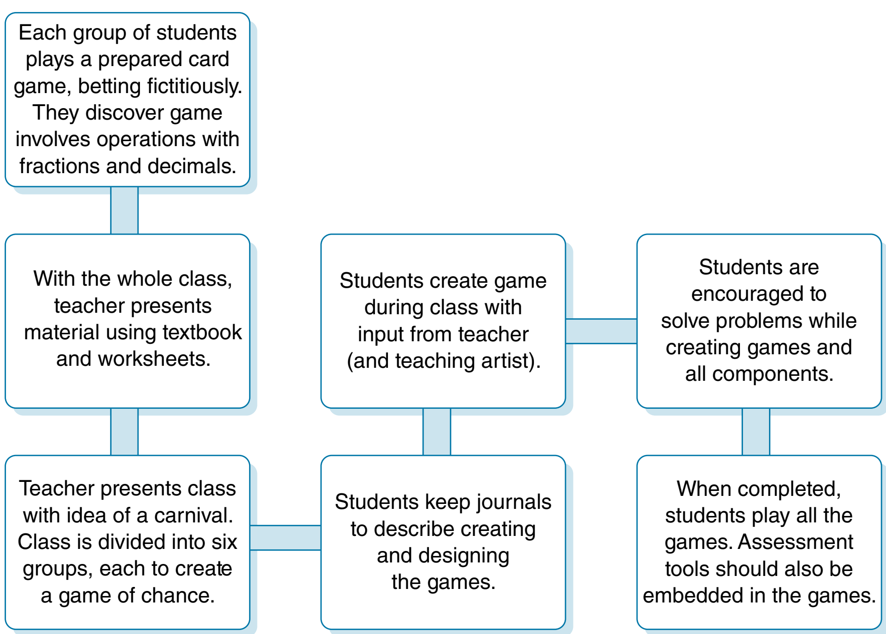


Grade 6 Mathematics: Multiplication and Division of Fractions and Decimals

Traditional approach: The students work alone, and the teacher uses the textbook to present material in a lecture format. Using worksheets, the students solve problems in multiplication and division of fractions and decimals, and then they are given a final test to evaluate their learning.



Arts-integrated approach: To add interest and engagement to this otherwise ordinary learning objective, the teacher (and teaching artist, if available) decides to use games of chance to teach multiplication and division of fractions and decimals. The students are to design and create their own games of chance. They keep written (or computer) journals describing the process. When the games are ready, the students play all of them one at a time, explaining the fractions and decimals involved. Assessment tools can be built into the games, if desired. The teacher, of course, can also use any other assessments the curriculum requires.



No doubt some teachers will feel that integrating the arts into their lessons on mathematics may be too much to take on for a variety of reasons. Teachers in the elementary grades are usually more receptive to arts integration because they recognize that they are already doing some of it, perhaps unwittingly, in many of their lessons. Secondary-level teachers, however, are more cautious. They often say there is too much content material to cover to take time for arts integration. "Remember, there are state tests coming in the spring," is a common refrain. Others believe they

Integrating the arts into mathematics instruction can enhance student engagement and motivation as well as improve student achievement.

are not creative enough themselves to provoke students to be creative. Experience has shown these concerns to be unfounded. Teachers unconsciously make creative decisions every day, from choosing what they will wear for the day to the instructional strategies they select for their classes. As for testing, both the Common Core State Standards for Mathematics and the Next Generation Science Standards are focusing more on developing creative problem solving and analytical skills, and so will the resulting assessments.

Places Where Arts Integration in Mathematics Is at Work

Here are just a few recent examples of how educators are finding ways to integrate the arts into their mathematics classes, to get their students involved in real-world applications, and even to open schools dedicated to the STEAM concept.

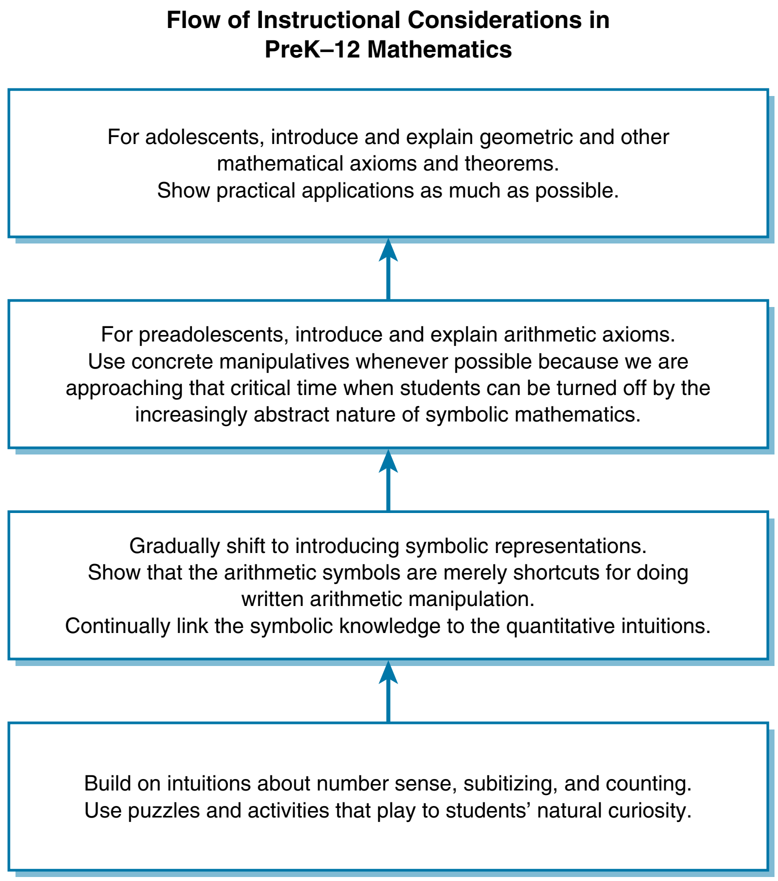
- **Rockford, IL:** Sixth-grade students at Kennedy Middle School with varying success in mathematics are designing and building different types of lamps to learn about ratios, proportions, and perimeters. Teachers report improved test scores in mathematics since this project started (Kravets, 2014).
- **Buffalo, WY:** Students at Clear Creek Middle School built sleds of cardboard and duct tape, and entered them in a tournament. The students used graph paper to design their sleds and calculate the surface area of various box designs. In science class, they learned about weight and friction on various types of snow, as well as the aerodynamics of the box designs (Stepenoff, 2014).
- **Daytona Beach, FL:** A teaching artist is working with teachers and fifth-grade students at Turie T. Small Elementary School, using song and dance to help the students remember important mathematics facts. Test scores in mathematics for these students have gone up dramatically since this program began (Trimble, 2014).
- **Jersey City, NJ:** Jersey City Public Schools will open a small high school in the fall of 2014 that will have a focus on STEAM. Its first class of 100 ninth graders will work with faculty from New Jersey City University in the STEAM subject areas (Jersey City Board of Education, 2014).

For more examples of mathematics/arts-integrated lessons, see the websites provided in the **Resources** section.

SIMPLIFIED INSTRUCTIONAL MODEL

Based on all I can gather from cognitive neuroscience, a reasonable model for teaching mathematics to children and adolescents proceeds through four major steps (Figure 8.2). Step 1 is to build on young children's intuitions about numbers, subitizing, quantitative manipulations, and counting.

Figure 8.2 This is a simplified model of instructional considerations for teaching preK–12 mathematics. The main considerations are to keep tying new information to intuitive concepts about number and quantity, and to include concrete manipulatives and practical applications as much as possible.



These innate talents are strongly rooted in developing neural networks and should be cultivated with concrete activities rather than stunted with paper worksheets. Activities and instruction should play to these students' natural curiosity with amusing number puzzles and problems.

Step 2 is to introduce students to symbolic notation in mathematics, emphasizing how it offers a powerful and convenient shortcut when manipulating quantities. It is important at this point to continue to tie the symbolic knowledge to the quantitative intuitions. In this way, the symbolic representations become part of the intuitive network instead of being memorized as a separate and unrelated language.

In Step 3, introduce the preadolescent brain to arithmetic axioms. Appropriate concrete manipulatives should be used here as much as possible, because we are moving into that critical time when students can be turned off by the increasingly abstract nature of symbolic mathematics. Later, in adolescence, the brain's frontal lobe becomes more adept at higher-order thinking and logic. So in Step 4, introduce and explain mathematical and geometric axioms and theorems. But it is still necessary to show practical applications whenever possible. Remember, when students understand and recognize practical uses for what they are learning, they can attach meaning and thus increase their chances of retention.

I certainly recognize that this model may be simplistic. On the other hand, one reason students get turned off to mathematics is that we often do not try hard enough to keep relating what they are experiencing in the classroom to concrete and practical applications. There are few school subjects in which teachers hear the lament, "Why do I have to know this?" more than in mathematics. That observation alone should be ample warning that we have to work harder at providing meaning.

REMEMBER ACTION RESEARCH

Sometimes teachers are reluctant to try out new strategies for fear that they will take up too much time or, even worse, not work. Several elementary teachers told me of their concerns about teaching the number-counting method based on the Asian system, outlined in Table 4.2 in Chapter 4. But everyone who did try it reported that this method helped students get a better understanding of the base-10 structure of mathematics. What these teachers did was a form of *action research.* They tried a strategy and assessed its effect on learning.

Action research is the systematic study of one's actions to determine the effect of those actions. In the classroom, it means implementing a strategy in a planned way, collecting and analyzing data resulting from using that strategy, and taking action based on the analysis. Action research can be done by just one teacher, but its impact and value grow significantly when it is the consistent effort of a teacher team, such as at a particular grade level, or subject-area group, such as the mathematics department in a middle or high school. The results should be reported to other teachers as part of a learning community. If several teachers carry out the same research, get similar results, and exchange data, then they will have evidence to support the continued use of the strategy.

CONCLUSION

As research in cognitive neuroscience expands, we are very likely to discover more about how the human brain grows, develops, and learns. These discoveries offer educators and parents exciting opportunities for deciding what kinds of learning experiences children and adolescents should have to develop to their full potential. Research not only leads to new ideas but often validates some past practices and questions others. Effective teaching strategies cut across all content areas. For the teaching of mathematics, I have suggested that the following notions be kept in mind:

- Everyone has the ability to do mathematical operations. We are born with it.
- Rote learning without meaning impedes long-term application of mathematical knowledge.

- Learning mathematics is easier when it makes sense and is meaningful to the learner.
- Learning mathematics is easier when the learner can connect mathematical operations and concepts to solving problems in the real world.
- Talking and writing about mathematics improves the depth of learning and recall.
- Learning mathematics involves a progression from the concrete to the representational to the abstract.
- People learn mathematics in different ways.

Regardless of all the instructional strategies we develop, all the curriculum reforms we write, and all the materials we buy, no component is as important as the teacher. How the teacher views the learning process will largely determine that teacher's instructional practice and, consequently, how well his or her students will learn mathematics. My purpose here was to present research from cognitive neuroscience to inform teachers of what we are discovering about the learning process in general and the learning of mathematics in particular.

And why is mathematics even worth learning? Our world is full of patterns. We find them in flowers, in snowflakes, in seashells, in the markings on animals such as zebras and leopards, in the distinctive songs of birds, and so on. If mathematics is truly the study of patterns, then teachers should help students recognize that the learning of mathematics not only will be useful in their future lives but also will give them a window into understanding the wonders and beauty of our magnificent world.

Chapter 8—Putting It All Together: Planning Lessons in PreK–12 Mathematics

QUESTIONS AND REFLECTIONS

Respond to the following questions, and jot down on this page key points, ideas, strategies, and resources you want to consider later. This sheet is your personal journal summary and will help jog your memory.

What questions should teachers consider when planning a lesson in mathematics? ____

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What are some guidelines for differentiating instruction in mathematics?

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Give an example of how a teacher could integrate the arts into a mathematics lesson/unit. ____

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Glossary

Abstraction. The process of extracting the underlying essence of a mathematical concept, removing any dependence on real-world objects to which it might originally have been connected, and generalizing it so it has wider applications as, for example, the axioms of geometry

Action research. The systematic implementation of an instructional strategy, followed by the collection and analysis of data to determine the effects of that strategy on learning

Algorithm. A process or set of rules used in calculations or other problem-solving operations

Arts integration. A teaching approach that uses skills from the arts to help students acquire another content area, such as mathematics (see also *STEAM*)

Associative memory. The brain's ability to detect patterns and make associations between working memory and past experiences

Associativity. A property of numbers whereby the sequence in which they are added or multiplied produces the same result. Thus, $(a + b) + c = a + (b + c)$, or $(a \times b) \times c = a \times (b \times c)$.

Cardinal principle (cardinality). The concept that the last number counted represents the size of the group in a collection

Cerebellum. A major part of the brain, located in the rear above the brain stem, that is largely responsible for coordinating muscle movement

Cerebrum. The largest of the major parts of the brain, it controls sensory interpretation, thinking, and memory.

Chunking. The ability of the brain to perceive a coherent group of items as a single item or chunk

Closure. The teaching strategy that allows learners quiet time in class to mentally reprocess what they have learned during a lesson

Cognitive overload. A condition where the flow of information exceeds the brain's ability to process and store it

Commutativity. A property of numbers whereby they can be added or multiplied in any order. Thus, $8 + 5 = 5 + 8$, or $4 \times 7 = 7 \times 4$.

Compensation. The idea that removing some items from one part of a collection and adding them to the other part leaves the whole quantity unchanged

Constructivism. A theory of learning stating that active learners use past experiences and chunking to construct sense and meaning from new learning, thereby building larger conceptual schemes

Corpus callosum. The bridge of nerve fibers that connects the left and right cerebral hemispheres and allows communication between them

Cortex. The thin layer of cells covering the cerebrum that contains all the neurons used for cognitive and motor processing

Covariation. The idea that a whole quantity increases (or decreases) if one of the parts is increased (or decreased)

Declarative memory. Knowledge of events and facts to which we have conscious access

Distributed practice. The repetition of a skill over increasingly longer periods of time to improve performance

Electroencephalograph (EEG). An instrument that charts fluctuations in the brain's electrical activity via electrodes temporarily attached to the scalp

Episodic memory. Knowledge of events in our personal history to which we have conscious access

Feedback. The information gathered about a person's performance that is shared with that person, with the goal of improving his or her performance

Formative assessment. Any process used during the learning episode to determine student progress toward achieving the learning objective

Frontal lobe. The front part of the brain that monitors higher-order thinking, directs problem solving, and regulates the excesses of the emotional (limbic) system

Functional magnetic resonance imaging (fMRI). An instrument that measures blood flow to the brain to record areas of high and low neuronal activity

Gray matter. The thin covering of the brain's cerebrum, also known as the cerebral cortex

Guided practice. The repetition of a skill in the presence of the teacher, who can give immediate and specific feedback

Immediate memory. A temporary memory in which information is processed briefly (in seconds) and subconsciously, then either blocked or passed on to working memory

Independent practice. The repetition of a skill on one's own outside the presence of the teacher

Limbic area. The structures at the base of the cerebrum that control emotions

Long-term memory. The areas of the cerebrum where memories are stored permanently

Massed practice. The repetition of a skill over short time intervals to gain initial competence

Mind-set. The beliefs one has about one's intelligence, talents, and personality. Those with a fixed mind-set believe their traits are inevitable and unchangeable. Those with a growth mind-set believe their traits can be developed through dedication and effort.

Mnemonic. A word or phrase used as a device for remembering information, patterns, rules, or procedures

Motivation. The influence of needs and desires on behavior

Motor cortex. The narrow band from ear to ear across the top of the brain that controls movement

Neural plasticity. The concept that the brain has an ability to change as a result of experiences

Neuron. The basic cell making up the brain and nervous system, consisting of a cell body, a long fiber (axon) that transmits impulses, and many shorter fibers (dendrites) that receive them

Nondeclarative memory. Knowledge of motor and cognitive skills to which we have no conscious access, such as riding a bicycle

Number sense. In its limited form, this refers to our ability to recognize that an object has been added or removed from a collection

Numerosity. The perception of approximate numerical quantities, such as more than and less than, without assigning an exact number

Occipital lobe. A brain area located in the back of the cerebrum that is responsible for processing mainly visual information

Parietal lobe. A brain area of the cerebrum, lying between the occipital and frontal lobes, that is involved in processing sensory information, including touch, taste, and movement

Positron emission tomography (PET) scanner. An instrument that traces the metabolism of radioactively tagged sugar in brain tissue, producing a color image of cell activity

Practice. The repetition of a skill to gain speed and accuracy

Prefrontal cortex. The area of the brain just behind the forehead that controls the planning, decision making, reasoning, and execution of behavior, and that integrates information in working memory

Primacy–recency effect. The phenomenon whereby one tends to remember best that which comes first in a learning episode and second best that which comes last

Prime time. The time in a learning episode when information or a skill is more likely to be remembered

Procedural memory. A form of nondeclarative memory that allows the learning of motor (e.g., riding a bicycle) and cognitive (e.g., learning to read) skills

Rehearsal. The reprocessing of information in working memory

Semantic memory. Knowledge of facts and data that may not be related to any event

STEAM. An acronym for the teaching approach that integrates the arts into lessons in science, technology, engineering, and mathematics (STEM)

STEM. The acronym for a nationwide initiative to increase instruction in science, technology, engineering, and mathematics

Subitizing. The ability to determine the number of objects at a glance, without counting

Summative assessment. An assessment given at the end of the learning unit to determine whether the student has accomplished the learning objective

Temporal lobe. A brain area of the cerebrum located behind each ear, just under the parietal lobe, that is involved in the processing and interpretation mainly of sound and spoken language

Transfer. The influence past learning has on the acquisition of new learning, and the degree to which it will be useful in the learner's future. If the past learning aids in the acquisition of new learning, it is called positive transfer; if it interferes, it is called negative transfer.

White matter. The support tissue that lies beneath the brain's gray matter

Working memory. The temporary memory wherein information is processed consciously

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Resources

There are literally thousands of websites devoted to various aspects of teaching and learning mathematics. Obviously, some are better than others. Deciding which sites to include here was not easy. Mainly, I relied on suggestions offered by teachers and supervisors of mathematics who assured me that the sites listed here were very helpful to them and their students. Many of these sites offer activities that can be downloaded to mobile devices.

Note: All websites were active at time of publication.

Algebra Help

www.algebrahelp.com

Here you will find a large collection of calculators, worksheets, and lessons focusing on concepts in algebra. The equation calculator will show how to solve an equation, and the worksheets will automatically grade the student's answers.

Art of Problem Solving

www.artofproblemsolving.com/Videos/index.php

This site has more than 300 videos on how to solve problems in prealgebra, algebra, counting, and probability, listed by subject and topic.

Arts Integration in Mathematics

www.mathactivities.net/art.htm

This attractive site is for Grades 2–4, and contains numerous lessons that integrate the arts with mathematics.

Ask Dr. Math

mathforum.org/dr.math/

This site began as a project at Swarthmore College that used university students to answer questions in mathematics. The site is now sponsored by Drexel University, and there are hundreds of volunteers from colleges around the world who answer mathematics questions. The Frequently Asked Questions page makes interesting reading, especially "A Crash Course in Symbolic Logic." There are links to many other mathematics sites as well.

Assessment in Math and Science

www.learner.org/resources/series93.html

This site is sponsored by the Annenberg Foundation and contains free (with registration) online video workshops on mathematics topics sorted by grade level.

Calc101.com Automatic Calculus and Algebra Help

www.calc101.com

This site offers free and password-accessible ($25 for a year) help for finding derivatives, integrals, graphs, matrices, determinants, and systems of linear equations. The solutions are provided in simple terms and use standard mathematics notation, and show the steps used.

Calculus Help.com

www.calculus-help.com

This fun site for calculus includes a problem of the week, tutorials, an interactive cheat sheet, and calculus music (the touching "Quadratic Formula Song," the catchy "Day Before Notebooks Are Due Blues," and even calculus holiday carols). The site updates often with new material.

Cliffs Notes for Mathematics

www.cliffsnotes.com/math

This content-rich site is for students in Grades 2–12 (basic mathematics through calculus). It reviews mathematical operations in all areas in clear, step-by-step directions, and provides quizzes for practice.

Coping With Math Anxiety

www.mathacademy.com/pr/minitext/anxiety/index.asp

This site takes a constructive look at math anxiety, its causes, its effects, and how students can learn to manage it so it no longer hinders their study of mathematics. It includes special strategies for studying mathematics, doing homework, and taking exams.

Education World

www.educationworld.com/a_lesson/archives/math_practice_4_you.shtml

This site is essentially for elementary-grade mathematics and has a large collection of math fact worksheets, lesson plans, and calculator lessons. It also has links to sites that offer downloadable PDFs on critical thinking, hands-on mathematics activities, and related crossword puzzles.

Geometry Junkyard

www.ics.uci.edu/~eppstein/junkyard/

Created by David Eppstein, a professor of computer science at the University of California–Irvine, this site contains clippings, web pointers, lecture notes, research excerpts, papers, abstracts, programs, problems, and other information related to discrete and computational geometry, presented in a unique and entertaining format.

Homework Help

www.math.com

This site offers practice lessons in many topics to help students with their mathematics homework. It also has different types of mathematical calculators and games.

Hooda Math

www.hoodamath.com/index.html

This site is designed to help K–12 students learn mathematical concepts through colorful and interesting games. Each game notes which of the Common Core State Standards for Mathematics is addressed.

Illuminations from National Council of Teachers of Mathematics

illuminations.nctm.org

This site contains hundreds of activities for teachers of mathematics for Grades preK–12. The resources include lesson plans, mathematics games and puzzles, and online discussions for special interest groups. Also included are downloadable apps for desktop and mobile devices.

Interactive Math Activities

www.shodor.org/interactivate/activities/

From the Shodor Education Foundation, this site includes an impressive list of graphs and games for geometry, algebra, calculus, probability, statistics, and modeling.

Interactive Mathematics Miscellany and Puzzles

www.cut-the-knot.org/content.shtml

This entertaining site presents hundreds of problems in all areas of mathematics in an interactive format.

Interactive Resources

www.globalclassroom.org/ecell00/javamath.html

This is a portal to interactive mathematics sites from around the world, designed for Grades 1–6. It includes algebra patterns and function, place value, money, arithmetic operations, time, graphing, measurement, fractions, decimals, and geometry.

Khan Academy

www.khanacademy.org

Started by Sal Khan, this site for teachers and students is supported by the Bill and Melinda Gates Foundation and other industry supporters. As a result, all the helpful videos and curriculum content are free to use, and range from kindergarten through college. All the major topics in the Common Core State Standards for Mathematics are covered.

Layered Curriculum

help4teachers.com

This is Kathie Nunley's site, which offers information about her methods for layering the curriculum in mathematics and other subject areas. It includes plenty of teacher tips and suggestions for flipping the classroom.

MacTutor History of Mathematics

www-groups.dcs.st-and.ac.uk/~history/index.html

Maintained by the University of St. Andrews in Scotland, this site is an excellent source of material on the history of mathematics. It has an extensive list of biographies of famous mathematicians and is rich in information on a wide variety of topics, including ancient Babylonian, Chinese, and modern-day mathematics.

Math Cats

www.mathcats.com

This is a site for elementary-grade students that uses attractive animation to teach arithmetic operations, conversions, measurement, estimation, geometry, spatial reasoning, probability, statistics, real-life mathematics, and other related topics.

Math Pickle

www.mathpickle.com/K-12/Videos.html

This site for teachers and students has videos with mini-lessons to explain mathematical concepts for Grades K–12. It also has numerous games and discussion boards for support.

The Math Playground

www.mathplayground.com

This graphically appealing site offers elementary and middle school students an entertaining way to learn word problems, logic games, and mathematics games. The site also contains printable worksheets, interactive quizzes, and practice facts.

Math VIDS (Video Instructional Development Source)—Struggling Learners

www.coedu.usf.edu/main/departments/sped/mathvids/

This site is for teachers of mathematics who have struggling learners. It offers videos of real teachers using research-supported instructional techniques (such as the concrete–representational–abstract approach), and accommodations and modifications that can be made for specific learning difficulties.

Mathematical Interactivities—Puzzles, Games, and Other Online Educational Resources

mathematics.hellam.net/

On this interesting site, students can play unique mathematical games, find out how to do number tricks, and more. Some games are interactive. Teachers should preview the games before assigning them, because some of the sites are not self-explanatory and do not provide adequate feedback.

Math2.org

www.math2.org

This site features reference tables for K–12 courses, an English/Spanish mathematics dictionary, a collection of mathematical theorems and formulas, and links to other, similar sites.

National Council of Teachers of Mathematics: Principles and Standards

www.nctm.org/resources/default.aspx?id=230

The official site of the National Council of Teachers of Mathematics offers K–12 teachers of mathematics numerous teaching tips, sample lessons and themes, and many other resources.

Online Quizzes

www.softschools.com/quizzes

Here is an extensive collection of quizzes in mathematics (as well as other subject areas) for grades preK through middle school. Also included are free online calculators, games, and flashcards.

Patrick JMT

patrickjmt.com

Put together by a university mathematics teacher, this content-rich site offers hundreds of videos showing how to solve problems in arithmetic, algebra, calculus, trigonometry, probability, statistics, and more.

PBS Learning Media

www.pbslearningmedia.org

This is a curriculum data bank and collection of teaching videos in all areas of preK–12 mathematics (and other subjects) provided by the Public Broadcasting Service, with links to books, media, and online professional development courses.

Practical Uses of Math and Science (PUMAS), NASA

pumas.jpl.nasa.gov

This site offers nearly 100 suggestions on how to show the practical and everyday uses of mathematics and science.

Professor Freedman's Math Help

www.mathpower.com

This site was created by Ellen Freedman, a professor from Camden County College in New Jersey. It provides information about basic mathematics, algebra, study skills, math anxiety, and learning styles, and specifically addresses the needs of the community college adult learner. There are musical videos illustrating different mathematical operations, tutorials, homework assignments, interactive games, and links to many more resources.

Purplemath

www.purplemath.com

This site provides numerous reviews and links to sites offering lessons and tutoring, quizzes and worksheets, a collection of downloadable information on mathematics, and a study skills survey.

STEAM (Science, Technology, Engineering, Arts, and Mathematics)

stemtosteam.org

This site is sponsored by the Rhode Island School of Design, which is providing resources to support the STEAM initiative. Included are research-based arguments for arts integration into the STEM areas, as well as links to related sites.

artsedge.kennedy-center.org/educators/how-to/growing-from-stem-to-steam

This site from the Kennedy Center offers numerous sample lessons on how to integrate the arts into the STEM subjects.

SuperKids Math Worksheet Creator

www.superkids.com/aweb/tools/math/index.shtml

This site helps teachers easily customize worksheets for arithmetic operations, fractions, percentages, greater than/less than, odd or even, rounding, averages, exponents, factorials, prime numbers, and telling time.

Virtual Manipulatives

nlvm.usu.edu/en/nav/siteinfo.html

This site is sponsored by Utah State University and contains a library of virtual manipulatives for preK–12 mathematics. These Java-based activities allow for exciting interactive approaches for learning mathematics concepts.

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Corwin is committed to improving education for all learners by publishing books and other professional development resources for those serving the field of PreK–12 education. By providing practical, hands-on materials, Corwin continues to carry out the promise of its motto: **“Helping Educators Do Their Work Better.”**