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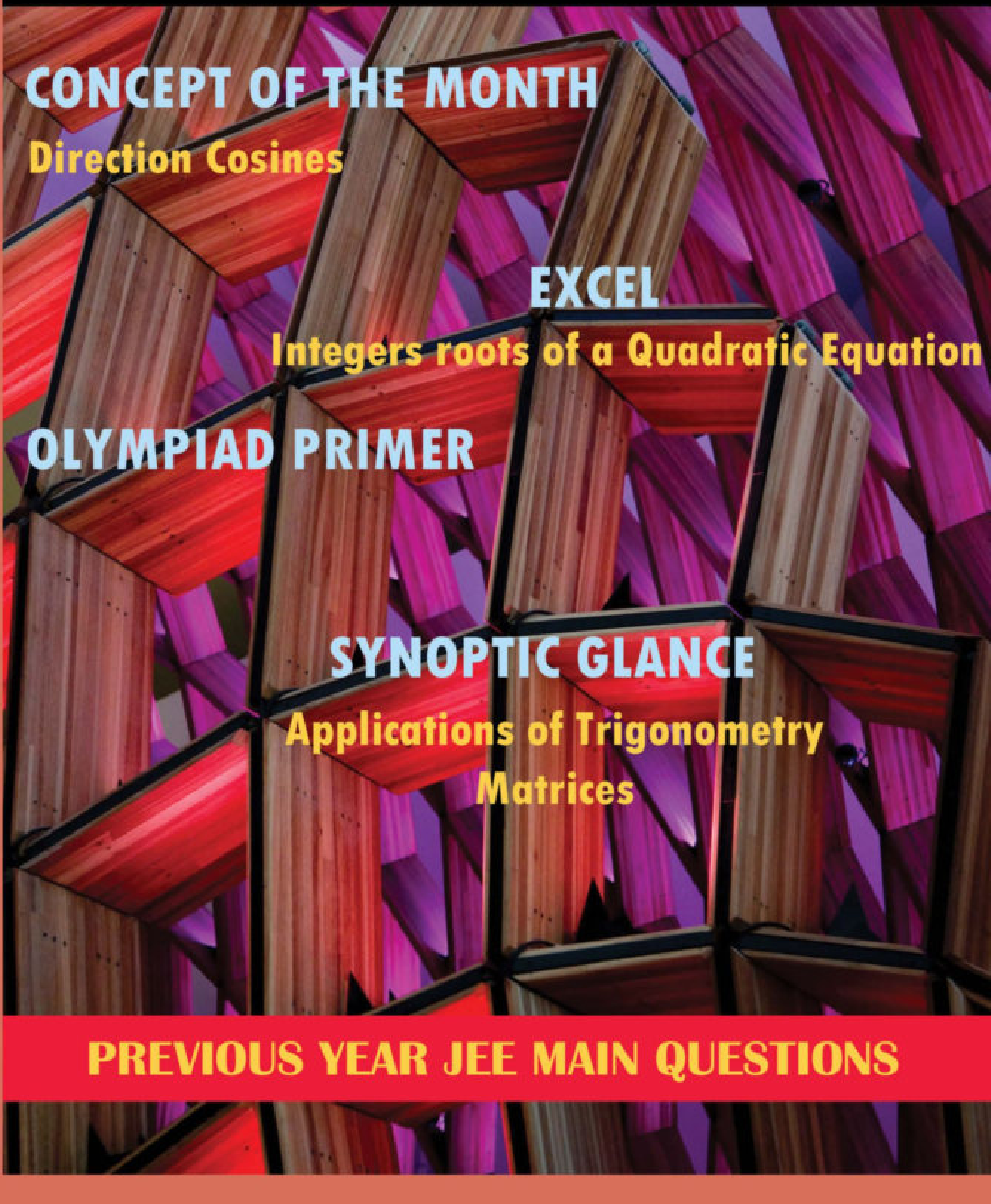

CONCEPT OF THE MONTH
Direction Cosines

EXCEL
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OLYMPIAD PRIMER

SYNOPTIC GLANCE
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AUGUST 2018

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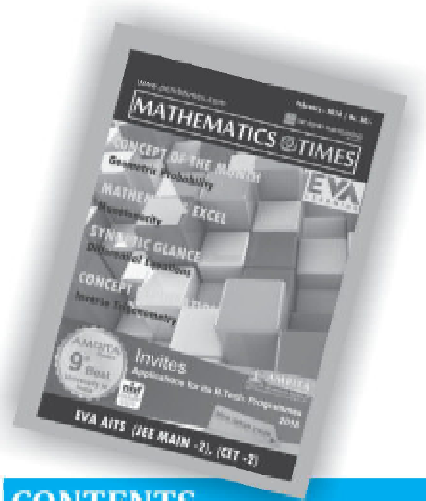

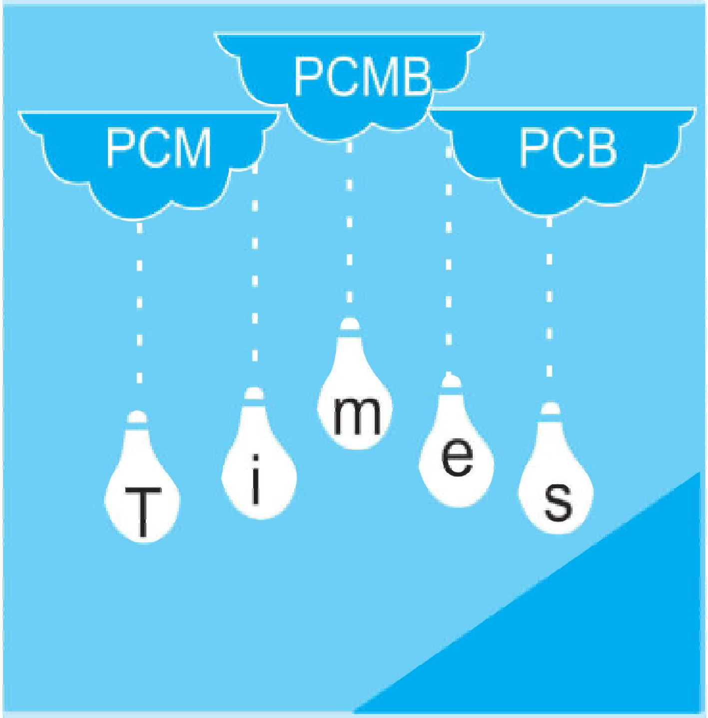

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Direction Cosines

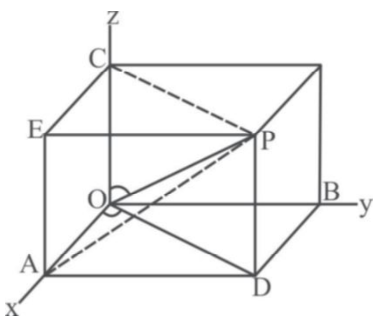
Concept of the month

This column is aimed at preparing students for all competitive exams like JEE, BITSAT etc. Every concept has been designed by highly qualified faculty to cater to the needs of the students by discussing the most complicated and confusing concepts in Mathematics.

By. DHANANJAYA REDDY THANAKANTI
(Bangalore)

Direction Cosines

In Figure.1 the lines OX ,OY and OZ are mutually perpendicular and lie along the positive directions of the vectors (not shown) **i**, **j**,and **k** of a unit orthogonal triad . These three lines are called *coordinate axes*. [They extend indefinitely on both sides of O , of course. The diagram shows only a



portion of each coordinate axes.] The line segment OP is a diagonal of the rectangular parallelepiped shown. There are many right angles in the diagram that are not readily recognized as such .

We wish to specify in a convenient way the direction of OP relative to **i**, **j**,and **k** . There are various possible ways . For example , we could state the angle through which the plane OPC has swung about OC from the plane AOC, and the angle in the plane OPC that the line OP makes with the line OC; that is,we could state the two angles AOD and COP. Alternatively , we could use other appropriate pairs of angles. In certain cases describing the direction in terms of two angles is both convenient and useful . Indeed, the two angles AOD and COP are those used in what are spherical polar coordinates.

There is a different method of specifying the direction of OP . It has the advantage of being symmetrical. But it uses *three* angles , and these angles , when first encountered , seem about as awkward a trio as one could imagine .

There are three angles AOP,BOP, and COP , that OP makes with the three axis OX,OY , and OZ. These angles , denoted respectively by α, β and γ , do not lie in one plane . Moreover, since any two of them suffice to determine the direction of OP, the three can not be independent of one another and , therefore , there must be a relation between them.

we may now be wondering what merits the angles α, β and γ can have that will outweigh the above disadvantages . But this is because we have, in a sense, told the story backwards . Mathematicians did not arbitrarily pick three queer angles to do the work of two seemingly more sensible ones. Let us look at the situation from different point view.

Consider OP as vector **V** with components (V_x ,V_y ,

V_z) relative to **i**, **j**,and **k**. Then if we know the components , we know the vector , and therefore , in particular , its direction . But if h is any number (greater than zero) the vector h**V** has the same direction as **V** . Let us specify the direction , then by means of the components of a vector of unit magnitude lying along **V**. This unit vector is $\left(\frac{1}{V}\right)$ **V**, where V is magnitude of **V**:

$$V = \sqrt{v_x^2 + V_y^2 + V_z^2}$$

The components of the unit vector , denoted by l, m, and n , are given by

$$l = \frac{v_x}{V},\quad m = \frac{v_y}{V},\quad n = \frac{v_z}{V};$$

and we easily see that

$$l^2 + m^2 + n^2 = 1,$$

so that any two of the quantities l,m, and n automatically detremine the third , to within a sign.

Having thus come upon these quantities l,m, and n, we now ask what they look like on diagram. let us look at triangle OAP in figure 1 . Since OA is perpendicular to the plane PDA , it is perpendicular to every line in that plane, and therefore to the line AP. So despite appearances to the contrary , angle OAP is right angle . In the right triangle OAP, the length of OA is V_x and the length of the hypotenuse OP is V . Therefore $\frac{V_x}{V}$, which is just l , is the cosine of the angle AOP that we have called α .

The coordinate planes OYZ, OZX, and OXY (called respectively the yz-plane , the zx-plane , and the xy-plane) separate the whole three dimensional space into eight regions called *octants* . We have here discussed only the case in which OP points into what is called the first octant. when OP points into other octant , some or all of the angles AOP ,BOP, and COP are obtuse and the corresponding cosines are negative.

The angles α, β, and γ that the line OP makes with the coordinate axis are called the *direction angles* of OP with respect to the reference frame; the cosines of these angles are called its *direction cosines* . In practice , one works much more with the direction cosines than directly with the direction angles.

Through the line segments OP and PO are the same, the directions OP and PO are opposite, and if the direction cosines of the direction OP are l,m, and n, those of direction PO are -l ,-m ,and -n . When we wish to stress that we are thinking of a given line (or line segment) as pointing in one rather than the other of the two directions associated with it, we call it a *directed* line (or line segment). Thus the x-axis , in its role as a coordinate axis is a directed line rather than just a line , though we often think of it as just a line - as when we say that a point is 5 unit away from it.

If a directed line or line segment does not pass through O, its direction can still be given in terms of direction can still be given in terms of direction cosines , since the direction is the same as that of a parallel directed line that does pass through O. Let points A and B have position vectors r_a and r_b relative to **i** , **j**, and **k** ,and let the coordinates of these points be (x_a, x_a, x_a), (x_b, x_b, x_b), Then the displacement $\overrightarrow{AB}$ has components $(x_b - x_a, y_b - y_s, z_b - z_a)$ and therefore its direction cosines are

$$\frac{x_b - x_a}{d},\quad \frac{y_b - y_a}{d},\quad \frac{z_b - z_a}{d}$$

Where

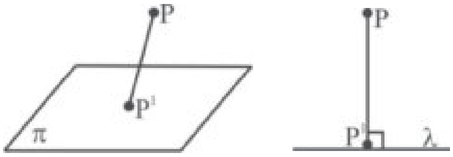
$$d = \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}$$

Orthogonal Projections

We can get an idea of the usefulness and convenience of direction cosines by deriving some fundamental formulas of analytical geometry .

In deriving these formulas by means of direction cosines , we must make use of the idea of an *orthogonal projection*. Given a point P and a plane π , drop a perpendicular from P to π to meet π at the point P'. Then P' is called orthogonal projection of P on the plane π . Given point P and a line λ ,the foot p', of the perpendicular from P

to λ is called the orthogonal projection of P on the line λ.



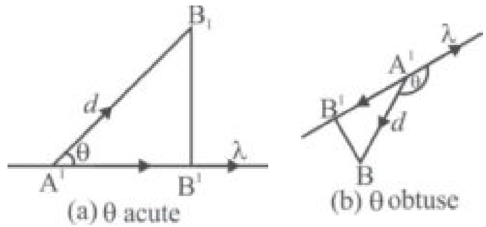
There are other types of projection, but it will be good for us to drop the word "orthogonal" when speaking of orthogonal projections. The idea of such projections seems so simple that one wonders how it could possibly worth considering . Yet it is actually a powerful mathematical concept , as we shall see.

If P traces out a curve , its projection, P', on a plane π traces out a curve called the projection on π of the original curve. If the curve is a plane curve and closed , the area enclosed by its projection on π is called the projection on π of the area enclosed by the original curve.

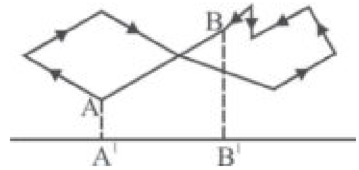
No matter how a point P may move , its projection , P', on a line λ cannot move off the line . Two simple theorems form the basis of the applications off projections on a line.

Theorem 1: *If a directed line segment AB of length d makes an angle θ, with a directed line λ , the length and sign of its projection on λ are given by $d\cos\theta$; the sign is positive if the projection points in the same direction as λ and negative if it points in the opposite direction.*

To prove this , pass planes through A and B that are perpendicular to λ and let them cut λ at A' and B' . Then A' and B' are the projections of A and B whether the AB is coplanar with λ , or not . If the segment AB is moved parallel to itself , with A and B remaining on the respective planes, the projection of A and B will be unaltered . So move AB to the position $A'B_1$, A' being the above mentioned projection of A. Then from the right triangle $A'B'B_1$ we see that since $\cos\theta$ is negative when θ is obtuse , the length and sign of the projection , $A'B'$, are given by $d\cos\theta$.

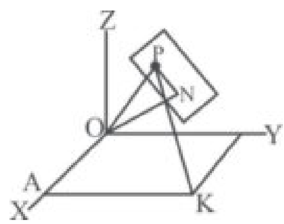


Theorem 2: *If the projection of A and B on line λ are A' and B' , and a point P starting at A moves on zigzag line ending at B , the algebraic sum of the projection of λ of the line segments forming the zigzags is just $A'B'$.*



As P traces out its zigzag path from A to B , its projection , P', moves to and fro on the line λ staring at A' and ending B'; and when, for example , P' retraces to the right ground previously traced out to the left , it cancels it. The *algebraic* sum of the projections is therefore $A'B'$. And that is that.

Let O be the origin and A the point $(x, 0, 0)$. Let ON be a directed line segment having direction cosines l, m and n . The length and sign of the projection on ON of the directed line segment OA are given by lx what ever the signs of l and x . Also K is the point $(x, y, 0)$, then the length and sign of the projection on ON of the directed line segment AK are similarly given by my (using Theorem.1).



To apply these two theorems, we first consider a plane such that ON , the perpendicular to it from the origin, has length p and direction cosines $l, m,$ and n. Let any point P on the plane have rectangular Cartesian coordinates (x, y, z). We wish to find an equation that x ,y and z must satisfy- an equation of the plane , as it called . Drop a perpendicular from P to the xy-plane meeting it at K. Draw KA parallel to the y-axis to meet the x-axis at A. Then $OA = x$, $AK = y$, and $KP = z$. if and only if P lies in the plane , PN will be perpendicular to ON . Therefore :

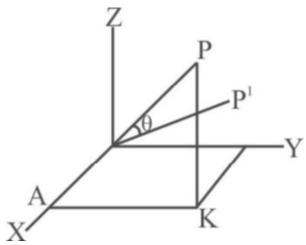
projection of OP on $ON = ON$

Using Theorem 2, we replace OP by zigzag $OAKP$. Then :

sum of projections of OA, AK, and KP on ON = ON. Using Theorem 1 , and remembering that length of ON is p, we have immediately

$$lx + my + nz = p,$$

which is the equation we sought.



Consider the problem of finding a formula for the angle between two directed lines having direction cosines $l, m\ n$ and l', m', n' respectively. [If two lines do not intersect , the angle between them is defined as the angle between lines parallel to them that do intersect. Actually there are infinitely many angles , both positive and negative , between two given lines. When we talk of "*the* angle" between them , we presumably have some specific one in mind. Here , as on previous occasions, we mean the smallest positive angle between the positive directions of the lines regarded as directed lines.]

Denote the angle between the two lines by θ, and for convenience (though it is not really necessary) , imagine the lines emanating from the origin. Take points P amd P' on the lines such that OP and OP' are of *unit* length . Then , by theorem 1, the length and sign of the projection of OP and OP' will be given by just $\cos\theta$. But we can also compute the length and sign of the projection by using the zigzag path $OAKP$ instead of OP. Since OP is of unit length, the coordinates of p are just (l, m, n) , so that $OA = l$, $AK = m$, and $KP = n$.

These line segments, being parallel to the coordinate axes , make with OP' angles whose cosines are respectively l', m', and n'.

Therefore the algebraic sum of the lengths of their projections on OP' is $ll' + mm' + nn'$. So we must have :

$$\cos\theta = ll' + mm' + nn'$$

Exercise

1. The angle between the lines whose direction cosines satisfy equations $l + m + n = 0$ and $l^2 = m^2 + n^2$, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

2. The angle between two diagonals of a cube is

(a) $\cos^{-1}\left(\frac{1}{3}\right)$ (b) $30°$

(c) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (d) $45°$

3. If the direction cosines of a vector of magnitude 3 are $\frac{2}{3}, \frac{-a}{3}, \frac{2}{3}$ and $a > 0$, then the vector is

(a) $2\hat{i} + \hat{j} + 2\hat{k}$ (b) $2\hat{i} - \hat{j} + 2\hat{k}$

(c) $2\hat{i} - 2\hat{j} + 2\hat{k}$ (d) $\hat{i} + 2\hat{j} + 2\hat{k}$

4. If the direction cosines of two lines are given by $l + m + n = 0$ and $l^2 - 5m^2 + n^2 = 0$, then the angle between them is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

5. If $A(3,4,5), B(4,6,3), C(-1,2,4)$ and $D(1,0,5)$ are such that the angle between the lines DC and AB is θ, then $\cos\theta$ is equal to

(a) $\frac{7}{9}$ (b) $\frac{2}{9}$ (c) $\frac{4}{9}$ (d) $\frac{5}{9}$

6. The angle between the lines, whose direction ratios are $(1,1,2)$, $(\sqrt{3}-1,-\sqrt{3}-1,4)$, is

(a) $\cos^{-1}\left(\frac{1}{65}\right)$ (b) $\frac{\pi}{6}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

7. The projection of any line on coordinate axes to be respectively 3, 4 and 5, then its length is

(a) 50 (b) 12

(c) $5\sqrt{2}$ (d) None of these

8. The direction cosines of the joining the points (4,3,-5) and (-2,1,-8) are

(a) $\left(\frac{2}{7},\frac{2}{7},\frac{3}{7}\right)$ (b) $\left(\frac{2}{7},\frac{3}{7},\frac{-6}{7}\right)$

(c) $\left(\frac{6}{7},\frac{3}{7},\frac{2}{7}\right)$ (d) None of these

9. The projection of a directed line segment on the coordinate axes are 12, 4, 3. The direction cosines of the line are

(a) $\frac{12}{13},\frac{4}{13},\frac{3}{13}$ (b) $\frac{12}{13},\frac{4}{13},-\frac{3}{13}$

(c) $-\frac{12}{13},\frac{4}{13},\frac{3}{13}$ (d) $\frac{12}{13},-\frac{4}{13},-\frac{3}{13}$

10. If (1,-2,-2) and (0,2,1) are direction ratios of two lines, then the direction cosines of a perpendicular to both the lines are

(a) $\left(\frac{1}{3},-\frac{1}{3},\frac{2}{3}\right)$ (b) $\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)$

(c) $\left(-\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)$ (d) $\left(\frac{2}{\sqrt{14}},-\frac{1}{\sqrt{14}},\frac{3}{\sqrt{14}}\right)$

11. The direction cosines of a line equally inclined to all the three rectangular coordinate axis are

(a) $\sqrt{3},\sqrt{3},\sqrt{3}$ (b) $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$

(c) 1, 1, 1 (d) None of these

12. How far from the origin is the plane whose equation is $3x+2y-6z=63$?

(a) 6 (b) 63 (c) 7 (d) 9

13. A line *AB* in three-dimensional space makes angle $45°$ and $120°$ with the positive *x*-axis. and the positive *y*-axis respectively. If *AB* makes an acute angle θ with the positive *z*-axis, then θ equals

(a) 30° (b) 45° (c) 60° (d) 75°

14. If a line makes angle α,β,γ with the coordinate axis, then

(a) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma-1=0$

(b) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma-2=0$

(c) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma+1=0$

(d) $\cos 2\alpha+\cos 2\beta+\cos 2\gamma+2=0$

15. If the co-ordinates of *A* and *B* be (1,2,3) and (7,8,7), then the projections of the line segment *AB* on the coordinate axis are

(a) 6, 6, 4 (b) 4, 6, 4

(c) 3, 3, 2 (d) 2, 3, 2

16. If projection of any line on co-ordinate axis 3, 4 and 5, then its length is

(a) 12 (b) 50 (c) $5\sqrt{2}$ (d) $3\sqrt{2}$

17. The angle between the lines $2x=3y=-z$ and $6x=-y=-4z$ is

(a) 90° (b) 0° (c) 30° (d) 45°

18. The angle between the lines whose direction cosines satisfy the equations $l+m+n=0$, $l^2+m^2-n^2=0$ is given by

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{5\pi}{6}$ (d) $\frac{\pi}{3}$

19. If direction cosines of two lines are proportional to (2, 3, -6) and (3, -4, 5), then the acute angle between them is

(a) $\cos^{-1}\left(\frac{49}{36}\right)$ (b) $\cos^{-1}\left(\frac{18\sqrt{2}}{35}\right)$

(c) 96° (d) $\cos^{-1}\left(\frac{18}{35}\right)$

20. If *A*, *B*, *C*, *D* are the points (2, 3, -1), (3, 5, -3), (1, 2, 3), (3, 5, 7) respectively, then the angle between *AB* and *CD* is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

ANSWER KEY

1. a	2. a	3. b	4. d	5. c
6. c	7. c	8. a	9. a	10. b
11. b	12. d	13. c	14. c	15. a
16. c	17. a	18. d	19. b	20. a

HINTS & SOLUTIONS

1.Sol: We know that, angle between two lines is

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\because \quad l+m+n=0 \Rightarrow l=-(m+n)$$

$$\Rightarrow \quad (m+n)^2=l^2 \quad \Rightarrow \quad m^2+n^2+2mn=m^2+n^2$$

$$\left[\because \quad l^2=m^2+n^2, \text{ given}\right]$$

$$\Rightarrow \quad 2mn=0$$

When $m=0$, then $l=-n$

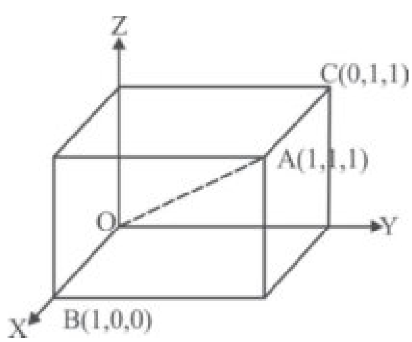
Hence, (l,m,n) is $(1,0,-1)$

When $n=0$, then $l=-m$

Hence, (l,m,n) is $(1,0,-1)$

$$\therefore \quad \cos\theta = \frac{1+0+0}{\sqrt{2}\times\sqrt{2}} = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

2.Sol: Let edge of a cube be 1 unit.
The diagonals of a cube are *OA* and *BC*.
So, DR's of diagonals *OA* are (1,1,1) and *BC* are (0-1,1,1), i.e., (-1,1,1).



Now, angle between diagonals,

$$\cos\theta = \frac{1(-1)+1(1)+1(1)}{\sqrt{1^2+1^2+1^2}\sqrt{(-1)^2+1^2+1^2}}$$

$$= \frac{1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}\sqrt{3}} = \frac{1}{3}$$

$$\therefore \quad \theta = \cos^{-1}(1/3)$$

3.Sol: Given direction cosines are $\frac{2}{3}, -\frac{a}{3}, \frac{2}{3}$

Then, direction ratios are 2, -*a*, 2
According to the equation,

$$3=\sqrt{2^2+(-a)^2+2^2} \Rightarrow 9=8+a^2$$

$$\Rightarrow \quad a^2=1 \Rightarrow a=\pm1 \Rightarrow a=1 \ [\because \ a>0]$$

So, the required vector is $2\hat{i}-\hat{j}+2\hat{k}$

4.Sol: Given direction cosines of two lines are

$l+m+n=0$ and $l^2-5m^2+n^2=0$

Also, $l^2+m^2+n^2=1$

$$(l_1,m_1,n_1)=\left(-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$$

$$(l_2,m_2,n_2)=\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}}\right)$$

$$\therefore \quad \cos\theta = |l_1l_2+m_1m_2+n_1n_2|$$

$$= \left|-\frac{2}{\sqrt{6}}\times\frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}}\times\frac{1}{\sqrt{6}}-\frac{2}{\sqrt{6}}\times\frac{1}{\sqrt{6}}\right|$$

$$\Rightarrow \quad \cos\theta = \left|-\frac{2}{6}+\frac{1}{6}-\frac{2}{6}\right| = \left|-\frac{3}{6}\right|$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = 60° = \frac{\pi}{3}$$

5.Sol: Given points are $A(3,4,5), B(4,6,3)$, C (1, 2, 4), and $D(1,0,5)$. Now, DR's of DC = (-2, 2, -1)

DR's of $AB=(1,2,-2)$

Let θ be the angle between *AB* and *DC*.

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}}$$

$$=\frac{-2\times1+2\times2-1\times(-2)}{\sqrt{(-2)^2+(2)^2+(-1)^2}\sqrt{(1)^2+(2)^2+(-2)^2}}$$

$$=\frac{-2+4+2}{\sqrt{4+4+1}\sqrt{1+4+4}}=\frac{4}{3\times3}=\frac{4}{9}$$

6.Sol: Since, angle between the lines whose direction ratios are (a_1,a_2,a_3) and (b_1,b_2,b_3), is given by θ, where

$$\theta=\cos^{-1}\left(\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}\right)$$

Given direction ratios are $(1,1,2)$ and $(\sqrt{3}-1,-\sqrt{3}-1,4)$.

Hence, angle between the lines, θ

$$=\cos^{-1}\left\{\frac{1\cdot(\sqrt{3}-1)+1\cdot(-\sqrt{3}-1)+2\cdot4}{\sqrt{1^2+1^2+2^2}\sqrt{\left(\sqrt{3}-1\right)^2+\left(-\sqrt{3}-1\right)^2+4^2}}\right\}$$

$$=\cos^{-1}\left\{\frac{\sqrt{3}-1-\sqrt{3}-1+8}{\sqrt{6}\sqrt{3+1+3+1+16}}\right\}=\cos^{-1}\left\{\frac{6}{\sqrt{6}\sqrt{24}}\right\}$$

$$=\cos^{-1}\left\{\sqrt{\frac{6}{24}}\right\}=\cos^{-1}\left(\frac{1}{2}\right)=\cos^{-1}\left(\cos\frac{\pi}{3}\right)=\frac{\pi}{3}$$

7.Sol: Required length $=\sqrt{(3)^2+(4)^2+(5)^2}$

$$=\sqrt{9+16+25}=\sqrt{50}=5\sqrt{2}$$

8.Sol: Let the points be $P=(4,3,-5)$ and $Q=(-2,1,-8)$.

Now, $|PQ|=\sqrt{(-2-4)^2+(1-3)^2+(-8+5)^2}$

$$=\sqrt{36+4+9}=\sqrt{49}=7$$

$\therefore$ DC's of line PQ are

$$l=\frac{x_2-x_1}{|PQ|},\ m=\frac{y_2-y_1}{|PQ|}\ \text{and}\ n=\frac{z_2-z_1}{|PQ|}$$

$$\therefore\quad l=\frac{2}{7},m=\frac{2}{7},n=\frac{3}{7}$$

9.Sol: DC's of line

$$=\left[\frac{12}{\sqrt{12^2+4^2+3^2}},\frac{4}{\sqrt{12^2+4^2+3^2}},\frac{3}{\sqrt{12^2+4^2+3^2}}\right]$$

$$=\left(\frac{12}{13},\frac{4}{13},\frac{3}{13}\right)$$

10.Sol: If (a_1,b_1,c_1) and (a_2,b_2,c_2) are direction ratios of two lines, then DC's of a perpendicular to both the lines are

$$\frac{b_1c_2-b_2c_1}{\sqrt{(b_1c_2-b_2c_1)^2+(a_2c_1-a_1c_2)^2+(a_1b_2-a_2b_1)^2}}$$

$$\frac{a_2c_1-a_1c_2}{\sqrt{(b_1c_2-b_2c_1)^2+(a_2c_1-a_1c_2)^2+(a_1b_2-a_2b_1)^2}}\ \text{and}$$

$$\frac{a_1b_2-a_2b_1}{\sqrt{(b_1c_2-b_2c_1)^2+(a_2c_1-a_1c_2)^2(b_1c_2-b_2c_1)^2+(a_2c_1-a_1c_2)^2}}$$

Putting the values of a_1,b_1,c_1 and a_2,b_2,c_2, we get $\frac{2}{3},-\frac{1}{3},\frac{2}{3}$

11.Sol: $l=m=n$ and $l^2+m^2+n^2=1$

$$\Rightarrow 3l^2=1\Rightarrow l=\pm\frac{1}{\sqrt{3}},m=\pm\frac{1}{\sqrt{3}},n=\pm\frac{1}{\sqrt{3}}$$

12.Sol: Rewrite the given equation as, say $300x+200y-600z=6300$, and thus have come to the conclusion that l is not 3 but 300, and similarly for m, n, and P. we have to remember that $l^2+m^2+n^2=1$. The numbers 3, 2, and -6 are not direction cosines but direction numbers.

Since $\sqrt{3^2+2^2+6^2}=7$, we divide. The given equations by 7 to obtain $\frac{3}{7}x+\frac{2}{7}y-\frac{6}{7}z=9$.

We may now make the identifications $l=\frac{3}{7}$,

$m=\frac{2}{7},n=\frac{6}{7}$, and therefore also, $p=9$.

13.Sol: $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

$\alpha=45°,\beta=120°$ put in equation (i)

$$\Rightarrow\ \frac{1}{2}+\frac{1}{4}+\cos^2\gamma=1\Rightarrow\cos^2\gamma=\frac{1}{4}\Rightarrow\gamma=60°$$

14.Sol: α, β, γ with coordinate axis

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ [By definition]

$\Rightarrow \quad 2\cos^2\alpha + 2\cos^2\beta + 2\cos^2\gamma = 2$

$\Rightarrow \quad 2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1 = 2 - 3$

$\Rightarrow \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0$

15.Sol: Here, $x_2 - x_1 = 6,\ y_2 - y_1 = 6,\ z_2 - z_1 = 4$ and d.c's of x, y, z -axis are (1,0,0), (0,1,0), (0,0,1) respectively.

Now, projection $= (x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n$

$\therefore$ Projections of line AB on co-ordinate axis are 6, 6, 4 respectively.

16.Sol: Let d be the length of line, then projection on x-axis = dl = 3 , projection on y-axis = dm = 4 and projection on z-axis = dn = 5.

Now, $d^2(l^2 + m^2 + n^2) = 50 \Rightarrow d^2(1) = 50$

$\Rightarrow \quad d = 5\sqrt{2}$

17.Sol: The given equations of lines are

$$2x = 3y = -z$$

$$\Rightarrow \quad \frac{x}{3} = \frac{y}{2} = \frac{z}{-6}$$

and $\quad 6x = -y = -4z$

$$\Rightarrow \quad \frac{x}{2} = \frac{y}{-12} = \frac{z}{-3}$$

Let θ be the angle between these two lines

$$\therefore \quad \cos\theta = \frac{(2)(3) + (2)(-12) + (-6)(-3)}{\sqrt{9+4+37}\ \sqrt{4+144+9}}$$

$$= \frac{6 - 24 + 18}{7\sqrt{157}} = 0 \quad \Rightarrow \quad \theta = 90^\circ$$

18.Sol: $l + m + n = 0,\ l^2 + m^2 - n^2 = 0$ and $l^2 + m^2 + n^2 = 1$.

Solving above equations, we get $m = \pm\dfrac{1}{\sqrt{2}}$, $n = \pm\dfrac{1}{\sqrt{2}}$ and $l = 0$.

$$\therefore \quad \theta = \frac{\pi}{3} \quad \text{or} \quad \frac{\pi}{2}$$

19.Sol: $$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$$\cos\theta = \left|\frac{(2)(3) + (3)(-4) + (-6)(5)}{\sqrt{2^2 + 3^2 + (-6)^2}\ \sqrt{3^2 + (-4)^2 + (5)^2}}\right|$$

$$\cos\theta = \frac{18\sqrt{2}}{35} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{18\sqrt{2}}{35}\right)$$

20.Sol:

D.r.'s of $AB \equiv (1,2,-2)$, D.r's of $CD \equiv (2,3,4)$

$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = 0$;

$\therefore \quad \cos\theta = 0 \quad \Rightarrow \quad \theta = \dfrac{\pi}{2}$

MATHEMATICS EXCEL

A Competitve Edge for JEE MAIN & ADVANCED

INTEGER ROOTS OF A QUADRATIC EQUATION

1. If $9z^2-30z+c$ is a perfect square for all integers z, what is the value of c?
(a) 10 (b) 9 (c) 5 (d) 25

2. Compute the number of positive integers a for which there exists an integer $b, 0 \le b \le 2002$, such that both of the polynomials x^2+ax+b and $x^2+ax+b+1$ have integer roots.
(a) 45 (b) 44 (c) 24 (d) 23

3. One of the roots of the equation $x^2+x+10=k$ $(k-1)$ is a positive integer. What is the sum of the possible integer values of k?
(a) -2 (b) 0 (c) 2 (d) 3

4. Find the number of integer values of n such that $x^2-6x-4n^2-32n=0$ has two integer roots.
(a) 1 (b) 2 (c) 3 (d) 4

5. Find the sum of all integer values of m such that the quadratic equation $x^2-(m-1)x+m+1=0$ has integer roots.
(a) 4 (b) 5 (c) 6 (d) 7

6. The zeros of the function $f(x)=x^2-px-580p$ are integers. What is the possible value of the prime number p?
(a) 29 (b) 30 (c) 31 (d) 39

7. The roots of the quadratic equation $x^2-2(m+1)x+m^2=0$ about x are integers. If $m^2-72m+720<0$, what is the sum of the possible integer values of m?
(a) 34 (b) 54 (c) 64 (d) 74

8. Find all possible values of integer r such that the following equation always has the two integer roots: $rx^2+(r+2)x+r-1=0$.
(a) 0 (b) 1 (c) 2 (d) 3

9. If m is an integer and $4<m<40$, find the greatest value of m such that $x^2-2(2m-3)x+4m^2-14m+8=0$ has integer roots.
(a) 12 (b) 24
(c) 36 (d) None of these

10. Find the sum of all posotive integers n for which $n^2-19n+99$ is a perfect square.
(a) 38 (b) 24 (c) 12 (d) 48

11. How many pairs of positive integers (a,b) are there such that GCD $(a,b)=1$ and $\frac{a}{b}+\frac{14b}{9a}$ is an integer ?
(a) 4 (b) 6 (c) 9 (d) 12

12. Compute the value of a such that the equation $(x-a)(x-8)-1=0$ has two integer roots.
(a) 8 (b) 24 (c) 16 (d) 12

13. The zeros of the function $f(x)=x^2-ax+2a$ are integers. What is the sum of the possible values of a?
(a) 7 (b) 8 (c) 16 (d) 17

14. Both roots of the equation $a^2x^2+ax+1-7a^2=0$ are integers. What is the sum of all possible positive values of a?
(a) 8 (b) $\frac{13}{6}$ (c) $\frac{11}{6}$ (d) 12

15. Find all possible values of rational number r such that the following equation always has the two integer roots: $rx^2+(r+2)x+r-1=0$.

(a) $-\frac{1}{3},-1$ (b) $\frac{1}{3},-1$ (c) $\frac{1}{3},1$ (d) $-\frac{1}{3},1$

16. For how many real numbers a does the quadratic equation $x^2+ax+6a=0$ have only integer roots for x ?

(a) 10 (b) 9 (c) 8 (d) 6

17. Compute the integer value of a such that the equation $x^2+(a-6)x+a=0$ has two integer roots. $a\neq 0$.

(a) 13 (b) 12 (c) 16 (d) 0

18. Find the sum of all integers n for which the quadratic equation $(x^2-x+1)n=-3x+1$ has integer solutions.

(a) -3 (b) -2 (c) -1 (d) 0

19. Find the sum of all integer values of m such that the quadratic equation $x^2+(m-17)x+m-2=0$ has two positive integer roots.

(a) 20 (b) 19 (c) -19 (d) 0

20. How many prime numbers of p satisfy the equation $p^2+x^2-2px-5p=1$ with two integral solutions?

(a) 1 (b) 2 (c) 3 (d) 4

ANSWER KEY

1. a	2. b	3. c	4. d
5. c	6. a	7. c	8. b
9. b	10. a	11. a	12. a
13. c	14. b	15. d	16. a
17. c	18. c	19. a	20. b

HINTS & SOLUTIONS

1.Sol: Since $9z^2-30z+c$ is a perfect square and z is an integer, c must also be an integer.

The discriminant of the quadratic must be zero.

That is $\Delta=(-30)^2-4\times 9\times c=0\Rightarrow c=25$

2.Sol: The discriminant of each quadratic must be a perfect square if each is to be factorable over the set of integers.

$$a^2-4b=m^2 \quad (1)$$

$$a^2-4(b+1)=n^2 \quad (2)$$

for integers m and n

(2) - (1):

$m^2-n^2=(m+n)(m-n)=4$

Since $m+n$ and $m-n$ are of the same parity, we have

$$m+n=2 \quad (3)$$

$$m-n=2 \quad (4)$$

(3) - (4):

$2n=0 \Rightarrow n=0$

(2) becomes: $a^2=4(b+1)$

Since a is a positive integer, we have $a=2\sqrt{b+1}$.

Any perfect square value of b will result a positive integer a.

Therefore $\left\lfloor\sqrt{2002}\right\rfloor=44$ perfect square values of b.

3.Sol: For x to be an integer, the discriminant

$\Delta=(-1)^2-4\times[10-k(k-1)]=4k^2-4k-39$

must be a square number.

Let $4k^2-4k-39=n^2 \Rightarrow 4k^2-4k-39-n^2=0$

Since k is an integer, $\Delta_k=(-4)^2-4\times 4(-39-n^2)$

$=16(40+n^2)$ must be a square number, or

$\Delta_k=40+n^2$ must be a square number.

Let $40+n^2=m^2 \Rightarrow m^2-n^2=40$

$$m+n=20 \quad (1)$$

$$m-n=2 \quad (2)$$

Solving (1) and (2), we get $n=9$.

$$m+n=10 \quad (3)$$

$$m-n=4 \quad (4)$$

Solving (3) and (4), we get $n=3$.

Case 1: $n=9$

We have $4k^2-4k-39-9^2=0$

$\Rightarrow 4k^2-4k-120=0$

$\Rightarrow k^2-k-30=0 \Rightarrow (k+5)(k-6)=0$

So $k=6$ or $k=-5$.

For $k=6, x^2+x+10=k(k-1)$

$$\Rightarrow x^2+x+10=30$$

$$x^2+x-20=0 \Rightarrow (x+5)(x-4)=0$$

$$x_1=4 \text{ and } x_2=-5$$

For $k=-5$, we get the same results ($x_1=4$ and $x_2=-5$).

Case 2: $n=3$

We have

$$4k^2-4k-39-3^2=0 \Rightarrow 4k^2-4k-48=0$$

$$\Rightarrow k^2-k-12=0 \Rightarrow (k+3)(k-4)=0$$

So $k=4$ or $k=-3$

For

$$k=4, x^2+x+10=k(k-1) \Rightarrow x^2+x+10=12$$

$$\Rightarrow x^2+x-2=0 \Rightarrow (x-1)(x+2)=0$$

$$x_1=1 \text{ and } x_2=-2$$

For $k=-3$, we get the same results ($x_1=1$ and $x_2=-2$)

So the answer is $6-5+4-3=2$.

4.Sol: Method 1: Since the roots of the quadratic equation with respect to x are integers, the discriminant

$$\Delta=(-6)^2-4(-4n^2-32n)=2^2(4n^2+32n+9)$$

must be a square number, or $4n^2+32n+9$ must be a square number.

Let $4n^2+32n+9=m^2$, where $m>0$.

Factoring gives us $(2n+8+m)(2n+8-m)=55$.

Since $55=1\times 55=5\times 11=(-1)\times(-55)=(-5)\times(-11)$, the values of n are $n_1=10$, $n_2=0$, $n_3=-18$, $n_4=-8$.

It is important to check all four values of n because $-6\pm\sqrt{\Delta}$ must be divisible by 2 so that the two roots are integers.

When $n=10$, $x^2-6x-4n^2-32n=0$ becomes $x^2-6x-720=0$. The two roots are

$$\frac{-6\pm\sqrt{36+4\times 720}}{2}=\frac{-6\pm 54}{2}=24,-30.$$

Similarly, for $n_2=0, n_3=-18, n_4=-8$, the roots are also integers.

There are 4 values of n.

Method 2:

Since the equation's roots are integers, the discriminant of the quadratic

$$\Delta=(-6)^2-4(-4n^2-32n)=2^2(4n^2+32n+9)$$

must be a square number, or $4n^2+32n+9$ must be a square number.

Let $4n^2+32n+9=m^2$, where $m>0$

$$\Rightarrow 4n^2+32n+9-m^2=0$$

Since n is an integer, the discriminant of $4n^2+32n+9-m^2=0$ with respect to n must be a square number:

$\Delta=32^2-4\times 4\times(9-m^2)=4^2(m^2+55)$ or m^2+55 must be a square number.

Let $m^2+55=s^2$, where $s>0$.

This equation can be rewritten as

$$s^2-m^2=55 \Rightarrow (s-m)(s+m)=1\times 55=5\times 11$$

$$=(-1)(-55)=(-5)(-11)$$

So m has 4 values: 3, -7, 27, -27. Since $m>0$, we can rule out -7 and -27, leaving $m=27$ or $m=3$.

When $m=3$, $4n^2+32n+9-3^2=0$

$$\Rightarrow 4n^2+32n=0 \Rightarrow 4n(n+8)=0$$

$$n=0 \text{ or } n=-8.$$

When $m=27$,

$$4n^2+32n+9-27^2=0 \Rightarrow 4n^2+32n-720=0$$

$$\Rightarrow 4n(n+8)=0$$

$$n=10 \text{ or } n=-18.$$

There are 4 values of n.

5.Sol: Since the equation has integer roots,

$\Delta=(m-1)^2-4(m+1)=m^2-6m-3$ must be a square number.

So we have $m^2-6m-3=k^2$, where k is an integer.

Now we have two ways to solve the problem.

Method 1:

We write $m^2-6m-3=k^2$ as

$m^2-2(3)k+3^2-3^2-3=k^2 \Rightarrow (m-3)^2-12=k^2$

$\Rightarrow (m-3)^2-k^2=12 \Rightarrow (m-3-k)(m-3+k)$

$=12=2\times 6$

We know that $(m-3+k)\geq(m-3-k)$ and two terms have the same parity.

So we have

1. $\begin{cases} m-3+k=6 \\ m-3-k=2 \end{cases}$ 2. $\begin{cases} m-3+k=-2 \\ m-3-k=-6 \end{cases}$

Solving we get: $m=7$ or $m=-1$.

The answer is $7-1=6$.

Method 2: We write $m^2-6m-3=k^2$ as

$m^2-6m-3-k^2=0$.

Since m is an integer,

$\Delta_m=(-6)^2-4(-3-k^2)=48+4k^2=4(12+k^2)$

must be a square number, or $12+k^2$ must be a square number. So we have $12+k^2=t^2$ (t is a positive integer).

Or $t^2-k^2=12 \Rightarrow (t-k)(t+k)=12$

We know that $(t+k)>(t-k)$ and two terms have the same parity.

We have

1. $\begin{cases} t+k=6 \\ t-k=2 \end{cases}$ 2. $\begin{cases} t+k=-2 \\ t-k=-6 \end{cases}$

Solving we get: $k=2$.

$m^2-6m-3=2^2 \Rightarrow m^2-6m-7=0$

$\Rightarrow (m+1)(m-7)=0$

Solving we get: $m=7$ or $m=-1$.

The answer is $7-1=6$.

6.Sol: Since the zeroes are integers, the discriminant of the quadratic

$$\Delta=(-p)^2-4(-580p)=p^2+4\times 580p$$

$$=p^2\left(1+\frac{4\times 580}{p}\right)$$

must be a square number, or p must be a factor of

$$4\times 580=2^4\times 5\times 29.$$

Since p is a prime number, we know that $p=2,5$, or 29.

We checked and only when $p=29$, the zeroes of the function $f(x)=x^2-px-580p$ are integers.

7.Sol: In order for the quadratic $x^2-2(m+1)x+m^2=0$ to have integer roots, its discriminant must be a square number.

In other words, $\Delta=[2(m+1)]^2-4m^2=4(2m+1)$ is a square number, or $2m+1$ is a square number.

From $m^2-72m+720<0$

$\Rightarrow (m-12)(m-60)<0$

$\Rightarrow 12<m<60$

Thus $5^2<2m+1<11^2$

Since $2m+1$ is odd, it can only be 49 or 81.

So $m=24$ or 40.

When $m=24$, the two roots of the quadratic equation are 32, and 18.

When $m=40$, the two roots of the quadratic equation are 50, and 32.

The answer is $24+40=64$.

8.Sol: **Method 1:** If $r=0$, the given equation can be written as $2x-1=0 \Rightarrow x=\frac{1}{2}$ (ignored since x is not an integer).

If $r\neq 0$, let two roots be x_1 and $x_2 (x_1\leq x_2)$.

$$x_1+x_2=-\frac{r+2}{r}$$

$$x_1x_2=\frac{r-1}{r}$$

Then $2x_1x_2-(x_1+x_2)=2\left(\frac{r-1}{r}\right)+\frac{r+2}{r}=3$

$\Rightarrow 4x_1x_2-2(x_1+x_2)+1=7$

$\Rightarrow (2x_1-1)(2x_2-1)=7$

Since x_1 and x_2 are integers with $x_1\leq x_2$, we get

$$\begin{cases}(2x_1-1)=1\\(2x_2-1)=7\end{cases} \Rightarrow \begin{matrix}x_1=1\\x_2=4\end{matrix}$$

So $x_1x_2=\dfrac{r-1}{r}=4 \Rightarrow r=-\dfrac{1}{3}$ (ignored since it is not an integer)

$$\begin{cases}(2x_1-1)=-7\\(2x_2-1)=-1\end{cases} \begin{matrix}x_1=-3\\x_2=0\end{matrix}$$

So $x_1x_2=\dfrac{r-1}{r}=0 \Rightarrow r=1$

So the answer is $r=1$.

Method 2:

If $r=0$, the given equation can be written as

$2x-1=0 \Rightarrow x=\dfrac{1}{2}$ (ignored since x is not an integer).

If $r\neq 0$, the discriminant of the given equation can be written as $\Delta_x=[-(r+2)]^2-4r(r-1)=-3r^2+8r+4$ and it is a square number.

Let $-3r^2+8r+4=n^2 \Rightarrow 3r^2-8r-4+n^2=0$

Since r is rational, $\Delta_r=(-8)^2-4\times 3(n^2-4)$ $=4(28-3n^2)$ must also be a square number, or $28-3n^2$ must be a square number.

Since $4(28-3n^2)\geq 0$, we have $3n^2\leq 28 \Rightarrow n^2 \leq \dfrac{28}{3}$.

So n^2 can be $=9,4,1,$ or 0. We see that if $n^2=0$, $4(28-3n^2)$ is not a square number.

So n^2 can only be 9, 4, or 1.

Case 1:

When $n^2=9,\ 3r^2-8r-4+n^2=0$

$\Rightarrow 3r^2-8r-4+9=0 \Rightarrow 3r^2-8r+5=0$

$\Rightarrow (3r-5)(r-1)=0$

Solving we get $r=\dfrac{5}{3}$ (ignored) or $r=1$.

For $r=1,\ rx^2+(r+2)x+r-1=0 \Rightarrow x^2+3x=0$

$x=0$ or $x=-3$.

Case 2:

When $n^2=4,\ 3r^2-8r-4+n^2=0$

$\Rightarrow 3r^2-8r-4+4=0 \Rightarrow 3r^2-8r=0$

$\Rightarrow r(3r-8)=0$

Solving we get $r=\dfrac{8}{3}$ or $r=0$ (both values are ignored).

Case 3: When $n^2=1,\ 3r^2-8r-4+n^2=0$

$\Rightarrow 3r^2-8r-4+1=0 \Rightarrow 3r^2-8r-3=0$

$\Rightarrow (3r+1)(r-3)=0$

Solving we get $r=-\dfrac{1}{3}$ (ignored) or $r=3$.

For $r=3,\ rx^2+(r+2)x+r-1=0$

$\Rightarrow 3x^2+(3+2)x+3-1=0 \Rightarrow 3x^2+5x+2=0$

$\Rightarrow (3x+2)(x+1)=0$

$x=-\dfrac{2}{3}$ or $x=-1$ (not all integer thus ignored)

So the answer is $r=1$.

9.Sol: Since the equation has integer roots, the discriminant should be a square number.

$\Delta=[-2(2m-3)]^2-4(4m^2-14m+8)$

$=4(2m+1)$

That is, $2m+1$ is a square number.

Since

$4<m<40,\ 8<2m<80 \Rightarrow 3^2<2m+1<9^2$

$3^2<2m+1$ can be $4^2,5^2,6^2,7^2,8^2$. Since $2m+1$ is an odd number, only 5^2 and 7^2 will work. So $2m+1=5^2$ or $2m+1=7^2$.

Thus $m=12$ or 24. The greatest value is 24.

We can check that when $m=12$, the two roots are 16 and 26; when $m=24$, the two roots are 38 and 52.

10.Sol: **Method 1:** If $n^2-19n+99=m^2$ for positive integers m and n, then $4m^2=4n^2-76n+396$

$=(2n-19)^2+35$. Thus $4m^2=(2n-19)^2+35$, or $(2m+2n-19)(2m-2n+19)=35$.

The sum of two factors is 4*m*, a positive integer, so the pair $(2m+2n-19,\ 2m-2n+19)$ can only be (1,35), (5,7), (7,5), or (35,1).

Subtract the second factor from the first to discover that $4n-38$ can be only -34, -2, 2, or 34, from which it follows that *n* can only be 1,9,10, or 18. The sum is 38.

Method 2:

Let $n^2-19n+99=m^2\ (m>0)$

Rewrite $n^2-19n+99=m^2$ as $n^2-19n+99-m^2=0$.

Since *n* is a positive integer, the discriminant of the quadratic equation with respect to $n, \Delta=(-19)^2-4\times(99-m^2)=4m^2-35$ must be a perfect square.

Let $4m^2-35=s^2\ (s>0)$.

$$4m^2-s^2=35 \Rightarrow (2m-s)(2m+s)=1\times 35=5\times 7$$

So $m=3$ or $m=9$

When $m=3,\ n^2-19n+99=3^2$

$$\Rightarrow n^2-19n+90=0$$

$$\Rightarrow (n-10)(n-9)=0$$

$n=9$ or $n=10$.

When $m=9,\ n^2-19n+99=9^2$

$$\Rightarrow n^2-19n+18=0$$

$$\Rightarrow (n-18)(n-1)=0$$

$n=18$ or $n=1$

The sum of all positive integers *n* is $9+10+18+1=38$.

11.Sol: Let $x=\frac{a}{b}$. The problem becomes equivalent to finding all the positive rational numbers *x* such that $x+\frac{14}{9x}=n$ for some integer *n*.

This equation can be rewritten into the quadratic equation $9x^2-9xn+14=0$, whose discriminant must be a square number in order for the root *x* to be a rational number.

$\Delta=(-9n)^2-4\times 9\times 14=m^2 \Rightarrow 9n^2-4\times 14=m^2$

$\Rightarrow 9n^2-m^2=2^3\times 7$

$\Rightarrow (3n-m)(3n+m)=2^3\times 7$

We know that $3n-m$ and $3n+m$ must both either be even or odd, and since their product is even, both should be even.

$3n-m$	$3n+m$
2	$2^2\times 7$
2^2	2×7

This gives us two pairs on *n* and *m*: (5,13) and (3, 5). Plugging them into the original quadratic $9x^2-9xn+14=0$ and solving for *x* gives us

$9x^2-9xn+14=0 \Rightarrow 9x^2-45x+14=0$

$$\Rightarrow x=\frac{14}{3} \text{ or } x=\frac{1}{3}$$

$9x^2-9xn+14=0 \Rightarrow 9x^2-27x+14=0$

$$\Rightarrow x=\frac{7}{3} \text{ or } x=\frac{2}{3}.$$

Therefore there are four pairs (*a*,*b*) that satisfy the given conditions, namely (1, 3), (2, 3), (7, 3), and (14, 3).

12.Sol: Method 1:

The given equation can be written as

$$x^2-(a+8)x+8a-1=0$$

Let x_1 and x_2 be the two integer roots.

Since $x_1+x_2=a+8,\ a$ must be an integer. Thus both $(x-a)$ and $(x-8)$ are integers.

So $(x-a)=(x-8)=(\pm 1)$. Thus $a=8$.

Method 2:

The given equation can be written as

$$x^2-(a+8)x+8a-1=0$$

Since $x_1+x_2=a+8$, *a* must be an integer.

Since the quadratic has two integer solutions, the discriminant must be a perfect square.

$[-(a-8)]^2-4(8a-1)=(a^2-16a+68)$

$$= (a-8)^2 + 4 = n^2$$

Or $(a-8)^2 + 4 = n^2 - (a-8)^2 = 4$

$\Rightarrow \quad (n+a-8)\,(n-a-8) = 4$

Since $(n+a-8)$ and $(n-a-8)$ are of the same parity, we have

$n+a-8=2$ (1)

$n-a-8=2$ (2)

(1) - (2): $2a=16 \quad \Rightarrow \quad a=8$

13.Sol: Method 1:

For x to be an integer, the discriminant $\Delta = (-a)^2$

$-4\times 2a = a^2 - 8a$ must be a square number.

$a^2 - 8a = n^2$ or $a^2 - 8a - n^2 = 0$

The discriminant $\Delta_a = (-8)^2 - 4\times 1(-n^2)$

$= 4\,(16+n^2)$ must be a square number, or

$$16 + n^2 = m^2 \,.$$

We then have $m^2 - n^2 = 16 \Rightarrow (m-n)(m+n) = 16$

Since $m-n$ and $m+n$ have the same parity and $m-n < m+n$, we have

$$\begin{cases} m-n=2 \\ m+n=8 \end{cases} \Rightarrow \quad n=3$$

$$\begin{cases} m-n=4 \\ m+n=4 \end{cases} \Rightarrow \quad n=0$$

$$\begin{cases} m-n=-8 \\ m+n=-2 \end{cases} \Rightarrow \quad n=3$$

$$\begin{cases} m-n=-4 \\ m+n=-4 \end{cases} \Rightarrow \quad n=0$$

When $n=0$, we have $a^2 - 8a - 0^2 = 0$

$\Rightarrow \quad a\,(a-8) = 0$

The solutions are $a=0$ or $a=8$

When $n=3$, we have $a^2 - 8a - 3^2 = 0$

$\Rightarrow \quad (a+1)\,(a-9) = 0$

The solutions are $a=-1$ or $a=9$

$$0+8-1+9=16$$

The answer is (c)

Method 2:

For x to be an integer, the discriminant $\Delta = (-a)^2$ $-4\times 2a = a^2 - 8a$ must be a square number.

In the expression $a^2 - 8a = n^2$ or $a^2 - 8a - n^2 = 0$

$\Rightarrow \quad a^2 - 2\times 4a + 4^2 - 4^2 - n^2 = 0$

$\Rightarrow \quad (a-4)^2 - n^2 = 16$

$\Rightarrow \quad (a-4a-n)(a-4a+n) = 16$

Since $a-4-n$ and $a-4+n$ have the same parity and $a-4-n \le a-4+n$, we have

$$\begin{cases} a-4-n=2 \\ a-4+n=8 \end{cases} \Rightarrow \quad a=9$$

$$\begin{cases} a-4-n=4 \\ a-4+n=4 \end{cases} \Rightarrow \quad a=8$$

$$\begin{cases} a-4-n=-8 \\ a-4+n=-2 \end{cases} \Rightarrow \quad a=-1$$

$$\begin{cases} a-4-n=-4 \\ a-4+n=-4 \end{cases} \Rightarrow \quad a=0$$

$9+8-1+90=16$

The answer is c.

14.Sol: Let x_1 and x_2 be the two integer roots.

We know by Vieta's Theorem that $x_1 + x_2 = -\dfrac{a}{a^2}$

$= -\dfrac{1}{a}$ must be an integer.

Let $\dfrac{1}{a} = n$, where n is a positive integer since a is positive.

The original equation becomes: $x^2 + nx + n^2 - 7 = 0$

For x to be an integer, the discriminant $\Delta = (-n)^2$ $-4\times(n^2-7) = -3n^2 + 28$ must be a square number and also $-3n^2 + 28 \ge 0$ or $n^2 \le \dfrac{28}{3}$.

So n^2 could be 9, 4, 1, or 0.

Only when $n^2 = 9,\, 4$, or 1 will make $-3n^2 + 28$ a square number.

When $n^2=9, x^2+nx+n^2-7=0$

$\Rightarrow x^2+3x+9-7=0 \Rightarrow x^2+3x+2=0$

$\Rightarrow \quad (x+1)(x+2)=0$

$x_1=-1$ and $x_2=-2$

When $n^2=4, x^2+nx+n^2-7=0$

$\Rightarrow x^2+2x+4-7=0 \Rightarrow x^2+2x-3=0$

$\Rightarrow \quad (x-1)(x+3)=0$

$x_1=1$ and $x_2=-3$

When $n^2=1, x^2+nx+n^2-7=0$

$\Rightarrow x^2+x+1-7=0 \Rightarrow x^2+x-6=0$

$\Rightarrow \quad (x-2)(x+3)=0$

$x_1=2$ and $x_2=-3$

So $a=1,\frac{1}{2},\frac{1}{3}$

The answer is $1+\frac{1}{2}+\frac{1}{3}=\frac{11}{6}$

15.Sol: $r=-\frac{1}{3}$ or $r=1$

Method 1:

If $r=0$, the given equation can be written as

$2x-1=0 \Rightarrow x=\frac{1}{2}$ (ignored since x is not an integer).

If $r\neq 0$, let two roots be x_1 and x_2 $(x_1\leq x_2)$.

$$x_1+x_2=-\frac{r+2}{r}$$

$$x_1x_2=\frac{r-1}{r}$$

Then $2x_1x_2-(x_1+x_2)=2\left(\frac{r-1}{r}\right)+\frac{r+2}{r}=3$

$\Rightarrow \quad 4x_1x_2-2(x_1+x_2)+1=7$

$\Rightarrow \quad (2x_1-1)(2x_2-1)=7$

Since x_1 and x_2 are integers with $x_1\leq x_2$, we get

$$\begin{cases}(2x_1-1)=1\\(2x_2-1)=7\end{cases} \Rightarrow \begin{matrix}x_1=1\\x_2=4\end{matrix}$$

So $x_1x_2=\frac{r-1}{r}=4 \Rightarrow r=-\frac{1}{3}$

$$\begin{cases}(2x_1-1)=-7\\(2x_2-1)=-1\end{cases} \quad \begin{matrix}x_1=-3\\x_2=0\end{matrix}$$

So $x_1x_2=\frac{r-1}{r}=0 \Rightarrow r=1$

So the answer is $r=-\frac{1}{3}$ or $r=1$

Method 2:

If $r=0$, the given equation can be written as

$2x-1=0 \Rightarrow x=\frac{1}{2}$ (ignored since x is not an integer).

If $r\neq 0$, let two roots be x_1 and x_2 $(x_1\leq x_2)$

We know by Vieta's Theorem that

$$x_1+x_2=-\frac{r+2}{r}=-1-\frac{2}{r}$$

Since x_1+x_2 is an integer, $\frac{2}{r}$ must be an integer as well.

Let $\frac{2}{r}=n \Rightarrow r=\frac{2}{n}$, where n is an integer.

The original equation becomes:

$$\frac{2}{n}x^2+\left(\frac{2}{n}+2\right)x+\frac{2}{n}-1=0$$

$\Rightarrow \quad 2x^2+(2+2n)x+2-n=0$

For x to be an integer, the discriminant

$\Delta=(2n+2)^2-4\times 2(2-n)=4(n^2+4n-3)$ must be a square number and also $n^2+4n+4-7$

$=(n+2)^2-7$ must be a square number.

Let $(n+2)^2-7=m^2 \Rightarrow (n+2)^2-m^2=7$

Thus we have

$$\begin{cases}(n+2-m)=1\\(n+2+m)=7\end{cases} \Rightarrow n=2$$

$$\begin{cases}(n+2-m)=-7\\(n+2+m)=-1\end{cases} \Rightarrow n=-6$$

When $n=2, r=\frac{2}{n}=1$. The two roots are $x_1=-3$ and $x_2=0$.

When $n=-6, r=\frac{2}{n}=-\frac{1}{3}$. The two roots of the quadratic equation are $x_1=-3$ and $x_2=0$.

16.Sol: Let two integer roots be x_1 and $x_2 (x_1 \le x_2)$. We know by Vieta's Theorem that

$$x_1+x_2=-\frac{a}{1}=-a$$

Since x_1+x_2 is an integer, a must be an integer as well.

For x to be an integer, the discriminant $\Delta=a^2-4\times 6a=a^2-24a=(a-12)^2-144$ must be a square number.

Let $(a-12)^2-144=m^2 \Rightarrow (a-12)^2-m^2=144$

$$(a-12-m)(a-12+m)=144$$

Since $a-12+m \ge a-12-m$ and they have the same parity, they both must be even.

Thus we have

$$\begin{cases} a-12-m=2 \\ a-12+m=72 \end{cases} \Rightarrow a=49$$

$$\begin{cases} a-12-m=4 \\ a-12+m=36 \end{cases} \Rightarrow a=32$$

$$\begin{cases} a-12-m=6 \\ a-12+m=24 \end{cases} \Rightarrow a=27$$

$$\begin{cases} a-12-m=8 \\ a-12+m=18 \end{cases} \Rightarrow a=25$$

$$\begin{cases} a-12-m=12 \\ a-12+m=12 \end{cases} \Rightarrow a=24$$

$$\begin{cases} a-12-m=-72 \\ a-12+m=-2 \end{cases} \Rightarrow a=-25$$

$$\begin{cases} a-12-m=-36 \\ a-12+m=-4 \end{cases} \Rightarrow a=-8$$

$$\begin{cases} a-12-m=-24 \\ a-12+m=-6 \end{cases} \Rightarrow a=-3$$

$$\begin{cases} a-12-m=-18 \\ a-12+m=-8 \end{cases} \Rightarrow a=-1$$

$$\begin{cases} a-12-m=-12 \\ a-12+m=-12 \end{cases} \Rightarrow a=0$$

It is not hard to see that the pair of integer solutions with the above values of a : (-42, -7), (-24, -8), (-18, -9), (-15, -10), (-12, -12), (-5, 30), (-4, 12), (-3, 6), (-2, 3), (0, 0).

17.Sol: For x to be an integer, the discriminant

$\Delta=(a-6)^2-4a=a^2-12a+36-4a$

$=a^2-16a+8^2-8^2+36=(a-8)^2-28$

must be a square number

Let $(a-8)^2-28=n^2 \Rightarrow (a-8)^2-n^2=28$

$\Rightarrow (a-8-n)(a-8+n)=28$

Since $(a-8-n) \ge (a-8+n)$ and of the parity, we have

$(a-8-n)=2$ (1)

$(a-8+n)=14$ (2)

Solving (1) and (2), we get $a=16$

$(a-8-n)=-14$ (3)

$(a-8+n)=-2$ (4)

Solving (3) and (4), we get $a=0$ (ignored since $a \ne 0$).

The answer is 16.

18.Sol: We re-write the equation $(x^2-x+1)n=-3x+1$ as $nx^2+(3-n)x+n-1=0 (n \ne 0)$.

Since the quadratic equation has integer solutions, $\Delta=(3-n)^2-4\times n\times(n-1)=9-3n^2-2n$ must be a square number.

Let $m^2=9-3n^2-2n \Rightarrow 3n^2+2n+m^2-9=0$

$\Delta_n=2^2-4\times 3\times(m^2-9)=4(28-3m^2)$ must be a square number, or $28-3m^2$ must be a squre number.

We see that when $m=\pm 1, \pm 2, \pm 3$, $28-3m^2$ is a square number.

$$3n^2+2n+(\pm1)^2-9=0 \Rightarrow 3n^2+2n-8=0$$
$$\Rightarrow (3n-4)(n+2)=0$$

Since n is an integer, $n=-2$

$$3n^2+2n+(\pm2)^2-9=0 \Rightarrow 3n^2+2n-5=0$$
$$\Rightarrow (3n+5)(n-1)=0$$

Since n is an integer, $n=1$.

$$3n^2+2n+(\pm3)^2-9=0 \Rightarrow 3n^2+2n=0$$
$$\Rightarrow n(3n+2)=0$$

Since n is an integer, and $(n\neq 0)$, no solution for this equation.

We checked and when $n=-2$ or $n=1$, the equation has integer solution: 1 and 2, or -2.

The answer is $-2+1=-1$.

19.Sol: By vieta's Theorem,

$$x_1+x_2=-(m-17)>0 \Rightarrow (m-17)<0 \Rightarrow m<17$$
$$x_1x_2=m-2>0 \Rightarrow m>2$$

Thus $2<m<17$

Since the equation has integer roots, $\Delta=(m-17)^2-4(m-2)=m^2-38m+297$ must be a square number.

So we have $m^2-38m+297=n^2$, where n is a positive integer.

Now we have two wyas to solve the problem.

Method 1:

We write $m^2-38m+297=n^2$ as $(m-19)^2-n^2=64 \Rightarrow (m-19-k)(m-19+k)=64$

We know that $(m-19+n)>(m-19-n)$ and two terms have the same parity.

So we have

1. $\begin{cases} m-19+n=16 \\ m-19-n=4 \end{cases}$ 2. $\begin{cases} m-19+n=32 \\ m-19-n=2 \end{cases}$

1. $\begin{cases} m-19+n=32 \\ m-19-n=2 \end{cases}$ 2. $\begin{cases} m-19+n=16 \\ m-19-n=4 \end{cases}$

3. $\begin{cases} m-19+n=8 \\ m-19-n=8 \end{cases}$ 4. $\begin{cases} m-19+n=-2 \\ m-19-n=-32 \end{cases}$

5. $\begin{cases} m-19+n=-4 \\ m-19-n=-16 \end{cases}$ 6. $\begin{cases} m-19+n=-8 \\ m-19-n=-8 \end{cases}$

Solving we get: $m=36,29,27,2,9$, and 11.

Since $2<m<17$, $m=9$ and 11.

We checked and with $m=9, x_1=1$ and $x_2=7$; with $m=11, x_1=3$ and $x_2=3$.

The answer is $9+11=20$.

Method 2:

We write $m^2-38m+297=n^2$ as $m^2-38m+297-n^2=0$.

Since m is an integer, $\Delta_m=(-38)^2-4(297-n^2)=256+4n^2=4(64+n^2)$ must be a square number, or $64+n^2$ must be a square number. So we have $64+n^2=t^2$ (t is a positive integer).

Or $t^2-n^2=64 \Rightarrow (t-n)(t+n)=64$.

We know that $(t+k)\geq(t-k)$ and two terms have the same parity.

We have

1. $\begin{cases} t+n=32 \\ t-n=2 \end{cases}$ 2. $\begin{cases} t+n=16 \\ t-n=4 \end{cases}$

3. $\begin{cases} t+n=8 \\ t-n=8 \end{cases}$ 4. $\begin{cases} t+n=-2 \\ t-n=-32 \end{cases}$

5. $\begin{cases} t+n=-4 \\ t-n=-16 \end{cases}$ 6. $\begin{cases} t+n=-8 \\ t-n=-8 \end{cases}$

Solving we get: $n=15$, 6, and 0.

If $n=15$, $m^2-38m+297-15^2=0$

$$\Rightarrow m^2-38m+72=0 \Rightarrow (m-36)(m-2)=0$$

Solving we get: $m=2$ or $m=36$

If $n=6$, $m^2-38m+297-6^2=0$

$$\Rightarrow m^2-38m+261=0 \Rightarrow (m-29)(m-9)=0$$

Solving we get: $m=9$ or $m=29$

If $n=0$, $m^2-38m+297-0^2=0$

$$\Rightarrow m^2-38m+297=0 \Rightarrow (m-27)(m-11)=0$$

Solving we get: $m=27$ or $m=11$

Since $2 < m < 17, m = 9$ and 11.

We checked and with $m = 9, x_1 = 1$ and $x_2 = 7$; with $m = 11, x_1 = 3$ and $x_2 = 3$.

The answer is $9 + 11 = 20$.

20.Sol: The given equation can be written as

$$x^2 - 2px + (p^2 - 5p - 1) = 0$$

Since the equation has two integral solutions, $\Delta = (-2p)^2 - 4(p^2 - 5p - 1)$ must be a square number.

$4p^2 - 4p^2 + 20p + 4 = 20p + 4 = 4(5p + 1)$

So $5p + 1$ must be a square number.

So we have $5p + 1 = n^2$, where n is a positive integer.

So $n^2 - 1 = 5p \Rightarrow (n-1)(n+1) = 5p$

We know that $p \geq 2$. So $n \geq 4$.

So we have

1. $\begin{cases} n-1=1 \\ n+1=5p \end{cases}$ 2. $\begin{cases} n-1=5 \\ n+1=p \end{cases}$ 3. $\begin{cases} n-1=p \\ n+1=5 \end{cases}$

Solving we get

$\begin{cases} n=2 \\ p=3/5 \end{cases}$ (ignored) $\begin{cases} n=6 \\ p=7 \end{cases}$ $\begin{cases} n=4 \\ p=3 \end{cases}$

When $p = 3$, the original equation becomes

$3^2 + x^2 - 2 \times 3x - 5 \times 3 = 1 \Rightarrow x^2 - 6x - 7 = 0$

$\Rightarrow (x-7)(x+1) = 0$

The solutions are $x = 7$ and $x = -1$.

When $p = 7$, the original equation becomes

$7^2 + x^2 - 2 \times 7x - 5 \times 7 = 1 \Rightarrow x^2 - 14x + 13 = 0$

$\Rightarrow (x-13)(x-1) = 0$

The solutions are $x = 13$ and $x = 1$

Thus the answer is 2.

RECREATIONAL MATHS

Gregory, Adam and Paul are athletes who competed in the downhill skiing event in the winter olympics. Gregory, Adam and Paul each finished in first, second or third. There were no ties. Each athlete is also from a different country. One is from Canada, one is from France and one is from Japan. Using the following clues, determine who placed first, second and third, and for which country each athlete was competing.

(1) Gregory was faster than Adam.

(2) Gregory is not canadian, and he did not finish in second place,

(3) The Japanese athlete was faster than the French athlete,

(4) Adam is not Japanese and he did not finish in third place,

(5) The canadian athlete was faster than French athlete.

Solution to the above problem will be published in the next month issue.

Previous years JEE MAIN Questions

RELATIONS & FUNCTIONS

[ONLINE QUESTIONS]

1. Let $f(x)=2^{10}x+1$ and $g(x)=3^{10}x-1$, if $(fog)(x)=x$, then x is equal to

[2017]

(a) $\frac{2^{10}-1}{2^{10}-3^{-10}}$ (b) $\frac{3^{10}-1}{3^{10}-2^{-10}}$

(c) $\frac{1-2^{10}}{3^{10}-2^{-10}}$ (d) $\frac{1-3^{-10}}{2^{10}-3^{-10}}$

2. The function $f:N\to N$ is defined by $f(x)=x-5\left[\frac{x}{5}\right]$, where N is the set of natural numbers and [x] denotes the greatest integer less than or equal to x, is

[2017]

(a) One-one but not onto
(b) One-one and onto
(c) Neither one-one nor onto
(d) Onto but not one-one

3. For $x\in R, x\neq 0, x\neq 1$, let $f_0(x)=\frac{1}{1-x}$ and $f_{n+1}(x)=f_0(f_n(X)), n=0,1,2,....$ T then the vlaue of $f_{100}(3)+f_1\left(\frac{2}{3}\right)+f_2\left(\frac{3}{2}\right)$ is equal to

[2016]

(a) $\frac{4}{3}$ (b) $\frac{1}{3}$ (c) $\frac{5}{3}$ (d) $\frac{8}{3}$

4. Let $A=\{x_1,x_2........,x_7\}$ and $B=\{y_1,y_2,y_3\}$ be two sets containing seven and three distinct elements respectively. Then the total number of functions $f:A\to B$ that are onto, if there exist exactly three elements x in A such that $f(x)=y_2$, is equal to :

[2015]

(a) $14.{}^7C_3$ (b) $16.{}^7C_3$ (c) $14.{}^7C_2$ (d) $12.{}^7C_2$

5. Let P be the relation defined on the set of all real numbers such that

$P=\{(a,b):\sec^2 a-\tan^2 b=1\}$ is:

[2014]

(a) Reflexive and transitive but not symmetric
(b) Reflexive and symmetric but not transitive
(c) Symmetric and transitive but not reflexive
(d) An equivalence relation

6. Let f be an odd function defined on the set of real numbers such that for $x\geq 0$.

$f(x)=3\sin x+4\cos x$. Then $f(x)$ at $x=-\frac{11\pi}{6}$ is equal to

[2014]

(a) $\frac{3}{2}-2\sqrt{3}$ (b) $\frac{3}{2}+2\sqrt{3}$

(c) $-\frac{3}{2}-2\sqrt{3}$ (d) $-\frac{3}{2}+2\sqrt{3}$

7. A relation on the set $A=\{x;|x|<3, x\in z\}$, when z is the set integer is defined by $R=\{(x,y:y=|x|, x\neq -1\}$. Then the number of elements in the power set of R is: **[2014]**
(a) 32 (b) 16 (c) 8 (d) 64

8. Let $f:R\to R$ be defined by $f(x)=\frac{|x|-1}{|x|+1}$ then f is: **[2014]**
(a) Both one-one and onto
(b) One-one but not onto
(c) Onto but not one-one
(d) Neither one-one nor onto

9. Let $A=\{1,2,3,4\}$ and $R:A\to A$. The correct relation defined by: $R=\{(1,1),(2,3),(3,4),(4,2)\}$. The correct statement is: **[2013]**
(a) R does not have an inverse
(b) R is not a one to one function
(c) R is an onto function
(d) R is not a function

10. Let $R=\{(3,3),(5,5),(9,9),(12,12),(5,12),(3,9),(3,12),$ $(3,12),(3,5)\}$ be a relation on the set $A=\{3,5,9,12\}$. Then R is **[2013]**
(a) Reflexive, symmetric but not transitive
(b) Symmetric, transitive but not reflexive
(c) An equivalence relation
(d) Reflexive, transitive but not symmetric

11. Let $R=\{(x,y):x,y\in n \text{ and } x^2-4xy+3y^2=0\}$, where n is the set of all natural numbers. Then the relation R is: **[2013]**
(a) Reflexive but neither symmetric nor transitive
(b) Symmetric and transitive
(c) Reflexive and symmetric
(d) Reflexive and transitive

12. Consider the function:
$f(x)=[x]+|1-x|, -1\le x\le 3$ where [x] is the greatest integer funciton.
Statement 1: f is not continuous at x = 0, 1, 2 and 3.

Statement 2: $f(x)=\begin{cases}-x & -1\le x<0\\ 1-x & 0\le x<1\\ 1+x & 1\le x<2\\ 2+x & 2\le x\le 3\end{cases}$

[2013]
(a) Statement 1 is true; Statement 2 is false,
(b) Statement 1 is true; Statement 2 is true; Statment 2 is not correct explanation for Statement 1.
(c) Statement 1 is true; Statement 2 is true; Statment it is a correct explanation for Statement 1.
(d) Statement 1 is false; Staement 2 is true.

[OFFLINE QUESTIONS]

1. The function $f:R\to\left[-\frac{1}{2},\frac{1}{2}\right]$ defined as $f(x)=\frac{x}{1+x^2}$, is: **[2017]**
(a) Neither injective nor surjective
(b) Invertible
(c) Injective but not surjective
(d) Surjective but not injective

2. If $f(x)+2f\left(\frac{1}{x}\right)=3x, x\neq 0$, and $S=\{x\in R: f(x)=f(-x)\}$; then S: [2016]
(a) Contains exactly one element
(b) Contains exaclty two elements
(c) Contains more than two elements
(d) Is an empty set

3. If $a\in R$ and the equation

$$-3(x-[x])^2+2(x-[x])+a^2=0$$

(where $[x]$ denotes the greatest integer $\le x$) has no integral solution, then all possible values of a lie in the interval: [2014]

(a) (-2,-1) (b) $(-\infty,-2)\cup(0,1)$

(c) $(-1,0)\cup(0,1)$ (d) (1,2)

ANSWER KEY

[ONLINE QUESTIONS]

1. c 2. c 3. c 4. a 5. d
6. a 7. b 8. d 9. c 10. d
11. a 12. a

[OFFLINE QUESTIONS]

1. d 2. b 3. c

HINTS & SOLUTIONS

[ONLINE QUESTIONS]

1.Sol: Given $f(x)=2^{10}x+1$

also $f(g(x))=x$

i.e., $f(3^{10}x-1)+1=x$

$\Rightarrow\ 2^{10}(3^{10}x-1)+1=x$

i.e., $6^{10}x-2^{10}+1=x$

$\left(6^{10}-1\right)x=2^{10}-1$

$\Rightarrow\ x=\dfrac{2^{10}-1}{6^{10}-1}$

i.e., $x=\dfrac{1-2^{-10}}{3^{10}-2^{-10}}$

2.Sol: Given $f(x)=x-5\left[\dfrac{x}{5}\right]$, where x ; $f(11)\,(-x)$

Now, $f(10)=10-5\left[\dfrac{10}{5}\right]=0\notin N$

$\therefore$ f is not onto

but $\left[\dfrac{x}{5}\right]$ is greatest integer function,

Therefore, $f(x)$ is many-one

3.Sol: $f_1(x)=f_0\left(f_0(x)\right)=\dfrac{1}{1-f_0(x)};f_0(x)\neq 1$

$$=\frac{1}{1-\dfrac{1}{1-x}}=\frac{1-x}{-x}\qquad x\neq 0$$

$$f_2(x)=f_0\left(f_1(x)\right)=\frac{1}{1-f_1(x)};f_1(x)\neq 1$$

$$=\frac{1}{1+\dfrac{1-x}{x}}=x$$

similarly

$f_3(x)=f_0(x)$

$f_4(x)=f_1(x)$ ....

$$f_{100}(3)+f_1\left(\frac{2}{3}\right)+f_2\left(\frac{3}{2}\right)=f_1(3)+f_1\left(\frac{2}{3}\right)+\frac{3}{2}$$

$$=1-\frac{1}{3}+1-\frac{3}{2}+\frac{3}{2}=\frac{5}{3}$$

4.Sol: Number of onto functions such that exactly three elements in $x\in A$ such that $f(x)=y_2$ is ${}^7c_3(2^4-2)\ =14\cdot{}^7c_3$

5.Sol: Given $P(a,b)=\sec^2 a-\tan^2 b=1$

if $P(a,a)=\sec^2 a-\tan^2 a=1,\forall a,b\in R$

Therefore, $P(a,b)$ is reflexive

Now, $P(a,b)=\sec^2 a-\tan^2 b=1$

$\Rightarrow 1+\tan^2 a-\sec^2 b-1=1$

i.e., $\tan^2 a-\sec^2 b=1=p(b,a)$

Therefore $P(a,b)$ is symmetric.

Finally, $P(a,b)=\sec^2 a-\tan^2 b=1$, and

$P(b,c)=\sec^2 b-\tan^2 c=1,\forall a,b,c\in R$

$$\Rightarrow\ \sec^2 a-\tan^2 c=\sec^2 a-\tan^2 b+\tan^2 b-\tan^2 c$$

$$=\left(\sec^2 a-\tan^2 b\right)+\sec^2 b-1-\tan^2 c$$

$$=\left(\sec^2 a-\tan^2 b\right)+\left(\sec^2 b-1-\tan^2 c\right)-1$$

Therefore, $P(a,c)$ is transitive.

6.Sol: Give $f(x)$ is odd function

i.e., $f(-x) = -f(x)$

$$\Rightarrow f\left(\frac{-11\pi}{6}\right) = -\left(3\sin\frac{11\pi}{6} + 4\cos\frac{11\pi}{6}\right)$$

$$= 3\sin\frac{\pi}{6} - 4\cos\frac{\pi}{6}$$

$$= 3\times\frac{1}{2} - \frac{4\sqrt{3}}{2}$$

$$= \frac{3}{2} - 2\sqrt{3}$$

7.Sol: Given $A = \{-2,-1,0,1,2\}$

and $R = \{(-2,2),(0,0),(1,1),(2,2)\}$

Total number of elements in the power set of R is

$$n(P(R)) = 2^4 = 16$$

8.Sol: Given $f(x) = \dfrac{|x|-1}{|x|+1}$

Rewriting the given function as

$$f(x) = \begin{cases} \dfrac{x-1}{x+1}; x \ge 0 \\ \dfrac{-(x+1)}{-x+1}; x < 0 \end{cases}$$

$f(x)$ is not one-one, since $f(1) = f(-1) = 0$

and $f(x) \ne 1$, Therefore it is not onto

Hence, $f(x)$ is neither one-one nor onto

12.Sol: Let $f(x) = [x] + |1-x|, -1 \le x \le 3$

Where $[x]$ = greatest integer function.

f is not continuous at $x = 0,1,2,3$

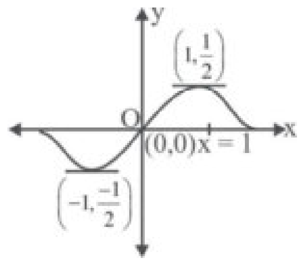
But in statement -2, *f(x)* is continuous at $x = 3$.
Hence, statement -1 is true and 2 is false.

[OFFLINE QUESTIONS]

1.Sol: Given $f(x) = \dfrac{x}{1+x^2}, \forall x \in R$

$$f'(x) = \frac{1-x^2}{(1+x)^2} < 0, \forall x \in R$$

Clearly from the graph, $f(x)$ is onto but not one-one.



2.Sol: Given that $f(x) + 2f\left(\dfrac{1}{x}\right) = 3x, x \ne 0$ (1)

$$\Rightarrow f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$$

solving 1 and 2, we get

$$f(x) = \frac{2}{x} - x$$

given $f(x) = f(-x)$

i.e., $\dfrac{2}{x} - x = \dfrac{-2}{x} + x$

$$\Rightarrow x^2 = 2$$

i.e., $x = \pm\sqrt{2}$

3.Sol: Method-1:

Given $-3(x-[x])^2 + 2(x-[x])^2 + a^2 = 0$

We know $x - [x] = \{x\}$

$$\Rightarrow -3\{x\}^2 + 2\{x\} + a^2 = 0$$

$$\Rightarrow \{x\} = \frac{-2 \pm \sqrt{4+12a^2}}{-6}$$

$$\{x\} = \frac{1 \pm \sqrt{1+3a^2}}{3}$$

We know $0 \le \{x\} < 1$

i.e., $0 \le \dfrac{1+\sqrt{1+3a^2}}{3} < 1$

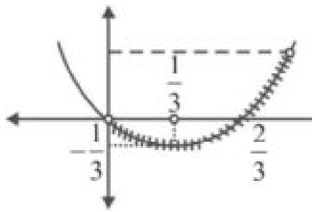
$$\Rightarrow -1 \le \sqrt{1+3a^2} < 2$$

i.e., $0 \le 3a^2 < 3$

i.e., $a \in (-1,1)$

Since the given equation has no integral solution, we get $a \in (-1,0) \cup (0,1)$

Method-2:



$a^2 = 3\{x\}^2 - 2\{x\}$ $\quad [\because x - [x] = \{x\}]$

Let $\{x\} = t$ $\quad \therefore t \in (0,1)$ $\quad$ As x is an integer

$\therefore \quad a^2 = 3t^2 - 2t \qquad f(t) = 3t\left(t - \frac{2}{3}\right)$

$\Rightarrow \quad a^2 = 3t\left(t - \frac{2}{3}\right)$

Clearly by graph $-\frac{2}{3} \le a^2 < 1$

$\therefore \quad a \in (-1,1) - \{0\}$ $\quad$ (As $x \ne$ integer)

Note: It should have been given that the solution exists else answer will be $a \in R - \{0\}$

VEDIC MATHEMATICS

SUMMATION OF SERIES

Below, we give Vedic method to find nth term and summation of n terms. The method is based on the Vedic sutra "**Sisyate Sesasamjnah**" means "**Remainders remains constant**". For explaining the method, let us give some example:

Example : Find nth term and sum of first n terms of the series: 2,5,8,11

The given series is in Arithmetic Progression (AP) and can be computed easily using the following formulas:

nth term = a + (n-1)d
= 2 + (n-1)3 = 2 + 3n – 3 = 3n-1
Sum = n/2 [2a + (n-1)d]
= n/2 [2 × 2 + (n-1)3]
= n × (3n+1)/2

In Vedic method, we compute the successive difference till we arrive constant term and then apply the following formulas:

nth term = First Diff + (n-1) × Second Diff + (n-1) × (n-2)/2 × Third Diff+ (n-1) × (n-2) × (n-3)/6 × Fourth Diff + ...

Summation = n × First Diff + n × (n-1)/2 × Second Diff+ n × (n-1) × (n-2)/6 × Third Diff+ n × (n-1) × (n-2) × (n-3)/24 × Fourth Diff + ...

Let us take the above series and apply the Vedic Method:

2 5 8 11
3 3 3

Very Important: Note that the difference 3 is constant in second line. As per VedicMethod, we need to compute the difference till we reach constant term.

First Diff is the first value in first line. The Second Diff is the first value in second line.
So, first Diff is 2 and second Diff is 3. Now, let us apply the vedic method:

nthterm = 2 + (n-1) × 3 = 3 × n – 1
Summation = n × 2 + n × (n-1) × 3/2 = n × (3n+1)/2

To arrive the same result using modern method, we need to do complicated computation. The formulas provided by Vedic Mathematics are easy to remember and have universal application.

Synopticglance

APPLICATIONS OF TRIGONOMETRY

Introduction

We know how to solve a right triangle: given two sides, or one side and one acute angle, we could find the remaining sides and angles. In each case we were actually given three pieces of information, since we already knew one angle was 90^o.

For a general triangle, which may or may not have a right angle, we will again need three pieces of information. The four cases are:

Case 1: One side and Two angles
Case 2: Two sides and one opposite angle
Case 3: Two sides and the angle between them
Case 4: Three sides

Note that if we were given all three angles we could not determine the sides uniquely; by similarity an infinite number of triangles have the same angles. In this chapter we will learn how to solve a general triangle in all four of the above cases. Though the methods described will work for right triangles, they are mostly used to solve oblique triangles, that is, triangles which do not have a right angle. There are two types of oblique triangles: an acute triangle has all acute angles, and an obtuse triangle has one obtuse angle.

As we will see, Cases 1 and 2 can be solved using the *law of sines*, Case 3 can be solved using either the *law of cosines* or the *law of tangents*, and Case 4 can be solved using the law of cosines.

General Triangles

(1) The Law of Sines

Theorem 1 (The law of Sine). If a triangle has sides of lengths a,b, and c opposite the angles A,B, and C, respectively, then

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (1)$$

Note that by taking reciprocals, equation (1) can be written as

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad (2)$$

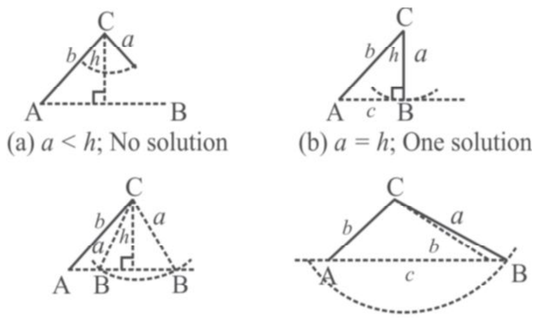
and it can also be written as a collection of three equations:

$$\frac{a}{b} = \frac{\sin A}{\sin B}, \frac{a}{c} = \frac{\sin A}{\sin C}, \frac{b}{c} = \frac{\sin B}{\sin C} \qquad (3)$$

Another way of stating the Law of sines is : *The sides of a triangle are proportional to the sines of their opposite angles.*

There is a way to determine how many solutions a triangle has in Case 2. For a triangle ΔABC, suppose that we know the sides a and b and the angle A. Draw the angle A and the side b, and imagine that the side a is attached at the vertex at C so that it can "swing" freely, as indicated by the dashed arc in the figure below.

If A is acute , then the altitude from C to $\overline{AB}$ has height $h = b \sin A$. As we can see in figure 1(a)-(c), there is no solution when $a < h$; there is exactly one solution - namely, a right triangle - when $a = h$; and there are two solutions when $h < a < b$. When $a \geq b$ there is only one solution, even though it appears from Figure 1-(d) that there may be two solutions, since the dashed arc intersects the horizontal line at two



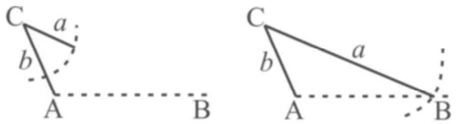
(a) $a < h$; No solution (b) $a = h$; One solution

(c) $h < a < b$; Two solutions (d) $a \geq b$; One solution

Fig-1: The ambiguous case when A is acute

points. However, the point of intersection to the left of A in Figure 1-(d) cannot be used to determine B, since that would make A an obtuse angle, and we assumed that A was acute.

If A is not acute (i.e., A is obtuse or a right angle), then the situation is simpler : there is no solution if $a \le b$, and there is exactly one solution if a > b (see in Figure 2).



(a) $a \le b$: No solution (b) $a > b$: One solution

Fig-2: The ambiguous case when $A \ge 90°$

Knowledge POOL

Table 1 summarises the ambiguous case of solving ΔABC when given a, A, and b. Of course, the letters can be interchanged, e.g. replace a and A by c and C, etc.

$0° < A < 90°$	$90° \le A \le 180°$
$a < b \sin A$: No solution	$a \le b$: No solution
$a = b \sin A$: One solution	$a > b$: One solution
$b \sin A < a < b$: Two solution	
$a \ge b$: One solution	

There is an interesting geometric consequence of the Law of Sines. In a right triangle the hypotenuse is the largest side. Since a right angle is the largest angle in a right triangle, this means that the largest side is opposite the largest angle. What the Law of Sines does is generalise this to any triangle:
In any triangle, the largest side is opposite the largest angle.

(2) The Law of Cosines

We will discuss how to solve a triangle in Case3: two sides and the angle between them.

Theorem 2 (Law of Cosines :). If a triangle has sides of lengths a, b, and c opposite the angles A,B, and C, respectively, then

$$a^2 = b^2 + c^2 - 2bc\cos A, \qquad (4)$$

$$b^2 = c^2 + a^2 - 2ca\cos B, \qquad (5)$$

$$c^2 = a^2 + b^2 - 2ab\cos C, \qquad (6)$$

The angle between two sides of a triangle is often called the included angle. Notice in the Law of Cosines that if two sides and their included angle are known (e.g. b, c, and A), then we have a formula for the square of the third side.
The Law of Cosines can also be used to solve triangles in Case 2 (two sides and one opposite angle), though it is less commonly used for that purpose than the Law of Sines.

(3) The Law of Tangents

We have shown how to solve a triangle in all four cases discussed at the beginning of this chapter. An alternative to the Law of Cosines for Case 3 (two sides and the included angle) is the Law of Tangents:

Theorem 3 (Law of Tangents). If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

$$\frac{a-b}{a+b} = \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} \qquad (7)$$

$$\frac{b-c}{b+c} = \frac{\tan\frac{1}{2}(B-C)}{\tan\frac{1}{2}(B+C)} \qquad (8)$$

$$\frac{c-a}{c+a} = \frac{\tan\frac{1}{2}(C-A)}{\tan\frac{1}{2}(C+A)} \qquad (9)$$

Note that since $\tan(-\theta) = -\tan\theta$ for any angle θ, we can switch the order of the letters in each of the above formulas. For example, we can rewrite formula (7) as

$$\frac{b-a}{b+a} = \frac{\tan\frac{1}{2}(B-A)}{\tan\frac{1}{2}(B+A)} \qquad (10)$$

and similarly for the other formulas. If a > b, then it is usually more convenient to use formula (7), while formula (10) is more convenient when b > a.

Note that in any triangle ΔABC, if $a = b$ then $A = B$, and so both sides of formula (7) would be 0 $(since \tan 0^o = 0)$. This means that the Law of Tangents is of no help in Case 3 when the two known sides are equal. For this reason, and perhaps also because of the somewhat unusual way in which it is used, the Law of Tangents seems to have fallen out of favour in trigonometry books lately. It does not seem to have any advantages over the Law of Cosines, which works even when the sides are equal, requires slightly fewer steps, and is perhaps more straightforward.

Knowledge POOL

Related to the Law of Tangents are Mollweide's equations

Mollweide's equations: For any triangle ΔABC,

$$\frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}, \text{ and} \qquad (11)$$

$$\frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}, \qquad (12)$$

Note that all six parts of a triangle appear in both of Mollweide's equations. For this reason, either equation can be used to check a solution of a triangle. If both sides of the equation agree (more or less), then we know that the solution is correct.

(4) The Area of a Triangle

In elementary geometry we learned that the area of a triangle is one-half the base times the height. We will now use that, combined with some trigonometry, to derive more formulas for the area when given various parts of the triangle.

Case 1. *Two sides and the included angle.*

Suppose that we have a triangle ΔABC, in which A can be either acute, a right angle, or obtuse, as in Figure 3. Assume that a, b, and c are known.

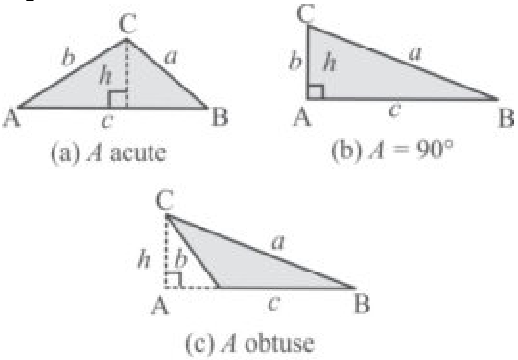


Fig-3: Area of ΔABC

In each case we draw an altitude of height h from the vertex at C to $\overline{AB}$, so that the area (which we will denote by the letter K) is given by $K=\frac{1}{2}hc$.

But we see that $h = b\sin A$ in each of the triangles (since $h = b$ and $\sin A=\sin 90^o=1$ in Figure3(b), and $h=b\sin(180^o-A)=b\sin A$ in Figure 3(c)). We thus get the following formula:

$$Area = K = \frac{1}{2}bc\sin A \qquad (13)$$

The above formula for the area of ΔABC is in terms of the known parts A, b, and c. Similar arguments for the angles B and C give us:

$$Area = K = \frac{1}{2}ac\sin B \qquad (14)$$

$$Area = K = \frac{1}{2}ab\sin C \qquad (15)$$

Notice that the height h does not appear explicitly in these formulas, although it is implicitly there. These formulas have the advantage of being in terms of parts of the triangle, without having to find h separately.

Case 2. *Three angles and any side*

Suppose that we have a triangle ΔABC in which one side, say, *a,* and all three angles are known. By the Law of Sines we know that

$$c=\frac{a\sin C}{\sin A}$$

so substituting this into formula (14) we get:

$$Area = K = \frac{a^2\sin B\sin C}{2\sin A} \qquad (16)$$

Similar arguments for the sides b and c give us:

$$Area = K = \frac{b^2\sin A\sin C}{2\sin B} \qquad (17)$$

$$Area = K = \frac{c^2\sin A\sin B}{2\sin C} \qquad (18)$$

Case 3. *Three sides*

Suppose that we have a triangle ΔABC in which all three sides are known. Then Heron's formula gives us the area:

Knowledge POOL

Heron's formula: For a triangle ΔABC with sides a,b, and c, let $s=\frac{1}{2}(a+b+c)$ (i.e., $2s=a+b+c$ is the perimeter of the triangle). Then the area K of the triangle is

$$Area = K = \sqrt{s(s-a)(s-b)(s-c)} \qquad (19)$$

Heron's formula is rewritten as :

For a triangle ΔABC with sides $a \ge b \ge c$, the area is:

$$Area = K = \frac{1}{4}\sqrt{(a+(b+c))(c-(a-b))(c+(a-b))(a+(b-c))} \qquad (20)$$

To use this formula, sort the names of the sides so that $a \ge b \ge c$. Then perform the operations inside the square root *in the exact order in which they appear in the formula, including the use of parentheses*.

Another formula for the area of a triangle given its three sides is given below:

For a triangle ΔABC with sides $a \ge b \ge c$, the area is:

$$A = K = \frac{1}{2}\sqrt{a^2c^2 - \left(\frac{a^2+c^2-b^2}{2}\right)^2} \qquad (21)$$

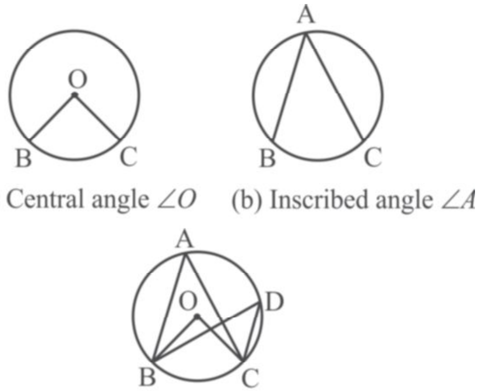
(5) Circumscribed and inscribed Circles

Recall from the Law of Sines that any triangle ΔABC has a common ratio of sides to sines of opposite angles, namely

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This common ratio has a geometric meaning: it is the diameter (i.e., twice the radius) of the unique circle in which ΔABC can be inscribed, called the **circumscribed circle** of the triangle.We review some elementary geometry properties.

A central angle of a circle is an angle whose vertex is the centre O of the circle and whose sides (called radii) are line segments from O to two points on the circle. In Figure 4(a), $\angle O$ is a central angle and we say that it intercepts the arc $\widehat{BC}$.



(a) Central angle $\angle O$ (b) Inscribed angle $\angle A$

(c) $\angle A = \angle D = \frac{1}{2}\angle O$

Fig-4: Types of angles in a circle

An **inscribed angle** of a circle is an angle whose vertex is a point A on the circle and whose sides are line segments (called **chords**) from A to two other points on the circle. In Figure 4(b), $\angle A$ is an inscribed angle that intercepts the arc $5.0pt\widehat{BC}$.

Theorem 4. If an inscribed angle $\angle A$ and a central angle $\angle O$ intercept the same arc, then $\angle A = \frac{1}{2}\angle O$.

Thus, inscribed angles which intercept the same are equal .

Theorem 5. For any triangle ΔABC , the radius R of its circumscribed circle is given by :

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad (15)$$

Corollary 5.1. For any triangle , the centre of its circumscribed circle is the intersection of the perpendicular bisectors of the sides.

Theorem 6. For a triangle ΔABC , let K be its area and let R be the radius of its circumscribed circle. Then

$$k=\frac{abc}{4R} \text{ and hence } R=\frac{abc}{4K}. \qquad (16)$$

Corollary 5.2. For a triangle ΔABC, let

$s=\frac{1}{2}(a+b+c)$. Then the radius R of its circumscribed circle is

$$R=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}. \qquad (17)$$

In addition to a circumscribed circle, every triangle has an **inscribed circle**, i.e., a circle to which the sides of the triangle are tangent, as in Figure 5.

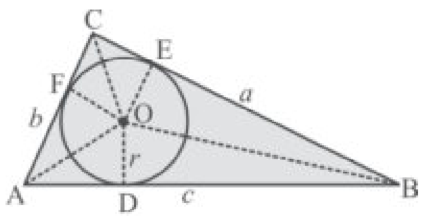


Fig-5: Inscribed circle for ΔABC

Let r be the radius of the inscribed circle, and let D,E, and F be the points on AB, BC, and AC, respectively, at which the circle is tangent. Then

$$\overline{OD}\perp\overline{AB}, \overline{OE}\perp\overline{BC}, \text{ and } \overline{OF}\perp\overline{AC}.$$

Thus, ΔOAD and ΔOAF are equivalent triangles, since they are right triangles with the same hypotenuse $\overline{OA}$ and with corresponding legs $\overline{OD}$ and $\overline{OF}$ of the same length r. Hence, $\angle OAD=\angle OAF$, which means that $\overline{OA}$ bisects the angle A. Similarly, $\overline{OB}$ bisects B and $\overline{OC}$ bisects C. We have thus shown:

IMPORTANT POINTS

For a triangle, the centre of its inscribed circle is the intersection of the bisectors of the angles.

Theorem 7. For any triangle ΔABC, let

$s=\frac{1}{2}(a+b+c)$. Then the radius r of its inscribed circle is

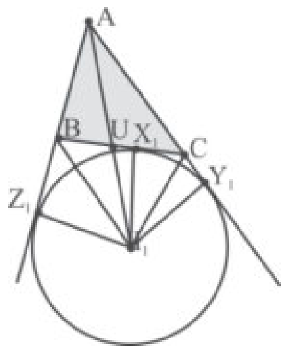
$$r=(s-a)\tan\frac{1}{2}A=(s-b)\tan\frac{1}{2}B=(s-c)\tan\frac{1}{2}C \qquad (18)$$

Theorem 8. For any triangle ΔABC, let

$s=\frac{1}{2}(a+b+c)$. Then the radius r of its inscribed circle is

$$r=\frac{K}{s}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad (19)$$

(I) Excircle, Excenter: Note that these notations cycle for all three ways to extend two sides (A1,B2,C3). I_1 is the excenter opposite A. It has two main properties:



(1) The angle bisectors of $\angle A, \angle Z_1BC, \angle Y_1CB$ are all concurrent at I_1.

(2) I_1 is the center of the excircle which is the circle tangent to BC and to the extensions of AB and AC. r_1 is the radius of the excircle.

(A) Properties:

(i) Elementary Length Formulae:

Theorem 9.

$$AY=AZ=s-a,\ BZ=BX=s-b,$$
$$CX=CY=s-c.$$

Theorem 10.

$$BX_1=BZ_1=s-c,\ CY_1=CX_1=s-b,$$
$$AY_1=AZ_1=s.$$

(B) Area Formulae:

Theorem 11.

$$[ABC]=K=rs=r_1(s-a)=r_2(s-b)$$
$$=r_3(s-c)$$

Where r_1,r_2 and r_3 are exradii

These are very useful when dealing with problems involving the inradius and the exradii. (Let R be the circumradius.)

$$\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$$

$$r_1+r_2+r_3-r=4R$$

$$s^2=r_1r_2+r_2r_3+r_3r_1.$$

$$[ABC]=\sqrt{rr_1r_2r_3}.$$

Here $[ABC]$ is the area of triangle.

(C) Radii RelationshipsComputing Lengths:

$$AI=r\cos ec\left(\frac{1}{2}A\right)$$

(ii) Projection Formula:

Theorem 12.

$a = b\cos C + c\cos B;$
$b = c\cos A + a\cos C;$
$c = a\cos B + b\cos A.$

(iii) Standard Results:

(I) Half-angle formulae:

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}};$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}};$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{(s-b)(s-c)}{\Delta};$$

$$\cot\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{s(s-a)}{\Delta}.$$

The expressions for

$$\sin\frac{B}{2},\cos\frac{B}{2},\tan\frac{B}{2},\cot\frac{B}{2},$$

$\sin\frac{C}{2},\cos\frac{C}{2},\tan\frac{C}{2},\cot\frac{C}{2}$ can be derived using symmetry.

Δ = area of triangle

$$ABC=\sqrt{s(s-a)(s-b)(s-c)}$$

$$\Delta=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C;$$

$$\Delta=\frac{abc}{4R}=rs.$$

(II) Values of sin A, cos A, cot A:

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\frac{2\Delta}{bc};$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc};$$

$$\cot A=\frac{\cos A}{\sin A}=\frac{b^2+c^2-a^2}{4\Delta}.$$

(III) m-n Theorem:

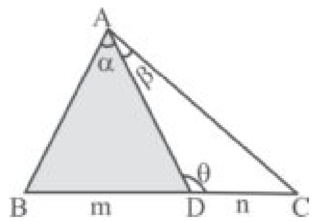
Consider a triangle ABC where D is a point dividing BC internally in the ration $m:n$.

$$\Rightarrow\ \frac{BD}{DC}=\frac{m}{n}$$

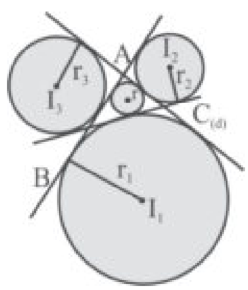
The segment AD makes angles α and β with sides AB and AC respectively.

Theorem: (1) $(m+n)\cot\theta=m\cot\alpha-n\cot\beta$

(2) $(m+n)\cot\theta=n\cot B-m\cot C$



(IV) Relation between inradius, sides, semiperimeter and area of the triangle:

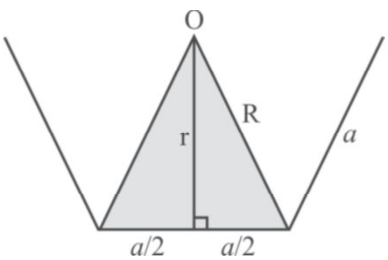


In radius	r	$\frac{\Delta}{s}$	$(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}$	$\frac{a\sin B/2.\sin C/2}{\cos A/2}$
Ex radius (opposite to A)	r_1	$r_1=\frac{\Delta}{s-a}$	$s\tan\frac{A}{2}$	$\frac{a\cos B/2.\cos C/2}{\cos A/2}$
Ex radius (opposite to B)	r_2	$r_2=\frac{\Delta}{s-b}$	$s\tan\frac{B}{2}$	$\frac{b\cos A/2.\cos C/2}{\cos B/2}$
Ex radius (opposite to C)	r_3	$r_3=\frac{\Delta}{s-c}$	$s\tan\frac{C}{2}$	$\frac{c\cos A/2.\cos B/2}{\cos C/2}$

(V) Regular n sides Polygon:

If the polygon has 'n' sides, Sum of the internal angles is $(n-2)\pi$ and each angle is $\frac{(n-2)\pi}{n}$.

a = side length; r = in radius; R = circum-radius



$$r=\frac{a}{2\tan\frac{\pi}{n}} \quad and \quad R=\frac{a}{2\sin\frac{\pi}{n}}$$

Area of polygen

$$=\frac{1}{4}na^2.\cot\left(\frac{\pi}{n}\right)=nr^2\tan\left(\frac{\pi}{n}\right)$$

$$=\frac{n}{2}R^2\sin\frac{2\pi}{n}.$$

(VI) More Results

(A) Distance of orthocentre from vertices of triangle AD, BE are altitudes and H is the orthocentre of a triangle ΔABC. As quadrilateral $CEHD$ is cyclic, $\angle EHA=\angle C$

from ΔAHE, $\quad AH\sin C=AE$

$\Rightarrow \quad AH\sin C=AB\cos A \quad$ [using ΔAHE]

$$\Rightarrow \quad AH=\frac{c\cos A}{\sin C}=\left(\frac{c}{\sin C}\right)\cos A$$

$$\Rightarrow \quad AH=2R\cos A$$

$\Rightarrow$ distances of orthocentre (H) from the vertices A, B and C are: $2R\cos A$, $2R\cos B$ and $2R\cos C$ respectively.

(B) Distance of orthocentre from sides of triangle

$DH = AD - AH$

$\Rightarrow DH=AB\sin B-2R\cos A$

$\Rightarrow DH=c\sin B-2R\cos A$

$\Rightarrow DH=2R\sin C\sin B+2R\cos(B+C)$

$(\because A=\pi-(B+C))$

$\Rightarrow DH=2R\cos B\cos C$

$\Rightarrow$ The distances of orthocentre (H) from the sides BC, CA and AB are: $2R\cos B\cos C$, $2R\cos C\cos A$ and $2R\cos A\cos B$ respectively.

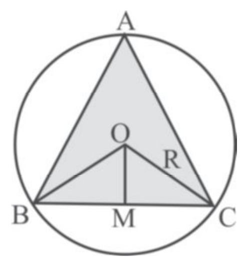
(C) Distance of circumcentre O from sides:

$$\angle BOC=2A$$

$$\Rightarrow \quad \angle COM=A$$

$$\Rightarrow \quad OM=R\cos A$$

$\Rightarrow$ distances of circumcentre from sides BC, CA and AB are $R\cos A$, $R\cos B$ and $R\cos C$ respectively.



(D) Important Theorem:
Theorem: The centroid, circumcentre and orthocentre in any triangle are collinear.
The centroid divides the line joining orthocentre and circumcentre in 2 : 1 internally.

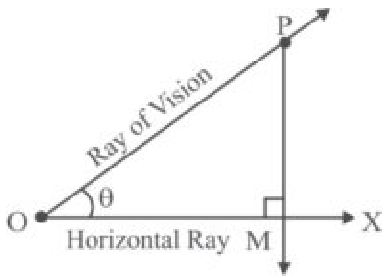
Heights and Distance

Angle of Elevation and Angle of Depression

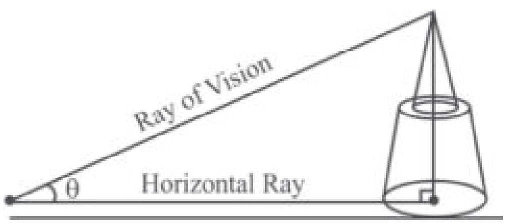
Horizontal Ray: A ray parallel to the surface of the earth emerging from the eye of an observer is called a horizontal ray.

Ray of Vision: The ray from the eye of an observer towards the object is called the ray of vision or ray of sight.

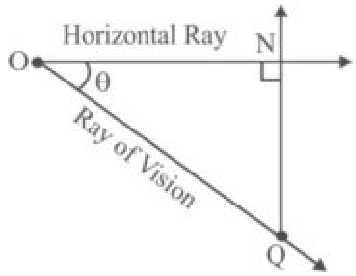
Angle of Elevation: If the object under observation is above an observer, but not directly above the observer, then the angle formed by the horizontal ray and the ray of sight in a vertical plane is called the angle of elevation. Here horizontal ray, observer and object are in the same vertical plane.



In figure the object P under observation is at a higher level than the observer O but not directly above O. Let $\overrightarrow{OM}$ be the horizontal ray in the vertical plane containg O and P. Then the union of the ray of vision $\overrightarrow{OP}$ and horizontal ray $\overrightarrow{OX}$ is $\angle POM$. If $m\angle POM = \theta$, then θ is called the measure of the angle of elevation $\angle POM$, of the object P at the point of observation O.



Angle of Depression: If the object under observation is at a lower level than an observer but not directly under the observer, then the angle formed by the horizontal ray and the ray of sight is called the angle of depression. Here horizontal ray, observer and the object are in the same vertical plane.



In figure the object under observation is at a lower level than the observer O but not directly under O. Let $\overrightarrow{ON}$ be the horizontal ray in the vertical plane containg O and Q. Then the union of the ray of vision $\overrightarrow{OQ}$ and horizontal ray $\overrightarrow{ON}$ is $\angle NOQ$.

Exercise

1. If the angles of a triangle are $30^\circ, 45^\circ$ and the included side is $\sqrt{3}+1$, then the remaining sides can be

(a) $2, \sqrt{2}$ (b) $2, 2\sqrt{3}$ (c) $\sqrt{2}, 4$ (d) $2, 4\sqrt{3}$

2. If the sides of a triangle have lengths 2,3 and 4, what is the radius of the circle circumscribing the triangle?

(a) 2 (b) $\frac{8}{\sqrt{15}}$ (c) $\frac{5}{2}$ (d) $\sqrt{6}$

3. In ΔABC, if $\sin A, \sin B$ are the roots of $c^2x^2 - c(a+b)x + ab = 0$ then $\sin C =$

(a) 0 (b) 1/2 (c) $1/\sqrt{2}$ (d) 1

4. If the angles A, B, C of a triangle are in A.P. and sides a, b, c are in G.P., then the a^2, b^2, c^2 are in

(a) A.P (b) G.P (c) H.P (d) A.G.P

5. Let a,b and c be the three sides of a triangle, then the equation $b^2x^2+\left(b^2+c^2-a^2\right)x+c^2=0$ has

(a) Real roots (b) Imaginary roots

(c) Equal roots (d) Real and equal roots

6. Two sides of a triangle are given by the roots of the equation $x^2-5x+6=0$ and the angle between the sides is $\pi/3$. Then the perimeter of the triangle is

(a) $5+\sqrt{2}$ (b) $5+\sqrt{3}$

(c) $5+\sqrt{5}$ (d) $5+\sqrt{7}$

7. If sides of a triangle are

$\sin\alpha, \cos\alpha, \sqrt{1+\sin\alpha\cos\alpha}$ for some $0<\alpha<\dfrac{\pi}{2}$ is greatest angle of triangle

(a) 60° (b) 150° (c) 120° (d) 90°

8. Let a,b,c be the lengths of three sides of ΔABC and $b^2=ac$. If $\underline{|B}=x$ and

$f(x)=\sin\left(4x-\dfrac{x}{6}\right)-\dfrac{1}{2}$, find the range of $f(x)$

(a) $\left(0,\dfrac{1}{2}\right)$ (b) $\left[0,\dfrac{1}{2}\right]$

(c) $\left(-1,\dfrac{1}{2}\right)$ (d) $\left[-1,\dfrac{1}{2}\right]$

9. In a ΔABC, $a=2b$ and $|A-B|=\dfrac{\pi}{3}$, then $\angle C$ is

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{2}$ (d) $\dfrac{\pi}{3}$

10. In a triangle ABC, $\underline{|BAC}=60^o, AB=2AC$. Point P is inside the triangle such that $PA=\sqrt{3}, PB=5, DC=2$. What is the area of triangle ABC ?

(a) 7 (b) $\dfrac{7\sqrt{3}+6}{2}$

(c) $7+2\sqrt{3}$ (d) $8+2\sqrt{3}$

11. The perimeter of a triangle ABC is 6 times the artimetic mean of the sines of its angles. If the side a is 1, then the angle A is

(a) 30° (b) 60° (c) 90° (d) 45°

12. The radius of the circumcircle of an isosceles triangle PQR is equal to $PQ(=PR)$, then the angle P is

(a) $\pi/6$ (b) $\pi/3$ (c) $\pi/2$ (d) $2\pi/3$

13. In ΔABC, if $r_1<r_2<r_3$ then

(a) $a<b<c$ (b) $a>b>c$

(c) $b<a<c$ (d) $a<c<b$

14. The area of a regular polygon of $2n$ sides inscribed in a circle is the geometric mean of the areas of the inscribed and circumscribed polygons of n sides. which is

(a) A.M (b) G.M

(c) H.M (d) Can not say

15. If I is the incentre of ΔABC then $AI=$

(a) $R\sin\dfrac{B}{2}\sin\dfrac{C}{2}$ (b) $\operatorname{cosec}\dfrac{A}{2}$

(c) $4R\sin\dfrac{A}{2}\sin\dfrac{C}{2}$ (d) $4R\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

16. In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that $AX:XB=1:2$ and $AY:YC=2:1$. If area of triangle AXY is 10 then what is the area of triangle ABC?

(a) 10 (b) 20 (c) 35 (d) 45

17. If $A=90^o$, then $(1-b\cos c)(b-\cos c)=$

(a) 0 (b) 1 (c) 2 (d) 3

18. If $3\tan\left(\theta-15^\circ\right)=\tan\left(\theta+15^\circ\right), 0<\theta<\pi$ then $\theta=$

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{6}$ (d) $\dfrac{\pi}{3}$

19. If $2\cos^2 x+47\cos x=20\sin^2 x$, then what is the value of $\cos x$?

(a) $\dfrac{4}{11}$ (b) $\dfrac{-5}{2}$ (c) $\dfrac{-4}{11}$ (d) $\dfrac{2}{11}$

20. The angle of elevation measured from two points A and B on a horizontal line from the foot of a tower are α and β. If $AB=d$, then the height of the tower is:

(a) $\left|\dfrac{d\sin\alpha\sin\beta}{\sin(\alpha-\beta)}\right|$ (b) $\left|\dfrac{d\sin\alpha\sin\beta}{\sin(\alpha+\beta)}\right|$

(c) $\left|\dfrac{d\sin\alpha+\sin\beta}{\sin(\alpha-\beta)}\right|$ (d) $\left|\dfrac{d\sin\alpha-\sin\beta}{\sin(\alpha-\beta)}\right|$

21. Points D, and E are taken on the side BC of the ΔABC, such that $BD=DE=EC$. If $\angle BAD=x$, $\angle DAE=y$ and $\angle EAC=z$, then the value of $\dfrac{\sin(x+y)\sin(y+z)}{\sin x\sin z}$ is

(a) 1 (b) 2
(c) 4 (d) None of these

22. In a ΔABC, if $\sin A\sin B=\dfrac{ab}{c^2}$, then the triangle is

(a) Equilateral (b) Isosceles
(c) Right angled (d) Obtuse angled

23. The equation $ax^2+bx+c=0$, where a,b,c are the sides of a ΔABC, and the equation $x^2+\sqrt{2}x+1=0$ have a common root. The measure of $\angle C$ is

(a) 90° (b) 45°
(c) 60° (d) None of these

24. The area (in sq units) of the triangle whose sides are $6,5,\sqrt{13}$, is

(a) $5\sqrt{2}$ (b) 9 (c) $6\sqrt{2}$ (d) 11'

25. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x. The maximum area enclosed by the park is

(a) $\sqrt{\dfrac{x^3}{8}}$ (b) $\dfrac{1}{2}x^2$ (c) πx^2 (d) $\dfrac{3}{2}x^2$

26. In a $\Delta PQR, P$ is the largest angle and $\cos P=\dfrac{1}{3}$. Further in circle of the triangle touches the sides PQ, QR and PR at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then, possible lengths (s) of the side (s) of the triangle is (are)

(a) 16 (b) 17 (c) 24 (d) 22

27. If A, A_1, A_2 and A_3 are the areas of the incircle and excircles, then $\dfrac{1}{\sqrt{A_1}}+\dfrac{1}{\sqrt{A_2}}+\dfrac{1}{\sqrt{A_3}}$ is equal to

(a) $\dfrac{1}{\sqrt{A}}$ (b) $\dfrac{2}{\sqrt{A}}$ (c) $\dfrac{3}{\sqrt{A}}$ (d) $\dfrac{4}{\sqrt{A}}$

28. The general value of x for the equation $9^{\cos x}-2\cdot 3^{\cos x}+1=0$ is

(a) $n\pi$ (b) $\dfrac{n\pi}{2}$
(c) $2n\pi$ (d) $(2n+1)\dfrac{\pi}{2}$

29. If PQR be a triangle of area Δ with $a=2, b=\dfrac{7}{2}$ and $c=\dfrac{5}{2}$, where a, b and c are the length of the sides of the triangle opposite to the angles at P, Q and R, respectively, then, $\dfrac{2\sin P-\sin 2P}{2\sin P+\sin 2P}$ is equal to

(a) $\dfrac{3}{4\Delta}$ (b) $\dfrac{45}{4\Delta}$ (c) $\left(\dfrac{3}{4\Delta}\right)^2$ (d) $\left(\dfrac{45}{4\Delta}\right)^5$

30. A vertical pole PO is standing at the centre O of a square $ABCD$. If AC subtends an $\angle 90^\circ$ at the top P of the pole, then the angle subtended by a side of the square at P is

(a) 30° (b) 45°
(c) 60° (d) None of these

ANSWER KEY

1. a	2. b	3. d	4. a	5. b
6. d	7. c	8. d	9. d	10. b
11. a	12. d	13. a	14. b	15. d
16. d	17. a	18. b	19. d	20. a
21. c	22. c	23. b	24. b	25. b
26. d	27. a	28. d	29. c	30. c

HINTS & SOLUTIONS

1.Sol: Using the sine rule, we can define the triangle, with the given information

i.e., $\frac{\sqrt{3}+1}{\sin 105^o}=\frac{b}{\sin 30^o}=\frac{c}{\sin 45^o}$

$\Rightarrow \quad c=2, b=\sqrt{2}$

2.Sol: Let vertex A be opposite to the side of length 2.

Then by law of cosines, we have $\cos A=\frac{7}{8}$. Thus

$\sin A=\sqrt{1-\left(\frac{7}{8}\right)^2}=\frac{\sqrt{15}}{8}$. Then by the extended law of sines, $R=\frac{1}{2}\frac{a}{\sin A}=\frac{1}{2}\frac{2}{\frac{\sqrt{15}}{8}}=\frac{8}{\sqrt{15}}$

3.Sol: We know, sum of the roots of a quadratic equation $ax^2+bx+c=0$ is $\frac{-b}{a}$ and that of product is $\frac{c}{a}$

i.e., $\sin A+\sin B=\frac{c(a+b)}{c^2}=\frac{a+b}{c}$

$\sin A\cdot\sin B=\frac{ab}{c^2}$.

$\Rightarrow \quad \sin^2 C=1$

$\Rightarrow \quad \sin C=1$

4.Sol: Given, $2B=A+C\Rightarrow 3B=\pi\Rightarrow B=\pi/3$

Also a,b,c in G.P. $\Rightarrow b^2=ac$

Now, $\cos B=\cos 60^{\circ}=\frac{1}{2}=\frac{c^2+a^2-b^2}{2ca}$

$\Rightarrow \quad ca=c^2+a^2-b^2$

$\Rightarrow \quad 2b^2=c^2+a^2$

$\Rightarrow \quad a^2,b^2,c^2$ are in A.P.

5.Sol: Given equation

$b^2x^2+\left(b^2+c^2-a^2\right)x+c^2=0$

rewrite the given equation as

$b^2x^2+(2bc\cos A)x+c^2=0$

Now, $D=(2bc\cos A)^2-4b^2c^2$

we have $|\cos A|<1$

i.e., $(2bc\cos A)^2<4b^2c^2$

$\Rightarrow \quad (2bc\cos A)^2-4b^2c^2<0$

$\therefore \quad D<0$

Hence, it has imaginary roots.

6.Sol: Let a, b be the roots of $x^2-5x+6x=0$

i.e., $a=2; b=3$

Also given that $\underline{|C}=\frac{\pi}{3}$

Now, $c^2=a^2+b^2-2ab\ \cos(C)$

$c^2=a^2+b^2-2ab\ \cos(C)$

$=4+9-12\left(\frac{1}{2}\right)$

$=13-6$

$=7$

$\Rightarrow \quad c=\sqrt{7}$

Now perimeter $=2S=a+b+c$

$=2+3+\sqrt{7}$

$=5+\sqrt{7}$

7.Sol: Given the sides of a triangle $a=\sin\alpha, b=\cos\alpha$ and $c=\sqrt{1+\sin\alpha\cos\alpha}$

$\Rightarrow \quad \underline{|c}$ is the greatest angle

$\therefore \quad \cos c=\frac{\sin^2\alpha+\cos^2\alpha-\left(\sqrt{1+\sin\alpha\cos\alpha}\right)^2}{2\sin\alpha\cdot\cos\alpha}$

$=\frac{-1}{2}$

$\Rightarrow \quad c=120^o$

8.Sol: The cosine rule gives

$\cos x=\frac{a^2+c^2-b^2}{2ac}\geq\frac{2ac-ac}{2ac}=\frac{1}{2}$

since $0<x<\pi$, so $0<x\leq\frac{\pi}{3}$ and

$\frac{-\pi}{6} < 4x - \frac{\pi}{6} \le \frac{7\pi}{6}$. Therefore

$\frac{-1}{2} \le \sin\left(4x - \frac{x}{6}\right) \le 1$

and the range of $f(x)$ is $\left[-1, \frac{1}{2}\right]$

9.Sol: Given $\frac{a}{b} = \frac{1}{2}$

Applying componendo and dividendo, we get

$\frac{a-b}{a+b} = \frac{-1}{3}$

using "The law of tangents" we have

$\tan\left(\left|\frac{A-B}{2}\right|\right) = \frac{a-b}{a+b}\cot\frac{c}{2}$

i.e., $\tan|30| = \frac{-1}{3}\cot\left(\frac{c}{2}\right)$

$\Rightarrow \quad \cot\left(\frac{c}{2}\right) = \sqrt{3}$

$\Rightarrow \quad \frac{c}{2} = \frac{\pi}{6}$

i.e., $c = \frac{\pi}{3}$

10.Sol: By consine rule, we have $a = \sqrt{3}b$

then ΔABC is a right angled triangle with sides b; a and 2b so $\lfloor ACB = 90^o$.

Let $\lfloor ACP = x \Rightarrow \lfloor BCP = 90 - x$

again by cosine rule, we have

$\cos x = \frac{1+b^2}{4b}$

and $\cos(90-x) = \frac{a^2-21}{4a}$

or $\sin x = \frac{a^2-21}{4a}$

now $\sin^2 x + \cos^2 x = 1$

$\left(\frac{a^2-21}{4a}\right)^2 + \left(\frac{1+b^2}{4b}\right) = 1$

simplifying it, we will get

$b^4 - 14b^2 + 37 = 0 \qquad (1)$

putting $a = \sqrt{3}b$

from this equation, we will get the value of b now the area of the triangle

$K = \frac{1}{2}bc\sin 60 = \frac{\sqrt{3}}{2}b^2$

so, putting the values of $b^2 = 7 + 2\sqrt{3}$ we will get the area of ΔABC.

$K = \frac{7\sqrt{3}+6}{2}$

11.Sol: Given that $a+b+c = 6 \times \frac{\sin A + \sin B + \sin C}{3}$

By law of sines, we have

$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

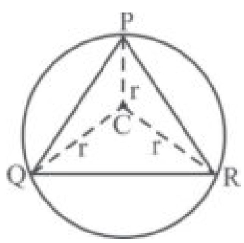
$\Rightarrow \quad \frac{\sin A}{a} = \frac{\sin A + \sin B + \sin C}{a+b+c}$

i.e., $\frac{\sin A}{a} = \frac{1}{2}$

$\Rightarrow \quad \sin A = \frac{1}{2} \qquad \{\text{given } a = 1\}$

$\therefore \quad A = \frac{\pi}{6}$

12.Sol:



In ΔPCR, we have $PC = CR = CQ = r$ also given that $PC = PR$

$\Rightarrow \quad PC = PR = CR$

$\therefore \quad \Delta PCR$ is an equilateral triangle i.e., $\angle CPR = 60^o$

similarly ΔPCQ is an equilateral triangle

$\angle CPQ = 60^o$

$\Rightarrow \angle QPR = \angle CPQ + \angle CPR$

$= 60^o + 60^o$

$= 120^o$

$= \dfrac{2\pi}{3}$

13.Sol: $r_1 < r_2 < r_3$

$\Rightarrow \quad \dfrac{\Delta}{s-a} < \dfrac{\Delta}{s-b} < \dfrac{\Delta}{s-c}$

i.e., $s-c < s-b < s-a$

$\Rightarrow \quad a < b < c$

14.Sol: Let a be the radius of the circle.

Then, S_1 = Area of regular polygon of n sides inscribed in the circle $= \frac{1}{2} na^2 \sin\left(\frac{2\pi}{n}\right)$

S_2 = Area of regular polygon of n sides circumscribing the circle $= na^2 \tan(\pi/n)$

S_3 = Area of regular polygon of $2n$ sides inscribed in the circle $= na^2 \sin\left(\frac{\pi}{n}\right)$

Therefore, geometric mean of S_1 and S_2

$= \sqrt{(S_1 S_2)} = na^2 \sin(\pi/n) = S_3$

15.Sol: Conceptual

16.Sol: Let side AB = c, AC = b and $\angle BAC = \theta$

Then $AX = \dfrac{c}{3}$ and $AY = \dfrac{2b}{3}$

Area of triangle $ABC = \dfrac{1}{2} bc \sin\theta$ and

Area of triangle $AXY = \dfrac{1}{2} \times \dfrac{2b}{3} \times \dfrac{c}{3} \sin\theta = 10$

Area $\Delta AXY = \dfrac{2}{9}$(Area ΔABC)

Area $\Delta ABC = \dfrac{9}{2}$(Area ΔAXY)

Area $\Delta ABC = \dfrac{9}{2} \times 10 = 45$

17.Sol: We have $a = b\cos c + c\cos B$

and $b = a\cos c + c\cos A$

also given that $A = 90^o$

i.e., $1 = b\cos c + c\cos B$

$b = \cos c$

$\Rightarrow \quad 1 - b\cos c = c\cos B$ and $b - \cos c = 0$

Now, $(1 - b\cos c)(b - \cos c) = 0$

18.Sol: Given $3\tan(\theta - 15^o) = \tan(\theta + 15^o)$

$\Rightarrow \quad \dfrac{\tan(\theta-15)}{\tan(\theta-15)} = \dfrac{3}{1}$

Applying componendo and dividendo, we get

$\dfrac{\tan(\theta+15)+\tan(\theta-15)}{\tan(\theta+15)-\tan(\theta-15)} = \dfrac{4}{2}$

$\Rightarrow \quad \dfrac{\sin 2\theta}{\sin 30} = 2$

i.e., $\sin 2\theta = 1$

$\therefore \quad 2\theta = \dfrac{\pi}{2}$ and $\theta = \dfrac{\pi}{4}$

19.Sol: Given $2\cos^2 x + 47\cos x = 20\sin^2 x$

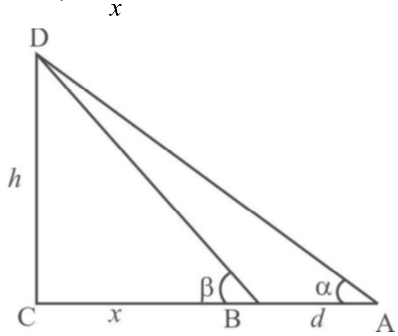
$\Rightarrow \quad 2\cos^2 x + 47\cos x = 20 - 20\cos^2 x$

$\Rightarrow \quad 22\cos^2 x + 47\cos x - 20 = 0$

put $\cos x = t$

i.e., $22t^2 + 47t - 20 = 0$

20.Sol: $\tan\beta = \dfrac{h}{x}$



$$\Rightarrow\ x = h\cot\beta$$

$$\tan\alpha = \frac{h}{x+d}$$

$$\Rightarrow\ x+d = h\cot\alpha$$

$$\Rightarrow\ h\cot\beta + d = h\cot\alpha$$

$$\Rightarrow\ h = \frac{d}{\cot\alpha - \cot\beta} = \frac{d\cdot\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$$

21.Sol: Using sine rule in $\Delta ADC, \frac{\sin(y+z)}{DC} = \frac{\sin C}{AD}$

In ΔABD, $\frac{\sin x}{BD} = \frac{\sin B}{AD}$

In ΔAEC, $\frac{\sin z}{EC} = \frac{\sin C}{AE}$

In ΔABE, $\frac{\sin(x+y)}{BE} = \frac{\sin B}{AE}$

$$\therefore\ \frac{\sin(x+y)\sin(y+z)}{\sin x\sin z} = \frac{BE}{AE}\times\frac{DC}{AD}\times\frac{AD}{BD}\times\frac{AE}{EC}$$

$$= \frac{2BD\times 2EC}{BD\times EC} = 4$$

22.Sol: Given, that $\sin A\sin B = \frac{ab}{c^2}$

$$\Rightarrow\ c^2 = \frac{ab}{\sin A\sin B}$$

$$\Rightarrow\ c^2 = \left(\frac{c}{\sin C}\right)^2 \qquad \text{\{The law of sine\}}$$

$$\Rightarrow\ \sin^2 C = 1$$

i.e., $C = 90^\circ$

Hence, ΔABC is a right angled triangle.

23.Sol: Clearly, the roots of $x^2+\sqrt{2}x+1=0$ are non-real complex. So, one root common implies both roots are common.

So, $\frac{a}{1} = \frac{b}{\sqrt{2}} = \frac{c}{1} = k.$

$$\therefore\ \cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{k^2+2k^2-k^2}{2\cdot k\cdot\sqrt{2}k} = \frac{1}{\sqrt{2}}.$$

24.Sol: $a=6, b=5$ and $c=\sqrt{13}$

$$\therefore\ \cos C = \frac{6^2+5^2-13}{2\times 6\times 5} = \frac{4}{5}$$

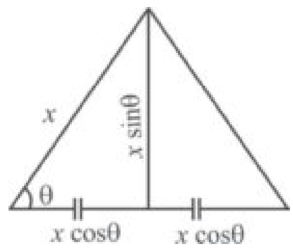
Now, $\sin C = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

$\therefore$ Area of ΔABC

$$= \frac{1}{2}ab\sin C = \frac{1}{2}\times 6\times 5\times\frac{3}{5} = 9 \text{ sq units}$$

25.Sol: Area $= 1/2\times\text{Base}\times\text{Altitude}$

$$=1/2\times(2x\cos\theta)\times(x\sin\theta) = 1/2x^2\sin 2\theta$$



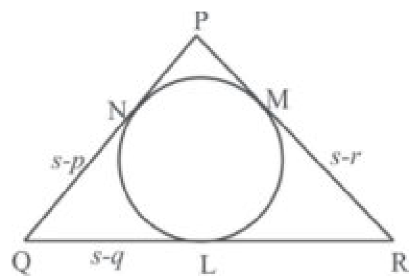
$\therefore$ Maximum area $= \frac{1}{2}x^2$

[since, maximum value of $\sin 2\theta$ is 1]

26.Sol: Let

$s-p=2k-2, s-q=2k, s-r=2k+2, k\in I, k>1$

On adding above equations, we get



$s = 6k$

$\therefore\ p=4k+2, q=4k, r=4k-2$

Now, $\cos P = \frac{1}{3}$

$$\Rightarrow\ \frac{q^2+r^2-p^2}{2qr} = \frac{1}{3}$$

$$\Rightarrow 3\left[(4k)^2+(4k-2)^2-(4k+2)^2\right]$$

$$=2(4k)(4k-2)$$

$$\Rightarrow\ 3\left[16k^2-4(4k)(2)\right]=8k(4k-2)$$

$$\Rightarrow\ 48k^2-96k=32k^2-16k$$

$$\Rightarrow\ 16k^2=80k\Rightarrow k=5$$

So, the sides are 22, 20 and 18

27.Sol: Area of a circle $=\pi\times(\text{Radius})^2$

$$\therefore\ A=\pi r^2, A_1=\pi r_1^2, A_2=\pi r_2^2, A_3=\pi r_3^2$$

Now, $\dfrac{1}{\sqrt{A_1}}+\dfrac{1}{\sqrt{A_2}}+\dfrac{1}{\sqrt{A_3}}=\dfrac{1}{r_1\sqrt{\pi}}+\dfrac{1}{r_2\sqrt{\pi}}+\dfrac{1}{r_3\sqrt{\pi}}$

$$=\frac{1}{\sqrt{\pi}}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$$

$$=\frac{1}{\sqrt{\pi}}\left(\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}\right)$$

$$=\frac{1}{\sqrt{\pi}}\left[\frac{3s-(a+b+c)}{\Delta}\right]$$

$$=\frac{1}{\sqrt{\pi}}\cdot\frac{3s-2s}{\Delta}$$

$$=\frac{1}{\sqrt{\pi}}\cdot\frac{s}{\Delta}=\frac{1}{r\sqrt{\pi}}=\frac{1}{\sqrt{A}}$$

28.Sol: Put $\quad 3^{\cos x}=a$

$$a^2-2a+1=0$$

$$a=1$$

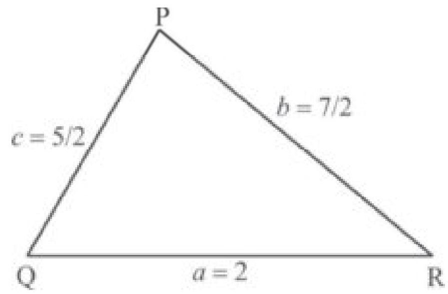
$$3^{\cos x}=3^{\circ}$$

$$x=(2n+1)\frac{\pi}{2}$$

29.Sol: $s=\dfrac{2+\dfrac{7}{2}+\dfrac{5}{2}}{2}=4$

$$\therefore\ \frac{2\sin P-\sin 2P}{2\sin P+\sin 2P}=\frac{2\sin P(1-\cos P)}{2\sin P(1+\cos P)}$$

$$=\frac{2\sin^2(P/2)}{2\cos^2(P/2)}=\tan^2(P/2)$$



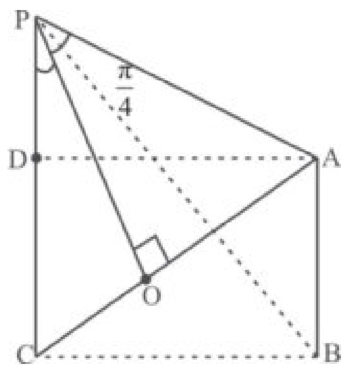
Now,

$$\tan^2\left(\frac{P}{2}\right)=\frac{(s-b)(s-c)}{s(s-a)}\times\frac{(s-b)(s-c)}{(s-b)(s-c)}$$

$$=\frac{\left[(s-b)(s-c)\right]^2}{\Delta^2}=\frac{\left(4-\frac{7}{2}\right)^2\left(4-\frac{5}{2}\right)^2}{\Delta^2}=\left(\frac{3}{4\Delta}\right)^2$$

30.Sol: Let $\angle OPA=\angle OPC=\dfrac{\pi}{4}$ and $OP=n$

So, $\qquad \dfrac{OA}{OP}=\tan 45^{\circ}=1$



$\therefore\qquad OA=n, AC=2n$

Also, $\dfrac{OP}{PA}=\cos\dfrac{\pi}{4}\Rightarrow PA=\sqrt{2}n$

and $\quad AC=\sqrt{2}AB\Rightarrow 2n=\sqrt{2}\cdot AB$

$\Rightarrow\qquad AB=\sqrt{2}n$

$\therefore\qquad PA=AB=PB\qquad$ [similarly]

So, ΔBPA is equilateral.

Hence, the required angle is 60°.

OLYMPIAD Primer

This article aims at solving entry level Math Olympiad (Pre-RMO in india). We have compiled some of the most useful results and tricks in geometry that helps in solving problems at this level.

1. (i) When each side of a triangle has a length which is a prime factor of 2001, how many different such triangles are there?
   (ii) How many isosceles triangles are there, such that each of its sides has an integral length, and its perimeter is 144?
2. In the figure below $AB = AC, \angle BAD = 30°$, and $AE = AD$. Then $\angle CDE$ equals:
   (a) 7.5° (b) 10° (c) 12.5° (d) 15°
   (e) 20°
3. As shown in the figure, in ΔABC, the angle bisectors of the exterior angles of $\angle A$ and $\angle B$ intersect opposite sides at D and E respectively, and $AD = AB = BE$. Then the size of angle A, in degrees , is
   (a) 10° (b) 11°
   (c) 12° (d) None of preceding
4. There are four points A, B, C, D on the plane, such that any three points are not colinear. Prove that in the triangles ABC, ABD, ACD and BCD there is at least one triangle which has an interior angle not greater than $45°$.
5. Given that in a right triangle the length of a leg of the right angle is 11 and the lengths of the other two sides are both positive integers. Find the perimeter of the triangle.
6. As shown in the figure, $\angle C = 90°, \angle 1 = \angle 2,$ $CD = 1.5$ cm, $BD = 25$ cm. Find AC.
7. In the figure, $\angle C = 90°, \angle A = 30°$, D is the mid-point of AB and $DE \perp AB$, $AE = 4$ cm Find BC.
8. In square $ABCD$. M is the midpoint of AD and N is the midpoint of MD. Prove that $\angle NBC = 2\angle ABM$.
9. Given that BE and CF are the altitudes of the ΔABC. P, Q are on BE and the extension of CF respectively such that $BP = AC, CQ = AB$. Prove that $AP \perp AQ$.
10. Given that ABC is an equilateral triangle of side 1, ΔBDC is isosceles with $DB = DC$ and $\angle BDC = 120°$. If points M and N are on AB and AC respectively such that $\angle MDN = 60°$, find the perimeter of ΔAMN.
11. In the square $ABCD$, E is the midpoint of AD, BD and CF intersect at F. Prove that $AF \perp BE$.
12. In the figure D, E are points on AB and AC such that $AE = 2EC$, and BE intersects CD at point F. Prove that $4EF = BE$.
13. As shown in the figure, in $\Delta ABC, \angle B = 2\angle C$, AD is perpendicular to BC at D and E is the midpoint of BC. Prove that $AB = 2DE$.
14. In the trapezium ABCD, $AB \parallel CD, \angle DAB = \angle ADC = 90°$, and the ΔABC is equilateral. Given that the midline of the trapezium EF = 0.75 a, find the length of the lower base AB in terms of a.
15. In $\Delta ABC, AD$ is the median on BC, E is on AD such that $BE = AC$. The line BE intersects AC at F. Prove that $AF = EF$.
16. In $\Delta ABC, \angle A : \angle B : \angle C = 1:2:4.$ Prove that

$$\frac{1}{AB} + \frac{1}{AC} = \frac{1}{BC}.$$

17. In ΔABC, *D, E* are on *BC* and *CA* respectively, and $BD : DC = 3:2,\ AE : EC = 3:4$. *AD* and *BE* intersect at *M*. given that the area of ΔABC is 1, find are of ΔBMD.

18. As shown in the figure, triangle *ABC* is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. find the area of triangle *ABC*.

19. In ΔABC, *M* is the midpoint of *BC*, *P*, *R* are on *AB*, *AC* respectively, *Q* is the point of intersection of *AM* and *PR*. If *Q* is the midpoint of *PR*, prove that $PR \parallel BC$.

20. In the given diagram below, *ABCD* is a parallelogram, *E, F* are two points on the sides *AD* and *DC* respectively, such that $AF = CE$. *AF* and *CE* intersect at *P*. Prove that PB bisects $\angle APC$.

HINTS & SOLUTIONS

1.Sol: (i) Since $2001 = 3 \times 23 \times 29$, the triangles with sides of the following lengths exist:

$\{3,3,3\}$; $\{23,23,23\}$; $\{29,29,29\}$;

$\{2,23,23\}$; $\{3,29,29\}$; $\{23,29,29\}$;

$\{23,23,29\}$

There are 7 possible triangles in total.

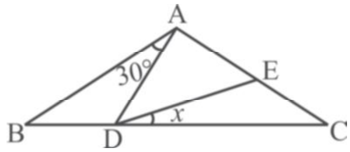
(ii) Suppose that each leg of the isosceles triangle has length *n*, then its base has a length $144 - 2n = 2(72 - n)$, i.e., the length of the base must be even.

(a) If $n \ge 1440 - 2n$ i.e., $3n \ge 144$, then $n \ge 48$. Since $2n \le 144 - 2 = 142$. i.e., $n \le 71$, we have $48 \le n \le 71$, there are 24 possible values for *n*.

(b) If $n < 144 - 2n$, then $n < 48$. From triangle inequality $2n > 144 - 2n$. i.e., $n > 36$, then $36 < n < 48$, so *n* has $47 - 36 = 11$ possible value. Thus, there are together $24 + 11 = 35$ possible isosceles triangles.

2.Sol: Let $\angle CDE = x$, then

$$x = \angle ADC - \angle ADE = \angle ADC - \angle AED$$
$$= \angle ADC - (x + \angle C)$$
$$\therefore x = \frac{1}{2}(\angle ADC - \angle C)$$
$$= \frac{1}{2}(\angle B + 30° - \angle C) = 15°$$



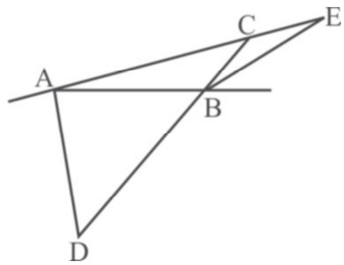
3.Sol: Let $\angle A = \angle E = \alpha$

$\angle D = \angle ABD = \beta$

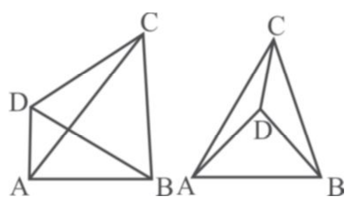
$\angle CBE = \gamma, \angle ACB = \delta$

Then $\beta = 2\gamma$ and $\beta = \alpha + \delta, \delta = \gamma + \alpha$, so $\beta = 2\alpha + \gamma$. From $2\gamma = \beta = 2\alpha + \gamma$, we obtain $\gamma = 2\alpha$, so $\beta = 4\alpha$.

$$\because \frac{1}{2}(180° - \alpha) + 2\beta = 180°$$
$$\therefore 4\beta - \alpha = 180°$$
$$16\alpha - \alpha = 180°$$
$$\alpha = 12°, \quad \therefore \angle A = 12°.$$



4.Sol: It suffices to discuss the two cases indicated by the following figures.

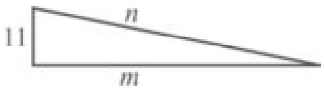


For case (a), since $\angle DAB + \angle ABC + \angle BCD + \angle CDA = 360°$, at least one of them is not less than $90°$. Assuming $\angle CDA \ge 90°$, then in ΔCDA, $\angle DCA + \angle CAD \le 90°$, so one of them is not greater than $45°$.

For case (b), since $\angle ADB + \angle ADC + \angle BDC = 360°$, one of the three angles is greater than $90°$, say $\angle ADB > 90°$, then $\angle DAB + \angle DBA$

$< 90°$, so one of $\angle DAB$ and $\angle DBA$ is less than $45°$.

5.Sol: From the given conditions we have



$$n^2 = m^2 + 11^2,$$

$$n^2 - m^2 = 11^2$$

$(n-m)(n+m) = 121 = 1 \cdot 121 = 11 \cdot 11$

therefore

$n - m = 1, n + m = 121$ or $n - m = 11, n + m = 11$

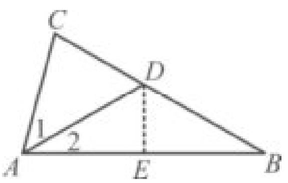
$\therefore n = 61, m = 60$. ($n = 11, m = 0$ is not acceptable).

Thus, the perimeter is $11 + 61 + 60 = 132$.

6.Sol: From D introduce $DE \perp AB$, intersecting AB at E.

When we fold up the plane that ΔCAD lies along the line AD, then C coincides with E, so

$AC = AE, \quad DE = CD = 1.5$ cm.



By applying Pythagora's Theorem to ΔBED,

$$BE = \sqrt{BD^2 - DE^2} = \sqrt{6.25 - 2.25} = 2(\text{cm})$$

Letting $AC = AE = x$ cm and applying Pythagora's Theorem to ΔABC leads the equation

$$(x+2)^2 = x^2 + 4^2$$

$$4x = 12, \quad \therefore x = 3$$

Thus $AC = 3$ cm.

7.Sol: Connect BE. Since ED is the perpendicular bisector of AB, $BE = AE$, so $\angle EBD = \angle EBA = \angle A = 30°, \angle CBE = 60° - 30° = 30°,$

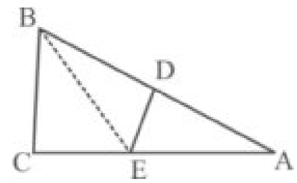
$$\therefore CE = \frac{1}{2} BE = DE = \frac{1}{2} AE = 2 \text{ cm.}$$

Now let $BC = x$ cm, then from Pythagora's Theorem,

$$(2x)^2 = x^2 + 6^2 \Rightarrow x^2 = 12$$

$$\Rightarrow x = \sqrt{12} = 2\sqrt{2}(\text{cm})$$

Thus, $BC = 2\sqrt{3}$ cm



8.Sol: Let $AB = BC = BC = CD = DA = a$. Let E be the midpoint of CD. Let the lines AD and BE intersect at F.

By symmetry, we have $DF = CB = a$. Since right triangles ABM and CBE are symmetric in the line BD, $\angle ABM = \angle CBE$.

It suffices to show $\angle NBE = \angle EBC$, and for this we only need to show $\angle NBF = \angle BFN$ since $\angle DFE = \angle EBC$.

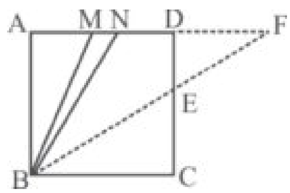
By assumption we have

$$AN = \frac{3}{4}a, \quad \therefore NB = \sqrt{\left(\frac{3}{4}a\right)^2 + a^2} = \frac{5}{4}a$$

On the other hand,

$$NF = \frac{1}{4}a + a = \frac{5}{4}a,$$

so, $NF = BN$, hence $\angle NBF = \angle BFN$.



9.Sol: From $AB \perp CQ$ and $BE \perp AC$

$$\angle ABE = \angle QCA$$

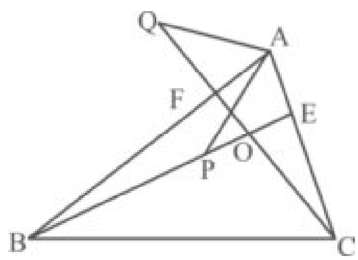
Since $AB = CQ$ and $BP = CA$

$\Delta ABP \cong \Delta QCA(S.A.S)$

$\therefore \angle BAP = \angle CQA$

$\therefore \angle QAP = \angle QAF + \angle \text{BAP}$

$= \angle QAF + \angle CQA = 180° - 90° = 90°$



10.Sol: $\because \angle DBC = \angle DCB = 30°$,

$\therefore DC \perp AC, DB \perp AB$. b

Extending AB and P such that $BP = NC$, then $\Delta DCN \cong \Delta DBP$ (S.S), therefore $DP = DN$.

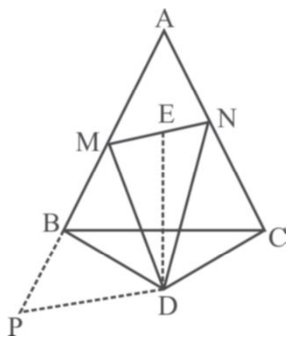
$\angle PDM = 60° = \angle MDN$ implies that

$\Delta PDM \cong \Delta MDN, (S.A.S)$

$\therefore PM = MN$

$\therefore MN = PM = BM + PM = BM + NC$

Thus, the perimeter of ΔAMN is 2.



Note: Here the congruence $\Delta PDM \cong \Delta MDN$ is obtained by rotating ΔDCN to the position of ΔDBP essentially.

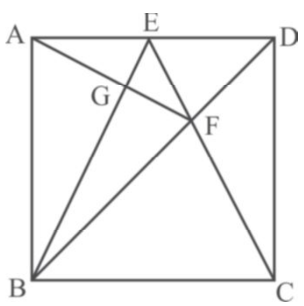
11.Sol: Let G be the point of intersection of AF and BE. It suffices to show

$\angle EAG = \angle ABG$

By symmetry we have

$\Delta ABE \cong \Delta DCE, \Delta ADF \cong \Delta CDF$

Therefore $\angle EAG = \angle DCF = \angle ABG$.



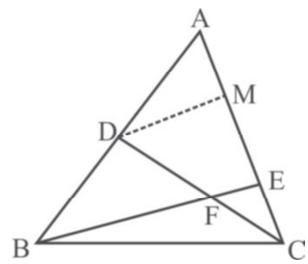
12.Sol: It is difficult to compare the lengths of EF and BE since they are on a same line. Here we can use a midline as a ruler to measure them.

Let M be the midpoint of AE. Connect DM. By applying the midpoint theorem to ΔABE and ΔCDM respectively, it follows that

$$DM = \frac{1}{2}BE,$$

$$EF = \frac{1}{2}DM$$

$\therefore EF = \frac{1}{4}BE$, i.e., $BE = 4EF$.

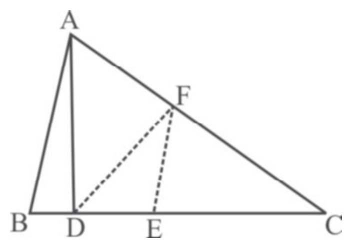


13.Sol: Let F be the midpoint of AC, connect EF, DF. By the midpoint theorem, $AB = 2EF$, it suffices to show $DE = EF$. Since DF is the median on hypotenuse AC of the right triangle ADC, $DF = FC = AF$, so $\angle CDF = \angle C$. Since $EF \parallel AB$,

$\angle CEF = \angle B = 2\angle C$,

$\therefore \angle DFE = \angle CEF - \angle CDF = \angle C$

$= \angle CDF$ hence $DE = EF$.



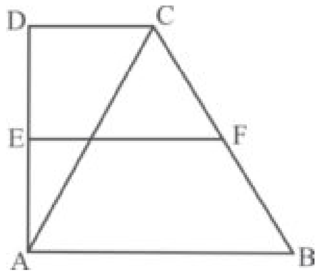
14.Sol: From the given conditions,

$\angle DAC = 30°, \therefore CD = \frac{1}{2}AC = \frac{1}{2}AB$.

By the midpoint theorem,

$$EF = \frac{1}{2}(CD + AB) = \frac{3}{4}AB,$$

$\therefore AB = a$.



15.Sol: From C introduce $CG \parallel AD$, intersecting the extension of BF at G.

$\because \angle EAF = \angle FCG,$

$\angle AEF = \angle FGC,$

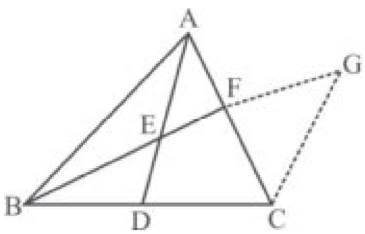
$\angle AFE = \angle GFC,$

$\therefore \Delta EAF \sim \Delta GCF(A.A.A).$

$$\therefore \frac{AF}{EF} = \frac{FC}{FG} = \frac{AF+FC}{EF+FG} = \frac{AC}{EG}.$$

By the midpoint theorem, $BE = EG$,

$\therefore EG = AC, AF = EF.$



16.Sol: If suffices to show $\frac{AB+AC}{AB} = \frac{AC}{BC}$. To prove it we construct corresponding similar triangles as follows.

Extending AB to D such that $BD = AC$. Extending BC to E such that $AC = AE$. Connect DE, AE.

Let $\angle A = \alpha, \angle B = 2\alpha, \angle C = 4\alpha$.

Then $7\alpha = 180°$

$\because \angle AEC = \angle ACE = 3\alpha$

$\angle CAE = \alpha = \angle CAB$,

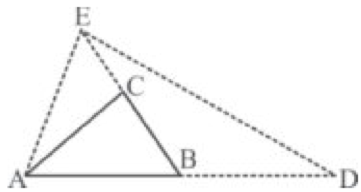
$\angle BAE = 2\alpha = \angle EBA.$

$\because \angle DBE = \angle BAE + \angle AEB = 5\alpha$

$$\therefore \angle EDA = \frac{1}{2}(180° - 5\alpha) = \alpha$$

$\therefore \Delta DAE \sim \Delta ABC(A.A.A).$

Thus, $\frac{AD}{AB} = \frac{AE}{BC}$, i.e., $\frac{AB+AC}{AB} = \frac{AC}{BC}$, as desired.



17.Sol: From E introduce $EN \parallel AD$, intersecting BC at N. Since $\frac{DN}{NC} = \frac{AE}{EC} = \frac{3}{4}, \frac{BD}{DC} = \frac{3}{2}$,

$$[ABE] = \frac{3}{7}[ABC] = \frac{3}{7}$$

$$\therefore [BEC] = \frac{4}{7}[ABC] = \frac{4}{7}$$

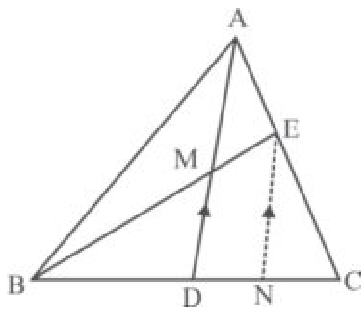
$\because BD : DN : NC = 21 : 6 : 8,$

$\therefore BN : NC = 27 : 8$ and

$BD : BN = 21 : 27 = 7 : 9,$

$$[BEN] = \frac{27}{35}[BEC] = \frac{27}{35} \cdot \frac{4}{7},$$

$$[BMD] = \left(\frac{7}{9}\right)^2 [BEN] = \frac{7^2 \cdot 27 \cdot 4}{9^2 \cdot 35 \cdot 7} = \frac{4}{15}.$$



18.Sol: $\frac{[CAP]}{[FAP]} = \frac{CP}{FP} = \frac{[CBP]}{[FBP]}$ yields

$$\frac{84+y}{40} = \frac{x+35}{30}, \qquad (1)$$

and $\frac{[CAP]}{[CDP]} = \frac{AP}{DP} = \frac{[BAP]}{[BDP]}$ yields

$$\frac{84+y}{x}=\frac{70}{35}=2, \qquad (2)$$

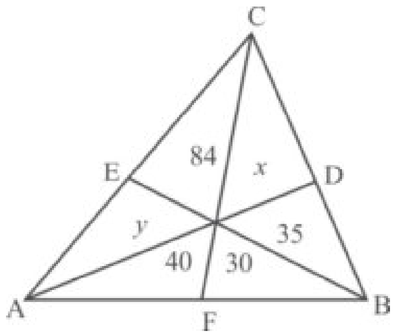
By $\frac{(1)}{(2)}$, it follows that $\frac{x}{40}=\frac{x+35}{60}$,

$\therefore 3x=2x+70$, i.e., $x=70$. Then by (2)

$y=140-84=56$.

Thus,

$[ABC]=84+56+40+30+35+70=315.$



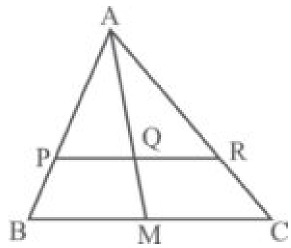
19.Sol: From that Q, M are the midpoints of PR and BC respectively.

$[APQ]=[ABQ]$. $[ABM]=[ACM]$

$$\therefore \frac{[APQ]}{[ABM]}=\frac{[ARQ]}{[ACM]},$$

$$\therefore \frac{AP\cdot AQ}{AB\cdot AM}=\frac{AQ\cdot AR}{AM\cdot AC},$$

i.e., $\frac{AP}{AB}=\frac{AR}{AC}$, $\therefore PQ \parallel BC$.



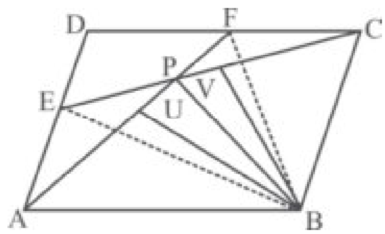
20.Sol: Connect BE, BF, make $BU\perp AF$ at U and $BV\perp CE$ at V. Then

$$[BAF]=[BCE]=\frac{1}{2}[ABCD]$$

Further, since $AF=CE$, we have

$BU=BV$, $\therefore \Delta BPU\cong \Delta BPV$,

$\therefore \angle BPA=\angle BPU=\angle BPV=\angle BPC.$



Synopticglance

MATRICES

Introduction

A matrix is a rectangular array of numbers, arranged in lines (rows) and columns. For instance, the left matrix has two lines and three columns, while the right matrix has three lines and two columns:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Formal Definition

A matrix is a table of m lines and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It is denotes by $A_{m \times n}$ matrices.

Properties

- If all elements are real, the matrix is called a real matrix.
- The size of a matrix is measured in the number of rows and columns the matrix has that is, if a matrix has m rows and n columns, it is said to be order $m \times n$. Matrices that have the same number of rows as columns are called square matrices.
- The elements of a matrix are specified by the line (row) and column they reside in.

  The numbers $a_{i,j}$ are called the elements of matrix A. i.e., $[a_{ij}]_{m \times n}$ or $(a_{ij})_{m \times n}$. This notation is especially convenient when the elements are related by some formula.
- The numbers $a_{j1}, a_{j2}, a_{j3}, \ldots a_{jm}$ form the j line of matrix A.
- A matrix of type m,n is denoted A(m,n) and it is the matrix with m lines and n columns.
- A matrix with only one line is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

- A matrix with exactly one column as a column vector

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

Some Special Matrix

Definition: If all the elements are zero, the matrix is is called a zero matrix or null matrix, denoted by $O_{m \times n}$.

Definition: A square matrix is the matrix that has an equal number of lines as columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Definition (principal diagonal): The principal diagonal is made of elements of the form a_{ii} of a square matrix $a_{11}, a_{22}, a_{33}, \ldots a_{nn}$.

Definition: Let $A = \left[a_{ij}\right]_{m \times n}$ be a square matrix.

(1) If $a_{ij} = 0$ for all i, j, then A is called a zero matrix.

(2) If $a_{ij} = 0$ for all $i < j$, then A is called a lower triangular matrix.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{12} & a_{22} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix}$$

Lower triangular matrix

(3) If $a_{ij} = 0$ for all $i > j$, then A is called a upper triangular matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & & & \vdots \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nm} \end{bmatrix}$$

Upper triangular matrix

Definition: Let $A = \left[a_{ij}\right]_{m\times n}$ be a square matrix.

If $a_{ij} = 0$ for all $i \neq j$, then A is called a diagonal matrix.

Definition: If A is diagonal matrix and

$a_{11} = a_{22} = \cdots = a_{nn} = m$, where m is any real number, then A is called a scalar matrix.

Definition: If A is a diagonal matrix and

$a_{11} = a_{22} = \cdots = a_{nn} = 1$, then A is called an identity matrix or a unit matrix, denoted by.

Example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Orthogonal Matrix

Any square matrix A of order n is said to be orthogonal, if $AA' = A'A = I_n$.

Idempotent Matrix

A square matrix A is called idempotent provided it satisfies the relation $A^2 = A$.

Involuntary Matrix

A square matrix such that $A^2 = I$ is called involuntary matrix.

Nilpotent matrix

A square matrix A is called a nilponent matrix if there exist a positive integer m such that $A^m = O$.

If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A.

Arithmetics of Matrices

Definition: Two matrices A and B are equal iff they are the same order and their corresponding elements are equal.

i.e. $\left[a_{ij}\right]_{m\times n} = \left[b_{ij}\right]_{m\times n} \Rightarrow a_{ij} = b_{ij}$ for all i, j

(1)Addition:

Definition: Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$.

Define A+B as the matrix $C = \left[C_{ij}\right]_{m\times n}$ of the same order such that $c_{ij} = a_{ij} + b_{ij}$ for all $i = 1, 2$, $\dots, m$ and $j = 1, 2, \dots, n$.

Definition: Let $A = \left[a_{ij}\right]_{m\times n}$. Then

$-A = \left[-a_{ij}\right]_{m\times n}$ and $A - B = A + (-B)$.

Properties of Matrix Addition

Theorem: Let A,B,C be matrices of the same order and O be the zero matrix of the same order. Then

(1) $A + B = B + A$

(2) $(A + B) + C = A + (B + C)$

(3) $A + (-A) = (-A) + A = O$

(4) $A + O = O + A$

(2)Multiplication

Definition (Matrix Multiplication):

Let $A = \left[a_{ik}\right]_{m\times n}$ and $B = \left[b_{kj}\right]_{n\times p}$.

Then the product AB is defined as then mxp matrix

$C = \left[c_{ij}\right]_{m\times p}$ where

$$c_{ij} = a_{i1}b_{ij} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$$

$$= \sum_{k=1}^{n} a_{ik}b_{kj}$$

i.e., $A = \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]_{m\times p}$

Note: In general, $AB \neq BA$ i.e. matrix multiplication is not commutative.

Properties of Matrix Multiplication

Theorem:

(1) $(AB)C = A(BC)$

(2) $A(B+C) = AB + AC$

(3) $(A+B)C = AC + BC$

(4) $AO = OA = O$

(5) $IA = AI = A$

(6) $k(AB) = (kA)B = A(kB)$

(7) $(AB)^T = B^T \cdot A^T$

IMPORTANT POINTS

- Since $AB \neq BA$; Hence, $A(B+C) \neq (B+C)A$ and $A(kB) \neq (kB)A$.
- $A^2 + kA = A(A+kI) = (A+kI)A$.
- $AB - AC = 0 \Rightarrow A(B-C) = 0$

  $\not\Rightarrow A = 0$ or B - C = 0

Definition(scalar Multiplication)

Let $A = \left[a_{ij}\right]_{m\times n}$, k is scalar. Then kA is the matrix $C = \left[C_{ij}\right]_{m\times n}$ defined by $c_{ij} = ka_{ij}, \forall i, j$

i.e,. $kA == \left[ka_{ij}\right]_{m\times n}$

Properties of scalar multiplication

Theorem: Let A, B be matrices of the same order and h,k be two scalars. Then

(1) $k(A+B) = kA + kB$

(2) $(k+h)A = kA + hA$

(3) $(hk)A = h(kA) = k(hA)$

Transpose of a Matrix

Definition: Let $A = \left[a_{ij}\right]_{m\times n}$. The transpose of A, denoted by A^T, or A', is defined by

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Properties of transpose

Theorem: Let A, B be two mxn matrices and k be a scalar, then

(1) $(A^T)^T = A$

(2) $(A+B)^T = A^T + B^T$

(3) $(kA)^T = k \cdot A^T$

(4) $(A \cdot B)^T = B^T \cdot A^T$

(1)Symmetric:

Definition: A square matrix A is called a symmetric matrix if $A^T = A$.

i.e, A symmetric matrix

$$\Leftrightarrow A^T = A \Leftrightarrow a_{ij} = a_{ji}, \forall i, j$$

Definition: A square matrix A is called a skew-symmtric matrix if $A^T = -A$.

i.e, A is skew-symmetric matrix

$$\Leftrightarrow A^T = -A \Leftrightarrow a_{ij} = -a_{ji}, \forall i, j$$

Properties of symmetric and skew-symmetric matrices

- If A is a symmetricmatrix, then
- $-A, kA, A^T, A^n, B^T AB$ are also symmetric marices, where $n \in N, k \in R$ and B is square matrix of order that of A
- If A is a skew-symmetric matrix, then

  (i) A^{2n} is a symmetric matrix for $n \in N$,

  (ii) A^{2n+1} is as a skew- symmetric matrix for $n \in$, N

  (iii) kA is also skew-symmetric matrix, where $k \in R$,

  (iv) $B^T AB$ is also skew-symmetric matrix where B is a square matrix of order that of A.
- If A,B are two symmetric matrices, then
- $A \pm B, AB + BA$ are also symmetric matrices,
- $AB - BA$ is a skew - symmetric matrix,
- AB is a symmetric matrix, when $AB = BA$.
- If A,B are two skew-symmetric matrices, then
- $A \pm B, AB - BA$ are skew-symmetric matrices,
- $AB + BA$ is a symmetric matrix.
- If A is a skew-symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.

Power of Matrices

Definition : For any square matrix A and any positive integer n, the symbol A^n denotes $\underbrace{A \cdot A \cdot A \cdots A}_{n \text{ factors}}$

IMPORTANT POINTS

- $(A+B)^2 = (A+B)(A+B)$
  $= AA + AB + BA + BB$
  $= A^2 + AB + BA + B^2$
- If $AB = BA$, then $(A+B)^2 = A^2 + 2AB + B^2$

Properties

Theorem of power of matrices:

(1) Let A be square matrix, then $(A^n)^T = (A^T)^n$.

(2) If $AB = BA$

(I) $(A+B)^n = A^n + C_1^n A^{n-1}B + C_2^n A^{n-2}B^2 + \ldots$
$+ C_{n-1}^n A^1 B^{n-1} + C_n^n A^{n-n}B^n$

(II) $(AB)^n = A^n B^n$

(III) $(A+I)^n = A^n + C_1^n A^{n-1} + C_2^n A^{n-2} + \ldots$
$+ C_{n-1}^n A^1 + C_n^n I$

Matrix Polynomial

If matrix A satisfies the polynomial

$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, then

$f(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n$.

Conjugate of a Matrix

A conjugate matrix is a matrix obtained from a given matrix A by taking the complex conjugate of each element of A.

Properties of a conjugate

- $(\overline{\overline{A}}) = A$
- $\overline{(A+B)} = \overline{A} + \overline{B}$
- $\overline{(\alpha A)} = \overline{\alpha}\overline{A}, \alpha$ being any number
- $\left(\overline{AB}\right) = \overline{A}\,\overline{B}$, A and B being conformble for multiplication

(1) Transpose Conjugate of a Matrix

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^θ. The conjugate of the transpose of A is the same as the transpose of the conjugate of A,

i.e., $\left(\overline{A'}\right) = \left(\overline{A}\right)' = A^\theta$.

If $A = [a_{ij}]_{m\times n}$ then $A^\theta = [a_{ji}]_{n\times m}$, where $b_{ji} = \overline{a}_{ij}$, i,e., the (j,i) th element of A^θ is equal to the conjugate of (i,j) th element of A.

Properties of Transpose Conjugate

- $(A^\theta)^\theta = A$
- $(A+B)^\theta = A^\theta + B^\theta$
- $(kA)^\theta = \overline{k}A^\theta$, k being any number
- $(AB)^\theta = B^\theta A^\theta$

IMPORTANT POINTS

Inverse of A Square Matrix

- If a,b,c are real numbers such that $ab = c$ and b is non-zero, then $a = \frac{c}{b} = cb^{-1}$ and b^{-1} is usually called the multiplicative inverse of b.
- If B, C are matrices $\frac{C}{B}$ is undefined.

Inverse Matrix

Definition: A square matrix A of order n is said to be non-singular or invertible if and only if there exists a square matrix B such that $AB = BA = I$.
The matrix B is called the multiplicative inverse of A, denoted by A^{-1}.

i.e., $AA^{-1} = A^{-1}A = I$

Definition: If a square matrix A has an inverse, A is said to be non- singular or invertible. Otherwise, it is called singular or non- invertible.

Theorem: The inverse of a non - singular matrix is unique.

Note:

(1) $I^{-1} = I$, so I is always non - singular.

(2) $OA = O \neq I$, so O is always singular.

(3) Since $AB = I$ implies $BA = I$.

Hence proof of either $AB = I$ or $BA = I$ is enough to assert that B is the inverse of A.

Theorem (Properties of Inverse): Let A, B be two non-singular matrices of the same order and be a scalar.

(1) $(A^{-1})^{-1} = A$

(2) A^T is a non-singular and $(A^T)^{-1} = (A^{-1})^T$

(3) A^n is a non-singular and $(A^n)^{-1} = (A^{-1})^n$

(4) λA is a non-singular and $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

(5) AB is a non-singular and $(AB)^{-1} = B^{-1}A^{-1}$

Unity matrix

A square matrix is said to be unity if $\overline{A}'A = I$ since $|\overline{A}'| = |A|$ and $|\overline{A}'A| = |\overline{A}'||A|$; therefore, if $\overline{A}'A = I$, we have $|\overline{A}'||A| = 1$.

Thus the determinant of unitary matrix is of unit modulus. For a matrix to be unitary it must be nonsingular.

System of Simultaneous Linear Equations

Consider the following system of n linear equations in n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$
$$\vdots \quad \vdots \qquad \vdots \quad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

The system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1}$$

or $AX = B$

Then $n \times n$ matrix A is called the coefficient matrix of the system of linear equations.

(1) Homogeneous and nonhomogeneous system of linear equations

A system of equations $AX = B$ is called a homogeneous system if $B = O$. Otherwise, it is called a nonhomogeneous system of equations.

(I) Solution of a System of Equations

Consider the sytem of equations $AX = B$. A set of values of the variables $x_1, x_2, \ldots x_n$ which simultaneously satisfies all the equations is called a solution of the system of equations.

(II) Consistent System

If the system of the equations has one or more solutions, then it is said to be consistent system of equations; otherwise it is an inconsistent system of equations.

(III) Solution of a nonhomogeneous system of linear equations

There are two methods of solving a nonhomogeneous system of simultaneous linear equations.

(A) Cramer's rule: Let us consider a system of equations

$$a_1x + b_1y + c_1z = d_1;$$
$$a_2x + b_2y + c_2z = d_2; \qquad \ldots\ldots. \quad (1)$$
$$a_3x + b_2y + c_3z = d_3;$$

Here $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

By Cramer's rule, we have,

$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}$ and $z = \frac{\Delta_3}{\Delta}$

IMPORTANT POINTS

Remarks

- $\Delta \neq 0$, then system will have unique finite solution, and so equations are consistent.

- $\Delta = 0$, and at least one of $\Delta_1, \Delta_2, \Delta_3$ be non zero, then the system has no solution i.e., equations are inconsistent.
- If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$ then equations will have infinite number of solutions, and at least one cofactor of Δ is non zero, i.e., equations are consistent.

(2)Matrix method

Consider the equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \quad \text{... (1)} \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

then, Eq. (1) is equivalent to the matrix equation

$$AX = D \quad \text{... (2)}$$

Multiplying both sides of Eq. (2) by the inverse matrix A^{-1}, we get $A^{-1}(AX) = A^{-1}D$ or

$$IX = A^{-1}D \quad [\because A^{-1}A = I]$$

$$X = A^{-1}D \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \text{... (3)}$$

where A_1, B_1, etc., are the cofactors of a_1, b_1, etc., in the determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} (\Delta \neq 0)$$

(i) If A is a nonsingular matrix, then the system of equations given by $AX = B$ has a unique solution is $X = A^{-1}B$.

(ii) If A is singular matrix, and $(\text{adj}A)D = 0$, then the system of the equations given by $AX = D$ is consistent with infinitely many solutions.

(iii) If A is a singular matrix and $(\text{adj}A)D \neq 0$, then the system of the equations given by $AX = D$ is inconsistent and has no solution.

Solution of Homogeneous system of linear equations

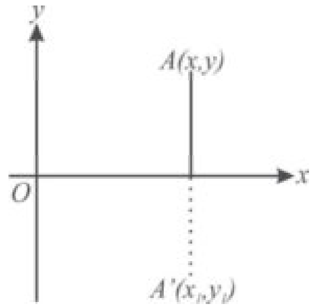
Let $AX = O$ be a homogeneous system of n linear equations with n unknowns. Now if A is nonsingular, then the system of equations will have a unique solution, i.e., trivial solution and if A is a singular, then the system of equations will have infinitely many solutions.

Matrices of Reflection and Rotation

(1)*Reflection matrix*

(I) Reflection in the x-axis

Let A be any point and A' be its image after reflection in the x-axis.



If the coordinates of A and A' are (x, y) and (x_1, y_1), respectively, then $x_1 = x$ and $y_1 = -y$

. These may be written as $\left.\begin{aligned} x_1 &= 1 \times x + 0 \times y \\ y_1 &= 0 \times x + (-1)y \end{aligned}\right\}$

Thus, system of equations in the matrix form will be

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of a point $A(x, y)$ in the x-axis.

Similarly, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ will describe the reflection of a point (x, y) in the y-axis.

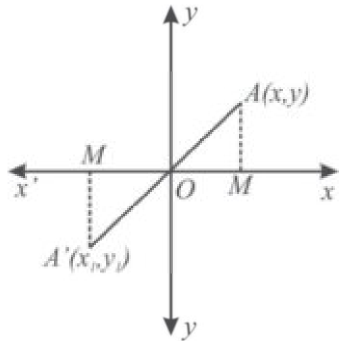
(II) Reflection through the origin

If $A'(x_1, y_1)$ is the image of $A(x, y)$ after reflection through the origin, then $\left.\begin{aligned} x_1 &= -x \\ y_1 &= -y \end{aligned}\right\}$

$$\Rightarrow x_1 = (-1)x + 0 \times y \text{ and}$$

$$y_1 = 0 \times x + (-1)y$$

Thus, the matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ describes the reflection of a point $A(x, y)$ through the origin.

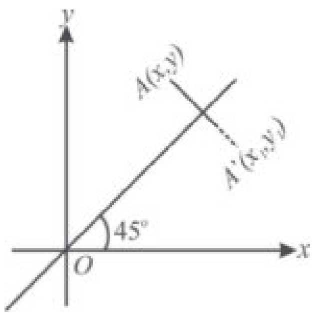


(III)Reflection in the line $\boldsymbol{y = x}$

In this case, $x_1 = 0 \times x + 1 \times y$,

$$y_1 = 1 \times x + 0 \times y$$

And the reflection matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.



(IV)Reflection in line $\boldsymbol{y = x \tan\theta}$

Considering the line $\boldsymbol{y = x \tan\theta}$ as shown in the fig, we have $x_1 = x\cos 2\theta + y\sin 2\theta$

($\because O$ is the midpoint of AA')

$$y_1 = x\sin 2\theta - y\cos 2\theta$$

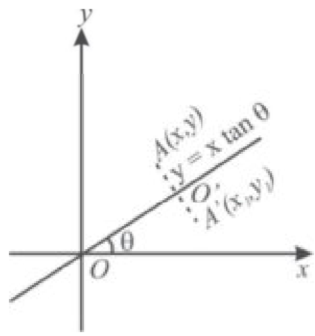
In matrix form, we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, the matrix

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

describes the reflection of a point (x, y) in the line $y = \tan\theta$.



IMPORTANT POINTS

- By putting $\theta = 0, \pi/2, \pi/4$ we can get the reflection matrices in the x-axis, y-axis, and the line $y = x$, respectively.

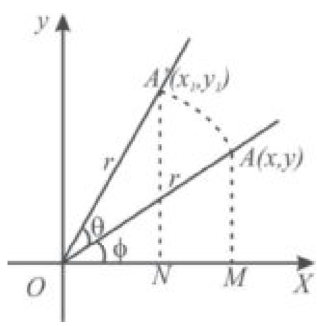
(V) Rotation through an angle $\boldsymbol{\theta}$

Let $A(x, y)$ be any point such that $OA = r$ and $\angle AOX = \phi$. Let OA rotate through an angle θ in the anticlockwise direction such that $A'(x_1, y_1)$ is the new position. Then $OA' = r$

(i) $y_1 = x\sin\theta + y\cos\theta$

In matrix form, we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



(2) Characteristic roots and characteristic vector of a square Matrix

Definition : Any nonzero vector, X, is said to be a characteristic vector of a matrix A, if there exists a number λ such that $AX = \lambda X$. And then λ is said to be a characteristic root of the matrix A corresponding to the characteristic vector X and vice versa. Characteristic roots (vectors) are also often called proper values, latent values or eigen values (vectors).

IMPORTANT POINTS

It will be useful to remember that

(i) A characteristic vector of a matrix cannot correspond two different characteristic roots.

(ii) A characteristic root of a matrix can correspond two different characteristic vectors. Thus, if

$$AX = \lambda_1 X, AX = \lambda_2 X, \lambda_2 \neq \lambda_1$$

$$\lambda_1 X = \lambda_2 X \Rightarrow (\lambda_1 - \lambda_2)X = O$$

But $X \neq O$ and $(\lambda_1 - \lambda_2) \neq 0$. And therefore $(\lambda_1 - \lambda_2)X \neq O$. Thus, we have a contradiction and as such we see the truth of statement (i).

But if $AX = \lambda X$, then also $A(kX) = \lambda(kX)$, so that kX is also a characteristic vector of A corresponding to the same characteristic root λ.

Thus, we have the truth of statement (ii).

(I) Echelon form of a matrix

A matix A is said to be in echelon form if

(A) Every row of A which has all its elements 0, occurs below row which has a nonzero element.

(B) The first non-zero element in each non-zero row is 1

(C) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

(II) Rank of a matrix

Let A be a matrix of order $m \times n$. If atleast one of its minors of order r is different from zero and all minors of order $(r+1)$ are zero, then the number r is called the rank of the matrix A and is denoted by $\rho(A)$.

IMPORTANT POINTS

- The rank of a zero matrix is zero and the rank of an identity matrix of order n is n.
- The rank of a non-singular matix of order n is n.
- The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Trace of a Matrix

(1) Diagonal Matrix: The elements of a square matrix A for which $i = j$, *i.e*, $a_{11}, a_{22}, a_{33}, \ldots a_m$ are called diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix A.

(2) Trace of a Matrix : The sum of diagonal eleents of a square matrix. A is called the trace of matrix A, which is denoted by trA.

$$trA = \sum_{i=1}^{n} a_n = a_{11} + a_{22} + \ldots a_m$$

Properties of trace of a matrix

Let $A = \left[a_{ij}\right]_{m\times n}$ and $B = \left[b_{ij}\right]_{m\times n}$ and λ be a scalar

- $tr(\lambda A) = \lambda tr(A)$
- $tr(A - B) = tr(A) - tr(B)$
- $tr(AB) = tr(BA)$
- $tr(A) = tr(A')$ or $tr(A^T)$
- $tr(I_m) = m$
- $tr(0) = 0$
- $tr(AB) \neq trA\, trB$

Exercise

1. If $A = \begin{bmatrix} x & y & z \end{bmatrix}, B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ and

$C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then ABC is

(a) a 1×1 matrix (b) not obtained

(c) a 3×3 matrix (d) none of these

2. If $B = \begin{bmatrix} 2+i & 3 & -1/2 \\ 3.5 & -1 & 2 \\ \sqrt{3} & 5 & 5/7 \end{bmatrix}$ and

$C = \begin{bmatrix} 1+x & x^3 & 3 \\ \cos x & \sin x+2 & \tan x \end{bmatrix}$ are two matrices. Then, consider the following statements

I. B has 3 rows and 3 columns.

II. C has 2 rows and 3 columns.

Choose the correct option.

(a) Only I is false

(b) Both I and II are true

(c) Both I and II are false

(d) Only II is false

3. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal, if they are, of the same order for all i and j, and

(a) $a_{ij} + b_{ij} = 0$ (b) $a_{ij} = -b_{ij}$

(c) $a_{ij} = b_{ji}$ (d) $a_{ij} = b_{ij}$

4. If D_1 and D_2 are two 3×3 diagonal matrices then

(a) D_1D_2 is a diagonal matrix

(b) $D_1 + D_2$ is a diagonal matrix

(c) $D_1^2 + D_2^2$ is a diagonal matrix

(d) $(a), (b), (c)$ are correct

5. If A and B are square matrix of same order, then

(a) $AB = BA$ (b) $A+B = A-B$

(c) $A-B = B-A$ (d) $A+B = B+A$

6. The number of diagonal matrix A of order n for which $A^3 = A$ is

(a) 2^n (b) 1 (c) 0 (d) 3^n

7. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\det(A^n - 1) = 1 - \lambda^n, n \in N$.

Then, the value of λ, is

(a) 0 (b) 1 (c) 2 (d) 3

8. If $abc = p$ and $A = \begin{bmatrix} a & b & c \\ c & a & a \\ b & c & a \end{bmatrix}$, such that a, b, c are the roots of the equation $x^3 \neq x^2 - p = 0$ if and only if A is

(a) Idempotent (b) Orthogonal

(c) Nilpotent (d) None of these

9. If P is an orthogonal matrix and $Q=PAP^T$ and $x=P^TQ^{1000}P$, then x^{-1} is, where A is involuntary matrix.

(a) I (b) A

(c) A^{100} (d) A^{1000}

10. Let $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A+I)^{50} - 50A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then the value of $a+b+c+d$ is

(a) 1 (b) 2

(c) 4 (d) None of these

11. If $A = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 1 & -2 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{bmatrix}$ and $2A + 3B - 5C = 0$ then C

(a) $\begin{bmatrix} 2 & 1 & \frac{6}{5} & \frac{7}{5} \\ 1 & \frac{7}{5} & 2 & \frac{3}{5} \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 1 & \frac{6}{5} & \frac{7}{5} \\ 1 & -\frac{7}{5} & 2 & \frac{3}{5} \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 1 & \frac{6}{5} & \frac{7}{5} \\ 1 & \frac{7}{5} & 2 & \frac{3}{5} \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & \frac{6}{5} & \frac{7}{5} \\ 1 & -\frac{7}{5} & 2 & \frac{3}{5} \end{bmatrix}$

12. If A and B are square matrices of order n, then $A - \lambda I$ and $B - \lambda I$ commute for every scalar λ, only if

(a) $AB = BA$ (b) $A = -B$

(c) $AB + BA = 0$ (d) None of these

13. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ then $AA^T =$

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 0 & -2 \\ 0 & 6 & 0 \\ -2 & 6 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

14. If three matrices A, B and C satisfy associative law, then

(a) $A(BC) = (AB)C$ (b) $A(BC) = (AC)B$

(c) $A + BC = AB + C$ (d) $A - BC = AC - B$

15. If A is square matrix of order 2, then $adj(adjA) =$

(a) A (b) I

(c) $|A| I$ (d) None of these

16. If A is a nilpotent matrix of index 2, then for any positive integer n, $A(1+A)^n$ is equal to

(a) A (b) A^n (c) A^{-1} (d) I_n

17. The addition to the elements of i^{th} column, the corresponding elements of j^{th} column multiplied by $k\left(k \neq 0\right)$ is denoted by

(a) $C_i \to C_j + kC_i$ (b) $C_i \to C_i - kC_j$

(c) $C_i \to C_i + kC_j$ (d) $C_i \to C_j - kC_i$

18. If $A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$ and $f(t) = t^2 - 3t + 7$,then

$f(A) + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix}$ is equal to

(a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

19. Which of the following statements is false:

(a) if $|A| = 0$, then $|adjA| = 0$

(b) adjoint of a diagonal matrix of order 3×3 is a diagonal matrix

(c) product of two upper triangular matrices is an upper triangular matrix

(d) $adj(AB) = adj(A)adj(B)$

20. If the system of equations

$ax + y + z = 0, x + by + z = 0, x + y + cz = 0$, $(a, b, c \neq 1)$

has non-trivial solution (non-zero solution), then

$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} =$

(a) 1 (b) -1 (c) 0 (d) none

21. $A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix}$, then the conjugate of A is

(a) $\begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$

(b) $\begin{bmatrix} 1+i & 1-i & 4 \\ -i & 3-2i & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 2+3i & 1-i & 4 \\ -i & 3-2i & 1 \end{bmatrix}$

(d) none

22. If the system of equations

$x + y + z = 6, x + 2y + \lambda z = 0, x + 2y + 3z = 10$

has no solution, then $\lambda =$

(a) 2 (b) 3 (c) 4 (d) $\frac{5}{2}$

23. If $A + B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $A - 2B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, then

$A =$

(a) $\frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ (d) none of these

24. The matrix 'X' in the equation $AX = B$, such that $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is given by

(a) $\begin{bmatrix} 0 & -1 \\ -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

25. The transformation 'orthogonal projection on X-axis' is given by the matrix

(a) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

26. The number of all possible matrices of order 3×3 with each entry 0 or 1 is
(a) 18 (b) 81 (c) 27 (d) 512

27. Let p be a non-singular matrix, and $I+p+p^2+...+p^n=0$ then $p^{-1}=$

(a) p^{n-1} (b) p^n (c) p^{n-2} (d) p^{n-3}

28. The values of m for which the system of equations $3x+my=m$ and $2x-5y=20$ has a solution satisfying the conditions $x>0, y>0$ are given by the set.

(a) $\{m \backslash m<-13/2\}$

(b) $\{m \backslash m>17/2\}$

(c) $\{m \backslash m<-13/2 \ or \ m>17/2\}$

(d) $\{m>30 \ or \ m<-15/2\}$

29. For a given $A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ which of the following statements holds good ?

(a) $A=A^{-1}, \forall \theta \in R$

(b) A is symmetric, for $\theta=(2n+1)\frac{\pi}{2}, n\in z$

(c) A is an orthogonal matrix, for $\theta \in R$

(d) A is Skew Symmetric, for $\theta=n\pi, n\in z$

30. $A=\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$ then $A^{26}=$

(a) I (b) -I (c) A (d) -A

31. If $A=\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ and $C=\begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$, then $3A-C$ is

(a) $\begin{bmatrix} 2 & 7 \\ 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & 2 \\ 6 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 8 & 5 \\ 6 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$

ANSWER KEY

1. a	2. b	3. d	4. d	5. d
6. d	7. c	8. b	9. a	10. b
11. d	12. a	13. a	14. a	15. a
16. a	17. c	18. b	19. d	20. a
21. a	22. d	23. c	24. d	25. c
26. d	27. b	28. d	29. c	30. b
31. d				

HINTS & SOLUTIONS

1.Sol: A is of order 1×3, B is of order 3×3, therefore, AB is of order 1×3 and since C is of order 3×1, therefore, $A(BC)=(AB)C$ is order of 1×1.

2.Sol: Conceptual

3.Sol: Two matrices $A=[a_{ij}]$ and $B=[B_{ij}]$ are said to be equal, if
(a) they are of the same order
(b) each element A is equal to the corresponding element of B, i.e., $a_{ij}=b_{ij}$ for all i and j

4.Sol: Conceptual

5.Sol: Matrix addition is commutative.

6.Sol: We know, if a diagonal matrix

$$A=\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_1 \end{bmatrix}$$

Satisfies $A^3=A$, then it follows that $A^3=A$, then it follows that $a_j^3=a_j$ for $j=1,2,3$.

$\Rightarrow a_j=0,\pm 1$, for all $j=\{1,2,3...\}$

$\therefore$ Each diagonal elements can be selected in three ways. Hence, the number of different matrices is 3^n.

7.Sol: Given $A=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Suppose $n=1$

$\Rightarrow |A-I|=1-\lambda$

$\Rightarrow \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}=1-\lambda$

$-1 = 1 - \lambda$

$\lambda = 2$

equating corresponding elements, we get $x = 0$.

8.Sol: Here

$$AA^T = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2+b^2+c^2 & ab+ac+bc & ab+bc+ca \\ ca+ab+bc & a^2+b^2+c^2 & cb+ba+ac \\ ab+bc+ac & bc+ac+ab & a^2+b^2+c^2 \end{bmatrix}$$

Given a, b, c i.e., $a+b+c = \pm 1$ and $ab+bc+ca = 0$, We have $a^2+b^2+c^2 = 1$

$$\therefore \quad AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= I$

Hence A is orthogonal.

9.Sol: Given $Q = PAP^T$

$X = P^T Q^{1000} P$

$\Rightarrow X = P^T (PAP^T)^{1000} P$

$= P^T (PAP^T)(PAP^T)^{999} P$

$= IAP^T (PAP^T)(PAP^T)^{998} P$

$= IAIAP^T (PAP^T)^{998} P$

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$A^{1000} = I\ (Involuntary)$

10.Sol: Given $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \quad A^n = 0, \forall n \geq 2$

Now $\quad (A+I)^{50} = I + 50A \quad$ {Since, for $n \geq 2, A^n = 0$ }

$\Rightarrow \quad (A+I)^{50} - 50A = I$

i.e., $\quad I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad$ {given}

$\therefore \quad a = d = 1 \quad b = c = 0$

Hence $\quad a+b+c+d = 2$

11.Sol: Given $2A + 3B - 5C = 0$

$\Rightarrow \quad 2A + 3B = 5C$

$$\text{Now } 5C = 2\begin{bmatrix} 2 & 1 & 3 & -1 \\ 1 & -2 & 2 & -3 \end{bmatrix} + 3\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 6 & -2 \\ 2 & -4 & 4 & -6 \end{bmatrix} + \begin{bmatrix} 6 & 3 & 0 & 9 \\ 3 & -3 & 6 & 9 \end{bmatrix}$$

$$5C = \begin{bmatrix} 10 & 5 & 6 & 7 \\ 5 & -7 & 10 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 & \frac{6}{5} & \frac{7}{5} \\ 1 & -\frac{7}{5} & 2 & \frac{3}{5} \end{bmatrix}$$

12.Sol: Given $(A - \lambda I)$ and $(B - \lambda I)$ communtative for every scalar λ.

i.e., $\quad (A-\lambda I)(B-\lambda I) = (B-\lambda I)(A-\lambda I)$

$\Rightarrow AB - \lambda(A+B) + \lambda^2 I^2 = BA - \lambda(B+A)I + \lambda^2 I^2$

$\Rightarrow \quad AB = BA$

13.Sol: $AA^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

14.Sol: The associative law: For any three matrices A,B and C, we have $(AB)C = A(BC)$, whenever both sides of equalities are defined.

15.Sol: $\because \quad adj(adjA) = |A|^{n-1}$, $\therefore$ for $n = 2$, we have $adj(adjA) = A$

Alternatively, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$adjA = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and hence $adj(adjA)$

$$= \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = A$$

16.Sol: $A^2 = 0.A^3 = A^4 = ... = A^n = 0$

$$A(I+A)^n = A(I+nA) = A + nA^2 = A$$

17.Sol: The addition to the elements of i^{th} column, the corresponding elements of j^{th} column multiplied by k is denoted by $C_i \to C_i - kC_j$

18.Sol: Given $A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$

$$f(A) = A^2 - 3A + 7I$$

$$= \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} - \begin{bmatrix} 3 & -6 \\ 12 & 15 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -6 \\ 12 & 9 \end{bmatrix}$$

Now $f(A) + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

19.Sol: We have $adj(AB) = adj(B)adj(A)$

but not $adj(AB) = adj(A)adj(B)$

20.Sol: Given system of equations has non-trivial solution

$\Rightarrow$ coefficient matrix is singular $\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$$\Rightarrow a(bc-1) - 1(c-1) + 1(1-b) = 0$$

$$\Rightarrow abc - a - c + 1 - b = 0$$

$$\Rightarrow abc = a + b + c - 2$$

$$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$$

$$= \frac{(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b)}{(1-a)(1-b)(1-c)}$$

$$\Rightarrow \frac{3 - 2(a+b+c)bc + ca + ab}{1 - (a+b+c) + ab + bc + ca - (a+b+c-2)} = 1$$

21.Sol: $\overline{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$

22.Sol: det $A = 0$

23.Sol: Given $A + B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$; $A - 2B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

$$\Rightarrow 2A + 2B = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}; \quad (1)$$

$$A - 2B = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad (2)$$

adding (1) and (2), we get

$$3A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

24.Sol: Given $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow |A| = 1 \neq 0$$

$\therefore$ A^{-1} exist, Hence $AX = B$

$$\Rightarrow X = A^{-1}B$$

Now $X = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$$

25.Sol: Under the given transformation, a point (x,y), is transformed to $(x,0)$ on the x-axis.

Now $x = 1, x + 0 = y$ and $0 = 0x + y$, therefore, the required matrix of the transformation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

26.Sol: In a 3×3 matrix, there are 9 elements and it is given that each element can take two values.

So, the total number of matrices having 0 and 1 as their elements is $2^9 = 512$

27.Sol: We know that $p^{-1}p = I$

$$\Rightarrow 1 + p + p^2 + p^3 + .. + p^n = 0$$

multiplying on both sides by p^{-1}

i.e., $p^{-1}\left(1 + p + p^2 + p^3 + ... + p^n\right) = p^{-1} \cdot 0$

$$\Rightarrow p^{-1} + \left(p^{-1}p\right) + \left(p^{-1}p \cdot p\right) + ... + \left(p^{-1}pp^{n-1}\right) = 0$$

$$p^{-1} + I + I \cdot p + I \cdot p^2 + ... + \left(I \cdot p^{n-1}\right) = 0$$

Therefore,

$$p^{-1} = -\left(I + p + p^2 + ... + p^{n-1}\right)$$

$$= -\left(-p^n\right) = p^n$$

28.Sol: $\Delta = \begin{vmatrix} 3 & m \\ 20 & -5 \end{vmatrix} = -15 - 2m$

$$\Delta_1 = \begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix} = -25m$$

$$\Delta_2 = \begin{vmatrix} 3 & m \\ 2 & 20 \end{vmatrix} = 60 - 2m$$

If $\Delta = 0, m = \dfrac{-15}{2}$ and system of equations is inconsistent.

By cramers rule $x > 0, y > 0 \Rightarrow m > 30$ or

$$m < -\frac{15}{2}$$

29.Sol: $A\,A^T = I$

30.Sol: $A^{26} = i^{26}I = -I$

31.Sol: Given $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Now $3A - C = 3\begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$$

Previous years JEE MAIN Questions

TRIGONOMETRY

[ONLINE QUESTIONS]

1. If m and M are the minimum and the maximum values of $4+\frac{1}{2}\sin^2 2x-2\cos^4 x, x\in R,$ then $M-m$ is equal to : **[2016]**

(a) $\frac{9}{4}$ (b) $\frac{15}{4}$ (c) $\frac{7}{4}$ (d) $\frac{1}{4}$

2. The number of $x\in[0,2\pi]$ for which $\left|\sqrt{2\sin^4 x+18\cos^2 x}-\sqrt{2\cos^4 x+18\sin^2 x}\right|=1$ is **[2016]**

(a) 2 (b) 6 (c) 4 (d) 8

3. If $\cos\alpha+\cos\beta=\frac{3}{2}$ and $\sin\alpha+\sin\beta=\frac{1}{2}$ and θ is the arithmetic mean of α and β, then $\sin 2\theta+\cos 2\theta$ is equal to : **[2015]**

(a) $\frac{3}{5}$ (b) $\frac{7}{5}$ (c) $\frac{4}{5}$ (d) $\frac{8}{5}$

4. If $2\cos\theta+\sin\theta=1\left(\theta\neq\frac{\pi}{2}\right)$, $7\cos\theta+6\sin\theta$ is equal to : **[2014]**

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{11}{2}$ (d) $\frac{46}{5}$

5. If $\operatorname{cosec}\theta=\frac{p+q}{p-q}(p\neq q\neq 0)$, then $\left|\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right|$ is equal to: **[2014]**

(a) $\sqrt{\frac{p}{q}}$ (b) $\sqrt{\frac{q}{p}}$ (c) $\sqrt{pq}$ (d) pq

6. The number of values of α in $[0,2\pi]$ for which $2\sin^3\alpha-7\sin^2\alpha+7\sin\alpha=2$, is : **[2014]**

(a) 9 (b) 4 (c) 3 (d) 1

7. Let $A=\{\theta:\sin(\theta)=\tan(\theta)\}$ and $B=\{\theta:\cos(\theta)=1\}$ be two sets. Then **[2013]**

(a) $A=B$
(b) $A\not\subset B$
(c) $B\not\subset A$
(d) $A\subset B$ and $B-A\neq\phi$

8. The number of solutions of the equation $\sin 2x-2\cos x+4\sin x=4$ in the interval $[0,5\pi]$ is: **[2013]**

(a) 3 (b) 5 (c) 4 (d) 6

9. **Statement -1:** The number of common solutions of the trigonometric equations $2\sin^2\theta-\cos 2\theta=0$ and $2\cos^2\theta-3\sin\theta=0$ in the interval $[0, 2\pi]$ is two.

Statement-2: The number of solutions of the equation, $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0,\pi]$ is two. **[2013]**

(a) Statement-1 is true; Statement-2 is true; Statement-2 is the correct explanation for Statement-1.
(b) Statement-1 is true; Statement-2 is true; Statement- 2 is not the correct explanation for Statement-1.
(c) Statement-1 is false; Statement-2 is true.
(d) Statement-1 is true; Statement-2 is false.

ANSWER KEY

1. a	**2.** d	**3.** a	**4.** b	**5.** b
6. c	**7.** b	**8.** a	**9.** b	

HINTS & SOLUTIONS

1.Sol: Given that

$$4+\frac{1}{2}\sin^2 2x - 2\cos^4 x$$

$$=4+\frac{1}{2}\sin^2 2x - \frac{1}{2}(1+\cos 2x)^2$$

$$=-\left(\cos^2 2x+\cos 2x-4\right)=\frac{17}{4}-\left(\cos 2x+\frac{1}{2}\right)^2$$

$\therefore$ Maximum value of the given expression is when $\cos 2x+\frac{1}{2}=0$

$\therefore$ Maximum value is $M=\frac{17}{4}$ and for minimum value, we have

$$-1\le\cos 2x\le 1$$

$$\Rightarrow \frac{-1}{2}\le\left(\cos 2x+\frac{1}{2}\right)\le\frac{3}{2}$$

$$\Rightarrow \frac{-9}{4}\le-\left(\cos 2x+\frac{1}{2}\right)^2\le\frac{-1}{4}$$

Now, $\frac{17}{4}-\left(\cos 2x+\frac{1}{2}\right)^2\ge\frac{17}{4}-\frac{9}{4}=\frac{8}{4}=2$

$\therefore$ minimum value is $m=2$

$$\therefore \quad m-m=\frac{17}{4}-2=\frac{9}{4}$$

2.Sol: $2\sin^4 x+18\cos^2 x=1+2\cos^4 x+18\sin^2 x$

$$+2\sqrt{2\cos^4 x+18\sin^2 x}$$

$$\Rightarrow 2\left(2\sin^2 x-\cos^2 x\right)+18\left(\cos^2 x-\sin^2 x\right)$$

$$=1+2\sqrt{2\cos^2 x+18\sin^2 x}$$

$$\Rightarrow 16\left(\cos^2 x-\sin^2 x\right)=1+2\sqrt{2\cos^4 x+18\sin^2 x}$$

$$\Rightarrow 16\cos 2x-1=2\sqrt{2\left(\frac{1+\cos 2x}{2}\right)^2+9(1-\cos 2x)}$$

$$\Rightarrow 256\cos^2 2x+1-32\cos 2x$$

$$=4\left\{\frac{1+2\cos 2x+\cos^2 x}{2}+9(1-\cos 2x)\right\}$$

$$=254\cos^2 2x=37$$

$$\Rightarrow \cos^2 2x=\frac{37}{254} \Rightarrow \cos 2x=\pm\sqrt{\frac{37}{254}} \quad [-1,1]$$

$\therefore$ It has clearly 8 solutions.

3.Sol: Given, $\cos\alpha+\cos\beta=\frac{3}{2}$ and

$$\sin\alpha+\sin\beta=\frac{1}{2}$$

$$\Rightarrow 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=\frac{3}{2} \qquad (1)$$

and $2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=\frac{1}{2} \qquad (2)$

from (1) and (2), we get

$$\tan\left(\frac{\alpha+\beta}{2}\right)=\frac{1}{3}$$

$$\Rightarrow \tan\theta=\frac{1}{3} \quad \{\because \alpha,\theta,\beta \text{ are in } A.P\}$$

Now, $\sin 2\theta+\cos 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}+\frac{1-\tan^2\theta}{1+\tan^2\theta}$

$$=\frac{\frac{2}{3}}{1+\frac{1}{9}}+\frac{1-\frac{1}{9}}{1+\frac{1}{9}} \quad =\frac{6}{2}+\frac{8}{10} \quad =\frac{7}{5}$$

4.Sol: Given that $2\cos\theta+\sin\theta=1$

$\Rightarrow \quad 2\cos\theta=1-\sin\theta$

$\Rightarrow \quad 4\cos^2\theta=1+\sin^2\theta-2\sin\theta$

$\Rightarrow \quad (\sin\theta-1)(5\sin\theta+3)=0 \Rightarrow \quad \sin\theta=\frac{-3}{5}$

$\therefore \quad \cos\theta=\frac{4}{5}$

Now, $7\left(\frac{4}{5}\right)-6\left(\frac{3}{5}\right)=\frac{10}{5}=2$

5.Sol: Given that $\operatorname{cosec}\theta=\frac{p+q}{p-q} \Rightarrow \quad \sin\theta=\frac{p-q}{p+q}$

i.e., $\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{p+q}{p-q} \quad \Rightarrow \quad \frac{1+\tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}=\frac{p+q}{p-q}$

Applying componendo and dividendo, we get

$$\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)^2=\frac{2p}{2q} \Rightarrow \left|\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right|=\sqrt{\frac{p}{q}}$$

i.e., $\left|\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right|=\sqrt{\frac{p}{q}} \Rightarrow \left|\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right|=\sqrt{\frac{q}{p}}$

6.Sol: Given that $2\sin^3\alpha-7\sin^2\alpha+7\sin\alpha-2=0$

observe the sum of coefficients is 0.

That is, $\sin\alpha=1$ is one solution.

Now,

$$2\sin^3\alpha-2\sin^2\alpha-5\sin^2\alpha+5\sin\alpha+2\sin\alpha-2=0$$

$\Rightarrow \quad (\sin\alpha-1)\left[2\sin^2\alpha-5\sin\alpha+2\right]=0$

$\therefore \quad \sin\alpha=1 \quad or \quad \sin\alpha=\frac{1}{2}$

$\therefore \quad \alpha=\frac{\pi}{2},\frac{\pi}{3},\frac{2\pi}{3}$

$\therefore \quad$ It has 3 solutions.

7.Sol: Let $A=\{\theta:\sin\theta=\tan\theta\}$

and $B=\{\theta:\cos(\theta)=1\}$

Now, $\quad A=\left\{\theta:\sin\theta=\frac{\sin\theta}{\cos\theta}\right\}$

$=\{\theta:\sin\theta(\cos\theta-1)=0\}$

$=\{\theta=0,\pi,2\pi,3\pi,\ldots\}$

For $B:\cos\theta=1 \Rightarrow \theta=\pi,2\pi,4\pi,\ldots$

This shows that A is not contained in $B.$ i.e., $A\not\subset B.$ but $B\subset A.$

8.Sol: $\sin 2x-2\cos x+4\sin x=4$

$\Rightarrow 2\sin x\cdot\cos x-2\cos x+4\sin x-4=0$

$\Rightarrow(\sin x-1)(\cos x+2)=0$

$\because \quad \cos x-2\neq 0, \qquad \therefore \quad \sin x=1$

$\therefore \quad x=\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2}$

9.Sol: $2\sin^2\theta-\cos 2\theta=0$

$\Rightarrow 2\sin^2\theta-\left(1-2\sin^2\theta\right)=0$

$\Rightarrow 4\sin^2\theta=1 \Rightarrow \sin\theta=\pm\frac{1}{2}$

$\therefore \quad \theta=\frac{\pi}{6},\frac{3\pi}{6},\frac{5\pi}{4},\frac{7\pi}{4},\theta\in[0,2\pi]$

$\therefore \quad \theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$

Now, $\quad 2\cos^2\theta-3\sin\theta=0$

$\Rightarrow 2\left(1-\sin^2\theta\right)-3\sin\theta=0$

$\Rightarrow -2\sin^2\theta-3\sin\theta+2=0$

$\Rightarrow \sin\theta(2\sin\theta-1)+2(2\sin\theta-1)=0$

$\Rightarrow \quad \sin\theta=\frac{1}{2},-2$

But $\sin\theta=-2,$ is not possible

$\therefore \quad \sin\theta=\frac{1}{2},-2 \quad \Rightarrow \theta=\frac{\pi}{6},\frac{5\pi}{6}$

Hence, there are two common solution, there each of the statement-1 and 2 are true but statement-2 is not a correct explanation for statement-1.